Front propagation for discrete periodic monostable equations*

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Abstract. This paper deals with the front propagation for discrete periodic monostable equations. We show that there is a minimal wave speed such that a pulsating traveling front solution exists if and only if the wave speed is above this minimal speed. Moreover, in comparing with the continuous case, we prove the convergence of discretized minimal wave speeds to the continuous minimal wave speed.

Keywords: front propagation, discrete periodic monostable equation, minimal wave speed, pulsating traveling front

1 Introduction

Front propagation occurs in many applied fields such as chemical kinetic, combustion theory, biological invasions, transport in porous media, etc. There have been many studies on traveling fronts in reaction-diffusion equations, since the pioneer works of Fisher [9] and Kolmogorov, Petrovsky and Piskunov [15] in 1937. Most of these works are concerned with traveling fronts propagating in homogeneous media. But, in many natural environments, for example, noise effects in biology and inhomogeneous porous media in transport theory, heterogeneities are often present. Therefore, it is very important to understand how these heterogeneities influence the properties of front propagation.

In this paper, we focus on the case of periodic environments. The typical example of a *continuous periodic* reaction-diffusion equation is in the form

$$u_t = \nabla \cdot (A(x)\nabla u) + g(x,u),\ x \in \mathbb{R}^N, \tag{1.1}$$

where the diffusion matrix A and the reaction term g are periodic in x. The question of propagation speed in periodic media was first studied by Gärtner and Freidlin [11] in 1979.

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See, also, the papers by Freidlin [10], Shigesada, Kawasaki and Teramoto [17], and Hudson and Zinner [13]. The existence of periodic traveling waves for the bistable reaction-diffusion equation with periodic coefficients was established in a series of papers by Xin [19, 21, 22] (see also the survey paper [23]). For more recent works, we refer the readers to [1, 2, 3, 4, 24] and the references cited therein.

In this paper, we study the following *periodic discrete* problem (P):

$$u_j'(t) = d_{j+1}u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1}+d_j)u_j(t) + f(j, u_j(t)),\ t \in \mathbb{R},\ j \in \mathbb{Z}, \tag{1.2}$$
$$u_j(t + N/c) = u_{j-N}(t),\ t \in \mathbb{R},\ j \in \mathbb{Z}, \tag{1.3}$$
$$u_j(t) \to 1 \text{ as } j \to -\infty, \quad u_j(t) \to 0 \text{ as } j \to +\infty, \quad \text{locally in } t \in \mathbb{R}, \tag{1.4}$$

where $d_j = d_{j-N}$ for all $j \in \mathbb{Z}$, N is a positive integer, c is a (nonzero) unknown constant (the wave speed), the function $f : \mathbb{Z} \times [0,1] \to \mathbb{R}$, $(j,s) \mapsto f(j,s)$ is of class C^1 in s for each $j \in \mathbb{Z}$ and it satisfies

$$\begin{cases} \forall\ j \in \mathbb{Z}, \quad f(j,0) = f(j,1) = 0, \\ \forall\ (j,s) \in \mathbb{Z} \times [0,1], \quad f(j,s) = f(j-N,s), \\ \forall\ j \in \mathbb{Z}, \quad f_s'(j,0) := \partial f/\partial s(j,0) > 0, \\ \forall\ (j,s) \in \mathbb{Z} \times (0,1), \quad 0 < f(j,s) \le f_s'(j,0)s, \\ \exists\ \alpha > 0,\ \exists\ \gamma \ge 0,\ \forall\ (j,s) \in \mathbb{Z} \times [0,1], \quad f(j,s) \ge f_s'(j,0)s - \gamma s^{1+\alpha}, \\ \exists\ \rho \in (0,1), \quad \forall\ j \in \mathbb{Z}, \quad \forall\ \rho \le s \le s' \le 1, \quad f(j,s) \ge f(j,s'). \end{cases} \tag{1.5}$$

The equation (1.2) is a spatial-discrete version of (1.1) in one space-dimensional case. It also comes directly from many biological models in a patchy environment (cf. [8, 18]). See also the book by Shigesada and Kawasaki [16]. For related works to (1.2) on homogeneous discrete media with monostable or bistable nonlinearities, we refer the readers to [5, 6, 7, 14, 25, 26] and the references cited therein.

Note that the assumption

$$\exists\ \alpha > 0,\ \exists\ \gamma \ge 0,\ \forall\ (j,s) \in \mathbb{Z} \times [0,1], \quad f(j,s) \ge f_s'(j,0)s - \gamma s^{1+\alpha}$$

could be replaced without loss of generality by

$$\exists\ \alpha > 0,\ \exists\ \ \beta > 0,\ \exists\ \gamma \ge 0,\ \forall\ (j,s) \in \mathbb{Z} \times [0,\beta], \quad f(j,s) \ge f_s'(j,0)s - \gamma s^{1+\alpha}.$$

Throughout the paper, the solutions $u = (u_j(t))_{(j,t)\in\mathbb{Z}\times\mathbb{R}}$ of (P) are assumed to range in $[0,1]$, namely $u_j : \mathbb{R} \mapsto [0,1]$ for each $j \in \mathbb{Z}$. We call a solution u of (P) a pulsating traveling front solution.

Introduce the $N \times N$ symmetric matrix $A := [a_{ij}]$ defined by

$$\begin{aligned} &a_{j,j} = -(d_{j+1} + d_j),\ j = 1, \cdots, N, \\ &a_{j,j+1} = a_{j+1,j} = d_{j+1},\ j = 1, \cdots, N-1, \\ &a_{1,N} = a_{N,1} = d_1, \\ &a_{i,j} = 0 \text{ if } |i-j| \ge 2 \text{ and } (i,j) \notin \{(1,N),(N,1)\}, \end{aligned}$$

and, for $\lambda \in \mathbb{R}$, denote by $A_\lambda := [a_{\lambda;i,j}]$ the $N \times N$ matrix defined by

$$\begin{aligned}
&a_{\lambda;j,j} = -(d_{j+1} + d_j),\ j = 1, \cdots, N,\\
&a_{\lambda;j,j+1} = d_{j+1}e^{-\lambda},\ j = 1, \cdots, N-1,\\
&a_{\lambda;j+1,j} = d_{j+1}e^{\lambda},\ j = 1, \cdots, N-1,\\
&a_{\lambda;1,N} = d_1e^{\lambda},\\
&a_{\lambda;N,1} = d_1e^{-\lambda},\\
&a_{\lambda;i,j} = 0 \text{ if } |i-j| \geq 2 \text{ and } (i,j) \notin \{(1,N),(N,1)\}.
\end{aligned}$$

In particular, $A_0 = A$. Lastly, call $D := [d_{i,j}]$ the diagonal $N \times N$ matrix defined by $d_{j,j} = f'_s(j,0)$ for all $j = 1, \cdots, N$.

Since the coefficients d_j are uniformly bounded from above and below by two positive constants, it especially follows that the Cauchy problem for (1.2), say with an initial condition $(u_j(0))_{j\in\mathbb{Z}}$ which ranges between 0 and 1, is well-posed, and that the parabolic maximum principle holds for the solutions of (1.2).

The following two theorems show that pulsating traveling front solutions of (P) exist if and only if the wave speed c is above a minimal wave speed c^*, where c^* is defined as in (1.6) below.

Theorem 1 *Let u be a C^1 solution of (1.2)-(1.4) with a speed $c \neq 0$. Then $c > 0$ and*

$$u'_j(t) > 0, \quad u_j(-\infty) = 0 < u_j(t) < 1 = u_j(+\infty)$$

for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$. Furthermore,

$$c \geq c^* = \min_{\lambda>0} \frac{M(\lambda)}{\lambda} > 0, \tag{1.6}$$

where $M(\lambda)$ is the largest real eigenvalue of the matrix $A_\lambda + D$.

Theorem 2 *For each $c \geq c^*$, there exists a solution u of (1.2)-(1.4) with speed c.*

Consider now the continuous problem

$$\partial_t u = \partial_x(d(x)\partial_x u) + g(x,u),\ t \in \mathbb{R},\ x \in \mathbb{R}. \tag{1.7}$$

The function d is assumed to be periodic with period $L > 0$, to be of class $C^{1,\beta}$ for some $\beta > 0$, and to satisfy $0 < \inf_{\mathbb{R}} d \leq \sup_{\mathbb{R}} d < +\infty$. The nonlinearity $g : \mathbb{R} \times [0,1] \to \mathbb{R}$ is assumed to be periodic with period L in its first variable and to be of class C^1. Furthermore, one assumes that $g(x,0) = g(x,1) = 0$ for all $x \in \mathbb{R}$, $0 < g(x,s) \leq \partial_s g(x,0)s$ for all $(x,s) \in \mathbb{R} \times (0,1]$ and that there exist $\alpha > 0$, $\delta \geq 0$ such that $g(x,s) \geq \partial_s g(x,0)s - \delta s^{1+\alpha}$ for all $(x,s) \in \mathbb{R} \times [0,1]$. It is known ([1], see also [4]) that equation (1.7) admits pulsating traveling solutions $u(t,x)$ such that $u(t + L/\gamma, x) = u(t, x-L)$ and $u(t,x) \to 1$ (resp. $\to 0$) as $x \to -\infty$ (resp. $x \to +\infty$) for each $t \in \mathbb{R}$, if and only if $\gamma \geq \gamma^*$, where $\gamma^* > 0$ is given by

$$\gamma^* = \min_{\lambda>0} \frac{k(\lambda)}{\lambda} \tag{1.8}$$

and $k(\lambda)$ is the principal eigenvalue of the operator $(d(x)\varphi')' - 2\lambda d(x)\varphi' + (-\lambda d'(x) + \lambda^2 d + \partial_s g(x,0))\varphi$ with periodicity L.

Approximate equation (1.7) by the discretized problem

$$\begin{aligned} u_j'(t) &= \frac{1}{h^2}\left(d((j+\frac{1}{2})h)(u_{j+1}(t)-u_j(t)) - d((j-\frac{1}{2})h)(u_j(t)-u_{j-1}(t))\right) \\ &\quad + g(jh, u_j(t)),\ t\in\mathbb{R},\ j\in\mathbb{Z}, \end{aligned} \tag{1.9}$$

where $h = L/N$, and N is a (large) integer. This discretized equation is of the type (1.2) with

$$\left\{ \begin{aligned} d_j &= h^{-2} d\left((j-\frac{1}{2})h\right) = \frac{N^2}{L^2} d\left((j-\frac{1}{2})\frac{L}{N}\right) =: d_j^h \\ f(j,s) &= g(jh,s) = g\left(j\frac{L}{N}, s\right) =: f^h(j,s). \end{aligned} \right.$$

It is immediate to check that the coefficients d_j^h are periodic with period N, so is f^h in the variable j. Furthermore, f^h satisfies (1.5). Therefore, for each $N \in \mathbb{N}\backslash\{0\}$, Theorems 1 and 2 assert the existence of solutions u of (1.9) and (1.3)-(1.4) if and only if $c \geq c_h^*$, where c_h^* is given by (1.6).

The following result connects the discretized minimal speeds c_h^* to the continuous one γ^*:

Theorem 3 *Under the above notations, one has*

$$hc_h^* \to \gamma^* \quad \text{as } N \to +\infty,\ \text{with } h = \frac{L}{N}.$$

After completing this work, we realized that the existence result of pulsating traveling fronts for all speeds $c \geq c^*$ (Theorem 2) was a consequence of a result of Hudson and Zinner [12]. However, the proofs are really different. The proof in [12] is based on the approximation of the equation in bounded domains (like in [26], or [1, 20] in the continuous case). Here, we directly attack the problem in the unbounded domain, by using suitable space-time global sub- and super-solutions. We choose to present this alternative approach, which also includes new Liouville type results for discrete time-dependent or stationary equations, since it has its own interest. Furthermore, in this paper, we also prove that the condition $c \geq c^*$ is not only a sufficient condition for the existence of pulsating traveling fronts, but it is also a necessary condition. We further prove that all such solutions are actually increasing in time. Lastly, we show the convergence of the renormalized discretized minimal speeds to the continuous minimal speed. For monostable nonlinearities, the convergence was known only in the case with constant diffusion coefficients, [13] (even if it was actually not previously known that these speeds were really the *minimal* speeds). We here generalize this convergence property to problem (1.2) where both the reaction and the diffusion are heterogeneous and periodic.

This paper is organized as follows. In Section 2, we study the lower bound of wave speeds and prove Theorem 1. The existence of pulsating traveling front solutions with speeds above the minimal speed (Theorem 2) is proved in Section 3. Finally, in Section 4, we give the proof of Theorem 3 and derive the convergence of discretized minimal speeds to the minimal continuous speed.

2 Lower bound for the speeds and monotonicity in time

This section is devoted to the proof of Theorem 1, which is itself divided into several lemmas.

Lemma 2.1 *Under the notation of Section 1, the matrix $A_\lambda + D$ has a largest real eigenvalue $M(\lambda)$, for each $\lambda \in \mathbb{R}$. Furthermore, the function $\lambda \mapsto M(\lambda)$ is convex, $M(0) > 0$, $M'(0) = 0$ and the minimum of $M(\lambda)/\lambda$ over all $\lambda > 0$ is achieved and is positive.*

Proof. It is easy to check that, for

$$\alpha > \max_{i\in\mathbb{Z}} \left(-d_i - d_{i+1} + d_{i+1}e^{-\lambda} + d_i e^{\lambda} + f_s'(i,0)\right),$$

the matrix $-A_\lambda - D + \alpha I$ is invertible and $(-A_\lambda - D + \alpha I)^{-1}$ satisfies the assumptions of the Krein-Rutman theorem in the space $M_{N,1}(\mathbb{R})$ of real column vectors of size N, with positive cone $K = \{X = (x_1, \cdots, x_N)^T,\ x_i > 0 \text{ for all } i\}$. Therefore, the matrix $-A_\lambda - D + \alpha I$ has a smallest real positive and simple eigenvalue (all other eigenvalues have larger real parts), and this eigenvalue is associated to an eigenvector $\phi^\lambda \in K$.

In other words, $A_\lambda + D$ has a largest real eigenvalue $M(\lambda)$, which satisfies $M(\lambda) \le \max_{i\in\mathbb{Z}} \left(-d_i - d_{i+1} + d_{i+1}e^{-\lambda} + d_i e^{\lambda} + f_s'(i,0)\right)$, whence

$$M(\lambda) \le C_1 \cosh(\lambda) + C_2$$

with $C_1 = 2\max_{i\in\mathbb{Z}} d_i > 0$ and $C_2 = \max_{i\in\mathbb{Z}} f_s'(i,0) > 0$.

