Ann. Inst. Henri Poincaré,

Vol. 14, n° 4, 1997, p. 457-498 *Analyse non linéaire*

Reaction-diffusion problems in cylinders with no invariance by translation. Part I: Small perturbations

by

François HAMEL
Université Paris VI, Laboratoire d'Analyse Numérique,
4, place Jussieu, 75252 Paris Cedex 05, France.

ABSTRACT. – This paper deals with existence and uniqueness of solutions (c, u) of reaction-convection-diffusion equations mainly derived from combustion models and set in infinite cylinders $\Sigma = \{(x_1, y) \in \mathbb{R} \times \omega\}$

$$\begin{cases} a(x_1, y, u, \nabla u)\Delta u - (c + \alpha(y))\partial_1 u + \vec{q}(x_1, y, u, \nabla u) \cdot \nabla u \\ \qquad +f(u) + g(x_1, y, u, \nabla u) = 0 \text{ in } \Sigma \\ \qquad \partial_\nu u = 0 \text{ on } \partial\Sigma \\ \qquad u(-\infty, \cdot) = 0, \quad u(+\infty, \cdot) = 1 \end{cases}$$

The functions a, α, $\vec{q}$, f and g are given. The section ω is a bounded smooth domain with outward unit normal ν. The existence of (c, u) is proved under various normalization conditions when the perturbative terms a, $\vec{q}$, g are close to $(1, \vec{0}, 0)$, and a continuity result as $(a, \vec{q}, g) \to (1, \vec{0}, 0)$ is stated.

Key words: Nonlinear PDE's, small perturbations, sliding method, implicit function theorem.

RÉSUMÉ. – Cet article traite de l'existence et de l'unicité de solutions (c, u) d'équations de réaction-convection-diffusion provenant essentiellement de modèles de combustion et posées dans des cylindres

A.M.S. Classification: 35 J 60, 35 Q 35, 47 F 05, 58 G 18, 76 R 99.

infinis $\Sigma = \{(x_1, y) \in \mathbb{R} \times \omega\}$

$$\begin{cases} a(x_1, y, u, \nabla u)\Delta u - (c + \alpha(y))\partial_1 u + \vec{q}(x_1, y, u, \nabla u) \cdot \nabla u \\ \qquad +f(u) + g(x_1, y, u, \nabla u) = 0 \text{ dans } \Sigma \\ \qquad \partial_\nu u = 0 \text{ sur } \partial\Sigma \\ \qquad u(-\infty, \cdot) = 0, \quad u(+\infty, \cdot) = 1 \end{cases}$$

Les fonctions a, α, $\vec{q}$, f et g sont données. La section ω est un domaine borné régulier de normale extérieure unitaire ν. L'existence de (c, u) est prouvée pour différentes conditions de normalisation quand les termes perturbatifs a, $\vec{q}$, g sont proches de $(1, \vec{0}, 0)$, et on énonce un résultat de continuité quand $(a, \vec{q}, g) \to (1, \vec{0}, 0)$.

1. INTRODUCTION

The paper deals with existence and uniqueness of solutions (c, u) of semilinear reaction-convection-diffusion equations

$$a(x_1, y, u, \nabla u)\Delta u - (c + \alpha(y))\partial_1 u + \vec{q}(x_1, y, u, \nabla u) \cdot \nabla u \\ + f(u) + g(x_1, y, u, \nabla u) = 0 \tag{1}$$

set in infinite cylinders $\Sigma = \{(x_1, y) \in \mathbb{R} \times \omega\}$. Homogenous Neumann boundary conditions are imposed on $\partial\Sigma$ as well as uniform Dirichlet conditions $u(-\infty, \cdot) = 0$, $u(+\infty, \cdot) = 1$ as $x_1 \to \pm\infty$. The given heterogeneous and nonlinear diffusion term $a(x_1, y, u, \nabla u)\Delta u$ is close to the uniform isotropic diffusion Δu. The given flow is the sum of a main divergence free, monodirectional flow $(\alpha(y), 0, \cdots, 0)$ and a small heterogeneous nonlinear multidirectional flow $\vec{q}(x_1, y, u, \nabla u)$. In the same way, the reaction term is the sum of a main source term $f(u)$ and a small heterogeneous one $g(x_1, y, u, \nabla u)$.

The unknowns of this problem are firstly the stationnary function u, only depending on the space variables (x_1, y) and not on the time t, which goes from 0 as $x_1 \to -\infty$ (the left) and 1 at $+\infty$ (the right). This means that there is steady transformation between two given states 0 and 1. The second unknown is the real c, a speed, which is added to the velocity field $(\alpha(y), 0, \cdots, 0) + \vec{q}$ and makes possible this steady transformation.

When $a \equiv 1$, $\vec{q} \equiv 0$, $g \equiv 0$, this problem is now well-known. The goal of this work is to set existence and uniqueness results for the equation (1) for general and non uniform coefficients a, $\vec{q}$, g depending on $(x_1, y, u, \nabla u)$, but close to $(1, \vec{0}, 0)$: they are called "small perturbations".

The nonlinear function f, which is the reaction or source term is assumed to have one the following two profiles on $[0,1]$ which are currently mentionned in the literature:

- first case: $\exists\theta \in (0,1)$, $f \equiv 0$ on $[0,\theta] \cup \{1\}$, $f > 0$ on $(\theta, 1)$ (ignition temperature case),

- second case $\exists\theta \in (0,1)$, $f(0) = f(\theta) = f(1) = 0$, $f < 0$ on $(0,\theta)$, $f > 0$ on $(\theta, 1)$ (bistable case).

These nonlinear reaction terms have two different physical meanings and correspond to two different physical motivations.

In the first case, equation (1) is motivated by combustion theory. Roughly speaking, the starting point is the thermo-diffusive model for curved deflagration flame in an infinite tube where a simple chemical reaction $\mathcal{A} \rightarrow \mathcal{B}$ takes place between two premixed gases $\mathcal{A}$ and $\mathcal{B}$, the Lewis number of the reactant $\mathcal{A}$ being equal to 1. The function u is the renormalized temperature of the mixture and $1 - u$ is the renormalized concentration of the reactant $\mathcal{A}$ (*see* the synthetic works of Berestycki, Buckmaster, Larrouturou, Ludford, Sivashinsky and Williams for instance [4], [11], [26], [28]).

In this model of one single stationnary, *i.e.* time-independant equation, the small perturbations $a - 1$, $\vec{q}$, g may take into account the basic physical phenomena of turbulence or may be due to small changes of density (*cf.* [10]). When there is no source term $f(u) + g(x_1, y, u, \nabla u)$, the medium has the velocity field $(\alpha(y), 0, \cdots, 0) + \vec{q}$. The profile of the unknown function u solution of (1) represents a stationnary and stabilized flame in the flow $(c + \alpha(y), 0, \cdots, 0) + \vec{q}$. The real c may thus be viewed as a flame speed.

More explicitely, in models of combustion, the real θ represents an ignition temperature below which no reaction happens. The source term f takes into account the mass action law and Arrhenius's law. The boundary condition $\partial_\nu u = 0$ on $\partial\Sigma$ means that there is no flow across the walls of the cylinder. The limits $u(-\infty, \cdot) = 0$ and $u(+\infty, \cdot) = 1$ mean that the fresh mixture is on the left and the burnt gases on the right.

The second case of profile f, called "bistable", comes from the study of growth of populations, gene developments or nerve propagation (*cf.* [2], [12]).

In dimension 1, *with* $(a, \alpha, \vec{q}, g) = (1, 0, \vec{0}, 0)$, equation (1) reduces to $u'' - cu' + f(u) = 0$. There are many results initiated by the works of Kolmogorov-Petrovsky-Piskunov, Zeldovic-Frank-Kamenetskii and Kanel' (*cf.* [12], [18], [19], [33]).

The multidimensional case *with* $(a, \vec{q}, g) = (1, \vec{0}, 0)$ corresponds to non planar solutions. It was especially studied by Berestycki, Larrouturou, Lions, Nirenberg and Vega (*cf.* [5], [6], [9], [27]). These authors proved that there exists a solution (c_0, u_0). With additional smoothness assumptions on f, the real c_0 is unique and the function u_0 is unique up to translation in x_1-direction. Such results are highly related to the invariance of the equation by translation with respect to x_1, one of the main tools being the sliding method.

In case $(a, \vec{q}, g) = (1, \vec{0}, 0)$, a system of two reaction-convection-diffusion equations of type (1) set in infinite cylinders Σ was studied in [7] for Lewis numbers close to 1. In one of these two equations, the diffusion term has the form Δu and in the other one, it is $d\Delta v$ where $d \simeq 1$. Existence and uniqueness results were proved. But the structure of the solutions is exactly the same as in the case of one single equation, due especially to the invariance in x_1 of the investigated system.

Similar problems were studied in the works of Xin and Papanicolaou (*cf.* [22], [29], [30], [31]) in periodic media $\mathbb{R} \times T$, where T is the unit torus in $\mathbb{R}^{N-1}$. The problem reads

$$\begin{aligned}(\nabla_y + \vec{k}\partial_s)(A(y)(\nabla_y + \vec{k}\partial_s)U) + \vec{b}(y) \cdot \\ \cdot (\nabla_y + \vec{k}\partial_s)U + c\partial_s U + f(U) = 0 \text{ in } \mathbb{R} \times T\end{aligned}$$

where the unknowns are the real c, which is a speed in the given direction $\vec{k}$, and the function $U(s, y)$ in $\mathbb{R} \times T$, periodic in $y \in T$. When f is of ignition temperature type, for free divergence velocity field $\vec{b}$ and for symmetric positive matrices $A(y)$, there is existence and uniqueness of (c, U) and monotonicity properties for U. This is proved by a very interesting continuation method. But, as above, the results are related to the invariance of this problem with respect to s. In the bistable case, results are obtained only for $\vec{b}(y) = (w(y), 0, \cdots, 0)$, w small and $a(y) \simeq Id$ by the use of Fourier transforms and the implicit function theorem.

At the end, let us notice some onedimensional works of Barrow-Bates and Hagan when the nonlinear terms f are perturbed [3], [13].

With respect to the above works, one the main interests of the present paper is the study of equations (1) in which the coefficients depend on the main space variable x_1. This seems to be the first study of such multidimensional reaction-convection-diffusion problems. In other words, we loose the very important property of invariance by translation in the x_1-direction, which implied uniqueness and monotonicity properties for the profiles solutions. If (c, u) is a solution of (1) and $x_0 \in \mathbb{R}$, then the pair $(c, (x_1, y) \mapsto u(x_1 + x_0, y))$ is *a priori* not a solution. Besides, all the coefficients are perturbed, the diffusion term a, the multidimensional convection term $\vec{q}$, which is multidirectional and not of divergence free, and lastly the reaction term $f + g$. At the end, these coefficients also depend on u and ∇u, this introduces new nonlinear phenomena. Various existence and uniqueness results are nevertheless proved for small perturbations $(a, \vec{q}, g)$ of $(1, \vec{0}, 0)$.

We mention that an equation similar to (1) is studied in [14] where $\vec{q} = (\beta(x_1), 0, \cdots, 0)$ and $f + g = f(x_1, u)$, and β and f are increasing in x_1. The new phenomenon is that the set of the speeds c solutions is an intervall and not a single point.

Lastly, some weaker results than those presented in this paper were announced in [15] and proved in [16]. In this paper, the proofs are completly different and the results are more general than those of [16]. The author thanks Professor J.-M. Roquejoffre for his suggestions in the advance of this work.

2. SETTING OF THE RESULTS

2.1. Some useful results and notations

We first set some results and introduce some notations which will be useful in the sequel.

Let $\Sigma = \mathbb{R} \times \omega = \{(x_1, y),\ x_1 \in \mathbb{R},\ y \in \omega\}$ be an infinite cylinder in $\mathbb{R}^N$ whose section ω is a bounded and smooth connected domain with outward unit normal ν. The variable y can also be denoted by $y = (x_2, \cdots, x_N)$. We set $\Sigma_\pm = \mathbb{R}_\pm \times \omega$. Let $\alpha(y)$ be a function defined in $\overline{\omega}$, of class $C^{2,\delta}$ for some $\delta > 0$.

