TRAVELING WAVE SOLUTIONS FOR A SINGULAR DIFFUSIVE PREY-PREDATOR MODEL WITH NONLOCAL DISPERSAL

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To Professor Yoshihisa Morita, with admiration to an esteemed mathematician

Abstract. We study a singular diffusive prey-predator system with nonlocal dispersal for which the carrying capacity of the predator is proportional to the density of prey. We show the existence of positive one-dimensional traveling waves connecting the predator-free state and the constant co-existence state. The set of admissible wave speeds is proved to be equal to the semi-infinite interval $[s^*, \infty)$, for some $s^* > 0$ which is characterized by a variational formula.

1. Introduction

We consider the following diffusive prey-predator model with nonlocal dispersal

$$\begin{cases} U_t(x,t) = \mathcal{N}_1[U(\cdot,t)](x) + [aU(1-U) - V](x,t),\ x \in \mathbb{R},\ t > 0, \\ V_t(x,t) = d\mathcal{N}_2[V(\cdot,t)](x) + [bV(1-V/U)](x,t),\ x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

where the unknown functions U, V stand for the population densities of prey and predator species at position x and time t, respectively, d, a, b are positive constants such that $1, d$ are diffusion coefficients and a, b are intrinsic growth rates of species U, V, respectively. The functional response of predator to prey is normalized to be 1. The prey obeys the logistic growth and its carrying capacity is normalized to be 1. However, the density of predator follows logistic dynamics with a varying carrying capacity proportional to the density of prey. Moreover, for $i = 1, 2$, $\mathcal{N}_i$ formulates the spatial nonlocal dispersal of individuals and is defined by

$$\mathcal{N}_i[u(\cdot,t)](x) := \int_{\mathbb{R}} J_i(y)u(x-y,t)dy - u(x,t),\ u = U, V,$$

where J_i is a probability density function satisfying the following conditions:

(H1) J_i is a nonnegative continuous function defined in $\mathbb{R}$;
(H2) $\int_{\mathbb{R}} J_i(y)dy = 1$ and $J_i(y) = J_i(-y)$ for all $y \in \mathbb{R}$;

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(H3) there exists $\hat{\lambda}_i \in (0,\infty]$ such that $\int_{\mathbb{R}} J_i(y)e^{\lambda y}dy < \infty$ for any $\lambda \in (0,\hat{\lambda}_i)$, and $\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy \to \infty$ as $\lambda \uparrow \hat{\lambda}_2$.

Note that there is always the predator-free state $(1,0)$, and there is the unique constant co-existence state (a^*,a^*), $a^* := 1-1/a \in (0,1)$, when $a>1$.

When the nonlocal dispersal in system (1.1) is replaced by the classical diffusion, the dynamical behaviors of the corresponding system were investigated in the survey paper [2]. See also [1] for more general reaction terms. In fact, system (1.1) without diffusion (the ODE system) arises in the control of introduced rabbits to protect native birds from introduced cat predation in an island (cf. [4]), when we consider the case without rabbits and control. For the detailed biological background of the full ODE system including the rabbits and control, we refer the reader to [4, 5]. However, it is reasonable and more realistic to take into account the influence of spatial movements of birds and cats, namely, the effect of diffusion. On the other hand, to model the long range movements and nonadjacent interactions of individuals it is more realistic to consider the nonlocal dispersal instead of the random movement with classical diffusion. This motivates us to study system (1.1).

The main purpose of this paper is to study the existence and nonexistence of traveling wave solutions to (1.1) connecting the predator-free state and the co-existence state. Here a solution (U,V) to (1.1) is called a traveling wave solution of (1.1), if there exist a constant $s \in \mathbb{R}$ (*the wave speed*) and a function (ϕ,ψ) (*the wave profile*) of class $C^1(\mathbb{R})$ such that

$$(U,V)(x,t) = (\phi,\psi)(z), \quad z := x - st.$$

We are interested in the traveling waves connecting the predator-free state $(1,0)$ to the co-existence state (a^*,a^*). Therefore, $\{s,\phi,\psi\}$ satisfies the following system of equations:

$$\begin{cases} \mathcal{N}_1[\phi](z) + s\phi'(z) + a\phi(z)\left[1-\phi(z)\right] - \psi(z) = 0, & z \in \mathbb{R}, \\ d\mathcal{N}_2[\psi](z) + s\psi'(z) + b\psi(z)\left[1 - \dfrac{\psi(z)}{\phi(z)}\right] = 0, & z \in \mathbb{R}, \end{cases} \tag{1.2}$$

together with the following asymptotic boundary conditions

$$(\phi,\psi)(\infty) = (1,0), \quad (\phi,\psi)(-\infty) = (a^*,a^*). \tag{1.3}$$

For convenience, for $i=1,2$, we set

$$I_i(\lambda) := \int_{\mathbb{R}} J_i(y)e^{\lambda y}dy, \quad \lambda \in \mathbb{R}.$$

Note that, by (H3), $I_i(\lambda) \in (0,\infty)$ if $0 \le \lambda < \hat{\lambda}_i$, $i=1,2$, and $I_2(\lambda) = \infty$ if $\lambda \ge \hat{\lambda}_2$. Also, due to the symmetry of J_i, the function I_i is even, and it is also strictly convex in $(-\hat{\lambda}_i,\hat{\lambda}_i)$. Then we introduce the quantity

$$s^* := \inf_{\lambda \in (0,\hat{\lambda}_2)} \frac{d[I_2(\lambda)-1]+b}{\lambda}. \tag{1.4}$$

Note that s^* is well-defined, the infimum is reached, and $s^*>0$.

We now state our main theorem as follows.