Let $\phi^\lambda = (\varphi_1^\lambda, \cdots, \varphi_N^\lambda)^T$ be an eigenvector (in K) of $A_\lambda + D$ with the eigenvalue $M(\lambda)$. Let $j \in \{1, \cdots, N\}$ be such that $m = \varphi_j^\lambda = \min_{i\in\{1,\cdots,N\}} \varphi_i^\lambda > 0$. One has

$$(-d_j - d_{j+1} + d_{j+1}e^{-\lambda} + d_j e^{\lambda} + f_s'(j,0))\ m \le M(\lambda)\ m, \tag{2.1}$$

whence

$$M(\lambda) \ge C_3 \cosh(\lambda) - C_1, \tag{2.2}$$

where $C_3 = 2\min_{i\in\mathbb{Z}} d_i > 0$. Furthermore, one also gets from (2.1) that

$$M(0) \ge \min_{i\in\mathbb{Z}} f_s'(i,0) > 0. \tag{2.3}$$

On the other hand, the min-max formulation of the largest eigenvalue $M(\lambda)$ of $A_\lambda + D$ reads

$$M(\lambda) = \min_{\phi\in K} \max_{i\in\{1,\cdots,N\}} \frac{((A_\lambda + D)\phi)_i}{(\phi)_i} = \min_{u\in K_{per}} \max_{i\in\mathbb{Z}} g(\lambda, u, i),$$

where $K_{per} = \{u = (u_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}},\ u_i > 0 \text{ and } u_{i-N} = u_i \text{ for all } i \in \mathbb{Z}\}$ and

$$g(\lambda, u, i) = \frac{-(d_i + d_{i+1})u_i + d_{i+1}e^{-\lambda}u_{i+1} + d_i e^{\lambda} u_{i-1} + f_s'(i,0)u_i}{u_i}.$$

Let us now prove that the function $\lambda \mapsto M(\lambda)$ is convex. Let $(\lambda^1, \lambda^2) \in \mathbb{R}^2$, $t \in [0,1]$, $(u^1, u^2) \in K_{per} \times K_{per}$. Call

$$\lambda = t\lambda^1 + (1-t)\lambda^2$$

and

$$u = (u_i)_{i\in\mathbb{Z}} = (e^{t\ln(u_i^1)+(1-t)\ln(u_i^2)})_{i\in\mathbb{Z}}.$$

Clearly, $u \in K_{per}$. It follows from the above characterization of $M(\lambda)$ that

$$\begin{aligned} M(\lambda) \le \max_{i\in\mathbb{Z}} \,[-(d_i+d_{i+1}) \; &+ d_{i+1}e^{-(t\lambda^1+(1-t)\lambda^2)+t\ln(u_{i+1}^1/u_i^1)+(1-t)\ln(u_{i+1}^2/u_i^2)} \\ &+ d_i e^{t\lambda^1+(1-t)\lambda^2+t\ln(u_{i-1}^1/u_i^1)+(1-t)\ln(u_{i-1}^2/u_i^2)} + f_s'(i,0)]. \end{aligned}$$

Since the coefficients d_i's are positive and the exponential function is convex, there holds

$$M(\lambda) \le \max_{i\in\mathbb{Z}} \,\left[tg(\lambda^1,u^1,i)+(1-t)g(\lambda^2,u^2,i)\right] \le t\max_{i\in\mathbb{Z}} g(\lambda^1,u^1,i) \;+\; (1-t)\max_{i\in\mathbb{Z}} g(\lambda^2,u^2,i)$$

because $0 \le t \le 1$. Since the functions u^1 and u^2 were arbitrary in K_{per}, one concludes that $M(\lambda) \le tM(\lambda^1)+(1-t)M(\lambda^2)$. Therefore, the function M is convex. In particular, it is continuous.

Let us now prove that $M'(0) = 0$. For each $\lambda \in \mathbb{R}$, call u^λ the unique element of K_{per} such that

$$-(d_i+d_{i+1})u_i^\lambda + d_{i+1}e^{-\lambda}u_{i+1}^\lambda + d_ie^\lambda u_{i-1}^\lambda + f_s'(i,0)u_i^\lambda = M(\lambda)u_i^\lambda \tag{2.4}$$

for all $i \in \mathbb{Z}$, with $\max_{i\in\mathbb{Z}} u_i^\lambda = 1$.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence converging to 0. By periodicity and boundedness, one can extract a subsequence $(u^{\lambda_{n'}})_{n'}$ such that $u_i^{\lambda_{n'}} \to \tilde{u}_i \in [0,1]$ for all $i \in \mathbb{Z}$, with $\tilde{u}_j = 1$ for some j. By continuity of the function M, the family $(\tilde{u}_i)_{i\in\mathbb{Z}}$ satisfies (2.4) with $\lambda = 0$, whence $(\tilde{u}_1,\cdots,\tilde{u}_N)^T$ is an eigenvector of $A_0 + D$ for the eigenvalue $M(0)$. By uniqueness, one concludes that $\tilde{u}_i = u_i^0$ for all $i \in \mathbb{Z}$ and that the whole sequence $(u_i^{\lambda_n})_{n\in\mathbb{N}}$ converges to u_i^0 as $n \to +\infty$, for all $i \in \mathbb{Z}$.

Next, multiply the equation (2.4) by u_i^0 and multiply the equation (2.4) with $\lambda = 0$ by u_i^λ. Substracting the two equations and summing over $i = 1,\ldots,N$ gives

$$\sum_{i=1}^N \left[d_{i+1}e^{-\lambda}u_{i+1}^\lambda u_i^0 + d_ie^\lambda u_{i-1}^\lambda u_i^0 - d_{i+1}u_{i+1}^0u_i^\lambda - d_iu_{i-1}^0u_i^\lambda\right] = (M(\lambda)-M(0))\sum_{i=1}^N u_i^\lambda u_i^0.$$

Divide by $\lambda\sum_{i=1}^N u_i^\lambda u_i^0$ (for $\lambda \ne 0$). By periodicity, one gets

$$\frac{\dfrac{e^{-\lambda}-1}{\lambda}\displaystyle\sum_{i=1}^N d_iu_i^\lambda u_{i-1}^0 + \dfrac{e^\lambda-1}{\lambda}\displaystyle\sum_{i=1}^N d_{i+1}u_i^\lambda u_{i+1}^0}{\displaystyle\sum_{i=1}^N u_i^\lambda u_i^0} = \frac{M(\lambda)-M(0)}{\lambda}.$$

Since $u_i^\lambda \to u_i^0$ for all $i \in \mathbb{Z}$ as $\lambda \to 0$, the left-hand side converges to 0 as $\lambda \to 0$. Therefore, the function M is differentiable at 0 and $M'(0) = 0$.

It especially follows that $M(\lambda) \ge M(0)$ for all $\lambda \in \mathbb{R}$. The conclusion of Lemma 2.1 is a consequence of (2.2), (2.3) and of the above properties. □

Lemma 2.2 *Let u be a C^1 solution of (1.2)-(1.4) with a speed $c \ne 0$. Then $0 < u_j(t) < 1$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$, $c > 0$, $u_j(t) \to 1$ as $t \to +\infty$, $u_j(t) \to 0$ as $t \to -\infty$ and $u_j'(t) \to 0$ as $t \to \pm\infty$, for all $j \in \mathbb{Z}$.*

Proof. By assumption, $0 \le u_j(t) \le 1$. Assume that there is $(j, t_0) \in \mathbb{Z} \times \mathbb{R}$ such that $u_j(t_0) = 0$. Therefore, $u_j'(t_0) = f(j, u_j(t_0)) = 0$. Since each coefficient d_i is positive, one infers that $u_{j-1}(t_0) = u_{j+1}(t_0) = 0$ from (1.2). By induction, $u_i(t_0) = 0$ for all $i \in \mathbb{Z}$, which contradicts (1.4).

Therefore, $u_j(t) > 0$ for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$. Replacing u with $1 - u$ leads with the same arguments to the conclusion that $u_j(t) < 1$ for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$.

Let k be any positive real number larger than $|N/c|$. Integrate the equation (1.2) over $[-k, k]$ and sum over $j = 1, \ldots, N$. One gets

$$\begin{aligned}\sum_{j=1}^{N}(u_j(k) - u_j(-k)) &= \int_{-k}^{k}[-d_1u_1(t) + d_{N+1}u_{N+1}(t) + d_1u_0(t) - d_{N+1}u_N(t)]dt \\ &\quad + \sum_{j=1}^{N}\int_{-k}^{k} f(j, u_j(t))dt \\ &= d_1\int_{-k-N/c}^{-k}(u_1(t) - u_0(t))dt + d_1\int_{k-N/c}^{k}(u_0(t) - u_1(t))dt \\ &\quad + \sum_{j=1}^{N}\int_{-k}^{k} f(j, u_j(t))dt\end{aligned} \tag{2.5}$$

because of (1.3) and the periodicity of the d_i's. But equations (1.3) and (1.4) imply that $u_j(t) \to 1$ as $t \to +\infty$ and $u_j(t) \to 0$ as $t \to -\infty$ (resp. $u_j(t) \to 0$ as $t \to +\infty$ and $u_j(t) \to 1$ as $t \to -\infty$) if $c > 0$ (resp. $c < 0$), for each $j \in \mathbb{Z}$. Since $f > 0$ in $\mathbb{Z} \times (0, 1)$, the passage to the limit as $k \to +\infty$ in (2.5) yields that each integral $\int_{-\infty}^{+\infty} f(j, u_j(t))dt$ converges, and that

$$N \ \mathrm{sgn}(c) = \sum_{j=1}^{N}\int_{-\infty}^{+\infty} f(j, u_j(t))dt,$$

where $\mathrm{sgn}(c)$ denotes the sign of c. Therefore, $c > 0$.

It then follows from (1.3-1.4) that $u_j(t) \to 1$ as $t \to +\infty$, $u_j(t) \to 0$ as $t \to -\infty$ and $u_j'(t) \to 0$ as $t \to \pm\infty$, for all $j \in \mathbb{Z}$. □

Lemma 2.3 *Let u be a C^1 solution of (1.2)-(1.4) with a speed $c \neq 0$. Then*

$$\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R}} \frac{|u_j'(t)|}{u_j(t)} < +\infty.$$

Proof. Since $0 \le f(j, s)/s \le \max_{i\in\mathbb{Z}} f_s'(i, 0)$ for all $(j, s) \in \mathbb{Z} \times (0, 1]$, it is enough to prove, from (1.2), that the quantities $u_{j+1}(t)/u_j(t)$ and $u_{j-1}(t)/u_j(t)$ are globally bounded. Because of (1.3), it is even enough to prove that the quantities $u_{j+1}(t)/u_{j+N}(t + N/c)$ and $u_{j-1}(t)/u_{j+N}(t + N/c)$ are globally bounded.

Let us work with the first quantity, $u_{j+1}(t)/u_{j+N}(t + N/c)$, the other one being dealt with the same way. Let $(j_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ be given. From the maximum principle applied to problem (1.2), and since $f \ge 0$, one immediately has that

$$u_{j_0+N}(t_0 + N/c) \ge v_{j_0+N}(t_0 + N/c),$$

where $(v_j(t))_{j\in\mathbb{Z}}$ satisfies (1.2) for $t > t_0$ with $f \equiv 0$ and $v_j(t_0) = u_j(t_0)$ for all $j \in \mathbb{Z}$. Since $u_j(t_0) \geq 0$ for all $j \in \mathbb{Z}$, one can even say that

$$u_{j_0+N}(t_0 + N/c) \geq v_{j_0+N}(t_0 + N/c) \geq w_{j_0+N}(t_0 + N/c), \tag{2.6}$$

where $(w_j(t))_{j\in\mathbb{Z}}$ satisfies (1.2) for $t > t_0$ with $f \equiv 0$, $w_{j_0+1}(t_0) = u_{j_0+1}(t_0)$ and $w_j(t_0) = 0$ for all $j \neq j_0 + 1$.

For all $i \in \mathbb{Z}$, call now $\gamma_i := z^i_{i+N}(N/c)$, where $(z^i_j(t))_{j\in\mathbb{Z}}$ satisfies (1.2) for $t > 0$ with $f \equiv 0$, $z^i_{i+1}(0) = 1$ and $z^i_j(0) = 0$ for all $j \neq i+1$. One has $z^i_j(t) \geq 0$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$, whence $\gamma_i \geq 0$. If $\gamma_i = 0$, then $(z^i_{i+N})'(N/c) = z^i_{i+N}(N/c) = 0$, and $z^i_{i+N-1}(N/c) = z^i_{i+N+1}(N/c) = 0$. By induction, $z^i_j(N/c) = 0$ for all $j \in \mathbb{Z}$. But

$$(z^i_j)'(t) \geq -C_1 z^i_j(t)$$

for all $j \in \mathbb{Z}$, where $C_1 = 2\max_{l\in\mathbb{Z}} d_l$. In particular,

$$\gamma_i = z^i_{i+1}(N/c) \geq e^{-C_1N/c} z^i_{i+1}(0) = e^{-C_1N/c} > 0,$$

which gives a contradiction.

As a consequence, each γ_i is positive. On the other hand, $\gamma_i = \gamma_{i-N}$ for all $i \in \mathbb{Z}$, since the coefficients d_i's satisfy the same property. Therefore, $\Gamma := \min_{i\in\mathbb{Z}} \gamma_i > 0$. Eventually, one has

$$w_{j_0+N}(t_0 + N/c) = \gamma_{j_0} u_{j_0+1}(t_0)$$

by linearity. Putting the last formula into (2.6) yields

$$u_{j_0+N}(t_0 + N/c) \geq \Gamma u_{j_0+1}(t_0),$$

whence $u_{j+1}(t)/u_{j+N}(t + N/c) \leq \Gamma^{-1}$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. This completes the proof of Lemma 2.3. □

Remark 2.1 The above arguments actually imply that, given a positive solution $(u_j(t))_{j\in\mathbb{Z}}$ of (1.2), there holds $\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R}} u_j(t)/u_{j+J}(t+T) < +\infty$ for any $J \in \mathbb{Z}$ and $T > 0$. This is a version of the Harnack inequality for discrete parabolic operators.

Under the additional property (1.3), it also follows that, for all bounded interval I and for all $J \in \mathbb{Z}$,

$$\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ \tau\in I} \frac{u_j(t)}{u_{j+J}(t+\tau)} < +\infty. \tag{2.7}$$

Lemma 2.4 *Let u be a C^1 solution of (1.2)-(1.4) with a speed $c \neq 0$. Then, under the notation of Lemma 2.1,*

$$\Lambda := \liminf_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ u_j(t)\to 0} \frac{u'_j(t)}{u_j(t)} > 0,$$

$M(\Lambda/c) = \Lambda$ and

$$c \geq \min_{\lambda>0} \frac{M(\lambda)}{\lambda}.$$

Proof. From Lemma 2.3, one knows that Λ is a real number. Let (j_n,t_n) be a sequence in $\mathbb{Z}\times\mathbb{R}$ such that $u_{j_n}(t_n)\to 0$ and

$$\frac{u'_{j_n}(t_n)}{u_{j_n}(t_n)}\to\Lambda \text{ as } n\to+\infty.$$

Because of (1.3), one can assume that $j_n\in\{1,\cdots,N\}$ for all n.

Call

$$u_j^n(t)=\frac{u_j(t+t_n)}{u_{j_n}(t_n)}.$$

Each function u_j^n is of class C^2 and satisfies

$$\begin{cases} (u_j^n)'(t) &= d_{j+1}u_{j+1}^n(t)+d_ju_{j-1}^n(t)-(d_{j+1}+d_j)u_j^n(t)+\dfrac{f(j,u_j(t+t_n))}{u_j(t+t_n)}u_j^n(t)\\ (u_j^n)''(t) &= d_{j+1}(u_{j+1}^n)'(t)+d_j(u_{j-1}^n)'(t)-(d_{j+1}+d_j)(u_j^n)'(t)+f'_s(j,u_j(t+t_n))(u_j^n)'(t). \end{cases}$$

Because of (2.7), for each $j\in\mathbb{Z}$, the functions u_j^n are locally bounded in $t\in\mathbb{R}$, uniformly in n. The previous equations for u_j^n then imply that, for each $j\in\mathbb{Z}$, the functions u_j^n are bounded in $C^2_{loc}(\mathbb{R})$, uniformly in $n\in\mathbb{N}$. Furthermore, (2.7) also yields $u_j(t+t_n)\to 0$ as $n\to+\infty$, for each $j\in\mathbb{Z}$ and locally in $t\in\mathbb{R}$.