Let f be a C^3 function defined in $[0, 1]$. We assume one of the following assumptions:

$$\exists \theta \in (0,1),\ f \equiv 0 \text{ on } [0, \theta] \cup \{1\},\ f > 0 \text{ on } (\theta, 1),\ f'(1) < 0 \tag{2}$$

$$\exists \theta \in (0,1),\ f(0) = f(\theta) = f(1) = 0,\ f < 0 \text{ on } (0, \theta),\ f > 0 \text{ on } (\theta, 1),$$
$$f'(0),\ f'(1) < 0 \text{ and } \omega \text{ is convex} \tag{3}$$

The first case is called "ignition temperature" case and the second one is called the "bistable" case. Besides, in each of these cases, f is extended outside $[0,1]$ by:

$$\begin{cases} f(s) = f'(0)s \text{ on }]-\infty,0] \\ f(s) = f'(1)(s-1) \text{ on } [1,+\infty[\end{cases}$$

If f satisfies (2) or (3), from results of Berestycki, Larrouturou, Lions, Nirenberg (*cf.* [5], [6], [9]), there exists a solution $(c_0, u_0) \in \mathbb{R} \times C^{2,\delta}_{loc}(\overline{\Sigma})$ of the problem

$$\begin{cases} \Delta u_0 - (c_0 + \alpha(y))\partial_1 u_0 + f(u_0) = 0 \text{ in } \Sigma \\ \partial_\nu u_0 = 0 \text{ on } \partial\Sigma \\ u_0(-\infty,\cdot) = 0,\ u_0(+\infty,\cdot) = 1 \end{cases} \tag{4}$$

where ∂_1 and ∂_ν are the partial derivatives with respect to x_1 and ν. In this paper, the limits as $x_1 \to \pm\infty$ are always uniform in $y \in \overline{\omega}$. Besides, we have $0 < u_0 < 1$ and $\partial_1 u_0 > 0$ in $\overline{\Sigma}$. The speed c_0 is unique and the function u_0 is unique up to any translation in the x_1-direction, that is if (c, v) is solution of (4), then $c = c_0$ and $v(x_1, y) = u_0(x_1 + \rho, y)$ in $\overline{\Sigma}$ for some $\rho \in \mathbb{R}$. The function u_0 has exponential behaviours as $x_1 \to \pm\infty$:

$$\begin{cases} u_0(x_1,y) = e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty \\ \nabla u_0(x_1,y) = \nabla(e^{\lambda x_1}\phi(y)) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty \\ u_0(x_1,y) = 1 - e^{\mu x_1}\psi(y) + o(e^{\mu x_1}) \text{ as } x_1 \to +\infty \\ \nabla u_0(x_1,y) = -\nabla(e^{\mu x_1}\psi(y)) + o(e^{\mu x_1}) \text{ as } x_1 \to +\infty \end{cases} \tag{5}$$

where the reals $\lambda > 0$ and $\mu < 0$ are unique and the continuous and positive functions ϕ and ψ on $\overline{\omega}$ are unique modulo normalization. They are solutions of

$$\begin{cases} \Delta\phi - \lambda(c_0 + \alpha(y))\phi + f'(0)\phi + \lambda^2\phi = 0 \text{ in } \overline{\omega} \\ \partial_\nu\phi = 0 \text{ on } \partial\omega \end{cases}$$

$$\begin{cases} \Delta\psi - \mu(c_0 + \alpha(y))\psi + f'(1)\psi + \mu^2\psi = 0 \text{ in } \overline{\omega} \\ \partial_\nu\psi = 0 \text{ on } \partial\omega \end{cases}$$

These behaviours are based on general results of Agmon-Nirenberg and Pazy ([1], [23]).

Hence, the integral $\int_{\Sigma_-} u_0(x_1, y)\, dx_1 dy$ is well defined and the function

$$\rho \mapsto \int_{\Sigma_-} u_0(x_1 + \rho, y)\, dx_1 dy$$

is continuous, increasing, goes to 0 as $\rho \to -\infty$ and $+\infty$ as $\rho \to +\infty$. Thus, for any $\tau \in \mathbb{R}^*_+$, there exists a unique real $\rho(\tau)$ such that the function $u_0(x_1 + \rho(\tau), y)$, which we note $u_{0,\tau}$ or $u_0^{\rho(\tau)}$ satisfies

$$\int_{\Sigma_-} u_{0,\tau} = \int_{\Sigma_-} u_0^{\rho(\tau)} = \tau$$

Besides, the function $\rho(\tau)$ is of class $C^2(\mathbb{R}_+^*)$ and we have in particular $\rho'(\tau) = 1/\int_\omega u_0(\rho(\tau), y)dy > 0$.

Some notations

We note $UC(\overline{\Sigma}, \mathbb{R}^k)$ the set of bounded and uniformly continuous functions defined in $\overline{\Sigma}$ with range in $\mathbb{R}^k$ (we often omit $\mathbb{R}^k$ by simplicity when there is no possible confusion), and $UC_0(\overline{\Sigma}, \mathbb{R}^k)$ the set of functions v of $UC(\overline{\Sigma}, \mathbb{R}^k)$ such that $||v|| \to 0$ as $x_1 \to -\infty$.

Let now $r > 0$ fixed once for all in $(0, \lambda/2)$. For any $\rho \in \mathbb{R}$, we set

$$w^\rho(x_1, y) = 1 + e^{-r(x_1+\rho)},$$

$$B^\rho = \{u \in UC_0(\overline{\Sigma}),\ w^\rho u \in UC_0(\overline{\Sigma})\}$$

and

$$D^\rho = \{u \in B^\rho,\ \Delta u \in B^\rho,\ \partial_\nu u = 0 \text{ on } \partial\Sigma\}$$

Actually, $D^\rho = D^0$ and $B^\rho = B^0$ for any $\rho \in \mathbb{R}$. The space B^ρ is a Banach space endowed with the norm

$$||u||_{B^\rho} = ||w^\rho u||_\infty$$

It is easy to check that, for any ρ, $\rho' \in \mathbb{R}$, the norms in B^ρ and $B^{\rho'}$ are equivalent and

$$\forall u \in B^\rho = B^{\rho'}, \quad ||u||_{B^\rho} \le e^{r(\rho'-\rho)_+}||u||_{B^{\rho'}} \tag{6}$$

where $x_+ = \max(x, 0)$ for any $x \in \mathbb{R}$.

From (5), by the choice of r and the standard elliptic estimates up to the boundary, we remark that

$$\forall \rho \in \mathbb{R},\ \forall \tau \in \mathbb{R}_+^*, \quad u_{0,\tau},\ \nabla u_{0,\tau},\ \partial_{ij} u_{0,\tau}\ \in B^\rho \tag{7}$$

The set of solutions (c, u) of (4) is the C^2 manifold $\{(c_0, u_{0,\tau}),\ \tau \in \mathbb{R}_+^*\}$ in $\mathbb{R} \times B^\rho$, for any $\rho \in \mathbb{R}$.

For any $\tau \in \mathbb{R}_+^*$, we define the operator L^τ in $B^{\rho(\tau)}$ by its domain $D^{\rho(\tau)}$ and its expression

$$L^\tau u = -\Delta u + (c_0 + \alpha(y))\partial_1 u - f'(u_{0,\tau})u$$

The operator L^τ is unbounded and closed. The space $D^{\rho(\tau)}$ is a Banach space with the norm

$$||u||_{D^{\rho(\tau)}} = ||u||_{B^{\rho(\tau)}} + ||L^\tau u||_{B^{\rho(\tau)}}$$

We remark that the spaces $D^{\rho(\tau)}$ do not depend of τ, but their norms actually depend on τ. But, if τ, $\tau' \in \mathbb{R}_+^*$, the norms in $D^{\rho(\tau)}$ and $D^{\rho(\tau')}$ are actually equivalent. Indeed, for any $u \in D^{\rho(\tau)} = D^{\rho(\tau')}$, we have

$$L^{\tau}u = L^{\tau'}u + (f'(u_0^{\rho(\tau')}) - f'(u_0^{\rho(\tau)}))u$$

Hence,

$$\begin{aligned}\|L^{\tau}u\|_{B^{\rho(\tau)}} &\leq \|L^{\tau'}u\|_{B^{\rho(\tau)}} + 2\|f'\|_{\infty}\|u\|_{B^{\rho(\tau)}}\\ &\leq e^{r(\rho(\tau')-\rho(\tau))_+}(\|L^{\tau'}u\|_{B^{\rho(\tau')}} + 2\|f'\|_{\infty}\|u\|_{B^{\rho(\tau')}})\end{aligned}$$

At the end,

$$\|u\|_{D^{\rho(\tau)}} \leq 2e^{r(\rho(\tau')-\rho(\tau))_+}(1 + \|f'\|_{\infty})\|u\|_{D^{\rho(\tau')}} \tag{8}$$

Besides, from standard elliptic estimates, there exists a constant C_0 such that if $u \in D^{\rho(\tau)}$, then

$$\|\nabla u\|_{B^{\rho(\tau)}} \leq C_0\|u\|_{D^{\rho(\tau)}}, \quad \|\Delta u\|_{B^{\rho(\tau)}} \leq C_0\|u\|_{D^{\rho(\tau)}} \tag{9}$$

Lastly, we note

$$Y^{\rho(\tau)} = \mathbb{R} \times D^{\rho(\tau)}$$

We now define the suitably choosen spaces for the small perturbations a, $\vec{q}$ and g. Let us set

$$\begin{gathered}A = \{u = u(x_1, y, s, p) \in UC(\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}),\\ \partial_{(s,p)}u \in UC(\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N, \mathcal{L}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}))\},\\ Q = A^N,\end{gathered}$$

and

$$G = \{g \in A,\ g(x_1, y, 0, p) = g(x_1, y, 1, p) = 0 \quad \forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N\}$$

We define $X = A \times Q \times G$ and the norm

$$\|(a, \vec{q}, g)\|_X = \|a\|_A + \|\vec{q}\|_Q + \|g\|_G$$

where

$$\begin{aligned}\|u\|_A = \|u\|_G &= \|u\|_{L^{\infty}(\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N)} + \|\partial_s u\|_{L^{\infty}(\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N)}\\ &\quad + \|\partial_p u\|_{L^{\infty}(\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N,\mathbb{R}))}\\ \|\vec{q}\|_Q &= \|q_1\|_A + \cdots + \|q_N\|_A\end{aligned}$$

The space X is a Banach space, as well as the product $X \times Y^{\rho(\tau)}$ endowed with the norm

$$\|(a, \vec{q}, g, c, v)\|_{X \times Y^{\rho(\tau)}} = \|(a, \vec{q}, g)\|_X + |c| + \|v\|_{D^{\rho(\tau)}}$$

2.2. Theorems

2.2.1. "Local" existence for small perturbations of the coefficients and the solutions

THEOREM 1. – *Let f of class $C^3([0,1])$ satisfying* (2) *or* (3), *and* $\{(c_0, u_{0,\tau}),\ \tau \in \mathbb{R}_+^*\}$ *be the set of solutions of* (4). *Let $\rho_0 \in \mathbb{R}$, and $\tau_0 \in \mathbb{R}_+^*$ be the unique real such that $\rho(\tau_0) = \rho_0$.*

Let I a bounded intervall in R_+^* such that $\inf I > 0$. There exist $\delta, \eta > 0$ such that, for any $\tau \in I$, if $\|(a-1, \vec{q}, g)\|_X < \delta$, there exists a unique pair (c, u) in Y^{ρ_0} solution of

$$\left\{\begin{array}{c} a(x_1, y, u, \nabla u)\Delta u - (c + \alpha(y))\partial_1 u + \vec{q}(x_1, y, u, \nabla u) \cdot \nabla u \\ +f(u) + g(x_1, y, u, \nabla u) = 0 \text{ in } \Sigma \\ \partial_\nu u = 0 \text{ on } \partial\Sigma \\ u(-\infty, \cdot) = 0, \quad u(+\infty, \cdot) = 1 \end{array}\right. \tag{10}$$

and

$$\int_{\Sigma_-} u \, dx_1 dy = \tau \quad \textit{(normalization condition)} \tag{11}$$

such that $\|(c - c_0, u - u_{0,\tau})\|_{Y^{\rho_0}} < \eta$.

Furthermore, the map

$$\Psi_I \ : \ I \times B_X((1, \vec{0}, 0), \delta) \to B_{Y^{\rho_0}}((c_0, 0), \eta)$$

$$(\tau, a, \vec{q}, g) \mapsto (c, u - u_{0,\tau})$$

is of class C^1.

In a general way, $B_E(x, r)$ denotes the open ball with center x and radius $r > 0$ in the Banach space E. The term "local" in the title of this theorem means that the normalization condition (11) is required only for bounded values of τ.

An example of application of this theorem is the existence of solutions (c, u) of equations of type (4), that is the existence of travelling waves $v(t, x_1, y) = u(x_1 + ct, y)$ of the reaction-diffusion equation

$$\partial_t v = \Delta v - \alpha(y)\partial_1 v + \tilde{f}(v)$$

when the nonlinear terms $\tilde{f}$ are small perturbations of ignition temperature or bistable nonlinearities. For instance, $\tilde{f}$ may be a function close to 0 on $[0,\theta]$, but which oscillates near 0. This is new with respect to the results of [9].

These theorems also prove the existence of travelling waves solutions of reaction-convection-diffusion equations

$$\partial_t v = (1+\epsilon^\beta h(y,v,\nabla v))\Delta v - \alpha(y)\partial_1 v + \epsilon^\gamma \vec{q}(y,v,\nabla v)\cdot\nabla v + f(v)$$

for $\epsilon > 0$ small enough ($\beta, \gamma > 0$). Here, the diffusion and convection terms may take into account models of turbulence with different scales.

For equation (4), we know that for any $h \in (0,1)$, there exists a unique solution (c,u) such that

$$\max_{\overline{\omega}}\ u(0,\cdot) = h$$

This function u is of the form $u(x_1,y) = u_0^{\mu(h)}(x_1,y) = u_0(x_1+\mu(h),y)$. A corrolary of theorem 2 is the following theorem, where the normalization condition (11) is replaced by another of "max" type.