Theorem 1.1. *Let a, b and d be given positive constants such that $a \geq 4$ and $d < b$. Then there exists a positive solution (ϕ, ψ) of* (1.2)-(1.3) *for each $s > s^*$. This existence also holds for $s = s^*$, if we further assume that J_2 has compact support. Moreover, there exist no positive solutions of* (1.2)-(1.3) *for $s < s^*$.*

Hereafter, a function (ϕ, ψ) is positive if $\phi, \psi > 0$ in $\mathbb{R}$. A corresponding theorem to Theorem 1.1, under the condition $a \geq 4$, on the traveling waves in the classical diffusion case was derived in [2]. Theorem 1.1 characterizes the minimal speed s^* in the nonlocal dispersal case, but it needs the extra condition $d < b$ in the construction of lower and upper solutions to (1.2).

Remark 1.1. In condition (H3), when $\hat{\lambda}_2 = \infty$, we must have

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}} J_2(y) e^{\lambda y} dy = \infty.$$

Indeed, due to conditions (H1) and (H2), there exist positive numbers $0 < \delta < y_0$ such that $J_2(y) \geq J_2(y_0)/2 > 0$ for all $y \in [y_0 - \delta, y_0 + \delta] \subset (0, \infty)$. This implies that

$$\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy \geq \delta J_2(y_0) e^{\lambda(y_0 - \delta)} \to \infty \text{ as } \lambda \to \infty.$$

Also, the infinite limit in the case when $\hat{\lambda}_2 < \infty$ is natural. This assumption covers the important case when $J_2(y) \sim C_2 e^{-\hat{\lambda}_2 y}$ as $y \to \infty$, for some $\hat{\lambda}_2 \in (0, \infty)$ and $C_2 > 0$. On the other hand, if $\hat{\lambda}_2 \in (0, \infty)$ is such that $\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy$ is finite for any $\lambda \in (0, \hat{\lambda}_2)$, infinite for any $\lambda > \hat{\lambda}_2$ and has a finite limit as $\lambda \uparrow \hat{\lambda}_2$ (for instance if $J_2(y) \sim C_2 e^{-\hat{\lambda}_2 y}/y^p$ as $y \to +\infty$ for some $C_2 > 0$ and $p > 1$), then the quantity s^* in (1.4) might not be reached in the open interval $(0, \hat{\lambda}_2)$, and the proofs (see especially Section 2.1) would not extend as such.

It is worth noting that there are some difficulties in the study of system (1.1). It is a non-monotone system which is lack of comparison principle and a singularity may occur in the V-equation when U reaches zero in a finite time which has been confirmed numerically in [4]. Thus system (1.1) is named as a *singular* prey-predator model due to the latter fact. This is in contrast to the system with U-equation in (1.1) replaced by

$$U_t(x,t) = \mathcal{N}_1[U(\cdot,t)](x) + [aU(1-U) - kUV](x,t), \ x \in \mathbb{R}, \ t > 0,$$

in which the functional response of predation is a linear function of prey. In this case, the strong maximum principle for the scalar equation gives that $U > 0$ for all $t > 0$. Hence no singularity can occur for the nonlinear term V/U in the V-equation.

The existence of wave profiles (ϕ, ψ) to (1.2)-(1.3) for all speeds $s > s^*$ is based on the construction of lower and upper solutions, which is new for this singular nonlocal system. The case $s = s^*$ is more involved, it requires a special care and uses the boundedness of the support of J_2 to control the convergence rates of suitable upper and lower solutions at $+\infty$. If the support of J_2 is not compact and if J_2 is approximated in a suitable sense by compactly supported kernels, the wave profiles with the minimal speeds associated to the approximated kernels might lose at the limit their simultaneous limiting conditions (1.3) at $+\infty$. It would definitely be a very interesting question to remove this compactness assumption on the support of J_2, but actually we do not exclude in general the non-existence of wave profiles with minimal speed when J_2 is not compactly supported. We leave this as an open question. In Section 2.2, whether s is larger than or equal to s^*, the characterization of the limiting state behind the front, that is, $(\phi,\psi)(-\infty)=(a^*,a^*)$ is carried out with a squeezing method and an argument by contradiction. The proof of the nonexistence of wave profiles for all speeds $s<s^*$ is also done by contradiction. Section 2 is devoted to the existence part in Theorem 1.1, while the nonexistence is shown in Section 3.

2. Existence of traveling waves

This section is devoted to the derivation of the existence of positive traveling waves, under the assumptions $a\geq 4$ and $d<b$. First, we give the definition of upper and lower solutions as follows.

Definition 2.1. Positive continuous functions $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ are called a pair of upper and lower solutions of (1.2) if $\underline{\phi}(z)\leq\overline{\phi}(z)$, $\underline{\psi}(z)\leq\overline{\psi}(z)$ for all $z\in\mathbb{R}$ and the following inequalities

$$\mathcal{U}_1(z):=\mathcal{N}_1[\overline{\phi}](z)+s\overline{\phi}'(z)+a\overline{\phi}(z)[1-\overline{\phi}(z)]-\underline{\psi}(z)\leq 0, \tag{2.1}$$

$$\mathcal{U}_2(z):=d\mathcal{N}_2[\overline{\psi}](z)+s\overline{\psi}'(z)+b\overline{\psi}(z)[1-\overline{\psi}(z)/\overline{\phi}(z)]\leq 0, \tag{2.2}$$

$$\mathcal{L}_1(z):=\mathcal{N}_1[\underline{\phi}](z)+s\underline{\phi}'(z)+a\underline{\phi}(z)[1-\underline{\phi}(z)]-\overline{\psi}(z)\geq 0, \tag{2.3}$$

$$\mathcal{L}_2(z):=d\mathcal{N}_2[\underline{\psi}](z)+s\underline{\psi}'(z)+b\underline{\psi}(z)[1-\underline{\psi}(z)/\underline{\phi}(z)]\geq 0 \tag{2.4}$$

hold for all $z\in\mathbb{R}\setminus E$, where E is some finite subset of $\mathbb{R}$.