Therefore, up to extraction of some subsequence, there exist some $C^1(\mathbb{R})$ functions v_j such that $u_j^n\to v_j$ as $n\to+\infty$, in $C^1_{loc}(\mathbb{R})$ and for all $j\in\mathbb{Z}$. The functions v_j satisfy

$$v'_j(t)=d_{j+1}v_{j+1}(t)+d_jv_{j-1}(t)-(d_{j+1}+d_j)v_j(t)+f'_s(j,0)v_j(t) \tag{2.8}$$

for all $j\in\mathbb{Z}$ and $t\in\mathbb{R}$. Furthermore, one can assume that $j_n\to J\in\{1,\cdots,N\}$ as $n\to+\infty$, whence $v_J(0)=1$. On the other hand, $v_j(t)\geq 0$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$, and v_j satisfies (1.3). It then follows from the strong maximum principle, as in Lemma 2.2, that $v_j(t)>0$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$. As a consequence, $v'_J(0)=\Lambda$ and

$$\forall\ (j,t)\in\mathbb{Z}\times\mathbb{R},\quad w_j(t):=\frac{v'_j(t)}{v_j(t)}\geq\Lambda.$$

Moreover, the passage to the limit $n\to+\infty$ yields

$$\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R}}\frac{|v'_j(t)|}{v_j(t)}\leq\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R}}\frac{|u'_j(t)|}{u_j(t)}<+\infty$$

from Lemma 2.3. From (2.8), each function v_j is of class C^2 and the function w_j satisfies

$$w'_j(t)=a_j(t)w_{j+1}(t)+b_j(t)w_{j-1}(t)-(a_j(t)+b_j(t))w_j(t),$$

where

$$a_j(t)=d_{j+1}v_{j+1}(t)/v_j(t)>0 \text{ and } b_j(t)=d_jv_{j-1}(t)/v_j(t)>0.$$

Remember that $w_j(t)\geq\Lambda$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$, and $w_J(0)=\Lambda$. Therefore, $w'_J(0)=0$, $w_{J-1}(0)=w_{J+1}(0)=\Lambda$, and $w_j(0)=\Lambda$ for all $j\in\mathbb{Z}$ by induction. Hence, $w_j(t)=\Lambda$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$ from the equation satisfied by w_j and from property (1.3) fulfilled by w_j.

In other words, $v_j'(t) = \Lambda v_j(t)$ and, because of (1.3), $v_j(t)$ can be written as

$$v_j(t) = e^{\Lambda t - \Lambda i/c} U_j,$$

where $U_j = U_{j-N} > 0$ for all $j \in \mathbb{Z}$. It is straightforward to see that

$$\Lambda U_j = d_{j+1} e^{-\Lambda/c} U_{j+1} + d_j e^{\Lambda/c} U_{j-1} - (d_{j+1} + d_j) U_j + f_s'(j,0) U_j$$

for all $j \in \mathbb{Z}$. From Lemma 2.1, one concludes that

$$M\left(\frac{\Lambda}{c}\right) = \Lambda$$

and $\Lambda > 0$. Therefore, $\mu := \Lambda/c > 0$, $M(\mu) = \mu c$ and

$$c \geq \min_{\lambda > 0} \frac{M(\lambda)}{\lambda}.$$

This completes the proof of Lemma 2.4. □

Lemma 2.5 *Let u be a C^1 solution of (1.2)-(1.4) with a speed $c \neq 0$. Then $u_j'(t) > 0$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$.*

Proof. Step 1. Because of (1.3) and Lemmas 2.2 and 2.4, there exists $A \geq 1$ such that

$$\left\{ \begin{array}{ll} \rho \leq u_j(t) < 1 & \text{for all } (j,t) \in \mathbb{Z} \times \mathbb{R} \text{ with } ct - j \geq A \\ 0 < u_j(t) \leq \rho/2, \quad u_j'(t) > 0 & \text{for all } (j,t) \in \mathbb{Z} \times \mathbb{R} \text{ with } ct - j \leq -A + 1, \end{array} \right. \tag{2.9}$$

where $\rho \in (0,1)$ is given in (1.5).

One claims that

$$\exists\, T \geq 0, \quad \forall\, T' \geq T, \quad \forall\, (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad (ct - j \leq -A) \Longrightarrow (u_j(t + T') \geq u_j(t)). \tag{2.10}$$

Assume it is not true. Then there is a sequence $(T_n)_{n \in \mathbb{N}} \to +\infty$ of positive numbers, and some points $(j_n, t_n) \in \mathbb{Z} \times \mathbb{R}$ such that $ct_n - j_n \leq -A$ and

$$u_{j_n}(t_n + T_n) < u_{j_n}(t_n). \tag{2.11}$$

Because of (1.3), one can assume that $1 \leq j_n \leq N$ for all n and that, up to extraction of some subsequence, $j_n = J \in \{1, \cdots, N\}$ for all n. It follows from Lemma 2.4 and (2.9) that $c(t_n + T_n) - j_n > -A$ for all n. Hence, the sequence $(t_n + T_n)_n$ is bounded from below. Up to extraction of some subsequence, two cases may occur :

Case 1 : $t_n + T_n \to T \in \mathbb{R}$ as $n \to +\infty$. Since $T_n \to +\infty$, one gets $t_n \to -\infty$. Hence, $u_{j_n}(t_n) \to 0$, whereas $u_{j_n}(t_n + T_n) \to u_J(T) > 0$ as $n \to +\infty$. This contradicts (2.11).

Case 2 : $t_n + T_n \to +\infty$ as $n \to +\infty$. Then $u_{j_n}(t_n + T_n) \to 1$ as $n \to +\infty$, whereas $u_{j_n}(t_n) \leq \rho/2 < 1$.

Therefore, both cases 1 and 2 are ruled out and the claim (2.10) is proved.

Step 2. Fix $T \geq 0$ as in (2.10) and let τ be any real number such that

$$\tau \geq \max(T, 2A/c).$$

One has $u_j(t+\tau)\ge u_j(t)$ as soon as $ct-j\le -A$, because of (2.10).

Let us now prove the same inequality for $ct-j\ge -A$. Since $0<u_j(t)<1$, it follows that $u_j(t+\tau)+\varepsilon\ge u_j(t)$ for all (j,t) with $ct-j\ge -A$ and for all $\varepsilon>0$ large enough. Call

$$\varepsilon^*=\inf\,\{\varepsilon>0,\quad u_j(t+\tau)\ge u_j(t)\text{ for all }(j,t)\text{ with }ct-j\ge -A\}.$$

One immediately has $\varepsilon^*\ge 0$ and $u_j(t+\tau)+\varepsilon^*\ge u_j(t)$ for all (j,t) such that $ct-j\ge -A$.

Assume now that $\varepsilon^*>0$. There exist then some sequences $(\varepsilon_n)_n$ and (j_n,t_n) such that $0<\varepsilon_n<\varepsilon^*$, $ct_n-j_n\ge -A$, $\varepsilon_n\to\varepsilon^*$ as $n\to+\infty$ and

$$u_{j_n}(t_n+\tau)+\varepsilon_n<u_{j_n}(t_n). \tag{2.12}$$

Because of (1.3), one can assume that $1\le j_n\le N$. Since $\varepsilon^*>0$ and $u_j(t)\to 1$ as $t\to+\infty$, for each $j\in\mathbb{Z}$, (2.12) implies that the sequence (t_n) is bounded from above. On the other hand, it is bounded from below because $ct_n-j_n\ge -A$.

Up to extraction of some subsequence, one can then assume that $(j_n,t_n)\to(J,T)\in\mathbb{Z}\times\mathbb{R}$ as $n\to+\infty$ (whence $j_n=J$ for n large enough), where $cJ-T\ge -A$. Passing to the limit as $n\to+\infty$ in (2.12) yields $u_J(T+\tau)+\varepsilon^*\le u_J(T)$. Since the opposite inequality holds as well, one gets that

$$u_J(T+\tau)+\varepsilon^*=u_J(T).$$

Denote

$$v_j(t)=u_j(t+\tau)+\varepsilon^*\text{ and }w_j(t)=v_j(t)-u_j(t)=u_j(t+\tau)+\varepsilon^*-u_j(t).$$

Let us extend $f(j,s)$ for $s>1$ by $f(j,s)=(s-1)f'_s(j,1)$ for all $j\in\mathbb{Z}$ and $s>1$. It follows from (1.5) that $f(j,\cdot)$ is of class $C^1([0,+\infty))$ for each $j\in\mathbb{Z}$ and that $f(j,s)\ge f(j,s')$ for all $j\in\mathbb{Z}$ and $\rho\le s\le s'<+\infty$.

For all (j,t) with $ct-j\ge -A$, one has $c(t+\tau)-j\ge -A+c\tau\ge A$ due to the choice of τ. Therefore, $u_j(t+\tau)\ge\rho$, $f(j,u_j(t+\tau))\ge f(j,u_j(t+\tau)+\varepsilon^*)$ and

$$v_j'(t)\ge d_{j+1}v_{j+1}(t)+d_jv_{j-1}(t)-(d_{j+1}+d_j)v_j(t)+f(j,v_j(t))$$

for all (j,t) with $ct-j\ge -A$. As a consequence, there are some continuous bounded functions c_j defined for $t\in[(j-A)/c,+\infty)$ for all $j\in\mathbb{Z}$, such that

$$w_j'(t)\ge d_{j+1}w_{j+1}(t)+d_jw_{j-1}(t)-(d_{j+1}+d_j)w_j(t)+c_j(t)w_j(t) \tag{2.13}$$

for all (j,t) with $ct-j\ge -A$.

Remember that $w_j(t)\ge 0$ if $ct-j\ge -A$, and $w_J(T)=0$, that is $u_J(T+\tau)+\varepsilon^*=u_J(T)$, with $cT-J\ge -A$. If $-A\le cT-J\le -A+1$, then $c(T+\tau)-J\ge A$ (as already emphasized) and

$$u_J(T+\tau)+\varepsilon^*>u_J(T+\tau)\ge\rho>\frac{\rho}{2}\ge u_J(T)$$

because of (2.9). Hence, $cT-J>-A+1$ $(>-A)$. As a consequence, $w_J'(T)=0$ and $w_{J+1}(T)=w_{J-1}(T)=0$ because of (2.13) (and since $cT-(J-1)\ge cT-(J+1)\ge -A$). By immediate induction, one gets $w_{J-k}(T)=0$ for all $k\in\mathbb{N}$. In other words,

$$u_{J-k}(T+\tau)+\varepsilon^*=u_{J-k}(T)\text{ for all }k\in\mathbb{N}.$$

Since $\varepsilon^* > 0$ and $u_j(t) \to 1$ as $j \to -\infty$, for each $t \in \mathbb{R}$, one reaches a contradiction as $k \to +\infty$.

As a consequence, $\varepsilon^* = 0$ and $u_j(t+\tau) \geq u_j(t)$ for all (j,t) such that $ct - j \geq -A$.

Eevntually,

$$u_j(t+\tau) \geq u_j(t) \text{ for all } (j,t) \in \mathbb{Z} \times \mathbb{R} \text{ and for all } \tau \geq \max(T, 2A/c).$$

Step 3. Set

$$\tau^* = \inf \{\tau > 0,\ u_j(t+\tau') \geq u_j(\tau) \text{ for all } (j,t) \in \mathbb{Z} \times \mathbb{R} \text{ and for all } \tau' \geq \tau\}. \tag{2.14}$$

Because of Step 2, τ^* is a nonnegative real number. One has immediately

$$v_j(t) := u_j(t+\tau^*) - u_j(t) \geq 0 \text{ for all } (j,t) \in \mathbb{Z} \times \mathbb{R}.$$

Assume now that $\tau^* > 0$. Each function v_j satisfies

$$\forall\, t \in \mathbb{R}, \quad v_j'(t) = d_{j+1} v_{j+1}(t) + d_j v_{j-1}(t) - (d_{j+1} + d_j) v_j(t) + b_j(t) v_j(t), \tag{2.15}$$

where b_j is a continuous function, and all b_j's are bounded in $L^\infty(\mathbb{R})$ norm by the Lipschitz constant of f with respect to the variable s.

If there exists $(J,T) \in \mathbb{Z} \times \mathbb{R}$ such that $v_J(T) = 0$, then $v_J'(T) = 0$, $v_{J+1}(T) = v_{J-1}(T) = 0$ and $v_j(T) = 0$ for all $j \in \mathbb{Z}$ by immediate induction. It follows from (2.15) that $v_j(t) = 0$ for all $j \in \mathbb{Z}$ and $t \geq 0$, and then for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ because of (1.3). In other words, $u_j(t+\tau^*) = u_j(t)$, whence

$$u_j(t + k\tau^*) = u_j(t)$$

for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ and for all $k \in \mathbb{Z}$. But τ^* is assumed to be positive, which yields $u_j(t+k\tau^*) \to 1$ as $k \to +\infty$ and $u_j(t+k\tau^*) \to 0$ as $k \to -\infty$, for each (j,t) (because of Lemma 2.2). That leads to a contradiction.

Therefore, $v_j(t) > 0$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$. By continuity of u_j, one infers that

$$\min_{\frac{j-A+1}{c} - \tau^* \leq t \leq \frac{j+A+1}{c} + \tau^*} v_j(t) = \min_{\frac{j-A+1}{c} - \tau^* \leq t \leq \frac{j+A+1}{c} + \tau^*} (u_j(t+\tau^*) - u_j(t)) > 0$$

for each $j \in \mathbb{Z}$. Because of (1.3), one even has

$$\min_{j \in \mathbb{Z}} \min_{\frac{j-A+1}{c} - \tau^* \leq t \leq \frac{j+A+1}{c} + \tau^*} (u_j(t+\tau^*) - u_j(t)) =: \delta > 0.$$

The continuity of each u_j and property (1.3) yield the existence of $\tau_* \in (0, \tau^*)$ such that

$$\forall\, \tau \in [\tau_*, \tau^*], \quad \min_{j \in \mathbb{Z}} \min_{\frac{j-A+1}{c} - \tau^* \leq t \leq \frac{j+A+1}{c} + \tau^*} (u_j(t+\tau) - u_j(t)) \geq \frac{\delta}{2} > 0.$$

Fix any $\tau \in [\tau_*, \tau^*]$. One then has

$$(-A + 1 - c\tau^* \leq ct - j \leq A + 1 + c\tau^*) \Longrightarrow \left(u_j(t+\tau) \geq u_j(t) + \frac{\delta}{2} \geq u_j(t) \right). \tag{2.16}$$

If $(j,t)\in\mathbb{Z}\times\mathbb{R}$ is such that $ct-j\le -A+1-c\tau^*$, then

$$ct-j\le c(t+\tau)-j\le ct-j+c\tau^*\le -A+1,$$

whence $u_j(t+\tau)\ge u_j(t)$ because of (2.9).