THEOREM 2. – *Let f satisfying the same assumptions as in theorem 1. For any $0 < a < b < 1$, there exist δ, $\eta_1 > 0$ such that for any $h \in [a,b]$, if $\|(a-1,\vec{q},g)\|_X < \delta$, then there exists a pair (c,u) in $Y^{\mu(h)}$ solution of* (10) *and the normalization condition*

$$\max_{\overline{\omega}}\ u(0,\cdot) = h \tag{12}$$

such that $\|(c-c_0, u-u_0^{\mu(h)})\|_{Y^{\mu(h)}} < \eta_1$.

Remark 2.1. – The same result holds if the normalization condition $\max_{\overline{\omega}}\ u(0,\cdot)$ is replaced by $\max_{\overline{\Sigma_-}}\ u$.

Remark 2.2. – Whereas theorem 1 delt with local existence and uniqueness for small perturbations of the coefficients and of the solutions, theorem 2 only ensures the existence of such solutions with this new normalization. A uniqueness result for this "max" normalization seems to be more difficult.

2.2.2. "Global" existence for small perturbations of the coefficients

THEOREM 3. – *Let f as in theorem 1 and assume moreover $g(x_1,y,s,p) \geq 0$ for $s \leq 0$ in the ignition temperature case. There exist δ, $\eta > 0$ such that, for any $\tau \in \mathbb{R}_+^*$, if $\|(a-1,\vec{q},g)\|_X < \delta$, then there exists a solution (c,u) of* (10) *in $\mathbb{R}\times D^0$ such that $\int_{\Sigma_-} u = \tau$* (11) *and $|c-c_0| < \eta$.*

The term "global" means that the normalization condition (11) may take any value $\tau \in \mathbb{R}_+^*$. On the other hand, we loose the uniqueness properties of the solutions u and the uniform bounds for u. But the bounds for c are kept.

2.2.3. Uniqueness results for small perturbations of the coefficients

From the previous theorems, we know the existence of solutions (c, u) of (10) for small perturbations of the coefficients. Besides, with suitable normalization conditions, the solutions are constructed to be closed to the solutions of equation (4). The aim of the following theorem is to show that, for suitable and small enough perturbations of the coefficients, all the solutions of (10) with suitable normalization condition are closed to the ones of (4).

THEOREM 4. – *Let f be a Lipschitz-continuous function defined in $[0, 1]$, but not necessarly C^3.*

Let $a_{ij}^n(x_1, y, s, p)$ ($1 \leq i, j \leq N$), $\vec{q}^n(x_1, y, s, p)$ and $g^n(x_1, y, s, p)$ be bounded, continuous and of class $C^{0,\alpha}$ functions defined in $\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N$ (for some $\alpha > 0$). The functions a_{11}^n, q_1^n and g^n are lipschitz continuous in s and p. Assume $\|a_{ij}^n(x_1, y, s, p) - \delta_{ij}\|_\infty$, $\|\vec{q}^n(x_1, y, s, p)\|_\infty$, $\|g^n(x_1, y, s, p)\|_\infty \to 0$ as $n \to +\infty$ and assume that, for any $n \in \mathbb{N}$, there exists a solution (c^n, u^n) of

$$\begin{cases} a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij}u^n - (c^n + \alpha(y))\partial_1 u^n \\ +\vec{q}^n(x_1, y, u^n, \nabla u^n) \cdot \nabla u^n + f(u^n) + g^n(x_1, y, u^n, \nabla u^n) = 0 \text{ in } \Sigma \\ \partial_\nu u^n = 0 \text{ on } \partial\Sigma \\ u^n(-\infty, \cdot) = 0, \quad u^n(+\infty, \cdot) = 1. \end{cases} \tag{13}$$

a) If f satisfies (2), is of class $C^{1,\delta}$ near 1 and if there exists $\mu > 0$ such that

$$\forall n,\ \forall s \leq \mu,\ \forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N,\ sg^n(x_1, y, s, p) \leq 0, \tag{14}$$

then

$$c^n \to c_0 \text{ as } n \to \infty.$$

If f satisfies (3) and is of class $C^{1,\delta}([0, 1])$ for some $\delta > 0$, then the same result holds.

b) Let f satisfy (2) and assume that there exists $\mu > 0$ such that

$$\forall n,\ \forall s \leq \mu,\ \forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N,\ g^n(x_1, y, s, p) = 0, \tag{15}$$

If

$$\int_{\Sigma_-} u^n = \tau^n \to \tau \in \mathbb{R}_+^* \text{ as } n \to \infty, \tag{16}$$

then

$$u^n \to u_{0,\tau} = u_0^{\rho(\tau)} \text{ in } D^{\rho(\tau)} \text{ and } W^{2,p}_{loc}(\Sigma) \text{ for any } p > 1.$$

c) With the assumptions of b), if $\max_{\overline{\Sigma_-}} u^n = h^n \to h \in (0,1)$ *as* $n \to \infty$,

then

$$u^n \to u_0^{\mu(h)} \text{ in } D^{\mu(h)} \text{ and } W^{2,p}_{loc}(\Sigma) \text{ for any } p > 1.$$

Remark 2.3. – For results b) and c) on convergence of the functions u^n to some solution of (4), the normalization conditions of type $\int_{\Sigma_-} u^n$ or $\max_{\Sigma_-} u^n$ are necessary. Otherwise, we could have $u^n \to 0$ or $u^n \to 1$: take for instance $a_{ij}^n = \delta_{ij}$, $\vec{q}^n = 0$, $g^n = 0$ and $(c^n, u^n) = (c_0, u_0(x_1 \pm n, y))$. Besides, for technical reasons, these results *b*) and *c*) do not hold clearly for a bistable function f.

Remark 2.4. – We are not able to get a priori monotonicity properties for the solutions u^n of (13). But, when $\|(a_{ij}^n - \delta_{ij}, \vec{q}^n, g^n)\|_\infty$ is small enough, the solutions u^n are close enough to some fixed solution of (4) in $C^{1,\delta}_{loc}(\overline{\Sigma})$ (with some suitable normalization condition for u^n). Hence, for any compact K in $\overline{\Sigma}$, the solutions u^n are increasing in x_1 in K for n large enough.

2.3. Comparison between existence and uniqueness results

From theorems 1 and 4, we conclude that if f satisfies (2) and $a_{ij}^n = \delta_{ij} a^n(x_1, y, s, p)$, $(a^n, \vec{q}^n, g^n) \in X$, $\|(a^n - 1, \vec{q}^n, g^n)\|_X \to 0$, then there exist solutions (c^n, u^n) of (13) and (16) (theorem 1). If (15) holds, then these solutions are the only ones (theorem 4 and formula (8)).

In theorems 1-3, we only consider matrices (a_{ij}) of the form $a\delta_{ij}$ because of the definition of the spaces $D^{\rho(\tau)}$ for which the useful properties of the operators L^τ are available.

2.4. Methods and structure of the paper

Existence theorem 1 is based on the uniform contraction mapping theorem, or a uniform implicit function theorem. We linearize equation (10) near $(c_0, u_{0,\tau})$ and have to use spectral properties of some operators taking L^τ into account, and especially their invertibility when $\int_{\Sigma_-} v$ is fixed. We apply some results of Roquejoffre and Sattinger similar to the ones of Krein-Rutman (*cf.* [20], [24], [25]). The same properties, joined with the application of an implicit function theorem, was used by Roquejoffre for the analysis of the nonlinear stability of the solutions of (4) [24]. Theorem 2 is a direct consequence of theorem 1 by a continuity argument. Theorem 3 comes from the study of equation (10) translated in x_1 by any step ρ.

The uniqueness results of theorem 4 are proved in a very different way. The boundedness of the speeds c^n is a consequence of comparison of u^n to onedimensional fixed functions, and the convergence of c^n to c_0 comes from comparison of u^n with travelling waves solutions of (4) for nonlinearities f_ϵ close to f. These comparisons can be made by a sliding method and the essential tools are the maximum principle and the Hopf lemma. By a study of the exponential decay of the function u^n as $x_1 \to -\infty$ when $\int_{\Sigma_-} u^n$ is bounded, we conclude to the convergence of u^n to some solution of (4).

The next two sections are respectively devoted to the proofs of existence theorems 1-3, and uniqueness theorem 4. The last section presents some open questions related to this work.

3. EXISTENCE RESULTS FOR SMALL PERTURBATIONS OF THE COEFFICIENTS

3.1. Local existence and uniqueness when $\int_{\Sigma_-} u$ is bounded

This section is devoted to the proof of theorem 1. Let f be of class $C^3([0,1])$ and $C^1(\mathbb{R})$ satisfying (1) or (2). Let $\rho_0 \in \mathbb{R}$. From section 2, ρ_0 is of the form $\rho_0 = \rho(\tau_0)$. Let I be a bounded intervall in $\mathbb{R}_+^*$ such that $\inf I > 0$.

For $a - 1$, $\vec{q}$, g small enough in X and for any $\tau \in I$, we look for solutions (c, u) of (10) in Y^{ρ_0}. Setting $u = u_{0,\tau} + v$, this is equivalent to solve the following equations:

$$F_1(\tau, a, \vec{q}, g, c, v) := \Delta v - (c_0 + \alpha(y))\partial_1 v + f(u_{0,\tau} + v) - f(u_{0,\tau})$$
$$+(a(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) - 1)\Delta(u_{0,\tau} + v)$$
$$-(c - c_0)\partial_1(u_{0,\tau} + v) + \vec{q}(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) \cdot \nabla(u_{0,\tau} + v)$$
$$+g(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) = 0 \text{ in } \Sigma,$$
$$\partial_\nu v = 0 \text{ on } \partial\Sigma,$$
$$v(-\infty, \cdot) = v(+\infty, \cdot) = 0$$

and

$$F_2(\tau, a, \vec{q}, g, c, v) := \int_{\Sigma_-} v = 0$$

In the following lemma, we actually prove that in order to solve the previous problem for $(c,v) \in Y^{\rho_0}$, it is enough to solve $F_1 = 0$ and $F_2 = 0$ when $a-1$, g and v are small enough in A, G and D^{ρ_0}. In other words, conditions $v(\pm\infty,\cdot) = 0$ and $\partial_\nu v = 0$ are redundant.

LEMMA 3.1. – *Let $g \in G$ and $a \in A$ such that $\|g\|_G \leq 1/2\ |f'(1)|$ and $\|a-1\|_A \leq 1/2$. There exists $\eta_0 > 0$ such that, if $v \in D^{\rho_0}$ satisfies $F_1(\tau, a, \vec{q}, g, c, v) = 0$ and $\|v\|_\infty < \eta_0$, then $v(\pm\infty,\cdot) = 0$ and $\partial_\nu v = 0$ on $\partial\Sigma$.*

Proof. – The conditions $v(-\infty,\cdot) = 0$ and $\partial_\nu v = 0$ are actually included in the definition of D^{ρ_0}. It only remains to prove $u(+\infty,\cdot) = 1$ where $u = u_{0,\tau} + v$ is solution of

$$a\Delta u - (c+\alpha(y))\partial_1 u + \vec{q}\cdot\nabla u + f(u) + g(x_1,y,u,\nabla u) = 0 \text{ in } \Sigma$$

We first remark that if $\|a-1\|_A \leq 1/2$, then $\|a-1\|_\infty \leq 1/2$, whence $a \in [1/2, 3/2]$. Since f is $C^1(\mathbb{R})$, $\|\partial_s g\|_\infty \leq 1/2\ |f'(1)|$ and $f'(1) < 0$, there exists $\eta_0 > 0$ such that

$$\begin{aligned} &2f'(1) < \partial_s g(x_1,y,s,p) + f'(s) < 1/4\ f'(1), \\ &\forall s \in [1-\eta_0, 1+\eta_0],\ (x_1,y,p) \in \overline{\Sigma} \times \mathbb{R}^N \end{aligned} \tag{17}$$

Let now $v = u - u_{0,\tau}$ such that $\|v\|_{D^{\rho_0}} < \eta_0$. This yields $\|v\|_\infty < \eta_0$, hence $-\eta_0 < u < u + \eta_0$ in $\overline{\Sigma}$ and

$$1-\eta_0 < \liminf_{x_1\to+\infty} u \leq \limsup_{x_1\to+\infty} u < 1+\eta_0 \tag{18}$$

Let us suppose that $l = \limsup_{x_1\to+\infty} u > 1$. Then there exist points $(x_n, y_n) \in \overline{\Sigma}$, $x_n \to +\infty$, such that $u(x_n,y_n) \to l \in (1, 1+\eta_0)$. In the compact $K = [-1,1] \times \overline{\omega}$, we define $u_n(x_1,y) = u(x_1+x_n,y)$. This function is solution of

$$a(x_1+x_n,y,u_n(x_1,y),\nabla u_n(x_1,y))\Delta u_n(x_1,y) - (c+\alpha(y))\partial_1 u_n(x_1,y)$$

$$\vec{q}(x_1+x_n,y,u_n(x_1,y),\nabla u_n(x_1,y))\cdot\nabla u_n(x_1,y)$$

$$+f(u_n(x_1,y)) + g(x_1+x_n,y,u_n(x_1,y),\nabla u_n(x_1,y)) = 0 \quad \text{in } K$$

Since $1/2 \leq a \leq 3/2$, $\vec{q}$ is bounded, u_n is bounded in $[-\eta_0, 1+\eta_0]$, f is continuous, $g(x_1,y,s,p)$ is lipschitz continuous in s and vanishes for $s = 0$ or 1, we deduce from the standard elliptic estimates that the (u_n)'s are bounded in $W^{2,p}(K)$ for any $1 < p < \infty$. From the Sobolev injections, it comes that for some subsequence that we rename (n), we have $u_n \to u_\infty$ in $C^{1,\alpha}(K)$ for any $\alpha \in (0,1)$.