The following result can be proved by applying Schauder's fixed point theorem (cf., e.g., [3, 7, 11] and references cited therein). We safely omit its proof.

Proposition 2.1. *Let $s>0$ be given. Let $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ be a pair of bounded upper and lower solutions of* (1.2)*. Then system* (1.2) *admits a positive solution (ϕ,ψ) of class $C^1(\mathbb{R})$ such that*

$$\underline{\phi}(z)\leq\phi(z)\leq\overline{\phi}(z),\ \underline{\psi}(z)\leq\psi(z)\leq\overline{\psi}(z),\ z\in\mathbb{R}.$$

2.1. **Upper and lower solutions.** To derive the existence of traveling waves, we need to find some suitable pairs of upper and lower solutions of (1.2). For this, we divide the analysis into two cases: $s>s^*$ and $s=s^*$. The main idea of the following construction is motivated by [2, 7].

2.1.1. *The case $s>s^*$.* For a given $s>s^*$, we consider the quantity

$$A(\lambda)=A(\lambda;s):=d[I_2(\lambda)-1]-s\lambda+b,\ \lambda>0.$$

It follows from the definition of s^* and the strict convexity of $A(\lambda)$ that there are two positive constants $\lambda_1<\lambda_2<\hat{\lambda}_2$ such that

$$A(\lambda_1)=A(\lambda_2)=0$$

and $A(\lambda)<0$ for all $\lambda\in(\lambda_1,\lambda_2)$. Also, set

$$B(\lambda)=B(\lambda;s):=[I_1(\lambda)-1]-s\lambda.$$

Since $B(0) = 0$ and $B'(0) = -s < 0$, we can choose a constant $\lambda_0 \in (0, \min\{\lambda_1, \hat{\lambda}_1\})$ small enough such that

$$B(\lambda_0) < 0.$$

Now, for a fixed constant $\mu \in (1, \min\{\lambda_2/\lambda_1, 2\})$, we choose

$$q > \max\Big\{1, \frac{2b}{-A(\mu\lambda_1)}\Big\}. \tag{2.5}$$

Then, the function $f(z) := e^{-\lambda_1 z} - qe^{-\mu\lambda_1 z}$ has exactly one zero $z_0 > 0$ and exactly one maximum point $z_M \in (z_0, \infty)$, and there holds $f(z_M) > 0$. Thus, using $a \geq 4 > 1$ and $d < b$, we can choose δ such that

$$0 < \delta < \min\Big\{f(z_M), \underbrace{1 - \frac{1}{a}}_{=a^*}, \frac{1}{2}\Big(1 - \frac{d}{b}\Big)\Big\},$$

and a point $z_1 \in (z_0, z_M)$ such that $f(z_1) = \delta$. Note that

$$d < b(1 - 2\delta). \tag{2.6}$$

Furthermore, the inequalities $z_1 > z_0 > 0$, $\lambda_1 > 0$ and $\mu > 1$ imply

$$e^{(\mu-1)\lambda_1 z_1} > e^{(\mu-1)\lambda_1 z_0} = q. \tag{2.7}$$

With these choices of μ, q, δ, we finally choose ε such that

$$0 < \varepsilon < \min\Big\{\frac{\delta}{1 + s\lambda_1 + a}, \frac{e^{(\mu-1)\lambda_1 z_1} - q}{(1 + s\lambda_1 + a)e^{(\mu-1)\lambda_1 z_1}}\Big\}. \tag{2.8}$$

Note that the constant ε is admissible, due to (2.7), and that $0 < \varepsilon < \delta < 1/2$.

Then we introduce the following bounded positive continuous functions

$$\begin{aligned} &\overline{\phi}(z) = \begin{cases} 1 - \varepsilon, & z \leq 0, \\ 1 - \varepsilon e^{-\lambda_1 z}, & z > 0, \end{cases} \qquad \underline{\phi}(z) = \begin{cases} \frac{1}{2}, & z \leq 0, \\ 1 - \frac{1}{2}e^{-\lambda_0 z}, & z > 0, \end{cases} \\ &\overline{\psi}(z) = \begin{cases} 1, & z \leq 0, \\ e^{-\lambda_1 z}, & z > 0, \end{cases} \qquad \underline{\psi}(z) = \begin{cases} \delta, & z \leq z_1, \\ e^{-\lambda_1 z} - qe^{-\mu\lambda_1 z}, & z > z_1. \end{cases} \end{aligned} \tag{2.9}$$

Lemma 2.2. *Assume that $a \geq 4$ and $d < b$. For $s > s^*$, the functions $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ defined in* (2.9) *are a pair of upper and lower solutions of* (1.2).

Proof. **Step 1.** Note that $\overline{\phi} \leq 1$ in $\mathbb{R}$. For $z < 0$, $\overline{\phi}(z) = 1 - \varepsilon$, whence $\mathcal{N}_1[\overline{\phi}](z) \leq \varepsilon$ and so

$$\mathcal{U}_1(z) \leq [\varepsilon + a(1 - \varepsilon)\varepsilon] - \delta < 0,\ z < 0,$$

due to $0 < \varepsilon < \delta/(1 + s\lambda_1 + a) < \delta/(1 + a)$. For $z > 0$, we have $\overline{\phi}(z) = 1 - \varepsilon e^{-\lambda_1 z}$ and so $\mathcal{N}_1[\overline{\phi}](z) \leq \varepsilon e^{-\lambda_1 z}$. Hence

$$\mathcal{U}_1(z) \leq \varepsilon e^{-\lambda_1 z} + \varepsilon s\lambda_1 e^{-\lambda_1 z} + a\varepsilon e^{-\lambda_1 z} - \underline{\psi}(z) = \varepsilon(1 + s\lambda_1 + a)e^{-\lambda_1 z} - \underline{\psi}(z),\ z > 0.$$