On the other hand, if $ct-j\ge A+c\tau^*$ ($>A$), then $c(t+\tau)-j\ge A$ and $u_j(t+\tau)\ge\rho$ because of (2.9). Denote

$$\varepsilon^*=\inf\left\{\varepsilon>0,\ u_j(t+\tau)+\varepsilon\ge u_j(t)\text{ for all }(j,t)\text{ with }ct-j\ge A+c\tau^*\right\}.$$

As in Step 2, ε^* is a nonnegative real number and one shall prove that it is zero. Assume that $\varepsilon^*>0$. With the same arguments as in Step 2, there exists $(J,T)\in\mathbb{Z}\times\mathbb{R}$ such that $cT-J\ge A+c\tau^*$ and

$$u_J(T+\tau)+\varepsilon^*=u_J(T),$$

whereas $u_j(t+\tau)+\varepsilon^*\ge u_j(t)$ for all (j,t) with $ct-j\ge A+c\tau^*$. The functions w_j's defined by

$$w_j(t)=u_j(t+\tau)+\varepsilon^*-u_j(t)$$

satisfy

$$(ct-j\ge A+c\tau^*)\Longrightarrow\left(w_j'(t)\ge d_{j+1}w_{j+1}(t)+d_jw_{j-1}(t)-(d_{j+1}+d_j)w_j(t)+d_j(t)w_j(t)\right),$$

for some continuous functions d_j which are uniformly bounded in $L^\infty([(j+A)/c+\tau^*,+\infty))$. Because of (2.16), one infers that

$$cT-J>A+1+c\tau^*\ (>A+c\tau^*),$$

whence $w_J'(T)=0$, $w_{J-1}(T)=w_{J+1}(T)=0$ and $w_{J-k}(T)=0$ for all $k\in\mathbb{N}$ by immediate induction. In other words, $u_{J-k}(T+\tau)+\varepsilon^*=u_{J-k}(T)$ for all $k\in\mathbb{N}$. Since ε^* is assumed to be positive, the limit as $k\to+\infty$ contradicts (1.4).

Therefore, $\varepsilon^*=0$, whence $u_j(t+\tau)\ge u_j(t)$ for all (j,t) with $ct-j\ge A+c\tau^*$.

One just proved that $u_j(t+\tau)\ge u_j(t)$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$ and for all $\tau\in[\tau_*,\tau^*]$. Since $\tau_*<\tau^*$, one gets a contradiction with the definition of τ^*in (2.14). As a consequence, $\tau^*=0$ and

$$u_j(t+\tau)\ge u_j(t)\text{ for all }(j,t)\in\mathbb{Z}\times\mathbb{R}\text{ and for all }\tau\ge 0.$$

Step 4. Eventually, $z_j(t):=u_j'(t)\ge 0$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$. But the functions z_j's satisfy

$$z_j'(t)=d_{j+1}z_{j+1}(t)+d_jz_{j-1}(t)-(d_{j+1}+d_j)z_j(t)+f_s'(j,u_j(t))z_j(t).$$

As in Lemma 2.2, the existence of $(J,T)\in\mathbb{Z}\times\mathbb{R}$ satisfying $z_J(T)=0$ would yield $z_j(t)=0$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$. This is impossible because $u_j(t)\to 1$ as $t\to+\infty$ and $u_j(t)\to 0$ as $t\to-\infty$, for each $j\in\mathbb{Z}$.

As a conclusion, $u_j'(t)>0$ for all $(j,t)\in\mathbb{Z}\times\mathbb{R}$.

This completes the proofs of Lemma 2.5 and Theorem 1. □

3 Existence of fronts

This section is devoted to the proof of Theorem 2. It is divided into several steps : for any given $c > c^*$, we first construct some suitable sub- and super-solutions of (1.2), we then solve a sequence of Cauchy problems starting at times $-n$ with $n \to +\infty$, lastly we prove a Liouville type result for time-global solutions which are trapped between the sub- and super-solutions of the first part. For any $c > c^*$, that provides the existence of a travelling front u solving (1.2)-(1.4). The case $c = c^*$ is obtained by passing to the limit as $c \to (c^*)^+$.

Step 1 : construction of sub- and super-solutions for any given $c > c^*$. We fix a speed $c > c^*$, where $c^* > 0$ is given by formula (1.6) of Theorem 1. Call $g(\lambda) = M(\lambda)/\lambda$ for $\lambda > 0$. Hence, $c^* = \min_{\lambda>0} g(\lambda)$. Note that $g(\lambda) \to +\infty$ as $\lambda \to 0^+, +\infty$ from the arguments in Lemma 2.1. Hence there exists a unique $\lambda^* \in (0, +\infty)$ such that $g(\lambda^*) = c^*$ and $g(\lambda) > c^*$ for all $0 < \lambda < \lambda^*$. Using the convexity of the function M, we can show that if $g(\lambda_1) = g(\lambda_2) = \gamma > c^*$ for some $0 < \lambda_1 < \lambda_2 < \lambda^*$, then $M(\lambda) \geq \gamma\lambda$ for all $\lambda \geq \lambda_2$, whence $g(\lambda^*) \geq \gamma > c^*$ and this is impossible. From this it can be easily deduced that the function g is decreasing in $(0, \lambda^*]$ (and non-decreasing in $[\lambda^*, +\infty)$).

For a fixed $c > c^*$, we choose the unique λ such that $0 < \lambda < \lambda^*$ and $g(\lambda) = c$. Then we can find $\mu \in (0, \lambda^*)$ such that

$$g(\mu) = \frac{M(\mu)}{\mu} < c \quad \text{and} \quad \lambda < \mu < \lambda(1+\alpha), \tag{3.1}$$

where $\alpha > 0$ is given in (1.5).

Let $(v_i)_{i\in\mathbb{Z}} \in K_{per}$ solve (2.4) with the parameter λ, and let $(w_i)_{i\in\mathbb{Z}} \in K_{per}$ solve (2.4) with the parameter μ.

Lemma 3.1 *The function $\overline{u} = (\overline{u}_j)_{j\in\mathbb{Z}}$ defined by*

$$\forall\ (j,t) \in \mathbb{Z}\times\mathbb{R}, \quad \overline{u}_j(t) = \min\left(e^{\lambda(ct-j)}v_j, 1\right)$$

is a super-solution of (1.2).

Proof. Since the constant 1 is a solution of (1.2), it is enough to prove that $\tilde{v}_j(t) = e^{\lambda(ct-j)}v_j$ satisfies

$$\tilde{v}_j'(t) \geq d_{j+1}\tilde{v}_{j+1}(t) + d_j\tilde{v}_{j-1}(t) - (d_{j+1}+d_j)\tilde{v}_j(t) + f(j, \tilde{v}_j(t))$$

for the (j,t)'s such that $e^{\lambda(ct-j)}v_j < 1$. This is an immediate consequence of (2.4) and of the inequality $0 \leq f(j,s) \leq f_s'(j,0)s$ for all $(j,s) \in \mathbb{Z}\times[0,1]$. □

Lemma 3.2 *There exists $A > 0$ large enough so that the function $\underline{u} = (\underline{u}_j)_{j\in\mathbb{Z}}$ defined by*

$$\forall\ (j,t) \in \mathbb{Z}\times\mathbb{R}, \quad \underline{u}_j(t) = \max\left(e^{\lambda(ct-j)}v_j - Ae^{\mu(ct-j)}w_j, 0\right)$$

is a sub-solution of (1.2).

Proof. Since $\mu > \lambda > 0$, there exists $A > 0$ large enough such that

$$(\underline{u}_j(t) > 0) \implies (ct - j \leq 0). \tag{3.2}$$

Since both (v_j) and (w_j) are positive and periodic, even if it means increasing A, one can assume without loss of generality that

$$\gamma v_j^{1+\alpha} + A\,[M(\mu) - c\mu]w_j \leq 0,\ \forall\, j \in \mathbb{Z},$$

where $\gamma \geq 0$ was given in (1.5).

With $A > 0$ being given as above, let us check that the function $\underline{u}$ defined in Lemma 3.2 is a sub-solution of (1.2). Since the constant 0 is a solution of (1.2), it is enough to prove that $e^{\lambda(ct-j)}v_j - Ae^{\mu(ct-j)}w_j$ is a sub-solution of (1.2) for the (j,t)'s such that $\underline{u}_j(t) > 0$. For such (j,t)'s, it follows from (1.5), (2.4), (3.1), and (3.2) that

$$\begin{array}{rcl}
\underline{u}_j'(t) - f(j, \underline{u}_j(t)) & - & d_{j+1}\underline{u}_{j+1}(t) - d_j\underline{u}_{j-1}(t) + (d_{j+1} + d_j)\underline{u}_j(t) \\
& = & f_s'(j,0)e^{\lambda(ct-j)}v_j - f(j, e^{\lambda(ct-j)}v_j - Ae^{\mu(ct-j)}w_j) \\
& & +[M(\mu) - c\mu - f_s'(j,0)]\ Ae^{\mu(ct-j)}w_j \\
& \leq & \gamma\left(e^{\lambda(ct-j)}v_j - Ae^{\mu(ct-j)}w_j\right)^{1+\alpha} \\
& & +[M(\mu) - c\mu]\ Ae^{\mu(ct-j)}w_j \\
& \leq & \gamma e^{\lambda(1+\alpha)(ct-j)}v_j^{1+\alpha} + [M(\mu) - c\mu]\ Ae^{\mu(ct-j)}w_j \\
& \leq & e^{\mu(ct-j)}\{\gamma v_j^{1+\alpha} + A[M(\mu) - c\mu]w_j\}.
\end{array}$$

One concludes from the choice of A that

$$\underline{u}_j'(t) - d_{j+1}\underline{u}_{j+1}(t) - d_j\underline{u}_{j-1}(t) + (d_{j+1} + d_j)\underline{u}_j(t) - f(j, \underline{u}_j(t)) \leq 0$$

for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ such that $\underline{u}_j(t) > 0$. That completes the proof of Lemma 3.2. □

Step 2 : solving a sequence of Cauchy problems. For each $n \in \mathbb{N}$, let $u^n = (u_j^n(t))_{j\in\mathbb{Z},\ t\geq -n}$ solve (1.2) with the initial condition

$$\forall\, j \in \mathbb{Z}, \quad u_j^n(-n) = \underline{u}_j(-n).$$

Since $0 \leq \underline{u}_j(t) \leq \overline{u}_j(t) \leq 1$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$, and since $\underline{u}$ (resp. $\overline{u}$) is a sub-solution (resp. a super-solution) of (1.2), the maximum principle yields

$$\forall\, n \in \mathbb{N},\ \forall\, j \in \mathbb{Z},\ \forall\, t \geq -n, \quad 0 \leq \underline{u}_j(t) \leq u_j^n(t) \leq \overline{u}_j(t) \leq 1.$$

In particular, one has that $u_j^n(-n+1) \geq \underline{u}_j(-n+1) = u_j^{n-1}(-n+1)$ for all $n \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{Z}$. It resorts from the maximum principle that $u_j^n(t) \geq u_j^{n-1}(t)$ for all $n \in \mathbb{N}\backslash\{0\}$, $j \in \mathbb{Z}$ and $t \geq -n+1$. For each $(j,t) \in \mathbb{Z} \times \mathbb{R}$, the sequence $(u_j^n(t))_{n\in\mathbb{N},\ n\geq|t|}$ is nondecreasing and bounded ; call $u_j(t)$ its limit as $n \to +\infty$. On the other hand, the functions $u_j^n(t)$ are uniformly bounded between 0 and 1, and the derivatives $(u_j^n)'(t)$ are then also uniformly bounded. Therefore, the convergence $u_j^n(t) \to u_j(t)$ as $n \to +\infty$ holds at least locally uniformly in t for each $j \in \mathbb{Z}$. For each n, we can integrate equation (1.2) in any given interval of time, and then pass to the limit as $n \to +\infty$. It follows that the functions u_j are of class C^1 and solve (1.2) for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$. Furthermore, the above estimates imply that

$$\forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad 0 \le \underline{u}_j(t) \le u_j(t) \le \overline{u}_j(t) \le 1. \tag{3.3}$$

Note that by (3.3) it follows immediately that $u_j(t) \to 0$ uniformly as $ct - j \to -\infty$.

Step 3 : a Liouville type result for solutions trapped between $\underline{u}$ and $\overline{u}$. This step is devoted to the proof of the following

Proposition 3.3 *Under the above notations, any C^1 solution $u = (u_j(t))$ of (1.2) satisfying (3.3) is a front, namely u solves (1.3) and (1.4). Furthermore, given $\underline{u}$ and $\overline{u}$ as above, u is unique.*

The proof itself is divided into several lemmas. Let us first observe that, by applying Krein-Rutman theory as in Section 2, there exists a unique principal eigenvalue $\lambda^{k,l}$ and a unique (up to multiplication) principal eigenfunction $\varphi^{k,l}$ solving

$$\begin{cases} d_{j+1}\varphi^{k,l}_{j+1} + d_j\varphi^{k,l}_{j-1} - (d_{j+1}+d_j)\varphi^{k,l}_j &= \lambda^{k,l}\varphi^{k,l}_j, \quad -k+l+1 \le j \le k+l-1 \\ \varphi^{k,l}_j &> 0, \qquad\qquad -k+l+1 \le j \le k+l-1 \\ \varphi^{k,l}_{\pm k+l} &= 0 \end{cases} \tag{3.4}$$

for any given $l \in \mathbb{Z}$ and $k \in \mathbb{N}\backslash\{0\}$.

Lemma 3.4 *Under the above notations, $\lambda^{k,l} \to 0$ as $k \to +\infty$, uniformly in $l \in \mathbb{Z}$.*

Proof. Observe first that, by uniqueness and periodicity of the coefficients d_j, the principal eigenvalues $\lambda^{k,l}$ are periodic in l with period N. It is then enough to prove that $\lambda^{k,l} \to 0$ as $k \to +\infty$, for each given $l \in \mathbb{Z}$.

Fix $l \in \mathbb{Z}$. In the first equation of (3.4), choosing j_0 such that $\varphi^{k,l}_{j_0} = \max_{-k+l \le j \le k+l} \varphi^{k,l}_j$ yields $\lambda^{k,l} \le 0$. On the other hand, we claim that

$$-\lambda^{k,l} = \min_{\phi \in E} R(\phi),\ R(\phi) := \frac{\displaystyle\sum_{-k+l+1 \le j \le k+l-1} (d_j + d_{j+1})\phi_j^2 - d_{j+1}\phi_{j+1}\phi_j - d_j\phi_j\phi_{j-1}}{\displaystyle\sum_{-k+l+1 \le j \le k+l-1} \phi_j^2}, \tag{3.5}$$

where $E = \{(\phi_j)_{-k+l \le j \le k+l},\ \phi_{\pm k+l} = 0,\ \exists\ j,\ \phi_j \neq 0\}$. This formula is the classical variational formulation of the first eigenvalue of self-adjoint operator. We check it here for the sake of completeness.

First, it is immediate to check that $R(\phi) \ge -C_1$ for all $\phi \in E$, where $C_1 = 2\max_{i \in \mathbb{Z}} d_i$. Let now $(\phi^n)_{n \in \mathbb{N}} = ((\phi^n_j)_{-k+l \le j \le k+l})_{n \in \mathbb{N}}$ be a sequence in E such that $R(\phi^n) \to \inf_{\phi \in E} R(\phi)$. Since $|\phi^n| = (|\phi^n_j|)_{-k+l \le j \le k+l} \in E$ and $R(\phi^n) \ge R(|\phi^n|)$, one can assume that $\phi^n_j \ge 0$ for all $n \in \mathbb{N}$ and $-k+l \le j \le k+l$. Up to normalization, one can also assume that $\sum_{j=-k+l+1}^{k+l-1} (\phi^n_j)^2 = 1$. Up to extraction of some subsequence, one can assume that $\phi^n_j \to \phi_j \ge 0$ as $n \to +\infty$ for all $-k+l \le j \le k+l$, with $\sum_{j=-k+l+1}^{k+l-1}(\phi_j)^2 = 1$ and $\phi = (\phi_j)_{-k+l \le j \le k+l} \in E$. Furthermore, $R(\phi^n) \to R(\phi)$, whence $R(\phi) = \min_{\varphi \in E} R(\varphi)$.