On the other side, since a and $\vec{q}$ are bounded and uniformly continuous in $\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N$, the functions

$$a_n(x_1, y) = a(x_1 + x_n, y, u_n(x_1, y), \nabla u_n(x_1, y))$$

and

$$\vec{q}_n(x_1, y) = \vec{q}(x_1 + x_n, y, u_n(x_1, y), \nabla u_n(x_1, y))$$

are bounded and uniformly equicontinuous in the compact K. From Ascoli's theorem, up to extraction of some subsequence, they converge in $L^\infty(K)$ to some continuous functions $a_\infty(x_1, y)$ and $\vec{q}_\infty(x_1, y)$. We obviously have $a_\infty \in [1/2, 3/2]$.

For n large enough, we have $1 - \eta_0 < u_n(x_1, y) < 1 + \eta_0$ in K from (18). The term $f(u_n) + g(x_1 + x_n, y, u_n, \nabla u_n)$ can be written

$$\begin{aligned} &f(u_n) + g(x_1 + x_n, y, u_n, \nabla u_n) \\ &\quad = \int_1^{u_n(x_1,y)} [\partial_s g(x_1 + x_n, y, s, \nabla u_n(x_1, y)) + f'(s)] ds \end{aligned}$$

Since f' is uniformly continuous on the compact $[1 - \eta_0, 1 + \eta_0]$ and $\partial_s g$ is uniformly continuous, the sequence of functions $c_n : K \times [1-\eta_0, 1+\eta_0] \to \mathbb{R}$, $(x_1, y, s) \mapsto \partial_s g(x_1 + x_n, y, s, \nabla u_n(x_1, y)) + f'(s)$ converge, up to extraction of some subsequence, in $L^\infty(K \times [1 - \eta_0, 1 + \eta_0])$ to some function $c(x_1, y, s)$ continuous such that $2f'(1) \leq c(x_1, y, s) \leq 1/4 f'(1)$ from (17). Hence, the function $K \to \mathbb{R}$, $(x_1, y) \mapsto f(u_n(x_1, y)) + g(x_1 + x_n, y, u_n(x_1, y), \nabla u_n(x_1, y))$ converge in $L^\infty(K)$ to a function $c_\infty(x_1, y)(u_\infty(x_1, y) - 1)$ where $2f'(1) \leq c_\infty(x_1, y) \leq 1/4 f'(1)$.

At the end, the function $u_\infty(x_1, y) \in W^{2,p}(K)$ satisfies

$$\begin{aligned} &a_\infty(x_1, y) \Delta u_\infty - (c + \alpha(y)) \partial_1 u_\infty + \vec{q}_\infty(x_1, y) \\ &\quad \cdot \nabla u_\infty + c_\infty(x_1, y)(u_\infty - 1) = 0 \text{ in } K \end{aligned}$$

and $u_\infty(0, \overline{y}) = l = \max_K u_\infty$ for some $\overline{y} \in \overline{\omega}$. From the previous equation and the Sobolev injections, we have Δu_∞ is continuous. Since $1 < l$ and $c \leq 0$, it comes from the strong maximum principle that $u_\infty \equiv l$ in K. This is impossible since $c_\infty \leq 1/4 f'(1) < 0$ in K.

We then conclude that $\operatorname{limsup}_{x_1 \to +\infty} u \leq 1$. With the same arguments, we have $\operatorname{liminf}_{x_1 \to +\infty} u \geq 1$. Finally, we get

$$u(+\infty, \cdot) = 1$$

This achieves the proof of lemma 3.1.

Definition and properties of the map $F = (F_1, F_2)$

Let $Z^{\rho_0} = B^{\rho_0} \times \mathbb{R}$, this is a Banach space with the product norm. Let F defined as follows:

$$F \;:\; I \times X \times Y^{\rho_0} \to Z^{\rho_0}$$

$$(\tau, a, \vec{q}, g, c, v) \mapsto (F_1(\tau, a, \vec{q}, g, c, v), F_2(\tau, a, \vec{q}, g, c, v))$$

This map is well defined. Indeed, firstly, for any $v \in D^{\rho_0}$, we have $v = o(e^{rx_1})$ as $x_1 \to -\infty$, whence $\int_{\Sigma_-} v$ converges. Moreover, from (7) and (9) and since a and $\vec{q}$ are bounded and uniformly continuous in all their arguments, each of the terms Δv, $(c_0+\alpha(y))\partial_1 v$, $a(x_1, y, u_{0,\tau}+v, \nabla(u_{0,\tau}+v))\Delta(u_{0,\tau}+v)$, $(c-c_0)\partial_1(u_{0,\tau}+v)$, $\vec{q}(x_1, y, u_{0,\tau}+v, \nabla(u_{0,\tau}+v)) \cdot \nabla(u_{0,\tau}+v)$ is in B^{ρ_0}. Lastly, the functions $(x_1, y) \mapsto f((u_{0,\tau}+v)(x_1, y))$, $f(u_{0,\tau}(x_1, y))$, $g(x_1, y, u_{0,\tau}+v, \nabla(u_{0,\tau}+v))$ are uniformly continuous and bounded on $\overline{\Sigma}$ (the function g is bounded and uniformly continuous in all its arguments). Since $f(0) = 0$, $g(x_1, y, 0, p) = 0$ $\forall(x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N$, since f' and $\partial_s g$ are bounded and $u_{0,\tau}$, $v = o(e^{rx_1})$ as $x_1 \to -\infty$, we conclude that $f(u_{0,\tau}+v)$, $f(u_{0,\tau})$, $g(x_1, y, u_{0,\tau}+v, \nabla(u_{0,\tau}+v))$ are in B^{ρ_0}.

A very clear but tedious calculation shows that the function F is of class C^1 in all its arguments: this uses in particular the facts that f' is uniformly continuous in $\mathbb{R}$ from its definition oustide $[0,1]$, that a, q_i, g are C^1 with respect to (s,p) and their derivatives ∂_s and ∂_p are bounded and uniformly continuous functions in $L^\infty(\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $L^\infty(\overline{\Sigma} \times \mathbb{R} \times \mathbb{R}^N, \mathcal{L}(\mathbb{R}^N, \mathbb{R}))$. For the C^1 dependance in τ, we need that $\Delta\partial_1 u_{0,\tau} \in B^{\rho_0}$ (this comes from Schauder's standard elliptic estimates since $L^\tau \partial_1 u_{0,\tau} = 0$ and $\partial_1 u_{0,\tau} \in B^{\rho_0}$) and we have already written that $\rho(\tau)$ is of class C^1.

We observe that $F(\tau, 1, \vec{0}, 0, c_0, 0) = (0,0)$ in Z^{ρ_0} for any $\tau \in I$. In order to prove the existence of solution (c, v) close to $(c_0, 0)$ of $F(\tau, a, \vec{q}, g, c, v) = (0,0)$, for $(a, \vec{q}, g)$ close to $(1, \vec{0}, 0)$ and for any $\tau \in I$, we apply a uniform mapping theorem. To do this we have to study the operator $\partial_{(c,v)} F(\tau, 1, \vec{0}, 0, c_0, 0)$.

LEMMA 3.2. – *For any $\tau \in \mathbb{R}_+^*$, the operator $\mathcal{F}^\tau = \partial_{(c,v)} F(\tau, 1, \vec{0}, 0, c_0, 0) \in \mathcal{L}(Y^{\rho_0}, Z^{\rho_0})$ is an isomorphism and*

$$\|(\mathcal{F}^\tau)^{-1}\|_{\mathcal{L}(Z^{\rho_0}, Y^{\rho_0})} \leq A(\tau)$$

where the function A remains bounded as τ and τ^{-1} are bounded, and $A \to +\infty$ as $\tau \to 0$ or $+\infty$.

Proof. – Let $\tau \in \mathbb{R}_+^*$. From the definition of F, we have

$$\mathcal{F}^\tau \; : \; Y^{\rho_0} \to Z^{\rho_0}$$

$$(c, v) \mapsto \left(-L^\tau v - c\partial_1 u_{0,\tau}, \int_{\Sigma_-} v \right)$$

where L^τ was defined in section 2 by its domain $D^{\rho(\tau)}$ and its expression

$$L^\tau \; : \; B^{\rho(\tau)} \to B^{\rho(\tau)}$$

$$v \mapsto -\Delta v + (c_0 + \alpha(y))\partial_1 v - f'(u_{0,\tau})v$$

Let $(w, \gamma) \in Z^{\rho_0}$. We have to solve the following system with unknowns (c, v):

$$-L^\tau v - c\partial_1 u_{0,\tau} = w \tag{19}$$

$$\int_{\Sigma_-} v = \gamma \tag{20}$$

Here, we use some important properties of the operator L^τ. We need in particular the assumption $0 < r < \lambda/2$ where $w^\rho = 1 + e^{-r(x_1+\rho)}$. The following assertions are proved in [8] and [24] in this multidimensional situation, and previously in the onedimensional case by Sattinger and Henry [25], [17].

a) The kernel $N(L^\tau)$ is onedimensional and spanned by $\partial_1 u_{0,\tau}$. This property is the analogous to the Krein-Rutman theorem for elliptic operators in bounded domains (*cf.* [20]).

b) The decomposition

$$B^{\rho(\tau)} = N(L^\tau) \oplus R(L^\tau)$$

holds in algebraic and topological sense, where $R(L^\tau)$ is the range of L^τ.

c) The kernel $N((L^\tau)^*)$ of the adjoint of L^τ is onedimensional, spanned by a linear form $e^{\tau *} \in B^{\rho(\tau)'}$ (the dual of $B^{\rho(\tau)}$). We may take $e^{\tau *}$ such that

$$< e^{\tau *}, \partial_1 u_{0,\tau} >_{(B^{\rho(\tau)'}, B^{\rho(\tau)})} = 1$$

d) The restriction M^τ of L^τ to $R(L^\tau)$ is an isomorphism between the Banach spaces $D^{\rho(\tau)} \cap R(L^\tau)$ and $R(L^\tau)$ endowed with the norm $\| \; \|_{B^{\rho(\tau)}}$.

Hence, equation (19) has a solution v if and only if $< e^{\tau *}, w + c\partial_1 u_{0,\tau} > = 0$, that is $c = - < e^{\tau *}, w >$. The set of solutions v of (19) is then

$$\{ -(M^\tau)^{-1}(w - < e^{\tau *}, w > \partial_1 u_{0,\tau}) + \nu \partial_1 u_{0,\tau}, \; \nu \in \mathbb{R} \}$$

Equation (20) determines ν in a unique way:

$$\nu = \frac{\gamma + \int_{\Sigma_-} (M^\tau)^{-1}(w - < e^{\tau *}, w > \partial_1 u_{0,\tau})}{\int_{\Sigma_-} \partial_1 u_{0,\tau}}$$

To summarize, $\mathcal{F}^\tau$ is a bijection and

$$\begin{aligned}(\mathcal{F}^\tau)^{-1}&(w,\gamma)\\ = \Bigg(&- < e^{\tau *}, w >, -(M^\tau)^{-1}(w - < e^{\tau *}, w > \partial_1 u_{0,\tau})\\ &+ \frac{\gamma + \int_{\Sigma_-} (M^\tau)^{-1}(w - < e^{\tau *}, w > \partial_1 u_{0,\tau})}{\int_{\Sigma_-} \partial_1 u_{0,\tau}} \partial_1 u_{0,\tau}\Bigg) \qquad (21)\end{aligned}$$

We now have to evaluate the norm of $(\mathcal{F}^\tau)^{-1}$ in $\mathcal{L}(Z^{\rho_0}, Y^{\rho_0})$. Firstly, from the definition of $e^{\tau *}$, we have

$$\begin{aligned}\|e^{\tau *}\|_{B^{\rho(\tau)'}} &= \sup_{w \in B^{\rho(\tau)} \setminus \{0\}} \frac{| < e^{\tau *}, w > |}{\|w\|_{B^{\rho(\tau)}}}\\ &= \frac{1}{\|\partial_1 u_0^{\rho(\tau)}\|_{B^{\rho(\tau)}}} = \frac{1}{\|\partial_1 u_0\|_{B^0}} \qquad (22)\end{aligned}$$

Secondly, we prove that the norms of the operators $(M^\tau)^{-1}$ in $\mathcal{L}(B^{\rho(\tau)})$ are independant of τ.

LEMMA 3.3. – *For any* $\tau \in \mathbb{R}_+^*$, *we have*

$$\|(M^\tau)^{-1}\|_{\mathcal{L}(B^{\rho(\tau)})} = \|(M^{\tau_0})^{-1}\|_{\mathcal{L}(B^{\rho_0})} = a,$$

where $\rho(\tau_0) = \rho_0$.

Proof of lemma 3.3. – From assertion *d*) above, we know that M^τ, the restriction of L^τ to $R(L^\tau)$ is an isomorphism from $D^{\rho(\tau)} \cap R(L^\tau)$ to $R(L^\tau)$. For any $\rho \in \mathbb{R}$ and for any function u defined in $\overline{\Sigma}$, we note u^ρ the function $u^\rho(x_1, y) = u(x_1 + \rho, y)$.