For $z \in (0, z_1)$, there holds $\underline{\psi}(z) = \delta$, which implies that

$$\mathcal{U}_1(z) \leq \varepsilon(1 + s\lambda_1 + a) - \delta < 0,$$

since $\varepsilon < \delta/(1+s\lambda_1+a)$. For $z > z_1$, we have $\underline{\psi}(z) = e^{-\lambda_1 z} - qe^{-\mu\lambda_1 z}$ and so

$$\mathcal{U}_1(z) \le e^{-\lambda_1 z}[\varepsilon(1+s\lambda_1+a)-1] + qe^{-\mu\lambda_1 z} < 0,$$

provided

$$q < [1-\varepsilon(1+s\lambda_1+a)]e^{(\mu-1)\lambda_1 z_1}. \tag{2.10}$$

But (2.10) holds due to (2.8). Therefore, (2.1) holds for all $z \in \mathbb{R}\setminus\{0, z_1\}$.

Step 2. For $z < 0$, since $\overline{\psi}(z) = 1$ and $\overline{\psi} \le 1$ in $\mathbb{R}$, we have $\mathcal{N}_2[\overline{\psi}](z) \le 0$. Hence

$$\mathcal{U}_2(z) \le b(1 - 1/\overline{\phi}(z)) \le 0,\ z < 0,$$

due to $0 < \overline{\phi} \le 1$ in $\mathbb{R}$. For $z > 0$, from $\overline{\psi}(z) = e^{-\lambda_1 z}$ and $\overline{\psi}(z-y) \le e^{-\lambda_1(z-y)}$ for all $y \in \mathbb{R}$ it follows that

$$\mathcal{U}_2(z) \le d[I_2(\lambda_1)-1]e^{-\lambda_1 z} - s\lambda_1 e^{-\lambda_1 z} + be^{-\lambda_1 z} = A(\lambda_1)e^{-\lambda_1 z} = 0,$$

due to $A(\lambda_1) = 0$. Therefore, (2.2) holds for all $z \in \mathbb{R}\setminus\{0\}$.

Step 3. For $z < 0$, we have $\underline{\phi}(z) = 1/2$. Since $\underline{\phi} \ge 1/2$ in $\mathbb{R}$, we obtain $\mathcal{N}_1[\underline{\phi}](z) \ge 0$ for $z < 0$. Then $\mathcal{L}_1(z) \ge a/4 - 1 \ge 0$ for $z < 0$, using $a \ge 4$. For $z > 0$, we have $\underline{\phi}(z) = 1 - e^{-\lambda_0 z}/2 \ge 1/2$ and $\overline{\psi}(z) = e^{-\lambda_1 z}$. Since $\underline{\phi}(z-y) \ge 1 - e^{-\lambda_0(z-y)}/2$ for all $y \in \mathbb{R}$ and $\underline{\phi} \ge 1/2$ in $\mathbb{R}$, we get, for $z > 0$,

$$\begin{aligned}
\mathcal{L}_1(z) &\ge [1 - e^{-\lambda_0 z} I_1(\lambda_0)/2] - (1 - e^{-\lambda_0 z}/2) + s\lambda_0 e^{-\lambda_0 z}/2 + (a/2)e^{-\lambda_0 z}/2 - e^{-\lambda_1 z}\\
&\ge \frac{1}{2} e^{-\lambda_0 z}\{-[I_1(\lambda_0) - 1 - s\lambda_0] + a/2 - 2\} \ge 0,
\end{aligned}$$

due to $e^{-\lambda_1 z} < e^{-\lambda_0 z}$, $B(\lambda_0) < 0$ and $a \ge 4$. Therefore, (2.3) holds for all $z \in \mathbb{R}\setminus\{0\}$.

Step 4. For $z < z_1$, $\underline{\psi}(z) = \delta$ and we compute

$$\mathcal{L}_2(z) \ge -d\delta + b\delta(1-2\delta) \ge 0,$$

using $\underline{\psi} \ge 0$, $\underline{\phi} \ge 1/2$ and (2.6). For $z > z_1$, since $\underline{\psi}(y) \ge e^{-\lambda_1 y} - qe^{-\mu\lambda_1 y}$ for all $y \in \mathbb{R}$, we compute

$$\begin{aligned}
\mathcal{L}_2(z) &\ge d\{[e^{-\lambda_1 z} I_2(\lambda_1) - qe^{-\mu\lambda_1 z} I_2(\mu\lambda_1)] - (e^{-\lambda_1 z} - qe^{-\mu\lambda_1 z})\}\\
&\qquad - s(\lambda_1 e^{-\lambda_1 z} - q\mu\lambda_1 e^{-\mu\lambda_1 z}) + b(e^{-\lambda_1 z} - qe^{-\mu\lambda_1 z})[1 - \underline{\psi}(z)/\underline{\phi}(z)]\\
&\ge e^{-\lambda_1 z} A(\lambda_1) - qe^{-\mu\lambda_1 z} A(\mu\lambda_1) - 2be^{-2\lambda_1 z}\\
&= \{-e^{-\mu\lambda_1 z} A(\mu\lambda_1)\}\left\{q - \frac{2be^{(\mu-2)\lambda_1 z}}{[-A(\mu\lambda_1)]}\right\} \ge 0,
\end{aligned}$$

using $\underline{\psi}(z) \le e^{-\lambda_1 z}$, $\underline{\phi} \ge 1/2$, $A(\lambda_1) = 0$, $A(\mu\lambda_1) < 0$, $\mu < 2$,

$$e^{(\mu-2)\lambda_1 z} \le e^{(\mu-2)\lambda_1 z_1} < 1,\ z > z_1 > 0,$$

and (2.5). Here we have used $(\alpha-\beta)^2 \le \alpha^2$ for any $\alpha \ge \beta > 0$, applied to $\alpha := e^{-\lambda_1 z}$ and $\beta := qe^{-\mu\lambda_1 z}$. Therefore, (2.4) holds for all $z \in \mathbb{R}\setminus\{z_1\}$.