Let now $\psi = (\psi_j)_{-k+l\leq j\leq k+l}$ be any test sequence in E. The sequence $\phi + t\psi$ is in E for $|t|$ small enough and $R(\phi + t\psi) \geq R(\phi)$, whence $\frac{d}{dt}R(\phi + t\psi)_{|t=0} = 0$. A straightforward calculation then gives that

$$\sum_{j=-k+l+1}^{k+l-1} 2(d_j + d_{j+1})\phi_j\psi_j - d_{j+1}\phi_{j+1}\psi_j - d_{j+1}\phi_j\psi_{j+1} - d_j\phi_{j-1}\psi_j - d_j\phi_j\psi_{j-1} - 2R(\phi)\phi_j\psi_j = 0.$$

Choosing $\psi_j = 1$ for $j = j_0$ and $\psi_j = 0$ for $j \neq j_0$, and doing that for any $j_0 \in \{-k+l+1, \ldots, k+l-1\}$ leads to

$$(d_j + d_{j+1})\phi_j - d_{j+1}\phi_{j+1} - d_j\phi_{j-1} = R(\phi)\phi_j$$

for all $j \in \{-k+l+1, \ldots, k+l-1\}$. Since $\phi_j \geq 0$ for all $j \in \{-k+l+1, \ldots, k+l-1\}$ and $\phi \in E$, the characterization of the principal eigenfunction for problem (3.4) implies that $R(\phi) = -\lambda^{k,l}$ and, up to multiplication, $\phi_j = \varphi_j^{k,l}$ for all $j \in \{-k+l, \ldots, k+l\}$. That completes the proof of the claim (3.5).

Choosing $(\phi_j)_{-k+l\leq j\leq k+l}$ with $\phi_j = 1$ for all $j \in \{-k+l+1, \ldots, k+l-1\}$ and $\phi_{\pm k+l} = 0$ as a test sequence in (3.5) implies that

$$-\lambda^{k,l} \leq \frac{d_{-k+l+1} + d_{-k+l+2} + d_{k+l-1} + d_{k+l}}{2k-1} \leq \frac{2C_1}{2k-1}.$$

Since $\lambda^{k,l} \leq 0$, one concludes that $\lambda^{k,l} \to 0$ as $k \to +\infty$. $\square$

Lemma 3.5 *Let $U = (U_j)_{j\in\mathbb{Z}}$ be a solution of*

$$\forall\, j \in \mathbb{Z}, \quad d_{j+1}U_{j+1} + d_jU_{j-1} - (d_{j+1} + d_j)U_j + f(j, U_j) = 0 \tag{3.6}$$

such that $0 \leq U_j \leq 1$ for all $j \in \mathbb{Z}$, and $U_{j_0} > 0$ for some $j_0 \in \mathbb{Z}$. Then $U_j = 1$ for all $j \in \mathbb{Z}$.

Proof. Let us first prove that $U_j > 0$ for all $j \in \mathbb{Z}$. Otherwise, there exists $i \in \mathbb{Z}$ such that $U_i = 0$. Then, $d_{i+1}U_{i+1} + d_iU_{i-1} = 0$ and since the real numbers U_j's are nonnegative and the coefficients d_j's are positive, one gets that $U_{i+1} = U_{i-1} = 0$. By immediate induction, it follows that $U_j = 0$ for all $j \in \mathbb{Z}$, which contradicts the positivity of U_{j_0}.

For all $k \in \mathbb{N}\backslash\{0\}$ and $l \in \mathbb{Z}$, let $\varphi^{k,l}$ be the unique principal eigenfunction of (3.4) such that $\max_{-k+l\leq j\leq k+l}\varphi_j^{k,l} = 1$. We set $\varphi_j^{k,l} = 0$ for $j \leq -k+l-1$ and $j \geq k+l+1$. Under the notations of Lemma 3.4, let $k_0 \in \mathbb{N}\backslash\{0\}$ be such that

$$\forall\, l \in \mathbb{Z}, \quad |\lambda^{k_0,l}| \leq \delta/2, \tag{3.7}$$

where $\delta = \min_{j\in\mathbb{Z}} f_s'(j,0) > 0$. Let $\varepsilon_0 > 0$ be such that

$$\forall\, j \in \mathbb{Z},\ \forall\, s \in [0, \varepsilon_0], \quad f(j,s) \geq \frac{\delta}{2}s.$$

For all $\varepsilon \in [0, \varepsilon_0]$, for all $l \in \mathbb{Z}$ and $j \in \{-k_0+l+1, \ldots, k_0+l-1\}$, one has

$$\begin{aligned} d_{j+1}\varepsilon\varphi_{j+1}^{k_0,l} + d_j\varepsilon\varphi_{j-1}^{k_0,l} - (d_{j+1} + d_j)\varepsilon\varphi_j^{k_0,l} + f(j, \varepsilon\varphi_j^{k_0,l}) &= \lambda^{k_0,l}\varepsilon\varphi_j^{k_0,l} + f(j, \varepsilon\varphi_j^{k_0,l}) \\ &\geq \left(\lambda^{k_0,l} + \frac{\delta}{2}\right)\varepsilon\varphi_j^{k_0,l} \\ &\geq 0 \end{aligned}$$

because $0 \leq \varepsilon\varphi_j^{k_0,l} \leq \varepsilon \leq \varepsilon_0$ and $|\lambda^{k_0,l}| \leq \delta/2$. Furthermore, the same inequality

$$d_{j+1}\varepsilon\varphi_{j+1}^{k_0,l} + d_j\varepsilon\varphi_{j-1}^{k_0,l} - (d_{j+1} + d_j)\varepsilon\varphi_j^{k_0,l} + f(j, \varepsilon\varphi_j^{k_0,l}) \geq 0 \tag{3.8}$$

holds immediately for all $j \leq -k_0 + l$ and $j \geq k_0 + l$.

Let now $l \in \mathbb{Z}$ be any integer. Since $U_j > 0$ for all $j \in \mathbb{Z}$, and $\varphi_j^{k_0,l} = 0$ for all $j \leq -k_0 + l$ and $j \geq k_0 + l$, there is $\eta_0 > 0$ such that $\eta\varphi_j^{k_0,l} \leq U_j$ for all $j \in \mathbb{Z}$ and $\eta \in [0, \eta_0]$. Call

$$\eta^* = \sup\{\eta \in (0, \varepsilon_0],\ \forall\, j \in \mathbb{Z},\ \eta\varphi_j^{k_0,l} \leq U_j\}.$$

One then has $0 < \min(\eta_0, \varepsilon_0) \leq \eta^* \leq \varepsilon_0$ and $\eta^*\varphi_j^{k_0,l} \leq U_j$ for all $j \in \mathbb{Z}$. Assume now that $\eta^* < \varepsilon_0$. There exists then $j_0 \in \mathbb{Z}$ such that $\eta^*\varphi_{j_0}^{k_0,l} = U_{j_0}$ (> 0), whence $j_0 \in \{-k_0 + l + 1, \ldots, k_0 + l - 1\}$. Call $v_j = U_j - \eta^*\varphi_j^{k_0,l}$ for all $j \in \mathbb{Z}$. From (3.6) and the above calculations for $\varepsilon\varphi^{k_0,l}$, one gets that

$$\forall\, j \in \mathbb{Z}, \quad d_{j+1}v_j + d_jv_{j-1} - (d_{j+1} + d_j)v_j + f(j, U_j) - f(j, \eta^*\varphi_j^{k_0,l}) \leq 0,$$

thus

$$\forall\, j \in \mathbb{Z}, \quad d_{j+1}v_j + d_jv_{j-1} - (d_{j+1} + d_j)v_j + b_jv_j \leq 0$$

for some coefficients b_j such that $\sup_{j\in\mathbb{Z}} |b_j| < +\infty$. But $v_j \geq 0$ for all $j \in \mathbb{Z}$, and $v_{j_0} = 0$. Therefore, $v_{j_0-1} = v_{j_0+1} = 0$, and $v_j = 0$ for all $j \in \mathbb{Z}$ by immediate induction. In other words, $U_j = \eta^*\varphi_j^{k_0,l}$ for all $j \in \mathbb{Z}$, whence $U_j = 0$ for $j \leq -k_0 + l$ and $j \geq k_0 + l$. But this is impossible because $U_j > 0$ for all $j \in \mathbb{Z}$.

Therefore, $\eta^* = \varepsilon_0$ and $U_j \geq \varepsilon_0\varphi_j^{k_0,l}$ for all $j \in \mathbb{Z}$ and $l \in \mathbb{Z}$. In particular,

$$1 \geq \underline{m} := \inf_{j\in\mathbb{Z}} U_j \geq \varepsilon_0 \inf_{j\in\mathbb{Z}} \varphi_j^{k_0,j}.$$

By uniqueness of the principal eigenfunctions $\varphi^{k,l}$ solving (3.4) (with the normalization $\max_{-k+l\leq j\leq k+l} \varphi_j^{k,l} = 1$), and by periodicity of the coefficients d_j (with period N), it resorts that the map $j \mapsto \varphi_j^{k_0,j}$ is periodic with period N. Consequently, $\underline{m} > 0$.

Let $(j_n)_{n\in\mathbb{Z}}$ be a sequence of integers such that $U_{j_n} \to \underline{m}$ as $n \to +\infty$. For each $n \in \mathbb{N}$, call $i_n \in N\mathbb{Z}$ and $J_n \in \{0, \ldots, N-1\}$ the integers such that $j_n = i_n + J_n$. Up to extraction of some subsequence, one can assume that $J_n = J$ for all $n \in \mathbb{N}$. Call $U_j^n = U_{i_n+j}$. Since $\underline{m} \leq U_j^n \leq 1$ for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, the diagonal extraction process implies that, up to extraction of some subsequence, $U_j^n \to V_j$ as $n \to +\infty$ for all $j \in \mathbb{Z}$. Furthermore, $\underline{m} \leq V_j \leq 1$ for all $j \in \mathbb{Z}$, and $V_J = \underline{m}$. On the other hand, since $i_n \in N\mathbb{Z}$ and the coefficients d_j's have period N, the sequence $(V_j)_{j\in\mathbb{Z}}$ still satisfies (3.6). At the point J, one has

$$0 = d_{J+1}\underbrace{V_{J+1}}_{\geq \underline{m}} + d_J\underbrace{V_{J-1}}_{\geq \underline{m}} - (d_{J+1} + d_J)\underline{m} + f(J, \underline{m}) \geq f(J, \underline{m}).$$

Since $0 < \underline{m} \leq 1$, one concludes from (1.5) that $\underline{m} = 1$. Therefore, $U_j = 1$ for all $j \in \mathbb{Z}$ and the proof of Lemma 3.5 is complete. □

Lemma 3.6 *Let $(u_j(t))_{j\in\mathbb{Z}}$ solve (1.2) for $t \geq 0$, with an initial condition $(u_j(0))_{j\in\mathbb{Z}}$ such that $0 \leq u_j(0) \leq 1$ for all $j \in \mathbb{Z}$ and $u_J(0) > 0$ for some $J \in \mathbb{Z}$. Then $u_j(t) \to 1$ as $t \to +\infty$ for all $j \in \mathbb{Z}$.*

Proof. The maximum principle implies that $0 \leq u_j(t) \leq 1$ for all $t \geq 0$ and $j \in \mathbb{Z}$. Assume now that there exists $t_0 > 0$ and $j_0 \in \mathbb{Z}$ such that $u_{j_0}(t_0) = 0$. Then $u'_{j_0}(t_0) = 0$, and $u_{j_0-1}(t_0) = u_{j_0+1}(t_0) = 0$. By immediate induction, $u_j(t_0) = 0$ for all $j \in \mathbb{Z}$. But $u'_j(t) \geq -(d_{j+1} + d_j)u_j(t) \geq -C_1 u_j(t)$ for all $t \geq 0$ and $j \in \mathbb{Z}$, where $C_1 = 2\max_{i\in\mathbb{Z}} d_i$. In particular, $u_J(t_0) \geq u_J(0)e^{-C_1 t_0} > 0$, and one has reached a contradiction. Therefore, $u_j(t) > 0$ for all $t > 0$ and $j \in \mathbb{Z}$.

Choose $k_0 \in \mathbb{N}$ large enough so that (3.7) holds and let $(\varphi_j^{k_0,0})_{-k_0\leq j\leq k_0}$ solve (3.4) with the normalization $\max_{-k_0\leq j\leq k_0} \varphi_j^{k_0,0} = 1$. As in the proof of Lemma 3.5, we set $\varphi_j^{k_0,0} = 0$ for all $|j| \geq k_0 + 1$. Let $\varepsilon_0 > 0$ be small enough so that (3.8) holds especially for $l = 0$, for all $j \in \mathbb{Z}$ and for all $\varepsilon \in [0, \varepsilon_0]$. Since $u_j(1) > 0$ for all $j \in \mathbb{Z}$, there exists $\varepsilon \in (0, \varepsilon_0]$ such that $u_j(1) \geq \varepsilon\varphi_j^{k_0,0}$ for all $j \in \mathbb{Z}$. Because of (3.8) and the maximum principle, one gets that

$$\forall\, t \geq 0,\ \forall\, j \in \mathbb{Z}, \quad u_j(t+1) \geq v_j(t) \geq \varepsilon\varphi_j^{k_0,0},$$

where $(v_j(t))_{j\in\mathbb{Z}}$ solves (1.2) for $t \geq 0$ with initial condition $v_j(0) = \varepsilon\varphi_j^{k_0,0}$ for all $j \in \mathbb{Z}$.

Since $v_j(h) \geq v_j(0)$ for all $h \geq 0$ and $j \in \mathbb{Z}$, the maximum principle yields $v_j(t+h) \geq v_j(t)$ for all $h \geq 0$, $t \geq 0$ and $j \in \mathbb{Z}$. Hence, $v_j(t)$ is nondecreasing in $t \geq 0$ for all $j \in \mathbb{Z}$. But $0 \leq v_j(t) \leq u_j(t+1) \leq 1$, whence $v_j(t) \to V_j \in [0, 1]$ as $t \to +\infty$, for all $j \in \mathbb{Z}$. By integration of the equation (1.2) satisfied by $v_j(t)$, between $t = n$ and $t = n + 1$, and then passing to the limit as $n \to +\infty$, it follows that the family $(V_j)_{j\in\mathbb{Z}}$ solves (3.6). Furthermore, $V_j \geq v_j(0) = \varepsilon\varphi_j^{k_0,0}$ for all $j \in \mathbb{Z}$. In particular, $V_0 \geq \varepsilon\varphi_0^{k_0,0} > 0$. Lemma 3.5 then yields $V_j = 1$ for all $j \in \mathbb{Z}$.

Therefore, for all $j \in \mathbb{Z}$,

$$\liminf_{t\to+\infty} u_j(t) \geq \lim_{t\to+\infty} v_j(t) = V_j = 1.$$

Since $u_j(t) \leq 1$ for all $t \geq 0$ and $j \in \mathbb{Z}$, one concludes that $u_j(t) \to 1$ as $t \to +\infty$ for all $j \in \mathbb{Z}$. □

Lemma 3.7 *Let u satisfy the assumptions of Proposition 3.3. Then $u_j(t) \to 1$ uniformly as $ct - j \to +\infty$.*

Proof. Owing to the definition of $\underline{u}$ in Lemma 3.2, there exists $B \geq 0$ such that

$$1 \geq \underline{u}_j(t) \geq \frac{\nu}{2} e^{\lambda(ct-j)}$$

for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$ such that $ct - j \leq -B$, where $\nu = \min_{i\in\mathbb{Z}} v_i > 0$ (remember that $(v_i)_{i\in\mathbb{Z}} \in K_{per}$ solves (2.4)). Therefore, $1 \geq \underline{u}_j((-B + j)/c) \geq \nu e^{-\lambda B}/2$ for all $j \in \mathbb{Z}$. From (3.3) and the maximum principle, it follows that

$$\forall\, i \in \mathbb{Z},\ \forall\, t \geq 0,\ \forall\, j \in \mathbb{Z}, \quad u_j\left(t + \frac{-B+i}{c}\right) \geq \tilde{u}^i_j(t), \tag{3.9}$$

where, for each $i \in \mathbb{Z}$, $\tilde{u}^i(t) = (\tilde{u}^i_j(t))_{j\in\mathbb{Z}}$ solves (1.2) for $t \geq 0$ with initial condition

$$\tilde{u}^i_j(0) = \begin{cases} \dfrac{\nu}{2} e^{-\lambda B} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Lemma 3.6 implies that $\tilde{u}^i_j(t) \to 1$ as $t \to +\infty$ for all $(i,j) \in \mathbb{Z}^2$. Furthermore, by uniqueness, $\tilde{u}^i_i(t)$ is periodic in i with period N for each $t \geq 0$.