Since $u_{0,\tau} = u_0^{\rho(\tau)}$, we have, for any $v \in D^{\rho(\tau)} = D^{\rho_0}$,

$$L^\tau v = -\Delta v + (c_0 + \alpha(y))\partial_1 v - f'(u_0(x_1 + \rho(\tau), y))v$$

But $u_0(x_1 + \rho(\tau), y) = u_{0,\tau_0}(x_1 + \rho(\tau) - \rho_0, y)$. Hence, we have

$$L^\tau(v) = [L^{\tau_0}(v^{\rho_0 - \rho(\tau)})]^{\rho(\tau) - \rho_0} \qquad (23)$$

Let now $w \in R(L^\tau)$. There exists $u \in D^{\rho(\tau)}$ such that $w = L^\tau u$, whence $w = [L^{\tau_0}(u^{\rho_0-\rho(\tau)})]^{\rho(\tau)-\rho_0}$. The function $w^{\rho_0-\rho(\tau)} = L^{\tau_0}(u^{\rho_0-\rho(\tau)})$ is in $R(L^{\tau_0})$. From assertion *d)* above applied to τ_0, let

$$v_0 = (M^{\tau_0})^{-1}(L^{\tau_0}(u^{\rho_0-\rho(\tau)})) = (M^{\tau_0})^{-1}(w^{\rho_0-\rho(\tau)}) \ \in R(L^{\tau_0})$$

This function v_0 is of the form $v_0 = L^{\tau_0}z$ where $z \in D^{\rho_0}$. Thus,

$$\begin{aligned} w &= (L^{\tau_0}v_0)^{\rho(\tau)-\rho_0} = (L^{\tau_0}(L^{\tau_0}z))^{\rho(\tau)-\rho_0} \\ &= L^\tau((L^{\tau_0}z)^{\rho(\tau)-\rho_0}) = L^\tau(L^\tau(z^{\rho(\tau)-\rho_0})) \end{aligned}$$

from (23) applied two times. From the definition of M^τ, we have

$$\begin{aligned} (M^\tau)^{-1}w &= L^\tau(z^{\rho(\tau)-\rho_0}) = (L^{\tau_0}z)^{\rho(\tau)-\rho_0} = v_0^{\rho(\tau)-\rho_0} \\ &= [(M^{\tau_0})^{-1}(w^{\rho_0-\rho(\tau)})]^{\rho(\tau)-\rho_0} \end{aligned}$$

Finally, with elementary arguments, we conclude

$$\|(M^\tau)^{-1}w\|_{B^{\rho(\tau)}} = \|(M^{\tau_0})^{-1}(w^{\rho_0-\rho(\tau)})\|_{B^{\rho_0}},$$

$$\begin{aligned} \|(M^\tau)^{-1}\|_{\mathcal{L}(R(L^\tau),B^{\rho(\tau)})} &= \sup_{w\neq 0,\in R(L^\tau)} \frac{\|(M^{\tau_0})^{-1}(w^{\rho_0-\rho(\tau)})\|_{B^{\rho_0}}}{\|w\|_{B^{\rho(\tau)}}} \\ &= \sup_{w\neq 0,\in R(L^\tau)} \frac{\|(M^{\tau_0})^{-1}(w^{\rho_0-\rho(\tau)})\|_{B^{\rho_0}}}{\|w^{\rho_0-\rho(\tau)}\|_{B^{\rho_0}}} \\ &= \|(M^{\tau_0})^{-1}\|_{\mathcal{L}(R(L^{\tau_0}),B^{\rho_0})} \end{aligned}$$

since $R(L^\tau)^{\rho_0-\rho(\tau)} = R(L^{\tau_0})$ from (23).

End of the proof of lemma 3.2. – We are now able to evaluate the norm of $(\mathcal{F}^\tau)^{-1}$, *i.e.* $\|(\mathcal{F}^\tau)^{-1}(w,\gamma)\|_{\mathbb{R}\times D^{\rho_0}}$ given by (21). From (22) and lemma 3.3, we have

$$\begin{aligned} |< e^{\tau *}, w >| &\leq \frac{1}{\|\partial_1 u_0\|_{B^0}}\|w\|_{B^{\rho(\tau)}} \\ &\leq \frac{1}{\|\partial_1 u_0\|_{B^0}} e^{r(\rho_0-\rho(\tau))_+}\|w\|_{B^{\rho_0}}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\|(M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau})\|_{B^{\rho(\tau)}} \\ &\quad \leq 2a\ \|w\|_{B^{\rho(\tau)}} \leq 2ae^{r(\rho_0-\rho(\tau))_+}\|w\|_{B^{\rho_0}}, \end{aligned} \tag{25}$$

$$\|(M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau})\|_{B^{\rho_0}} \leq 2a\ e^{r|\rho(\tau)-\rho_0|}\|w\|_{B^{\rho_0}}.$$

Besides,

$$\begin{aligned} &L^{\tau_0}((M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau})) \\ &\quad = w- < e^{\tau *}, w > \partial_1 u_{0,\tau} \\ &\qquad + (f'(u_0^{\rho(\tau)}) - f'(u_0^{\rho_0}))(M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau}) \end{aligned}$$

From (24) and the previous inequality, we get

$$\begin{aligned} &\|L^{\tau_0}((M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau}))\|_{B^{\rho_0}} \\ &\quad \leq [1 + e^{r(\rho_0-\rho(\tau))_+} + 4a\|f'\|_\infty e^{r|\rho(\tau)-\rho_0|}]\ \|w\|_{B^{\rho_0}} \end{aligned}$$

Let us now get an upper bound in D^{ρ_0} for the last term of (21): $\nu\partial_1 u_{0,\tau}$. First of all, we have

$$\int_{\Sigma_-} \partial_1 u_{0,\tau} = \int_\omega u_0(\rho(\tau), y)dy \geq |\omega| \min_{\overline{\omega}}\ u_0(\rho(\tau), \cdot).$$

From (25), it comes

$$\begin{aligned} &\int_{\Sigma_-} (M^\tau)^{-1}(w- < e^{\tau *}, w > \partial_1 u_{0,\tau}) \\ &\quad \leq 2a\ e^{r(\rho_0-\rho(\tau))_+}\|w\|_{B^{\rho_0}}|\omega| \int_{-\infty}^0 \frac{1}{1+e^{-r(x_1+\rho(\tau))}}dx_1 \\ &\quad = 2a|\omega|\frac{\ln(1+e^{r\rho(\tau)})}{r}e^{r(\rho_0-\rho(\tau))_+}\|w\|_{B^{\rho_0}} \end{aligned}$$

Thus,

$$|\nu| \leq \frac{1}{|\omega| \min_{\overline{\omega}}\ u_0(\rho(\tau), \cdot)}\left[|\gamma| + 2a|\omega|\frac{\ln(1+e^{r\rho(\tau)})}{r}e^{r(\rho_0-\rho(\tau))_+}\|w\|_{B^{\rho_0}}\right],$$

$$\|\partial_1 u_{0,\tau}\|_{B^{\rho_0}} \leq e^{r(\rho(\tau)-\rho_0)_+}\|\partial_1 u_0\|_{B^0},$$

and

$$L^{\tau_0}\partial_1 u_{0,\tau} = (f'(u_{0,\tau}) - f'(u_{0,\tau_0}))\partial_1 u_{0,\tau},$$

hence

$$\|L^{\tau_0}\partial_1 u_{0,\tau}\|_{B^{\rho_0}} \leq 2\|f'\|_\infty e^{r(\rho(\tau)-\rho_0)_+}\|\partial_1 u_0\|_{B^0}.$$

Summarizing all the previous inequalities in the definition (21) of $(\mathcal{F}^\tau)^{-1}$, we get after a straightforward calculation:

$$\|(\mathcal{F}^\tau)^{-1}\|_{\mathcal{L}(Z^{\rho_0}, Y^{\rho_0})} \leq A(\tau)$$

where

$$A(\tau) = 1 + \left(1 + \frac{1}{\|\partial_1 u_0\|_{B^{\rho_0}}}\right) e^{r(\rho_0 - \rho(\tau))_+}$$
$$+ \frac{1}{|\omega| \min_{\overline{\omega}} u_0(\rho(\tau), \cdot)} (1 + 2\|f'\|_\infty) \|\partial_1 u_0\|_{B^0} e^{r(\rho(\tau) - \rho_0)_+}$$
$$+ \left[2a + 4a\|f'\|_\infty + \frac{2a(1 + 2\|f'\|_\infty)}{\min_{\overline{\omega}} u_0(\rho(\tau), \cdot)} \|\partial_1 u_0\|_{B^0} \frac{\ln(1 + e^{r\rho(\tau)})}{r}\right]$$
$$\times e^{r|\rho(\tau) - \rho_0|}$$

It is clear that A is a continuous function of τ and that $A(\tau)$ remains bounded as τ and τ^{-1} remain bounded in $\mathbb{R}_+^*$. Furthermore, since $u_0(-\infty, \cdot) = 0$, $\rho(0^+) = -\infty$ and $\rho(+\infty) = +\infty$, we conclude

$$A(\tau) \to +\infty \text{ as } \tau \to 0^+ \text{ or } +\infty$$

This achieves the proof of lemma 3.2.

In order to apply a uniform contraction mapping theorem, we set

$$G : \mathbb{R}_+^* \times X \times Y^{\rho_0} \to Y^{\rho_0}$$

$$(\tau, a, \vec{q}, g, c, v) \mapsto (c, v) - (\mathcal{F}^\tau)^{-1} F(\tau, a, \vec{q}, g, c + c_0, v)$$

The operators $\mathcal{F}^\tau$ are actually of class C^1 with respect to τ in $\mathcal{L}(Y^{\rho_0}, Z^{\rho_0})$. This is easy to check and uses the fact that $\rho(\tau)$ is of class C^2 and f is of class $C^3([0, 1])$. Hence, from the result of lemma 3.2 and straightforward arguments, the operators $(\mathcal{F}^\tau)^{-1}$ are also of class C^1 of τ in $\mathcal{L}(Z^{\rho_0}, Y^{\rho_0})$. Finally, since F is of class C^1, we get that the map G is of class C^1 from $\mathbb{R}_+^* \times X \times Y^{\rho_0}$ to Y^{ρ_0}.

Moreover, we have

$$G(\tau, 1, \vec{0}, 0, 0, 0) = (0, 0) \qquad \text{and} \qquad \partial_{(c,v)} G(\tau, 1, \vec{0}, 0, 0, 0) = 0_{\mathcal{L}(Y^{\rho_0})}$$

for any $\tau \in \mathbb{R}_+^*$.

Hence, with the notations of lemma 3.1, in order to achieve the proof of theorem 1, is is enough to prove the existence of reals $\delta, \eta > 0$, $\delta \in (0, \min(1/2, 1/2|f'(1)|))$ and $\eta \in (0, \eta_0)$ such that, if $\tau \in I$, $\|(a - 1, \vec{q}, g)\|_X < \delta$, $\|(c, v)\|_{Y^{\rho_0}} < \eta$, then

$$\|\partial_{(c,v)} G(\tau, a, \vec{q}, g, c, v)\|_{\mathcal{L}(Y^{\rho_0})} \leq 1/2, \tag{26}$$

$$\|\partial_{(a,\vec{q},g)} G(\tau, a, \vec{q}, g, c, v)\|_{\mathcal{L}(X, Y^{\rho_0})} \leq C \tag{27}$$

and $C\delta < 1/2\ \eta$.

Once the previous inequalities are proved, from the uniform contraction mapping theorem, for any $(\tau, a, \vec{q}, g)$ in $I \times B_X((1, \vec{0}, 0), \delta)$, there exists a unique pair (c, v) in $B_{Y^{\rho_0}}((0,0), \eta)$ solution of $G(\tau, a, \vec{q}, g, c, v) = (0, 0)$, *i.e.* $F(\tau, a, \vec{q}, g, c + c_0, v) = (0, 0)$ in Z^{ρ_0}. In other words, $(c + c_0, u + u_{0,\tau})$ is solution of (10) with the normalization condition (11).