Lastly, it is clear by construction that $1/2 \le \underline{\phi} \le \overline{\phi} < 1$ and $0 < \underline{\psi} \le \overline{\psi} \le 1$ in $\mathbb{R}$. The lemma is thereby proved. $\square$

2.1.2. *The case $s = s^*$.* In this case, we have $\lambda_1 = \lambda_2 > 0$, i.e., λ_1 is a double root of

$$A(\lambda; s^*) = 0,$$

with necessarily $\lambda_1 < \hat{\lambda}_2$. In this case, we also have

$$d \int_{\mathbb{R}} J_2(y) y e^{\lambda_1 y} dy = s^*. \tag{2.11}$$

Suppose that J_2 has a compact support such that its support is contained in $[-S, S]$ for some $S \in (0, \infty)$. We also choose a small positive constant $\lambda_0 \in (0, \min\{\lambda_1, \hat{\lambda}_1\})$ such that

$$B(\lambda_0; s^*) < 0.$$

First, we choose a fixed constant $h > \lambda_1 e$ large enough such that the equation $hze^{-\lambda_1 z} = 1$ has exactly two roots z_1, z_2 with

$$0 < z_1 < \frac{1}{\lambda_1} < z_2 < \infty \quad \text{and} \quad z_2 - z_1 > S.$$

Note that $hze^{-\lambda_1 z} > 1$ for all $z \in (z_1, z_2)$, and that the map $z \mapsto hze^{-\lambda_1 z}$ is decreasing in $[z_2, \infty)$. Secondly, we choose $z_3 > z_2$ large enough such that

$$he^{-\lambda_0 z_3} \leq \frac{a(\lambda_1 - \lambda_0)e}{4}. \tag{2.12}$$

Thirdly, for any positive constant q, one can check that the function

$$g(z) := (hz - q\sqrt{z})e^{-\lambda_1 z}, \quad z \geq 0,$$

has a zero $z_0 := (q/h)^2 \in (0, \infty)$ such that $g > 0$ in (z_0, ∞), $g(z_0) = 0$ and g has a unique maximal point $z_M \in (z_0, \infty)$. Note that $g(z_M) > 0$. Furthermore, g is increasing in $[z_0, z_M]$ and decreasing in $[z_M, \infty)$. We choose a constant q large enough such that $z_0 > z_2$ and

$$q > \frac{16bh^2 \max_{z>0}\{z^2(z+S)^{3/2} e^{-\lambda_1 z}\}}{d \int_{\mathbb{R}} J_2(y) y^2 e^{\lambda_1 y} dy}. \tag{2.13}$$

Then we choose z_4 in (z_0, z_M) such that $g(z_4) = \delta$, for a fixed δ satisfying

$$0 < \delta < \min\left\{g(z_M), a^*, \frac{1}{2}\Big(1 - \frac{d}{b}\Big)\right\},$$

so that (2.6) still holds. Lastly, with the above choices of h, q, δ, we choose ε such that

$$0 < \varepsilon < \frac{\min\{\delta, hz_4 - q\sqrt{z_4}\}}{1 + s^*\lambda_1 + a}. \tag{2.14}$$

Note that $0 < \varepsilon < \delta < 1/2$.

Then we define the bounded positive continuous functions

$$\begin{aligned}
&\overline{\phi}(z) = \begin{cases} 1 - \varepsilon, & z \leq 0, \\ 1 - \varepsilon e^{-\lambda_1 z}, & z > 0, \end{cases} \qquad \underline{\phi}(z) = \begin{cases} \frac{1}{2}, & z \leq z_3, \\ 1 - \frac{1}{2}e^{-\lambda_0(z - z_3)}, & z > z_3, \end{cases} \\
&\overline{\psi}(z) = \begin{cases} 1, & z \leq z_2, \\ hze^{-\lambda_1 z}, & z > z_2, \end{cases} \qquad \underline{\psi}(z) = \begin{cases} \delta, & z \leq z_4, \\ (hz - q\sqrt{z})e^{-\lambda_1 z}, & z > z_4. \end{cases}
\end{aligned} \tag{2.15}$$

Lemma 2.3. *Assume that $a \geq 4$, $d < b$, and that J_2 has a compact support such that its support is contained in $[-S, S]$ for some $S \in (0, \infty)$. For $s = s^*$, the functions $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ defined in (2.15) are a pair of upper and lower solutions of (1.2).*

Proof. As before, we also divide our proof into 4 steps.

Step 1. Note that $\overline{\phi} \leq 1$ in $\mathbb{R}$. For $z < 0$, we have $\mathcal{U}_1(z) \leq [\varepsilon + a(1-\varepsilon)\varepsilon] - \delta \leq 0$, due to (2.14). For $z \in (0, z_4)$, we have

$$\mathcal{U}_1(z) \leq \varepsilon(1 + s^*\lambda_1 + a)e^{-\lambda_1 z} - \delta \leq 0,$$

using $\varepsilon < \delta/(1 + s^*\lambda_1 + a)$. For $z > z_4$, we have $\underline{\psi}(z) = (hz - q\sqrt{z})e^{-\lambda_1 z}$ and so

$$\mathcal{U}_1(z) \leq \{\varepsilon(1 + s^*\lambda_1 + a) - (hz - q\sqrt{z})\}e^{-\lambda_1 z} \leq 0,$$

due to (2.14) and $hz - q\sqrt{z} \geq hz_4 - q\sqrt{z_4}$ for all $z \geq z_4 > z_0$. Therefore, (2.1) holds for all $z \in \mathbb{R} \setminus \{0, z_4\}$.