Let now $\varepsilon > 0$ be fixed. From the above arguments, there exists $T_0 \geq 0$ such that $\tilde{u}^j_j(t) \geq 1-\varepsilon$ for all $t \geq T_0$ and for all $j \in \mathbb{Z}$. Let now $(j,t) \in \mathbb{Z} \times \mathbb{R}$ be any couple such that $ct - j \geq cT_0 - B$. Then $t - (-B+j)/c \geq T_0 \geq 0$ and it follows from (3.9) that

$$u_j(t) = u_j\left(t - \frac{-B+j}{c} + \frac{-B+j}{c}\right) \geq \tilde{u}^j_j\left(t - \frac{-B+j}{c}\right) \geq 1-\varepsilon.$$

Since $0 \leq u_j(t) \leq 1$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$, the conclusion of Lemma 3.7 follows. □

Lemma 3.8 *Let u satisfy the assumptions of Proposition 3.3. Then*

$$\forall\ K \geq 0, \qquad \inf_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ |ct-j|\leq K} u_j(t) > 0. \tag{3.10}$$

Proof. Assume that the conclusion does not hold for some $K \geq 0$. Since $u_j(t)$ is always nonnegative, there exists a sequence $(j_n,t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z} \times \mathbb{R}$ such that $u_{j_n}(t_n) \to 0$ as $n \to +\infty$, and $|ct_n - j_n| \leq K$. Write j_n as $j_n = I_n + J_n$ with $I_n \in N\mathbb{Z}$ and $J_n \in \{0,\ldots,N-1\}$. Up to extraction of some subsequence, one can assume that $J_n = J$ for all n. By periodicity of the coefficients d_j and of f with respect to j, the functions u^n defined by

$$\forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad u^n_j(t) = u_{I_n+j}(t+t_n)$$

solve (1.2). Furthermore, $0 \leq u^n_j(t) \leq 1$ and the functions $t \mapsto (u^n_j)'(t)$ are uniformly bounded. Therefore, up to extraction of some subsequence, one has $u^n_j(t) \to U_j(t)$ as $n \to +\infty$ for all $j \in \mathbb{Z}$ and locally uniformly in t. The functions $t \mapsto U_j(t)$ are continuous and, by writing (1.2) in the integral form, it follows that the functions U_j are of class C^1 and solve (1.2).

On the other hand, $0 \leq U_j(t) \leq 1$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ and $U_J(0) = 0$. Therefore, $U'_J(0) = 0$ and $U_{J-1}(0) = U_{J+1}(0) = 0$. By immediate induction, one gets that

$$\forall\ j \in \mathbb{Z}, \quad U_j(0) = 0. \tag{3.11}$$

The bounds (3.3) imply that

$$\forall\ (n,j) \in \mathbb{N} \times \mathbb{Z},\ u^n_j(0) = u_{I_n+j}(t_n) \geq \underline{u}_{I_n+j}(t_n) \geq e^{\lambda(ct_n-I_n-j)}v_{I_n+j} - Ae^{\mu(ct_n-I_n-j)}w_{I_n+j}.$$

But $|ct_n - I_n| \leq K + J$ for all $n \in \mathbb{N}$, whence

$$\forall\ (n,j) \in \mathbb{N} \times \mathbb{Z},\ u^n_j(0) \geq \nu\ e^{-\lambda(K+J+j)} - A\omega\ e^{\mu(K+J-j)},$$

where $\nu = \min_{j\in\mathbb{Z}} v_j \in (0,+\infty)$ and $\omega = \max_{j\in\mathbb{Z}} w_j \in (0,+\infty)$. Thus, there exists $j_0 \in \mathbb{Z}$ such that $\inf_{n\in\mathbb{N}} u^n_j > 0$ for all $j \geq j_0$, whence $U_j > 0$ for all $j \geq j_0$. This contradicts (3.11). Therefore, (3.10) holds for all $K \geq 0$. □

Remark 3.1 *The same arguments as the ones used in the proof of Lemma 3.8 imply that, for any u satisfying the assumptions of Proposition 3.3,*

$$\forall\ K \geq 0, \qquad \sup_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ |ct-j|\leq K} u_j(t) < 1.$$

Lemma 3.9 *Let u satisfy the assumptions of Proposition 3.3. Then there exists $\tau_0 \in \mathbb{R}$ such that*

$$\forall\ \tau \geq \tau_0,\ \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad u_j^{\tau}(t) := u_j(t+\tau) \geq u_{j-N}(t).$$

Proof. From (3.3) and Lemma 3.7, there exists $B_1 \geq 0$ such that, for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$,

$$\begin{cases} (ct - j \geq B_1) & \Longrightarrow \quad (u_j(t) \geq \rho), \\ (ct - j \leq -B_1 + 1) & \Longrightarrow \quad \left[\begin{array}{l} \left(\dfrac{1}{2}e^{\lambda(ct-j)}v_j \leq u_j(t) \leq e^{\lambda(ct-j)}v_j \leq \rho\right) \\ \text{and } \left(\dfrac{1}{2}e^{\lambda(ct-j+N)}v_j \leq u_{j-N}(t) \leq e^{\lambda(ct-j+N)}v_j \leq \rho\right) \end{array} \right], \end{cases}$$

where $\rho \in (0,1)$ is given in (1.5). From Lemma 3.8, there exists then $\delta > 0$ such that $u_j(t) \geq \delta$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ such that $|ct - j| \leq B_1$. Let now $B_2 \geq B_1$ be such that $u_{j-N}(t) \leq \min(\delta, \rho)$ as soon as $ct - j \leq -B_2$.

Let $\tau_0 \geq 0$ be such that $c\tau_0 \geq B_2 + B_1$ and $e^{\lambda c \tau_0}/2 \geq e^{\lambda N}$ and let us check that the conclusion of Lemma 3.9 follows with this choice of τ_0. Fix any $\tau \geq \tau_0$.

If $ct - j \geq -B_1$, then $c(t+\tau) - j \geq -B_1 + c\tau \geq B_2 \geq B_1$, whence $u_j^{\tau}(t) \geq \rho$. Furthermore, if $-B_1 \leq ct - j \leq -B_1 + 1$, then $u_{j-N}(t) \leq \rho$. With the same arguments as in the proof of Lemma 2.5, it then follows that $u_j^{\tau}(t) \geq u_{j-N}(t)$ for all (j,t) such that $ct - j \geq -B_1$.

If $-B_2 \leq ct - j \leq -B_1$, then $u_{j-N}(t) \leq \rho$ and $c(t+\tau) - j \geq -B_2 + c\tau \geq B_1$, whence $u_j^{\tau}(t) \geq \rho$ and $u_j^{\tau}(t) \geq u_{j-N}(t)$.

If $ct - j \leq -B_2$ and $c(t+\tau) - j \geq -B_1$, then $u_j^{\tau}(t) \geq \min(\delta, \rho) \geq u_{j-N}(t)$.

Lastly, if $ct - j \leq -B_2$ and $c(t+\tau) - j \leq -B_1$, then

$$u_j^{\tau}(t) \geq \frac{1}{2}e^{\lambda(c(t+\tau)-j)}v_j \geq e^{\lambda N}e^{\lambda(ct-j)}v_j \geq u_{j-N}(t).$$

Eventually, $u_j^{\tau}(t) \geq u_{j-N}(t)$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$ and for all $\tau \geq \tau_0$. □

Let us now turn to the

Proof of Proposition 3.3. With the notations of Lemma 3.9, one shall now decrease τ and call

$$\tau_* = \inf\ \{\tau \in \mathbb{R},\ \forall\ \tau' \geq \tau,\ \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R},\ u_j^{\tau'}(t) \geq u_{j-N}(t)\}.$$

One has that $\tau_* \leq \tau_0$ and $\tau_* \in \mathbb{R}$ (because $u_j(t) \to 0$ as $t \to -\infty$ for all $j \in \mathbb{Z}$ from (3.3), and $u_j(t) > 0$ for all (j,t) from Lemma 3.7). By continuity,

$$\forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad u_j^{\tau_*}(t) \geq u_{j-N}(t).$$

Let us now assume that

$$\tau_* > \frac{N}{c}.$$

Call $\tau_{**} = (N/c + \tau_*)/2\ \in\ (N/c, \tau_*)$. One claims that there exists $D \geq 0$ such that

$$\forall\ \tau \in [\tau_{**}, \tau_*],\ \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}, \quad (ct - j \leq -D)\ \Longrightarrow\ (u_j^{\tau}(t) \geq u_{j-N}(t)). \tag{3.12}$$

Assume not. Then there are some sequences $(\tau_n, t_n, j_n)_{n\in\mathbb{N}}$ with $\tau_n \in [\tau_{**}, \tau_*]$, $(j_n, t_n) \in \mathbb{Z}\times\mathbb{R}$, $ct_n - j_n \to -\infty$ as $n \to +\infty$ and $u^{\tau_n}_{j_n}(t_n) < u_{j_n-N}(t_n)$ for all $n \in \mathbb{N}$. The bounds (3.3) then give

$$e^{\lambda(ct_n+c\tau_n-j_n)}v_{j_n} - Ae^{\mu(ct_n+c\tau_n-j_n)}w_{j_n} \le u^{\tau_n}_{j_n}(t_n) \le u_{j_n-N}(t_n) \le e^{\lambda(ct_n-j_n+N)}v_{j_n}.$$

Thus,

$$e^{\lambda c\tau_n}v_{j_n} - Ae^{(\mu-\lambda)(ct_n-j_n)+\mu c\tau_n}w_{j_n} \le e^{\lambda N}v_{j_n}.$$

Since $\mu > \lambda$, $ct_n - j_n \to -\infty$, since the τ_n's are bounded and the v_j's and w_j's are bounded from above and below by positive constants, the passage to the limit as $n \to +\infty$ in the above inequality yields that $e^{\lambda c\tau} \le e^{\lambda N}$, where $\tau \in [\tau_{**}, \tau_*]$ is the limit of some subsequence of the sequence (τ_n). That is impossible since $\lambda > 0$, $c > 0$ and $\tau_{**} > N/c$ by assumption. Consequently, claim (3.12) is proved.

On the other hand, as in Lemma 3.9, there is $B_1 \ge 0$ such that $u_j(t) \ge \rho$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$ such that $ct - j \ge B_1$.

With B_1 and D as above, two and only two cases may occur :

Case 1 : $\inf_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ -D\le ct-j\le B_1+1} \big(u^{\tau_*}_j(t) - u_{j-N}(t)\big) > 0$. Since the functions $t \mapsto u_j(t)$ are globally Lipschitz continuous, uniformly with respect to j, it follows in this case that there exists $\eta \in (0, \tau_* - \tau_{**}]$ such that

$$\forall\, \tau \in [\tau_* - \eta, \tau_*],\ \forall\, (j,t) \in \mathbb{Z}\times\mathbb{R},\quad (-D \le ct - j \le B_1 + 1) \Longrightarrow (u^{\tau_*}_j(t) \ge u_{j-N}(t)). \tag{3.13}$$

Let τ be any shift in $[\tau_* - \eta, \tau_*]$ (whence $\tau \ge \tau_* - \eta \ge \tau_{**} \ge 0$). If $ct - j \ge B_1$, then $c(t+\tau) - j \ge B_1$ and $u^{\tau}_j(t) \ge \rho$ (where $\rho \in (0,1)$ is given in (1.5)). Furthermore, if $B_1 \le ct - j \le B_1 + 1$, then $u^{\tau}_j(t) \ge u_{j-N}(t)$ from (3.13). It then follows as in Step 2 of the proof of Lemma 2.5 that $u^{\tau}_j(t) \ge u_{j-N}(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$ such that $ct - j \ge B_1$.

Lastly, since $\tau \in [\tau_* - \eta, \tau_*] \subset [\tau_{**}, \tau_*]$, (3.12) implies that $u^{\tau}_j(t) \ge u_{j-N}(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$ such that $ct - j \le -D$.

One concludes that $u^{\tau}_j(t) \ge u_{j-N}(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$ and for all $\tau \in [\tau_* - \eta, \tau_*]$ with $\eta > 0$. This contradicts the minimality of τ_*. Thus, case 1 is ruled out.

Case 2 : $\inf_{(j,t)\in\mathbb{Z}\times\mathbb{R},\ -D\le ct-j\le B_1+1} \big(u^{\tau_*}_j(t) - u_{j-N}(t)\big) = 0$. There exists then a sequence $(j_n, t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}\times\mathbb{R}$ such that $-D \le ct_n - j_n \le B_1 + 1$ and $u^{\tau_*}_{j_n}(t_n) - u_{j_n-N}(t_n) \to 0$ as $n \to +\infty$. Write j_n as $j_n = I_n + J_n$ with $I_n \in N\mathbb{Z}$ and $J_n \in \{0, \ldots, N-1\}$. Up to extraction of some subsequence, one can assume that $J_n = J \in \{0, \ldots, N-1\}$ for all n. As in the proof of Lemma 3.8, the functions $t \mapsto u^n_j(t) = u_{I_n+j}(t+t_n)$ converge as $n \to +\infty$, up to extraction of some subsequence, locally uniformly in t and for all $j \in \mathbb{Z}$, to some functions $t \mapsto U_j(t)$ solving (1.2). Furthermore, $0 \le U_j(t) \le 1$ and $U^{\tau_*}_j(t) \ge U_{j-N}(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$, and $U^{\tau_*}_J(0) = U_{J-N}(0)$.

The nonnegative functions $t \mapsto z_j(t) = U^{\tau_*}_j(t) - U_{j-N}(t)$ solve

$$z_j'(t) = d_{j+1}z_{j+1}(t) + d_j z_{j-1}(t) - (d_{j+1} + d_j)z_j(t) + b_j(t)z_j(t),$$

with $\sup_{(j,t)\in\mathbb{Z}\times\mathbb{R}} |b_j(t)| < +\infty$. Furthermore, $z_J(0) = 0$, whence $z_J'(0) = 0$ and $z_{J-1}(0) = z_{J+1}(0) = 0$. By immediate induction, one gets that

$$\forall\, j \in \mathbb{Z},\quad U^{\tau_*}_j(0) - U_{j-N}(0) = z_j(0) = 0. \tag{3.14}$$

On the other hand, the bounds (3.3) imply that

$$e^{\lambda(c(t+t_n)-I_n-j)}v_j - Ae^{\mu(c(t+t_n)-I_n-j)}w_j \le u_j^n(t) = u_{I_n+j}(t+t_n) \le e^{\lambda(c(t+t_n)-I_n-j)}v_j$$

for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. Since $-D+J \le ct_n - I_n = ct_n - j_n + J \le B_1 + 1 + J$, one can assume up to extraction of some subsequence that $ct_n - I_n \to \sigma \in \mathbb{R}$ as $n \to +\infty$. The passage to the limit as $n \to +\infty$ in the above inequalities leads to

$$e^{\lambda(ct-j+\sigma)}v_j - Ae^{\mu(ct-j+\sigma)}w_j \le U_j(t) \le e^{\lambda(ct-j+\sigma)}v_j$$

for all (j,t). In particular, together with (3.14), one gets that

$$e^{\lambda(c\tau_*-j+\sigma)}v_j - Ae^{\mu(c\tau_*-j+\sigma)}w_j \le U_j^{\tau_*}(0) = U_{j-N}(0) \le e^{\lambda(N-j+\sigma)}v_j$$

for all $j \in \mathbb{Z}$. Therefore,

$$\forall\ j \in \mathbb{Z}, \quad e^{\lambda c\tau_*}v_j - Ae^{(\mu-\lambda)(\sigma-j)+\mu c\tau_*}w_j \le e^{\lambda N}v_j$$

and the passage to the limit as $j \to +\infty$ leads to $c\tau_* \le N$. That contradicts our assumption and case 2 is then ruled out too.