Proof of (26). – We have

$$\partial_{(c,v)} G(\tau, a, \vec{q}, g, c, v) \ : \ Y^{\rho_0} \to Y^{\rho_0},$$

$$\begin{aligned}(\gamma, w) \mapsto (\gamma, w) &- (\mathcal{F}^\tau)^{-1} \partial_{(c,v)} F(\tau, a, \vec{q}, g, c + c_0, v) \cdot (\gamma, w) \\ &= (\mathcal{F}^\tau)^{-1} [\partial_{(c,v)} F(\tau, 1, \vec{0}, 0, c_0, 0) - \partial_{(c,v)} F(\tau, a, \vec{q}, g, c + c_0, v)] \cdot (\gamma, w)\end{aligned}$$

It is easy to check that

$$\begin{aligned}&[\partial_{(c,v)} F(\tau, 1, \vec{0}, 0, c_0, 0) - \partial_{(c,v)} F(\tau, a, \vec{q}, g, c + c_0, v)] \cdot (\gamma, w) \\ &= (c\partial_1 w - \gamma \partial_1 v - \ [\partial_s a(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v))w \\ &\quad + \partial_p a(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) \cdot \nabla w] \Delta(u_{0,\tau} + v) \\ &\quad - \ (a(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) - 1)\Delta w \\ &\quad - \ [\partial_s \vec{q}(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v))w \\ &\quad + \ \partial_p \vec{q}(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) \cdot \nabla w] \cdot \nabla(u_{0,\tau} + v) \\ &\quad - \ \vec{q}(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) \cdot \nabla w \\ &\quad - \partial_s g(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v))w \\ &\quad - \ \partial_p g(x_1, y, u_{0,\tau} + v, \nabla(u_{0,\tau} + v)) \cdot \nabla w, \ 0)\end{aligned}$$

From lemma 3.2 and (9), we conclude that

$$\begin{aligned}&\|\partial_{(c,v)} G(\tau, a, \vec{q}, g, c, v)\|_{\mathcal{L}(Y^{\rho_0})} \\ &\le A(\tau)[C_0|c| + C_0\|v\|_{D^{\rho_0}} + (\|\partial_s a\|_\infty + C_0\|\partial_p a\|_\infty) \\ &\quad \times (\|\Delta u_0\|_\infty + C_0\|v\|_{D^{\rho_0}}) + C_0\|a - 1\|_\infty \\ &\quad + (\|\partial_s \vec{q}\|_\infty + C_0\|\partial_p \vec{q}\|_\infty)(\|\nabla u_0\|_\infty + C_0\|v\|_{D^{\rho_0}}) \\ &\quad + C_0\|\vec{q}\|_\infty + \|\partial_s g\|_\infty + C_0\|\partial_p g\|_\infty] \qquad .\end{aligned}$$

From the definition of the norm in X, assertion (26) is now clear.

Remark 3.1. – From the properties of the function A, the positive reals δ and η constructed by this method go to 0 as $\tau \to 0^+$ or $+\infty$.

Proof of (27). – We have

$$\partial_{(a,\vec{q},g)} G(\tau, a, \vec{q}, g, c, v) \ : \ X \to Y^{\rho_0}$$

$$(\tilde{a},\tilde{q},\tilde{g}) \mapsto (\mathcal{F}^{\tau})^{-1}(\tilde{a}(x_1,y,u_{0,\tau}+v,\nabla(u_{0,\tau}+v))\Delta(u_{0,\tau}+v)$$
$$+\tilde{q}(x_1,y,u_{0,\tau}+v,\nabla(u_{0,\tau}+v))\cdot\nabla(u_{0,\tau}+v)$$
$$+\tilde{g}(x_1,y,u_{0,\tau}+v,\nabla(u_{0,\tau}+v)),\ 0)$$

Thus, by lemma (3.2) and formula (9), and since $\tilde{g}(x_1,y,0,p) = 0$ $\forall(x_1,y,p) \in \overline{\Sigma}\times\mathbb{R}^N$, we get

$$\begin{aligned}&\partial_{(a,\vec{q},g)}G(\tau,a,\vec{q},g,c,v)\cdot(\tilde{a},\tilde{q},\tilde{g})\\&\quad\le A(\tau)[\|\tilde{a}\|_\infty(\|\Delta u_0\|_{B^0}e^{r(\rho(\tau)-\rho_0)_+}+C_0\|v\|_{D^{\rho_0}})\\&\quad+\|\tilde{q}\|_\infty(\|\nabla u_0\|_{B^0}e^{r(\rho(\tau)-\rho_0)_+}+C_0\|v\|_{D^{\rho_0}})\\&\quad+\|\partial_s\tilde{g}\|_\infty(\|u_0\|_{B^0}e^{r(\rho(\tau)-\rho_0)_+}+\|v\|_{D^{\rho_0}})]\end{aligned}$$

Hence, if $\tau\in I$, $\|(a-1,\vec{q},g)\|_X<\delta$, $\|v\|_{D^{\rho_0}}<\eta$,

$$\|\partial_{(a,\vec{q},g)}G(\tau,a,\vec{q},g,c,v)\|_{\mathcal{L}(X,Y^{\rho_0})}\le C$$

Lastly, we can choose $\delta>0$ small enough such that $C\delta<1/2\eta$. This achieves the proof of theorem 1.

Remark 3.2. – The smoothness assumption $f\in C^3([0,1])$ was crucial in the proof. The linearized operator L^{τ_0} takes f' into account, and a continuous dependance of $\|(M^{\tau_0})^{-1}\|$ with respect to f' does not seem to be clear.

Remark 3.3. – Since the functions $u_{0,\tau}$ are of class C^1 in D^{ρ_0} with respect to τ, we infer that for any $(a,\vec{q},g)\in B_X((1,\vec{0},0),\delta)$, the set of solutions (c,u) of (10) contains the C^1 manifold $\{\Psi_I(\tau,a,\vec{q},g)+(0,u_{0,\tau}),\ \tau\in I\}$. Unfortunately, since $A(\tau)\to+\infty$ as $\tau\to0^+$ or $+\infty$, this result cannot be extended with the same method to the intervall $I=\mathbb{R}_+^*$.

3.2. Local existence when $\max_{\overline{\omega}}\ u(0,\cdot)$ is bounded: proof of theorem 2

Let $0<a<b<1$ and $\rho_0\in\mathbb{R}$. We recall that for any $h\in(0,1)$, $\mu(h)$ is defined by $\max_{\overline{\omega}}\ u_0(\mu(h),\cdot)=h$. In other words, $u_0^{\mu(h)}=u_0(\cdot+\mu(h),\cdot)$ satisfies $\max_{\overline{\omega}}\ u_0^{\mu(h)}(0,\cdot)=h$.

Let $a'=a/2$ and $b'=(1+b)/2$ and $I=\rho^{-1}(\mu([a',b']))=[\tau_{min},\tau_{max}]\subset\mathbb{R}_+^*$. From theorem 1, there exist $\delta,\ \eta>0$ such that if $(\tau,a,\vec{q},g)\in I\times B_X((1,\vec{0},0),\delta)$, there exists a unique solution (c,u) of (10) such that $(c,u-u_{0,\tau})\in B_{\mathbb{R}\times D^{\rho_0}}((c_0,0),\eta)$. From the proof of the previous section, we can choose δ and η small enough in such a way that $\eta < \min(a - a', b' - b)$.

Let now $\|(a-1, \vec{q}, g)\|_X < \delta$. The function

$$M \; : \; I \to B_{\mathbb{R}\times D^{\rho_0}}((0,0),\eta) \to \mathbb{R}$$

$$\tau \mapsto \Psi_I(\tau, a, \vec{q}, g) = (c(\tau), v(\tau)) \mapsto \max_{\overline{\omega}} \, (u_{0,\tau} + v(\tau))(0,\cdot)$$

is continuous on the intervall I. We have $\|v(\tau)\|_\infty \leq \|v(\tau)\|_{D^{\rho_0}} < \eta < a - a'$ and

$$\max_{\overline{\omega}} \; u_{0,\tau_{min}}(0,\cdot) = \max_{\overline{\omega}} \; u_0^{\mu(a')}(0,\cdot) = a',$$

whence $M(\tau_{min}) < a$. In the same way, $M(\tau_{max}) > b$. Hence, for any $h \in [a,b]$, there exists $\tau \in I$ such that $M(\tau) = h$. In other words, there exists a pair $(c,u) = (c(\tau), u_{0,\tau} + v(\tau))$ solution of (10) such that $\max_{\overline{\omega}} \; u(0,\cdot) = h$, $|c - c_0| < \eta$ and

$$\|u - u_{0,\tau}\|_{D^{\rho_0}} < \eta \tag{28}$$

Thus, $\max_{\overline{\omega}} \; u_0^{\mu(h)-\eta} = h - \eta < \max_{\overline{\omega}} \; u_0^{\rho(\tau)} < h + \eta = \max_{\overline{\omega}} \; u_0^{\mu(h)+\eta}$, that is to say

$$\mu(h - \eta) < \rho(\tau) < \mu(h + \eta)$$

Since μ is lipschitz-continuous and $\|\mu\|_{lip} \leq \|\partial_1 u_0\|_\infty$, we have

$$|\mu(h) - \rho(\tau)| \leq \|\partial_1 u_0\|_\infty \eta$$

From (28), it comes

$$\begin{aligned}\|u - u_0^{\mu(h)}\|_{D^{\rho_0}} &< \eta + \|u_0^{\mu(h)} - u_0^{\rho(\tau)}\|_{D^{\rho_0}} \\ &\leq \eta + \|\eta\partial_1 u_0\|_\infty \|\partial_1 u_0\|_{B^0} e^{r\max((\mu(h)-\rho_0)_+,(\rho(\tau)-\rho_0)_+)} \\ &\quad + \eta\|\partial_1 u_0\|_\infty (2C_0 + \|f'\|_\infty)\|\partial_1 u_0\|_{D^0} e^{r\max((\mu(h)-\rho_0)_+,(\rho(\tau)-\rho_0)_+)}\end{aligned}$$

Since $\rho(\tau)$, and thus $\mu(h)$ are bounded, the right hand side is bounded by a constant only depending on a and b. Besides, the norms in D^{ρ_0} and $D^{\mu(h)}$ are uniformly equivalent from (8) because $\mu(h)$ remains bounded. Finally, this gives the existence of a real η_1 in the assertion of theorem 2.

Remark 3.4. – The same arguments also hold for other normalization conditions like $\max_{\overline{\Sigma_-}} \; u = h$ or when the max is replaced by the min.

3.3. Global existence for small perturbations of the coefficients

Let I_0 be a fixed intervall in $\mathbb{R}_+^*$ such that $\inf I_0 > 0$ and the interior of I_0 is not empty. Let $\rho_0 = 0$. By theorem 1, there exist $\delta,\ \eta > 0$ such that if $\|(a-1, \vec{q}, g)\|_X < \delta$ and $\tau \in I_0$, then there exists a solution (c, u) of (10) in $\mathbb{R} \times D^0$ such that $\int_{\Sigma_-} u = \tau$ and $\|(c - c_0, u - u_{0,\tau})\|_{\mathbb{R}\times D^0} < \eta$. Besides, the map

$$\Phi\ :\ (\tau, a, \vec{q}, g) \mapsto (c, u)$$

is of class C^1.

Let τ_0 in the interior of I_0 and $\tau \in \mathbb{R}_+^*$. Let $(a, \vec{q}, g) \in B_X((1, \vec{0}, 0), \delta)$. For any $t \in \mathbb{R}$, we note $a^t(x_1, y, s, p) = a(x_1 + t, y, s, p)$ and in the same way $\vec{q}^{\,t}$ and g^t. We obvioulsy have $(a^t, \vec{q}^{\,t}, g^t) \in B_X((1, \vec{0}, 0, \delta)$, hence we can set $(c_t, u_t) = \Phi(\tau_0, a^t, \vec{q}^{\,t}, g^t)$.

In the proof of theorem 1, we choosed $\delta > 0$ small enough such that $\|a - 1\|_\infty \leq 1/2$ and, in the case where f satisfies (3) (bistable case), $\|\partial_s g\|_\infty \leq 1/2|f'(0)|$. In the case where f satisfies (2) (ignition temperature), we only consider functions g such that $g(x_1, y, s, p) \geq 0$ if $s \leq 0$. Hence, we always have $f(s) + g(x_1, y, s, p) \geq 0$ if $s \leq 0$. Since $u_t \to 0$ and 1 as $x_1 \to \pm\infty$, it comes from the strong maximum principle and the Hopf lemma that $u_t > 0$ in $\overline{\Sigma}$. Besides, we can choose (δ, η) small enough such that $\eta < 1/2\ |\omega|$.

Since $u_t = o(e^{rx_1})$ as $x_1 \to -\infty$ and $u_t \to 1$ as $x_1 \to +\infty$, in the same way as for u_0, the function $k\ :\ \nu \mapsto \int_{\Sigma_-} u_t(x_1 + \nu, y)$ is an increasing and continuous bijection from $\mathbb{R}$ to $\mathbb{R}_+^*$. Hence, there exists a unique real $\nu(t)$ such that

$$\int_{\Sigma_-} u_t(x_1 + \nu(t), y)\ dx_1 dy = \tau$$

We note $u_t^{\nu(t)} = u_t(x_1 + \nu(t), y)$. The pair $(c_t, u_t^{\nu(t)})$ is solution of

$$\begin{cases} a^{t+\nu(t)}(x_1, y, u_t^{\nu(t)}, \nabla u_t^{\nu(t)})\Delta u_t^{\nu(t)} - (c_t + \alpha(y))\partial_1 u_t^{\nu(t)} \\ \quad + \vec{q}^{\,t+\nu(t)}(x_1, y, u_t^{\nu(t)}, \nabla u_t^{\nu(t)}) \cdot \nabla u_t^{\nu(t)} \\ \quad + f(u_t^{\nu(t)}) + g^{t+\nu(t)}(x_1, y, u_t^{\nu(t)}, \nabla u_t^{\nu(t)}) = 0 \quad \text{in } \Sigma, \\ \partial_\nu u_t^{\nu(t)} = 0 \text{ on } \partial\Sigma, \\ u_t^{\nu(t)}(-\infty, \cdot) = 0,\ u_t^{\nu(t)}(+\infty, \cdot) = 1 \\ \int_{\Sigma_-} u_t^{\nu(t)} = \tau \end{cases} \tag{29}$$

In order to achieve the proof of theorem 3, it is enough to show the existence of $t \in \mathbb{R}$ such that $t + \nu(t) = 0$, indeed the pair $(c_t, u_t^{\nu(t)})$ will then be solution of (10) and $\int_{\Sigma_-} u_t^{\nu(t)} = \tau$, with $|c_t - c_0| < \eta$. To do this, we prove the following lemma:

LEMMA 3.4. – *The function* $\mathbb{R} \to \mathbb{R}$, $\quad t \mapsto \nu(t)$ *is continuous and bounded.*

Proof. – The function $\mathbb{R} \to X$, $\quad t \mapsto (a^t, \vec{q}, g^t)$ is continuous since a, $\vec{q}$, g are uniformly continuous as well as their derivatives with respect to s and p. Besides, when if f is of ignition temperature type, f satisfies (2), the property $g(x_1, y, s, p) \leq 0$ for $s \leq 0$ is preserved for g^t. In the same way, $\|\partial_s g\|_\infty \leq 1/2|f'(0)|$ is preserved for g^t (for the bistable case).