Step 2. For $z < z_2$, since $\overline{\psi}(z) = 1$ and $\overline{\psi} \leq 1$ in $\mathbb{R}$, we have $\mathcal{N}_2[\overline{\psi}](z) \leq 0$. Hence $\mathcal{U}_2(z) \leq 0$ for $z < z_2$, since $0 < \overline{\phi} \leq 1$ in $\mathbb{R}$. For $z > z_2$, we have $\overline{\psi}(z) = hze^{-\lambda_1 z}$ and

$$\int_{\mathbb{R}} J_2(y)\overline{\psi}(z-y)dy = \int_{-S}^{S} J_2(y)\overline{\psi}(z-y)dy \leq \int_{\mathbb{R}} J_2(y)h(z-y)e^{-\lambda_1(z-y)}dy,$$

since $z - y > z_2 - S > z_1$ for all $y \in [-S, S]$ gives $\overline{\psi}(z-y) \leq h(z-y)e^{-\lambda_1(z-y)}$. Hence we get, for $z > z_2$,

$$\mathcal{U}_2(z) \leq \{d[I_2(\lambda_1) - 1] - s^*\lambda_1 + b\}(hze^{-\lambda_1 z}) - \left[d\int_{\mathbb{R}} J_2(y)ye^{\lambda_1 y}dy - s^*\right](he^{-\lambda_1 z}) = 0,$$

using $A(\lambda_1; s^*) = 0$ and (2.11). Therefore, (2.2) holds for all $z \in \mathbb{R} \setminus \{z_2\}$.

Step 3. For $z < z_3$, we have $\underline{\phi}(z) = 1/2$. Hence $\mathcal{L}_1(z) \geq a/4 - 1 \geq 0$ for $z < z_3$, using $\underline{\phi} \geq 1/2$ in $\mathbb{R}$, $\overline{\psi} \leq 1$ in $\mathbb{R}$, and $a \geq 4$. For $z > z_3$, using $\underline{\phi}(z) = 1 - e^{-\lambda_0(z-z_3)}/2$, $\underline{\phi}(z-y) \geq 1 - e^{-\lambda_0(z-y-z_3)}/2$ for all $y \in \mathbb{R}$, and $\underline{\phi} \geq 1/2$ in $\mathbb{R}$, we get

$$\begin{aligned}
\mathcal{L}_1(z) &\geq \frac{1}{2}e^{-\lambda_0(z-z_3)}\{-[I_1(\lambda_0) - 1 - s^*\lambda_0] + a/2\} - hze^{-\lambda_1 z}, \\
&\geq \frac{1}{2}e^{-\lambda_0(z-z_3)}\{a/2 - 2hze^{-\lambda_1 z}e^{\lambda_0(z-z_3)}\} \geq 0,
\end{aligned}$$

due to $B(\lambda_0; s^*) < 0$ and (2.12). Here we have also used

$$2hze^{-\lambda_1 z}e^{\lambda_0(z-z_3)} \leq \frac{2he^{-\lambda_0 z_3}}{(\lambda_1 - \lambda_0)e}, \ \forall\, z > 0.$$

Therefore, (2.3) holds for all $z \in \mathbb{R} \setminus \{z_3\}$.

Step 4. For $z < z_4$, $\underline{\psi}(z) = \delta$ implies that

$$\mathcal{L}_2(z) \geq -d\delta + b\delta(1 - 2\delta) \geq 0,$$

using $\underline{\psi} \geq 0$, $\underline{\phi}(z) \geq 1/2$ and (2.6). Recall that $z_4 > z_0 > z_2 > z_1 + S > S$ and $J_2(y) = 0$ for $y \in \mathbb{R} \setminus [-S, S]$. Now, let $z > z_4$. Using $z - y > z_4 - y > z_2 - S > z_1 > 0$ for $y \in [-S, S]$ and

$$\underline{\psi}(\xi) \geq (h\xi - q\sqrt{\xi})e^{-\lambda_1 \xi} \text{ for all } \xi \geq 0,$$

we get that

$$
\begin{aligned}
&\int_{\mathbb{R}} J_2(y)\underline{\psi}(z-y)dy = \int_{-S}^{S} J_2(y)\underline{\psi}(z-y)dy \\
\geq\ & \int_{-S}^{S} J_2(y)[h(z-y)-q\sqrt{z-y}]e^{-\lambda_1(z-y)}dy \\
=\ & hze^{-\lambda_1 z}\int_{-S}^{S} J_2(y)e^{\lambda_1 y}dy - he^{-\lambda_1 z}\int_{-S}^{S} J_2(y)ye^{\lambda_1 y}dy - qe^{-\lambda_1 z}\int_{-S}^{S} J_2(y)\sqrt{z-y}e^{\lambda_1 y}dy \\
=\ & hze^{-\lambda_1 z}I_2(\lambda_1) - he^{-\lambda_1 z}\int_{\mathbb{R}} J_2(y)ye^{\lambda_1 y}dy - qe^{-\lambda_1 z}\int_{-S}^{S} J_2(y)\sqrt{z-y}e^{\lambda_1 y}dy.
\end{aligned}
$$