One concludes that the assumption $\tau_* > N/c$ can not hold. Thus, $\tau_* \le N/c$ and

$$\forall\ (j,t) \in \mathbb{Z}\times\mathbb{R}, \quad u_j\left(t+\frac{N}{c}\right) \ge u_{j-N}(t).$$

The same type of proof (defining $\tau^* = \sup\{\tau \in \mathbb{R},\ \forall \tau' \le \tau,\ \forall\ (j,t) \in \mathbb{Z}\times\mathbb{R},\ u_j(t+\tau') \le u_{j-N}(t)\}$, and proving that $\tau^* > -\infty$ and $\tau^* \ge N/c$) leads to the opposite inequality. Therefore, (1.3) is proved, namely

$$\forall\ (j,t) \in \mathbb{Z}\times\mathbb{R}, \quad u_j\left(t+\frac{N}{c}\right) = u_{j-N}(t).$$

Together with Lemma 3.7 and the bounds (3.3), (1.4) follows as well.

Lastly, the same arguments as above imply that, given two solutions u and v satisfying the assumptions of Proposition 3.3, one can slide in time u with respect to v, and v with respect to u, to prove that $u_j(t) \ge v_j(t)$ and $v_j(t) \ge u_j(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$.

That completes the proof of Proposition 3.3. □

Step 4 : conclusion of the proof of Theorem 2. It follows from the previous steps that, for any $c > c^*$, there exists a solution $u = (u_j(t))_{(j,t)\in\mathbb{Z}\times\mathbb{R}}$ of (1.2)-(1.4). It only remains to prove here that there is a solution for the limiting case $c = c^*$ as well.

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $c_n > c^*$ and $c_n \to c^*$ as $n \to +\infty$. For each n, there exists a solution $u^n = (u_j^n(t))_{(j,t)\in\mathbb{Z}\times\mathbb{R}}$ of (1.2)-(1.4) with the speed c_n. Furthermore, $0 \le u_j^n(t) \le 1$ and, from Theorem 1, each function $t \mapsto u_j^n(t)$ is increasing.

As in the proof of Lemma 3.8, one can assume, up to extraction of some subsequence, that $u_j^n(t) \to u_j(t)$ as $n \to +\infty$, locally uniformly in t for each $j \in \mathbb{Z}$. The functions $t \mapsto u_j(t)$ are of class C^1 and solve (1.2). Furthermore, $0 \le u_j(t) \le 1$ and $u_j'(t) \ge 0$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. The equality (1.3) also follows by passage to the limit, since $c_n \to c^*$.

On the other hand, because of (1.3)-(1.4) and $c_n > c^* > 0$, each (continuous) function $t \mapsto u_i^n(t)$ satisfies $u_i^n(t) \to 1$ (resp. 0) as $t \to +\infty$ (resp. $t \to -\infty$). One could then have assumed, up to normalization, that $u_0^n(0) = 1/2$. Thus, $u_0(0) = 1/2$.

Lastly, since $0 \leq u_j(t) \leq 1$ and $u_j'(t) \geq 0$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$, one has $u_j(t) \to U_j^{\pm}$ as $t \to \pm\infty$, where $(U_j^{\pm})_{j\in\mathbb{Z}}$ solve (3.6) with $0 \leq U_j^{\pm} \leq 1$ for all $j \in \mathbb{Z}$. Moreover, $U_0^+ \geq u_0(0) = 1/2 \geq U_0^-$. Lemma 3.5 implies then that $U_j^+ = 1$ and $U_j^- = 0$ for all $j \in \mathbb{Z}$. In other words, $u_j(t) \to 1$ (resp. 0) as $t \to +\infty$ (resp. $t \to -\infty$). The limits (1.4) then follow from (1.3) and the positivity of c^*.

The proof of Theorem 2 is now complete. □

4 Convergence to the minimal continuous speed for problems (1.9)

This section is devoted to the

Proof of Theorem 3. Let $N \in \mathbb{N}\backslash\{0\}$ and call $h = L/N$. Under the notations of Section 1, Theorem 1 asserts that

$$hc_h^* = \min_{\mu>0} \frac{LM^h(\mu)}{\mu} > 0,$$

where, for each μ, $M^h(\mu)$ is the unique real number such that there exists $u = (u_j)_{j\in\mathbb{Z}} \in K_{per}$ solving

$$\forall\ j \in \mathbb{Z}, \quad -(d_j^h + d_{j+1}^h)u_j + d_{j+1}^h e^{-\mu/N} u_{j+1} + d_j^h e^{\mu/N} u_{j-1} + (f^h)_s'(j,0)u_j = M^h(\mu)u_j. \quad (4.1)$$

Lemma 4.1 *One has*

$$\limsup_{N\to+\infty,\ h=L/N} hc_h^* < +\infty.$$

Proof. Set

$$d_0 = \min_{x\in\mathbb{R}} d(x)\ > 0, \quad D = \max_{x\in\mathbb{R}} d(x), \quad D' = \max_{x\in\mathbb{R}} |d'(x)|, \quad G = \max_{x\in\mathbb{R}} \partial_s g(x,0).$$

For given $h = L/N$ and $\mu > 0$, let $j_0 \in \mathbb{Z}$ be such that $u_{j_0} = \max_{j\in\mathbb{Z}} u_j$, where $(u_j)_{j\in\mathbb{Z}} \in K_{per}$ solves (4.1). By choosing j_0 in (4.1) and dividing by $u_{j_0} > 0$, one gets that

$$M^h(\mu) \leq \max_{j\in\mathbb{Z}}\ [d_j^h(e^{\mu/N} - 1) + d_{j+1}^h(e^{-\mu/N} - 1)]\ +\ G.$$

But, for each $j \in \mathbb{Z}$, one has

$$\begin{aligned} d_j^h(e^{\mu/N}-1) + d_{j+1}^h(e^{-\mu/N}-1) &= \frac{N^2}{L^2}\left\{\left[d((j-\frac{1}{2})\frac{L}{N}) - d(j\frac{L}{N})\right](e^{\mu/N}-1)\right. \\ &\qquad + \left[d((j+\frac{1}{2})\frac{L}{N}) - d(j\frac{L}{N})\right](e^{-\mu/N}-1) \\ &\qquad \left. + d(j\frac{L}{N})(e^{\mu/N} + e^{-\mu/N} - 2)\right\} \\ &\leq \frac{N^2}{L^2}\left\{\frac{LD'}{2N}(e^{\mu/N} - e^{-\mu/N}) + D \times (e^{\mu/N} + e^{-\mu/N} - 2)\right\}. \end{aligned}$$

For each $\mu > 0$, there holds $hc_h^* \le LM^h(\mu)/\mu$, whence

$$\begin{aligned}\limsup_{N\to+\infty,\ h=L/N} hc_h^* &\le \frac{L}{\mu} \times \limsup_{N\to+\infty,\ h=L/N} M^h(\mu)\\ &\le \frac{L}{\mu}\left(\frac{\mu D'}{L}+\frac{\mu^2 D}{L^2}+G\right) = D' + \frac{D\mu}{L}+\frac{GL}{\mu}.\end{aligned}$$

Since this holds for all $\mu > 0$, one gets that

$$\limsup_{N\to+\infty,\ h=L/N} hc_h^* \le \min_{\mu>0}\left(D'+\frac{D\mu}{L}+\frac{GL}{\mu}\right) = D' + 2\sqrt{DG}$$

and the proof of Lemma 4.1 is complete. □

Call now

$$\gamma = \liminf_{N\to+\infty,\ h=L/N} hc_h^*.$$

One has $0 \le \gamma \le \limsup_{N\to+\infty,\ h=L/N} hc_h^* < +\infty$. Let $(N_k)_{k\in\mathbb{N}}$ be a sequence of integers such that $N_k \to +\infty$ and $h_k c_{h_k}^* \to \gamma$ as $k \to +\infty$, where $h_k = L/N_k$. For each k, let $\mu_k > 0$ be such that $h_k c_{h_k}^* = LM^{h_k}(\mu_k)/\mu_k$. Therefore,

$$LM^{h_k}(\mu_k)/\mu_k \to \gamma \text{ as } k \to +\infty.$$

Lemma 4.2 *The sequence $(\mu_k)_{k\in\mathbb{N}}$ is bounded from below and above by two positive constants, namely*

$$0 < \liminf_{k\to+\infty} \mu_k \le \limsup_{k\to+\infty} \mu_k < +\infty.$$

Proof. From the arguments of Lemma 2.1, it follows that

$$M^{h_k}(\mu_k) \ge M^{h_k}(0) \ge \min_{j\in\mathbb{Z}} (f^{h_k})'_s(j,0) \ge \min_{x\in\mathbb{R}} \partial_s g(x,0) > 0.$$

Since $M^{h_k}(\mu_k)/\mu_k \to \gamma/L \in \mathbb{R}_+$ as $k \to +\infty$, it follows then that $\liminf_{k\to+\infty} \mu_k > 0$.

Define $m = \limsup_{k\to+\infty} \mu_k/N_k \in [0,+\infty]$. Let $u^k \in K_{per}$ (with period N_k) be the solution (unique up to multiplication) of (4.1) with $\mu = \mu_k$, $h = h_k$ and $N = N_k$. By choosing in (4.1) an integer $i_k \in \mathbb{Z}$ such that $u^k_{i_k} = \min_{i\in\mathbb{Z}} u^k_i > 0$ and dividing by $u^k_{i_k}$, one gets that

$$M^{h_k}(\mu_k) \ge d^{h_k}_{i_k}(e^{\mu_k/N_k}-1) + d^{h_k}_{i_k+1}(e^{-\mu_k/N_k}-1). \tag{4.2}$$

Hence,

$$\frac{L^2 M^{h_k}(\mu_k)}{N_k^2} \ge d((i_k-\frac{1}{2})\frac{L}{N_k})(e^{\mu_k/N_k}-1) + d((i_k+\frac{1}{2})\frac{L}{N_k})(e^{-\mu_k/N_k}-1).$$

Since the function d is periodic, continuous and positive, one can assume, up to extraction of some subsequence, that $d((i_k-1/2)L/N_k) \to \delta > 0$ and $d((i_k+1/2)L/N_k) \to \delta$. If $m \in (0,+\infty)$ and assuming, up to extraction of some subsequence, that $\mu_k/N_k \to m$, then

$$\liminf_{k\to+\infty} \frac{L^2 M^{h_k}(\mu_k)}{N_k^2} \ge 2\delta\ (\cosh m\ -\ 1) > 0.$$

But

$$\frac{L^2 M^{h_k}(\mu_k)}{N_k^2} = \frac{L^2 M^{h_k}(\mu_k)}{\mu_k} \times \frac{\mu_k}{N_k} \times \frac{1}{N_k} \ \rightarrow\ L\gamma \times m \times 0 = 0 \text{ as } k \to +\infty.$$

Therefore, the case $m \in (0, +\infty)$ is ruled out. If $m = +\infty$ and assuming, up to extraction of some subsequence, that $\mu_k/N_k \to +\infty$, then

$$\liminf_{k \to +\infty} \frac{L^2 M^{h_k}(\mu_k)}{e^{\mu_k/N_k} N_k^2} \geq \delta > 0.$$

But

$$\frac{L^2 M^{h_k}(\mu_k)}{e^{\mu_k/N_k} N_k^2} = \frac{L^2 M^{h_k}(\mu_k)}{\mu_k} \times \frac{\mu_k}{N_k} e^{-\mu_k/N_k} \times \frac{1}{N_k} \to L\gamma \times 0 \times 0 = 0 \text{ as } k \to +\infty.$$

Therefore, the case $m = +\infty$ is ruled out too. One concludes that $m = 0$. In other words, $\mu_k/N_k \to 0$ as $k \to +\infty$.

Let $\varepsilon \in (0, 1/2)$ be fixed. With the same notations as above, and since $\mu_k/N_k \to 0$ as $k \to +\infty$, it follows from (4.2) that, for k large enough,

$$\begin{aligned}
\frac{M^{h_k}(\mu_k)}{\mu_k} &\geq \frac{1}{\mu_k} \left[\frac{N_k^2}{L^2} d((i_k - \frac{1}{2})\frac{L}{N_k}) \left(\frac{\mu_k}{N_k} + (\frac{1}{2} - \varepsilon)(\frac{\mu_k}{N_k})^2 \right) \right. \\
&\qquad \left. + \frac{N_k^2}{L^2} d((i_k + \frac{1}{2})\frac{L}{N_k}) \left(-\frac{\mu_k}{N_k} + (\frac{1}{2} - \varepsilon)(\frac{\mu_k}{N_k})^2 \right) \right] \\
&\geq \frac{1}{\mu_k} \left[-\frac{D'\mu_k}{L} + \frac{d_0(1 - 2\varepsilon)\mu_k^2}{L^2} \right].
\end{aligned}$$

But the left-hand side of the above inequality is bounded as $k \to +\infty$, whence

$$\limsup_{k \to +\infty} \mu_k < +\infty.$$

That completes the proof of Lemma 4.2. □

From Lemma 4.2, one can then assume, up to extraction of some subsequence, that

$$\mu_k \to \Lambda \in (0, +\infty) \ \text{ as } k \to +\infty. \tag{4.3}$$

For each $k \in \mathbb{N}$, call now $u^k = (u_j^k)_{j \in \mathbb{Z}} \in K_{per}$ the unique solution of (4.1) with $h = h_k$, $N = N_k$ and $\mu = \mu_k$, assuming, up to normalization, that $\max_{j \in \mathbb{Z}} u_j^k = 1$. Namely, u^k satisfies

$$-(d_j^{h_k} + d_{j+1}^{h_k})u_j^k + d_{j+1}^{h_k} e^{-\mu_k/N_k} u_{j+1}^k + d_j^k e^{\mu_k/N_k} u_{j-1}^k + (f^{h_k})'_s(j, 0)u_j^k = M^{h_k}(\mu_k)u_j^k \tag{4.4}$$

for all $j \in \mathbb{Z}$. Let $\varphi_k : \mathbb{R} \to \mathbb{R}$ be the piecewise linear function defined by:

$$\begin{cases} \varphi_k(x) = u_j^k & \text{if } x = \dfrac{jL}{N_k}, \ \ j \in \mathbb{Z}, \\ \varphi_k(x) = u_j^k + \dfrac{N_k}{L}\left(x - \dfrac{jL}{N_k}\right)(u_{j+1}^k - u_j^k) & \text{if } \dfrac{jL}{N_k} < x < \dfrac{(j+1)L}{N_k}, \ \ j \in \mathbb{Z}. \end{cases} \tag{4.5}$$

Therefore, $0 < \varphi_k(x) \leq 1$ for all $x \in \mathbb{R}$, and each function φ_k belongs to H^1_{per}, where H^1_{per} denotes the space of H^1_{loc} functions which are periodic with period L, equiped with the usual H^1 norm in $(0, L)$.