From theorem 1, the function $t \mapsto (c_t, u_t)$ solution of (10) and $\int_{\Sigma_-} u^t = \tau_0$ with $(a^t, \vec{q}^t, g^t)$ is then continuous from $\mathbb{R}$ to $\mathbb{R} \times D^0$. For the continuity of the function ν, it only remains to prove that the function

$$D^{0\prime} = \{u \in D^0,\ u > 0 \text{ in } \overline{\Sigma}\} \to \mathbb{R}$$

$$u \mapsto \mu(u),$$

where $\mu(u)$ is the unique real such that $\int_{\Sigma_-} u^{\mu(u)} = \tau$, is continuous. Let $u_n \to u$ in D^0 as $n \to \infty$ such that $u_n \in D^{0\prime}$ and $u \in D^{0\prime}$. There exists a sequence $\epsilon_n \to 0$ such that, for any n,

$$-\frac{\epsilon_n}{1 + e^{-rx_1}} + u \leq u_n \leq u + \frac{\epsilon_n}{1 + e^{-rx_1}} \text{ in } \overline{\Sigma}$$

Let $\epsilon > 0$. We have

$$-\frac{\epsilon_n}{1 + e^{-r(x_1 + \mu(u) - \epsilon)}} + u^{\mu(u)-\epsilon} \leq u_n^{\mu(u)-\epsilon} \leq u^{\mu(u)-\epsilon} + \frac{\epsilon_n}{1 + e^{-r(x_1 + \mu(u) - \epsilon)}}$$

This yields

$$-\frac{\epsilon_n \ln(1 + e^{r(\mu(u)-\epsilon)})}{r} + \int_{\Sigma_-} u^{\mu(u)-\epsilon}$$
$$\leq \int_{\Sigma_-} u_n^{\mu(u)-\epsilon} \leq \int_{\Sigma_-} u^{\mu(u)-\epsilon} + \frac{\epsilon_n \ln(1 + e^{r(\mu(u)-\epsilon)})}{r}$$

The right hand side of these inequalities goes to $\int_{\Sigma_-} u^{\mu(u)-\epsilon} < \tau$ as $n \to +\infty$. Hence, for n large enough, we have

$$\mu(u) - \epsilon < \mu(u_n)$$

In the same way, we have, for n large enough:

$$\mu(u) + \epsilon > \mu(u_n)$$

This proves that the function μ is continuous, so the function $t \mapsto \nu(t)$ is continuous.

Let us now prove that this function ν is bounded. We know that for any $t \in \mathbb{R}$, $\|u_t - u_{0,\tau_0}\|_{D^0} < \eta$. Hence,

$$-\frac{\eta}{1+e^{-rx_1}} + u_{0,\tau_0} < u_t < u_{0,\tau_0} + \frac{\eta}{1+e^{-rx_1}}$$

This gives

$$\begin{aligned} -\frac{\eta \ln(1+e^{r\nu(t)})}{r} + \int_{\Sigma_-} u_{0,\tau_0}^{\nu(t)} < \tau \\ = \int_{\Sigma_-} u_t^{\nu(t)} < \int_{\Sigma_-} u_{0,\tau_0}^{\nu(t)} + \frac{\eta \ln(1+e^{r\nu(t)})}{r} \end{aligned} \tag{30}$$

The right hand side goes to 0 as $\nu(t) \to -\infty$. Since $\tau > 0$, there exists a constant $A(\tau, \eta)$ such that

$$\forall t \in \mathbb{R}, \quad \nu(t) \geq A(\tau, \eta)$$

Since $u_{0,\tau}(+\infty, \cdot) = 1$, the integral $\int_{\Sigma_-} u_{0,\tau_0}^{\nu(t)}$ is greater than $1/2|\omega|\nu(t)$ if $\nu(t)$ is large enough. But we choosed η small enough such that $\eta < 1/2|\omega|$. Hence the left hand side of (30) goes to $+\infty$ as $\nu(t) \to +\infty$, so there exists a constant $B(\tau, \eta)$ such that

$$\forall t \in \mathbb{R}, \quad \nu(t) \leq B(\tau, \eta)$$

This achieves the proof of lemma 3.4. The function $t \mapsto t + \nu(t)$ is then continuous and its range is $\mathbb{R}$ since ν is bounded. Thus, there exists $t_0 \in \mathbb{R}$ such that $t_0 + \nu(t_0) = 0$. From (29), the pair $(c_{t_0}, u_{t_0}^{\nu(t_0)})$ is solution of (10) and $\int_{\Sigma_-} u_{t_0}^{\nu(t_0)} = \tau$ for the perturbation $(a, \vec{q}, g)$. Moreover, $|c_{t_0} - c| < \eta$.

Remark 3.5. – Let $a = 1/2|f'(0)|$ if f satisfies (3) and $+\infty$ if f satisfes (2). If we note (δ_I, η_I) the pair (δ, η) constructed in theorem 1 for any intervall $I \subset \mathbb{R}_+^*$ such that $\inf I > 0$, then theorem 3 actually holds for $\delta = \min(\sup_I \ \delta_I, a)$.

4. UNIQUENESS OF SOLUTIONS (c, u) FOR SMALL PERTURBATIONS OF THE COEFFICIENTS: PROOF OF THEOREM 4

In this part, we assume that f is a lipschitz-continuous function defined on $[0, 1]$, of class $C^{1,\delta}$ near 0 and 1, and such that $f(0) = f(1) = 0$.

This function is extended outside $[0,1]$ as in section 2. Let $a_{ij}^n(x_1,y,s,p)$, $\vec{q}^n(x_1,y,s,p)$, $g^n(x_1,y,s,p)$ be bounded, continuous functions defined on $\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N$. Besides, they are assumed to be of class $C^{0,\delta}$ with respect to (x_1,y) and a_{11}^n, q_1^n and g^n are assumed to be lipschitz-continuous with respect to (s,p). We assume that $\|a_{ij}^n-\delta_{ij}\|_\infty$, $\|\vec{q}^n\|_\infty$, $\|g^n\|_\infty\to 0$ as $n\to\infty$. Assume there exists a solution (c^n,u^n) of (13) for any n. For n large enough, the matrices $(a_{ij}^n)_{1\le i,j\le N}$ are elliptic; by the smoothness assumptions and the standard elliptic estimates, it comes that the u^n's are in $W_{loc}^{2,p}(\Sigma)$ and even in $C^{2,\alpha}(\overline{\Sigma})$. We always assume in the sequel that n is large enough in such a way that the previous properties are satisfied.

4.1. Convergence of the speeds c^n to c_0

LEMMA 4.1. – *There exists a real K such that $\forall\, n,\ |c^n|\le K$*

Proof. – Let $1+\eta^n$ be the unique real ≥ 1 such that $f(1+\eta^n)=-\|g^n\|_\infty$. We have $\eta^n\to 0$ as $n\to\infty$, and

$$f(s)+g^n(x_1,y,s,p)<0\quad \forall s>1+\eta^n,\ \forall(x_1,y,p)\in\overline{\Sigma}\times\mathbb{R}^N$$

In the same way, if f satisfies (3) (bistable nonlinearity), there exists $\epsilon^n\ge 0$, $\to 0^+$, such that $f(s)+g^n(x_1,y,s,p)>0\quad\forall s<-\epsilon^n$, $\forall(x_1,y,p)\in\overline{\Sigma}\times\mathbb{R}^N$. If f satisfies (2) (ignition temperature case), then $f(s)+g^n(x_1,y,s,p)\ge 0\ \forall s\le 0$, $\forall(x_1,y,p)\in\overline{\Sigma}\times\mathbb{R}^N$ by (14). From the maximum principle and the Hopf lemma, we get

$$\begin{cases} 0\le u^n\le 1+\eta^n \text{ if } f \text{ satisfies (2)}\\ -\epsilon^n\le u^n\le 1+\eta^n \text{ if } f \text{ satisfies (3)}\end{cases}\tag{31}$$

We now define a fixed function $\tilde{f}\ge f$ on an intervall $[\epsilon,1+\epsilon]$ for ϵ small enough. If f satisfies (2), let $0<\epsilon<1/2\min(\mu,\theta)$. We set

$$\tilde{f}=\begin{cases} 0 \text{ on } [\epsilon,2\epsilon]\\ f+(u-2\epsilon)(1+\epsilon-u) \text{ on } [2\epsilon,1]\\ (u-2\epsilon)(1+\epsilon-u) \text{ on } [1,1+\epsilon]\end{cases}$$

If f satisfies (3), let $0<\epsilon<1/2\ \theta$. We set

$$\tilde{f}=\begin{cases} 0 \text{ on } [\epsilon,2\epsilon]\\ (u-2\epsilon)(1+\epsilon-u) \text{ on } [2\epsilon,\theta]\\ f+(u-2\epsilon)(1+\epsilon-u) \text{ on } [\theta,1]\\ (u-2\epsilon)(1+\epsilon-u) \text{ on } [1,1+\epsilon]\end{cases}$$

In each of these cases, the function $\tilde{f}$ is of ignition temperature type. Hence, there exists a unique pair (k, v) solution of

$$\begin{cases} v'' - kv' + \tilde{f}(v) = 0 \text{ in } \mathbb{R} \\ v(-\infty) = \epsilon, \quad v(+\infty) = 1 + \epsilon \end{cases} \tag{32}$$

The real k is positive. The function v is unique up to translation, and $v' > 0$ in $\mathbb{R}$.

Since $\|a_{ij}^n - \delta_{ij}\|_\infty$ and $\|g^n\|_\infty \to 0$ as $n \to \infty$, we then infer that, for n large enough, we have $\min_{\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N} a_{11}^n > 0$ and

$$\frac{f(s) + g^n(x_1, y, s, p)}{a_{11}^n(x_1, y, s, p)} \le \tilde{f}(s) \quad \forall s \in [\epsilon, 1+\epsilon], \, \forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N \tag{33}$$

For any $\rho \in \mathbb{R}$, the function $v^\rho(x_1, y) := v(x_1 + \rho)$ satisfies

$$\begin{aligned} Av^\rho : &= a_{ij}^n \partial_{ij} v^\rho - (c^n + \alpha(y))\partial_1 v^\rho + \vec{q}^n \cdot \nabla v^\rho + f(v^\rho) + g(x_1, y, v^\rho, \nabla v^\rho) \\ &= a_{11}^n(x_1, y, v^\rho, \nabla v^\rho)\left[v^{\rho\prime\prime} - \frac{c^n + \alpha(y) - q_1^n}{a_{11}^n} v^{\rho\prime} + \frac{f(v^\rho) + g^n}{a_{11}^n}\right] \end{aligned}$$

Let us now assume that $\dfrac{c^n + \min_{\overline{\omega}} \alpha - \max_{\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N} q_1^n}{\max_{\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N} a_{11}^n} \ge k > 0$. Since $v' \ge 0$, $a_{11}^n > 0$, and from (32) and (33), we infer that for n large enough,

$$\forall \rho \in \mathbb{R}, \quad Av^\rho \le a_{11}^n(x_1, y, v^\rho, \nabla v^\rho)[v^{\rho\prime\prime} - kv^{\rho\prime} + \tilde{f}(v^\rho)] = 0 \text{ in } \Sigma \tag{34}$$

On the other side, let n large enough such that $\eta^n < \epsilon$ and the previous inequalities hold. From the limits of u^n and v as $x_1 \to \pm\infty$, there exists a real ρ_1 such that $v^{\rho_1} > u^n$ in Σ. Sliding v^{ρ_1} to the right, there exists a real ρ such that

$$v^\rho \ge u^n \text{ in } \overline{\Sigma} \text{ with equality somewhere}$$

Let $z = v^\rho - u^n$. From (34) and (13), we have

$$\begin{aligned} 0 &\ge Av^\rho - Au^n \\ &= a_{ij}^n(x_1, y, v^\rho, \nabla v^\rho)\partial_{ij} v^\rho - a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij} u^n \\ &\quad - (c^n + \alpha(y))\partial_1 z \;+\; \vec{q}^n(x_1, y, v^\rho, \nabla v^\rho) \cdot \nabla v^\rho \\ &\quad - \vec{q}^n(x_1, y, u^n, \nabla u^n) \cdot \nabla u^n \\ &\quad + f(v^\rho) - f(u^n) \;+\; g^n(x_1, y, v^\rho, \nabla v^\rho) - g^n(x_1, y, u^n, \nabla u^n) \end{aligned}$$

The first term $B = a_{ij}^n(x_1, y, v^\rho, \nabla v^\rho)\partial_{ij}v^\rho - a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij}u^n$ is of the form $B = B_1 + B_2 + B_3$ where

$$B_1 = a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij}z$$

$$B_2 = v^{\rho\prime\prime}[a_{11}^n(x_1, y, v^\rho, \nabla v^\rho) - a_{11}^n(x_1, y, u^n, \nabla v^\rho)]$$

$$B_3 = v^{\rho\prime\prime}[a_{11}^n(x_1, y, u^n, \nabla v^\rho) - a_{11}^n(x_1, y, u^n, \nabla u^n)]$$

Since v'' is bounded and a_{11}^n is lipschitz-continuous with respect to s and p, there exist bounded functions b_2 and $\vec{b_3}$ such that

$$B = a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij}z + \vec{b_3}(x_1, y) \cdot \nabla z + b_2(x_1, y)z \text{ in } \Sigma$$

The other terms of $Av^\rho - Au^n$ are treated in the same way. Finally, there exist bounded functions $\vec{b}(x_1, y)$ and $c(x_1, y)$ such that

$$0 \geq a_{ij}^n(x_1, y, u^n, \nabla u^n)\partial_{ij}z + \vec{b}(x_1, y) \cdot \nabla z + c(x_1, y)z \text{ in } \Sigma$$

From the strong maximum principle and the Hopf lemma, we conclude that $z \equiv 0$ in Σ. This is clearly impossible because of the behaviours of u^n and v as $x_1 \to \pm\infty$.