Then, using $\underline{\phi} \geq 1/2$, $A(\lambda_1; s^*) = 0$ and (2.11), we obtain

$$
\begin{aligned}
\mathcal{L}_2(z) \ \geq\ & hze^{-\lambda_1 z}\{d[I_2(\lambda_1)-1] - s^*\lambda_1 + b\} - he^{-\lambda_1 z}\left\{d\int_{\mathbb{R}} J_2(y)ye^{\lambda_1 y}dy - s^*\right\} \\
& -dqe^{-\lambda_1 z}\left\{\int_{-S}^{S} J_2(y)\sqrt{z-y}e^{\lambda_1 y}dy - \sqrt{z}\right\} + qs^*\left[\lambda_1\sqrt{z} - \frac{1}{2\sqrt{z}}\right]e^{-\lambda_1 z} \\
& -bq\sqrt{z}e^{-\lambda_1 z} - 2bh^2z^2e^{-2\lambda_1 z} \\
:=\ & e^{-\lambda_1 z}[qQ_1(z) - Q_2(z)],
\end{aligned}
$$

where

$$
\begin{aligned}
&Q_1(z) := -d\left\{\int_{-S}^{S} J_2(y)\sqrt{z-y}e^{\lambda_1 y}dy - \sqrt{z}\right\} + s^*\left[\lambda_1\sqrt{z} - \frac{1}{2\sqrt{z}}\right] - b\sqrt{z}, \\
&Q_2(z) := 2bh^2z^2e^{-\lambda_1 z}.
\end{aligned}
$$

Note that, using $A(\lambda_1; s^*) = 0$ and (2.11), we get

$$
\begin{aligned}
Q_1(z) \ =\ & d\int_{-S}^{S} J_2(y)[\sqrt{z} - \sqrt{z-y}]e^{\lambda_1 y}dy - \frac{d}{2\sqrt{z}}\int_{-S}^{S} J_2(y)ye^{\lambda_1 y}dy \\
=\ & d\int_{-S}^{S}\left[\sqrt{z} - \sqrt{z-y} - \frac{y}{2\sqrt{z}}\right]J_2(y)e^{\lambda_1 y}dy \\
=\ & d\int_{-S}^{S}\frac{1}{2\sqrt{z}(\sqrt{z}+\sqrt{z-y})^2}J_2(y)y^2e^{\lambda_1 y}dy \geq \frac{d}{8(\sqrt{z+S})^3}\int_{\mathbb{R}} J_2(y)y^2e^{\lambda_1 y}dy.
\end{aligned}
$$

Hence we conclude that

$$
\mathcal{L}_2(z) \geq \frac{de^{-\lambda_1 z}}{8(\sqrt{z+S})^3}\left(\int_{\mathbb{R}} J_2(y)y^2e^{\lambda_1 y}dy\right)\left\{q - \frac{16bh^2[z^2(z+S)^{3/2}e^{-\lambda_1 z}]}{d\int_{\mathbb{R}} J_2(y)y^2e^{\lambda_1 y}dy}\right\} \geq 0
$$

for $z > z_4$, due to (2.13). Therefore, (2.4) holds for all $z \in \mathbb{R}\setminus\{z_4\}$.

One can also check that $1/2 \leq \underline{\phi} \leq \overline{\phi} < 1$ and $0 < \underline{\psi} \leq \overline{\psi} \leq 1$ in $\mathbb{R}$, and the lemma is thus proved. $\square$

2.2. **Tail limits for $s \geq s^*$.** First, for any $s \geq s^*$, the existence of a $C^1(\mathbb{R})$ solution (ϕ, ψ) to (1.2) such that $1/2 \leq \underline{\phi} \leq \phi \leq \overline{\phi} < 1$ and $0 < \underline{\psi} \leq \psi \leq \overline{\psi} \leq 1$ in $\mathbb{R}$ follows from Lemmas 2.2-2.3 and Proposition 2.1. Moreover, by the construction of upper and lower solutions, it is clear that $(\phi, \psi)(\infty) = (1, 0)$.

Next, we claim that the limit $(\phi, \psi)(-\infty)$ exists, by applying the method used in [16]. For the reader's convenience, we provide an outline as follows. To show the limit exists, we let

$$\phi^- := \liminf_{z \to -\infty} \phi(z), \quad \phi^+ := \limsup_{z \to -\infty} \phi(z), \quad \psi^- := \liminf_{z \to -\infty} \psi(z), \quad \psi^+ := \limsup_{z \to -\infty} \psi(z).$$

By the construction of upper-lower solutions, we have

$$1/2 \leq \phi^- \leq \phi^+ \leq 1 \ \text{ and } \ 0 < \delta \leq \psi^- \leq \psi^+ \leq 1,$$

for some $\delta \in (0, a^*)$, in both cases $s > s^*$ and $s = s^*$.

Now, from (1.2) and the positivity of s, both functions ϕ and ψ have bounded derivatives ϕ' and ψ', and by differentiating (1.2), the functions ϕ and ψ are of class $C^2(\mathbb{R})$ with bounded second-order derivatives (ϕ and ψ are actually of class $C^\infty(\mathbb{R})$). From the definition of ψ^+, there exists a sequence $(z_n)_{n \in \mathbb{N}}$ diverging to $-\infty$ such that $\psi(z_n) \to \psi^+$, $\psi'(z_n) \to 0$, and $\phi(z_n) \to \gamma$ as $n \to \infty$, for some $\gamma \in [\phi^-, \phi^+] \subset [1/2, 1]$. By evaluating the second equation in (1.2) at z_n and passing to the limit as $n \to \infty$, one gets, by the same argument as that for [16, (2.6)], that $b\psi^+(1 - \psi^+/\gamma) \geq 0$, whence

$$\psi^+ \leq \phi^+.$$

We omit the details here. Similarly, one has

$$\phi^- \leq \psi^-.$$

On the other hand, by considering a sequence $(\zeta_n)_{n \in \mathbb{N}}$ diverging to $-\infty$ such that $\phi(\zeta_n) \to \phi^-$, $\phi'(\zeta_n) \to 0$, and $\psi(\zeta_n) \to \Gamma$ as $n \to \infty$, for some $\Gamma \in [\psi^-, \psi^+]$, by evaluating the first equation in (1.2) at ζ_n and passing to the limit as $n \to \infty$, one gets that $a\phi^-(1 - \phi^-) - \Gamma \leq 0$, whence