Lemma 4.3 *The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in H^1_{per}.*

Proof. Multiply equation (4.4) by Lu^k_j/N_k and sum over $j=1,\ldots,N_k$. Since the coefficients $d^{h_k}_j$ and u^k_j are periodic in j with period N_k, one gets that

$$\begin{aligned}&\frac{L}{N_k}\sum_{j=1}^{N_k}\Big[-d^{h_k}_j((u^k_j)^2+(u^k_{j-1})^2)+d^{h_k}_j u^k_j u^k_{j-1}(e^{-\mu_k/N_k}+e^{\mu_k/N_k})+\partial_s g(jh_k,0)(u^k_j)^2\Big]\\ &=\frac{L}{N_k}\sum_{j=1}^{N_k}M^{h_k}(\mu_k)(u^k_j)^2.\end{aligned}$$

Owing to the definitions of φ_k and $d^{h_k}_j$, it follows that

$$\begin{aligned}d_0\int_0^L(\varphi_k'(x))^2dx &\le \sum_{j=1}^{N_k}\int_{(j-1)L/N_k}^{jL/N_k} d((j-\frac{1}{2})\frac{L}{N_k})(\varphi_k')(x))^2dx\\ &= \frac{L}{N_k}\sum_{j=1}^{N_k}d^{h_k}_j(u^k_j-u^k_{j-1})^2\\ &= \frac{L}{N_k}\sum_{j=1}^{N_k}\Big[d^{h_k}_j u^k_j u^k_{j-1}(e^{-\mu_k/N_k}+e^{\mu_k/N_k}-2)\\ &\qquad\qquad +(\partial_s g(jh_k,0)-M^{h_k}(\mu_k))(u^k_j)^2\Big],\end{aligned}$$

where $0<d_0=\min_{x\in\mathbb{R}}d(x)$. Since $N_k\to+\infty$, $\mu_k\to\Lambda\in(0,+\infty)$ (whence $0\le M^{h_k}(\mu_k)\to\gamma\Lambda/L$) as $k\to+\infty$, there exists a constant C such that $0\le e^{-\mu_k/N_k}+e^{\mu_k/N_k}-2\le C/(N_k)^2$ for all $k\in\mathbb{N}$. Remember also that $0\le u^k_j\le 1$ for all j. Therefore,

$$d_0\int_0^L(\varphi_k'(x))^2dx\le\frac{CD}{L}+LG.$$

Since each function φ_k ranges in $[0,1]$, the conclusion of Lemma 4.3 follows. □

Up to extraction of some subsequence, one can then assume that $\varphi_k\rightharpoonup\varphi\in H^1_{per}$ weak, and $\varphi_k\to\varphi$ in $C^{0,\eta}(\mathbb{R})$ for all $0\le\eta<1/2$ as $k\to+\infty$.

Lemma 4.4 *The function φ is of class $C^{2,\beta}(\mathbb{R})$ (where $\beta>0$ is such that $d\in C^{1,\beta}(\mathbb{R})$), and it satisfies*

$$(d\varphi')'-2\lambda d\varphi'-\lambda d'\varphi+\partial_s g(x,0)\varphi+\lambda^2 d\varphi=\gamma\lambda\varphi\quad\text{in }\mathbb{R},\tag{4.6}$$

where $\lambda=\Lambda/L>0$ and Λ is given in (4.3). Furthermore, $0<\varphi(x)\le 1$ for all $x\in\mathbb{R}$ and $\max_{\mathbb{R}}\varphi=1$.

Proof. Since each function φ_k is L-periodic and the convergence of the functions φ_k to φ is uniform in $\mathbb{R}$, and since $0\le\min_{\mathbb{R}}\varphi_k\le\max_{\mathbb{R}}\varphi_k=1$ by definition, it follows that $0\le\varphi\le 1$ and $\max_{\mathbb{R}}\varphi=1$.

Let now ψ be any function in H^1_{per} (without loss of generality, one can then assume that ψ is continuous). Multiply (4.4) by $L\psi(jL/N_k)/N_k = h_k\psi(jh_k)$ and sum over $j = 1, \ldots, N_k$. Since $u^k_j = \varphi_k(jh_k)$, one gets that

$$
\begin{aligned}
&-\sum_{j=1}^{N_k} h_k(d^{h_k}_j + d^{h_k}_{j+1})\varphi_k(jh_k)\psi(jh_k) \\
&+\sum_{j=1}^{N_k} \Big(h_k d^{h_k}_{j+1} e^{-\mu_k/N_k}\varphi_k((j+1)h_k)\psi(jh_k) + h_k d^{h_k}_j e^{\mu_k/N_k}\varphi_k((j-1)h_k)\psi(jh_k)\Big) \\
&+\sum_{j=1}^{N_k} h_k\partial_s g(jh_k,0)\varphi_k(jh_k)\psi(jh_k) \;=\; \sum_{j=1}^{N_k} h_k M^{h_k}(\mu_k)\varphi_k(jh_k)\psi(jh_k).
\end{aligned}
\tag{4.7}
$$

Since $M^{h_k}(\mu_k) \to \gamma\Lambda/L = \gamma\lambda$, the right-hand side converges to $\gamma\lambda\displaystyle\int_0^L \varphi(x)\psi(x)dx$ as $k \to +\infty$. Similarly, the last term of the left-hand side converges to $\displaystyle\int_0^L \partial_s g(x,0)\varphi(x)\psi(x)dx$ as $k \to +\infty$.

Call now I_k the sum of the first two terms of the left-hand side of (4.7). Because of the N_k-periodicity of all the coefficients involved in (4.7), one can write $I_k = II_k + III_k$ with

$$
\left\{
\begin{aligned}
II_k &= -\sum_{j=1}^{N_k} h_k d^{h_k}_j(\varphi_k(jh_k) - \varphi_k((j-1)h_k))(\psi(jh_k) - \psi((j-1)h_k)) \\
III_k &= \sum_{j=1}^{N_k} h_k d^{h_k}_j(e^{\mu_k/N_k} - 1)\varphi_k((j-1)h_k)\psi(jh_k) \\
&\quad +\sum_{j=1}^{N_k} h_k d^{h_k}_j(e^{-\mu_k/N_k} - 1)\varphi_k(jh_k)\psi((j-1)h_k).
\end{aligned}
\right.
$$

Since φ'_k is constant in each interval $((j-1)h_k, jh_k)$ and is equal to $h_k^{-1}(\varphi_k(jh_k) - \varphi_k((j-1)h_k))$, and since $d^{h_k}_j = h_k^{-2}d((j-1/2)h_k)$, the term II_k can be written

$$
II_k = -\sum_{j=1}^{N_k} d((j-\frac{1}{2})h_k)\int_{(j-1)h_k}^{jh_k} \varphi'_k(x)\psi'(x)dx.
$$

But $\varphi'_k \rightharpoonup \varphi'$ in $L^2(0,L)$ weak, and d is (at least) uniformly continuous. Therefore,

$$
II_k \to -\int_0^L d(x)\varphi'(x)\psi'(x)dx \quad \text{as } k \to +\infty.
$$

Moreover, one knows that $\mu_k \to \Lambda \in (0,+\infty)$, and $N_k \to +\infty$ as $k \to +\infty$. Remember also that the functions φ_k are uniformly bounded (by 1), and note that ψ is bounded as well.

Therefore, the term III_k can be written as

$$\begin{aligned}
III_k &= \sum_{j=1}^{N_k} h_k^{-1} d((j-\frac{1}{2})h_k)\frac{\mu_k}{N_k}(\varphi_k((j-1)h_k)\psi(jh_k) - \varphi_k(jh_k)\psi((j-1)h_k)) \\
&\quad + \sum_{j=1}^{N_k} h_k^{-1} d((j-\frac{1}{2})h_k)\frac{\mu_k^2}{2N_k^2}(\varphi_k((j-1)h_k)\psi(jh_k) + \varphi_k(jh_k)\psi((j-1)h_k)) \\
&\quad + O(N_k^{-1}) \ \text{ as } k \to +\infty \\
&= \frac{\mu_k}{L}\sum_{j=1}^{N_k} d((j-\frac{1}{2})h_k) \\
&\qquad\qquad \times[(\varphi_k((j-1)h_k) - \varphi_k(jh_k))\psi(jh_k) + \varphi_k(jh_k)(\psi(jh_k) - \psi((j-1)h_k))] \\
&\quad + \frac{\mu_k^2}{2L^2}\sum_{j=1}^{N_k} h_k d((j-\frac{1}{2})h_k)(\varphi_k((j-1)h_k)\psi(jh_k) + \varphi_k(jh_k)\psi((j-1)h_k)) \\
&\quad + O(N_k^{-1}) \ \text{ as } k \to +\infty.
\end{aligned}$$

Since the functions φ_k, d and ψ are uniformly equi-continous, and since $\varphi_k' \rightharpoonup \varphi'$ in $L^2(0,L)$ weak, one concludes that

$$III_k \to \lambda \int_0^L d(x)(-\varphi'(x)\psi(x) + \varphi(x)\psi'(x))dx + \lambda^2 \int_0^L d(x)\varphi(x)\psi(x)dx \ \text{ as } k \to +\infty.$$

Eventually,

$$\int_0^L d\varphi'\psi' - \lambda\int_0^L d\varphi'\psi + \lambda\int_0^L d\varphi\psi' + \lambda^2\int_0^L d\varphi\psi + \int_0^L \partial_s g(x,0)\varphi\psi = \gamma\lambda\int_0^L \varphi\psi$$

for all $\psi \in H^1_{per}$. Elliptic regularity theory and the fact that d is of class $C^{1,\beta}(\mathbb{R})$ imply that $\varphi \in C^{2,\beta}(\mathbb{R})$ and satisfies (4.6). Since $0 \le \varphi$ and $\max_{\mathbb{R}} \varphi = 1$, the strong maximum principle then implies that $\varphi > 0$ in $\mathbb{R}$. That completes the proof of Lemma 4.4. □

Lemma 4.5 *There holds*

$$\liminf_{N\to+\infty,\ h=L/N} hc_h^* = \gamma \ge \gamma^*,$$

where $\gamma^* > 0$ is the minimal speed for the pulsating traveling fronts of (1.7).

Proof. It follows from Lemma 4.4 that φ is the first eigenfunction of the operator defined by the left-hand side of (4.6), whence $k(\lambda) = \gamma\lambda$ (the first eigenvalue of this operator). Since $\lambda > 0$, one concludes that $\gamma = k(\lambda)/\lambda \ge \gamma^*$ because of (1.8). □

Lemma 4.6 *There holds*

$$\Gamma := \limsup_{N\to+\infty,\ h=L/N} hc_h^* \le \gamma^*.$$

Proof. From Lemma 4.1, one knows that Γ is finite. Let now $(N_k)_{k\in\mathbb{N}}$ be a sequence of integers such that $N_k \to +\infty$ and $h_k c_{h_k}^* \to \Gamma$ as $k \to +\infty$, with $h_k = L/N_k$.

Let λ' be any arbitrary positive real number and call $\mu = \lambda' L > 0$. One knows from Theorem 1 that

$$h_k c^*_{h_k} \leq \frac{L M^{h_k}(\mu)}{\mu} \tag{4.8}$$

for all $k \in \mathbb{N}$. With the same arguments as in the proof of Lemma 4.1, one can prove that the nonnegative sequence $(M^{h_k}(\mu))_{k\in\mathbb{N}}$ is bounded. Up to extraction of some subsequence, one can then assume that $LM^{h_k}(\mu)/\mu \to \gamma' \in \mathbb{R}_+$ as $k \to +\infty$.

For each $k \in \mathbb{N}$, let now $u^k = (u^k_j)_{j\in\mathbb{N}} \in K_{per}$ solve (4.1) with $h = h_k$ and $N = N_k$, assuming that $\max_{j\in\mathbb{Z}} u^k_j = 1$. Then, define φ^k as in (4.5). With the same arguments as in Lemmas 4.3 and 4.4, one can prove that the functions φ_k are bounded in H^1_{per} and that they converge, up to extraction of some subsequence, in H^1_{per} weak and in $C^{0,\eta}(\mathbb{R})$ for all $0 \leq \eta < 1/2$, to a positive and L-periodic function φ solving (4.6) with λ' and γ'. In other words, one concludes that $\gamma'\lambda' = k(\lambda')$.

Passing to the limit as $k \to +\infty$ in (4.8) yields

$$\Gamma \leq \gamma' = \frac{k(\lambda')}{\lambda'}.$$

But λ' was any arbitrary positive number. One then concludes from (1.8) that $\Gamma \leq \gamma^*$. hfill□

The above Lemmas 4.5 and 4.6 complete the proof of Theorem 3.

Remark 4.1 Fix any speed $c > \gamma^*$ and let $(N_k)_{k\in\mathbb{N}}$ be a sequence of integers such that $N_k \to +\infty$ as $k \to +\infty$. Set $h_k = L/N_k$. Because of Theorem 3, one knows that $c/h_k > c^*_{h_k}$ for k large enough, and assume that this is true for all k without loss of generality. For each $k \in \mathbb{N}$, let $(u^k_j(t))_{(j,t)\in\mathbb{Z}\times\mathbb{R}}$ solve (1.9) with the periodicity and limiting conditions (1.3)-(1.4), and with the speed c/h_k. In particular,

$$\forall\ (j,t) \in \mathbb{Z}\times\mathbb{R}, \quad u^k_j(t + N_k/(c/h_k)) = u^k_j(t + L/c) = u^k_{j-N_k}(t). \tag{4.9}$$

Up to shift in time, assume that $u^k_0(0) = 1/2$. One also knows from Theorem 1 that each function u^k_j is increasing in t, and that $0 < u^k_j(t) < 1$.

Define now $U^k(t,x)$, for all $(t,x) \in \mathbb{R}^2$, as follows

$$\begin{cases} U^k(t,x) = u^k_j(t) & \text{if } x = \dfrac{jL}{N_k}, \quad j \in \mathbb{Z}, \\ U^k(t,x) = u^k_j(t) + \dfrac{N_k}{L}\left(x - \dfrac{jL}{N_k}\right)(u^k_{j+1}(t) - u^k_j(t)) & \text{if } \dfrac{jL}{N_k} < x < \dfrac{(j+1)L}{N_k}, \quad j \in \mathbb{Z}. \end{cases}$$

Up to extraction of some subsequence, and from parabolic regularity, one can assume that the functions U^k converge locally uniformly in $\mathbb{R}^2$ to a classical solution U of (1.7). Furthermore, $0 \leq U(t,x) \leq 1$ for all $(t,x) \in \mathbb{R}^2$, and $U(t + L/c, x) = U(t, x - L)$ by passage to the limit in (4.9). Lastly, $U(0,0) = 1/2$ and U is nondecreasing in time. Therefore, $U(t,x) \to U_\pm(x)$ as $t \to \pm\infty$, where $U_\pm$ solve $(dU'_\pm)' + g(x, U_\pm) = 0$ in $\mathbb{R}$, and $0 \leq U_- \leq U_+ \leq 1$. Since $0 \leq U_-(0) \leq 1/2 \leq U_+(0) \leq 1$, one concludes with the results in [3] that $U_- \equiv 0$ and $U_+ \equiv 1$. As a consequence, $U(+\infty, x) = 1$ and $U(-\infty, x) = 0$ for all $x \in \mathbb{R}$, whence $U(t, -\infty) = 1$ and $U(t, +\infty) = 0$ for all $t \in \mathbb{R}$. In other words, U is a pulsating traveling front with the effective speed c for equation (1.7).

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