$$a\phi^-(1 - \phi^-) \leq \psi^+. \tag{2.16}$$

Similarly, one has

$$a\phi^+(1 - \phi^+) \geq \psi^-. \tag{2.17}$$

Since $\psi^- > 0$ and $\phi^+ \leq 1$, one infers that $\phi^+ < 1$, whence $\psi^+ \leq \phi^+ < 1$. Therefore, $a\phi^-(1 - \phi^-) \leq \psi^+ < 1$ and, since $a \geq 4$ and $\phi^- \geq 1/2$, one gets that $\phi^- > 1/2$. Finally,

$$1/2 < \phi^- \leq \psi^- \leq \psi^+ \leq \phi^+ < 1. \tag{2.18}$$

Let now

$$D := \big\{\nu \in [0, 1] \mid l(\nu) < \phi^- \leq \phi^+ < r(\nu)\big\},$$

where

$$l(\nu) := \nu a^* + (1 - \nu)/2, \quad r(\nu) := \nu a^* + 1 - \nu, \quad \nu \in [0, 1].$$

Note that $1/2 < a^* < 1$. Hence $l(\nu)$ is increasing and $r(\nu)$ is decreasing in $\nu \in [0, 1]$, and $l(1) = r(1) = a^* = 1 - 1/a$. Secondly, due to (2.18), $0 \in D$ and so, by continuity of $l(\nu)$ and $r(\nu)$ with respect to ν, $\nu_0 := \sup D$ is well-defined such that $\nu_0 \in (0,1]$. For contradiction, we assume that $\nu_0 < 1$. Then, by passing to the limit,

$$l(\nu_0) \le \phi^- \le \phi^+ \le r(\nu_0), \tag{2.19}$$

and either $\phi^- = l(\nu_0)$ or $\phi^+ = r(\nu_0)$. Then we divide the discussion into two cases.

Case 1: $\phi^- = l(\nu_0)$. From (2.16) and (2.18)-(2.19), one knows that

$$al(\nu_0)[1 - l(\nu_0)] = a\phi^-(1-\phi^-) \le \psi^+ \le \phi^+ \le r(\nu_0),$$

whence

$$\omega := al(\nu_0)[1-l(\nu_0)] - r(\nu_0) \le 0.$$

But a straightforward calculation yields

$$\omega = (1-\nu_0)\left[\frac{a}{4} - 1 + \frac{\nu_0(a-2)^2}{4a}\right],$$

whence $\omega > 0$, since $0 < \nu_0 < 1$ and $a \ge 4$. Therefore, case 1 is ruled out.

Case 2: $\phi^+ = r(\nu_0)$. Similarly, from (2.17)-(2.19), one knows that

$$ar(\nu_0)[1 - r(\nu_0)] = a\phi^+(1-\phi^+) \ge \psi^- \ge \phi^- \ge l(\nu_0),$$

whence

$$\theta := ar(\nu_0)[1-r(\nu_0)] - l(\nu_0) \ge 0.$$

But a straightforward calculation yields

$$\theta = -(1-\nu_0)\left[\frac{1}{2} - \frac{\nu_0}{a}\right],$$

whence $\theta < 0$, since $0 < \nu_0 < 1$ and $a \ge 4 > 2$. Therefore, case 2 is ruled out too.

These contradictions entail that $\nu_0 = 1$ and so the limit $\phi(-\infty)$ exists such that $\phi(-\infty) = a^*$. Finally, it follows from (2.18) that $\psi(-\infty) = a^*$. This proves that $(\phi,\psi)(-\infty) = (a^*, a^*)$ and the proof of the existence part of Theorem 1.1 is thus complete. □

3. Nonexistence

In this section, we provide a proof of the nonexistence part of Theorem 1.1 as follows. Although it is almost the same as that in [7], for the reader's convenience we give the details here.

Theorem 3.1. *If $s < s^*$, then* (1.2)-(1.3) *does not admit any positive solution (ϕ,ψ).*

Proof. We argue by contradiction by assuming that (1.2) admits a positive solution (ϕ,ψ) satisfying (1.3) and for some wave speed $s_0 < s^*$. From the continuity and positivity of ϕ together with (1.3), there is $\varepsilon > 0$ such that $\phi \ge \varepsilon > 0$ in $\mathbb{R}$. Then the ψ-equation of (1.2) gives that

$$-s_0\psi'(z) \ge d\mathcal{N}_2[\psi](z) + b\psi(z)[1-\psi(z)/\varepsilon],\ z \in \mathbb{R}.$$

Hence the $C^1(\mathbb{R}\times\mathbb{R})$ function $V(x,t) := \psi(x - s_0 t)$ satisfies $L[V](x,t) \ge 0$, where

$$L[V](x,t) := V_t(x,t) - \left\{d\mathcal{N}_2[V(\cdot,t)](x) + bV(x,t)[1-V(x,t)/\varepsilon]\right\},\ x\in\mathbb{R},\ t>0, \tag{3.1}$$

and $V(x,0) = \psi(x) \not\equiv 0$ for $x \in \mathbb{R}$.

Next, since $\hat{s} := (s_0 + s^*)/2 < s^*$ and $V(\cdot, 0) \geq 0$ with $\liminf_{x\to-\infty} V(x, 0) > 0$, it follows from the comparison principle and the spreading dynamics for the nonlocal logistic equation $L[v] = 0$ (cf. [13]) that

$$\liminf_{t\to\infty} \psi(\hat{s}t - s_0 t) = \liminf_{t\to\infty} V(\hat{s}t, t) \geq \varepsilon > 0.$$

However, $z := \hat{s}t - s_0 t = (s^* - s_0)t/2 \to \infty$ as $t \to \infty$, a contradiction to $\psi(\infty) = 0$. Thus the theorem is proved. $\square$

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