Hence, this proves that $\dfrac{c^n + \min_{\overline{\omega}} \alpha - \max_{\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N} q_1^n}{\max_{\overline{\Sigma}\times\mathbb{R}\times\mathbb{R}^N} a_{11}^n} < k$. In other words,

$$c^n \leq K_1$$

for n large enough, where K_1 is independant of n.

In the same way, we can define, for $\epsilon > 0$ fixed and small enough, a function $\underline{f}$ on $[-\epsilon, 1 - \epsilon]$ such that $\underline{f} = 0$ on $\{-\epsilon\} \cup [1 - 2\epsilon, 1 - \epsilon]$, $\underline{f} < 0$ on $(-\epsilon, 1 - 2\epsilon)$, and such that the inequality

$$\frac{f(s) + g^n(x_1, y, s, p)}{a_{11}^n(x_1, y, s, p)} \geq \underline{f}(s) \quad \forall s \in [-\epsilon, 1 - \epsilon],\ \forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N$$

holds for n large enough. With the same arguments as above, we conclude the existence of K_2 independant of n such that $c^n \geq -K_2$. This achieves the proof of lemma 4.1.

End of the proof of the convergence of c^n to c_0

We argue in several steps.

a) From lemma 4.1, there exists a subsequence that we rename (n) and a real c such that

$$c^n \to c \text{ as } n \to +\infty$$

b) Let $\mu^n = \|g^n\|_\infty \to 0$ as $n \to \infty$, and $\eta^n > 0$ such that $u^n \leq 1 + \eta^n$ in $\overline{\Sigma}$. From the beginning of the proof of lemma 4.1, we can choose η^n in such a way that $\eta^n \to 0$ as $n \to \infty$. From the profile of f near 1, there exists $\alpha > 0$, fixed once for all small enough such that

$$\begin{cases} f(s) \geq f'(1)/2\ (1-s) & \forall 1-\alpha \leq s \leq 1 \\ f(s) \geq -3/2 f'(1)\ (s-1) & \forall 1 \leq s \leq 1+\alpha \end{cases}$$

For n large enough, we have $\eta^n \leq \alpha$ and $0 \leq \alpha^n := 2\eta^n - 2/f'(1)\ \mu^n \leq \alpha$. It is easy to check that

$$f(s) - \mu^n \geq -f'(1)/2\ (1 - \alpha^n - s) \quad \forall\ 1 - \alpha \leq s \leq 1 + \eta^n \tag{35}$$

c) Since $u^n(-\infty, \cdot) = 0$, $u^n(+\infty, \cdot) = 1$ and $0 < 1 - \alpha < 1$, there exists a unique real τ^n such that

$$\min_{\overline{\Sigma_+}} u^n(\cdot + \tau^n, \cdot) = \min_{\overline{\omega}} u^n(\tau^n, \cdot) = 1 - \alpha$$

We set $v^n = u^n(x_1 + \tau^n, y)$. The functions (v^n)'s satisfy the equations:

$$\begin{aligned} &a_{ij}^n(x_1 + \tau^n, y, v^n, \nabla v^n)\partial_{ij} v^n - (c^n + \alpha(y))\partial_1 v^n \\ &\quad + \vec{q}^n(x_1 + \tau^n, y, v^n, \nabla v^n) \cdot \nabla v^n \\ &\quad + f(v^n) + g(x_1 + \tau^n, y, v^n, \nabla v^n) = 0 \text{ in } \Sigma \end{aligned}$$

Since $\|a_{ij}^n - \delta_{ij}\|_\infty$, $\|\vec{q}^n\|_\infty$, $\|g^n\|_\infty \to 0$ as $n \to \infty$ and since the speed (c^n)'s are bounded and the (v^n)'s remain in $[-1/2, 3/2]$ for n large enough, we conclude from the standard elliptic estimates that, up to extraction of some subsequence, we have $v^n \to v$ in $W_{loc}^{2,p}(\Sigma)$ $(\forall p > 1)$ and v is solution of

$$\begin{cases} \Delta v - (c + \alpha(y))\partial_1 v + f(v) = 0 \text{ in } \Sigma \\ \partial_\nu v = 0 \text{ on } \partial\Sigma \\ \min_{\overline{\omega}} v(0, \cdot) = 1 - \alpha = \min_{\overline{\Sigma_+}} v \\ 0 < v < 1 \text{ in } \overline{\Sigma} \text{ (from the strong maximum principle)} \end{cases} \tag{36}$$

d) We note L^n the elliptic operator (for n large enough):

$$\begin{aligned} L^n u = &a_{ij}^n(x_1 + \tau^n, y, v^n, \nabla v^n)\partial_{ij} u - (c^n + \alpha(y))\partial_1 u \\ &+ \vec{q}^n(x_1 + \tau^n, y, v^n, \nabla v^n) \cdot \nabla u + f'(1)/2\ w \end{aligned}$$

From (35), we have, $\forall 1-\alpha \leq s \leq 1+\eta^n$, $\forall (x_1, y, p) \in \overline{\Sigma} \times \mathbb{R}^N$,

$$f(s) + g^n(x_1 + \tau^n, y, s, p) \geq -f'(1)/2\ (1 - \alpha^n - s)$$

In $\overline{\Sigma_+}$, we have $1 - \alpha \leq v^n \leq 1 + \eta^n$. Hence

$$L^n(1 - \alpha^n - v^n) \geq 0 \text{ in } \Sigma_+$$

We now look for a supersolution for L^n of the form $w = e^{\mu x_1}$, $\mu < 0$. We have, for n large enough,

$$\begin{aligned} &L^n e^{\mu x_1} \\ &\quad = e^{\mu x_1}\ [a_{11}^n(x_1 + \tau^n, y, v^n, \nabla v^n)\mu^2 - (c^n + \alpha(y))\mu \\ &\qquad + q_1^n(x_1 + \tau^n, y, v^n, \nabla v^n)\mu + f'(1)/2] \\ &\quad \leq e^{\mu x_1}[3/2\mu^2 + (-c^n - \alpha(y) + q_1^n(x_1 + \tau^n, y, v^n, \nabla v^n))\mu + f'(1)/2] \end{aligned}$$

There exists a constant b such that $|c^n + \alpha(y) - q_1^n(x_1 + \tau^n, y, v^n, \nabla v^n)| \leq b$ $\forall (x_1, y) \in \overline{\Sigma}$. Let μ be the negative root of $3/2X^2 - bX + f'(1)/2 = 0$, it exists since $f'(1) < 0$. Then we have, for n large enough,

$$L^n e^{\mu x_1} \leq 0 \text{ in } \Sigma$$

Let $z^n = 1 - \alpha^n - v^n - (\alpha - \alpha^n)e^{\mu x_1}$. We get

$$L^n z^n \geq 0 \text{ in } \Sigma_+$$

and $z^n(0, \cdot) = 1 - \alpha^n - v^n(0, \cdot) - (\alpha - \alpha^n) = 1 - \alpha - v^n(0, \cdot) \leq 0$, $z^n(+\infty, \cdot) = -\alpha^n \leq 0$. Since $f'(1) < 0$, it comes from the maximum principle and the Hopf lemma that $z^n \leq 0$ in Σ_+, that is to say

$$v^n \geq 1 - \alpha^n - (\alpha - \alpha^n)e^{\mu x_1} \text{ in } \Sigma_+$$

We recall that $\alpha^n \to 0$ as $n \to \infty$. Passing to the limit $n \to \infty$, this yields

$$v \geq 1 - \alpha e^{\mu x_1} \text{ in } \Sigma_+$$

e) For any $\theta > \epsilon > 0$, we define a new function f_ϵ on $[-\epsilon, 1]$ such that:
- if f is of ignition temperature case (2),

$$\begin{cases} f_\epsilon = 0 \text{ on } [-\epsilon, 0] \\ f_\epsilon = f \text{ on } [0, 1] \end{cases}$$

- if f is of bistable type (3),

$$f_\epsilon(s) = f(s) - f(-\epsilon)\rho(s/\epsilon) \text{ in } [-\epsilon, 1]$$

where ρ is a fixed $C^\infty(\mathbb{R})$ function such that $\rho \equiv 1$ on $]-\infty, -1]$, $\rho \equiv 0$ on $[1, +\infty[$ and $0 < \rho < 1$ on $(-1, 1)$. The function f_ϵ is of bistable type on $[-\epsilon, 1]$ and is still of class $C^{1,\delta}([-\epsilon, 1])$.

In any case, we have $f_\epsilon \leq f$ and there exists a pair (c_ϵ, v_ϵ), unique up to translation in x_1 for v_ϵ, solution of

$$\begin{cases} \Delta v_\epsilon - (c_\epsilon + \alpha(y))\partial_1 v_\epsilon + f_\epsilon(v_\epsilon) = 0 \text{ in } \Sigma \\ \partial_\nu v_\epsilon = 0 \text{ on } \partial\Sigma \\ v_\epsilon(-\infty, \cdot) = -\epsilon, \quad v_\epsilon(+\infty, \cdot) = 1 \end{cases}$$

From results of [9], we have $c_\epsilon < c_0$ and $c_\epsilon \to c_0$ as $\epsilon \to 0^+$. *Stricto sensu*, this was only proved for case (2), but this can be easily extended for the bistable case, with the same ideas in § 6 of [9]: the functions (v_ϵ)'s, after two suitable normalizations, converge to two functions v and w solutions of $\Delta u - (c' + \alpha(y))\partial_1 u + f(u) = 0$ in Σ with $v(-\infty, \cdot) = 0$ and $w(+\infty, \cdot) = 1$ and $c' = \lim_{\epsilon\to 0} c_\epsilon$. After comparison of v and w to the function u_0 by a sliding method similar to the one used in the proof of lemma 4.1, we conclude that $c' = c_0$.

f) We now compare this function v_ϵ to the function v constructed in *c*). Let us assume $c_\epsilon > c$. We have $v_\epsilon(+\infty, \cdot) = v(+\infty, \cdot) = 1$ and v_ϵ, $v < 1$. From the results of [1], [23], [9], it comes

$$v(x_1, y) = 1 - e^{\nu x_1}\psi(y) + o(e^{\nu x_1}) \text{ as } x_1 \to +\infty$$

$$v_\epsilon = 1 - e^{\nu_\epsilon x_1}\psi_\epsilon(y) + o(e^{\nu_\epsilon x_1}) \text{ as } x_1 \to +\infty$$

where $\nu < \nu_\epsilon < 0$ and ψ, $\psi_\epsilon > 0$ on $\overline{\omega}$. Hence, after translation of the function v_ϵ to the right and then to the left, there exists a real τ such that

$$v_\epsilon^\tau = v_\epsilon(x_1 + \tau, y) \leq v(x_1, y) \text{ in } \overline{\Sigma} \text{ with equality somewhere}$$

The function $z = v_\epsilon^\tau - v \leq 0$ satisfies

$$\Delta z - (c + \alpha(y))\partial_1 z + f_\epsilon(v_\epsilon^\tau) - f_\epsilon(v) = (c_\epsilon - c)\partial_1 v_\epsilon^\tau + f(v) - f_\epsilon(v) \geq 0$$

since $f \geq f_\epsilon$, $c_\epsilon > c$ and $\partial_1 v_\epsilon > 0$. But f_ϵ is lipschitz-continous, whence

$$\begin{cases} \Delta z - (c + \alpha(y))\partial_1 z + c(x_1, y)z \geq 0 \text{ in } \Sigma \\ \partial_\nu z = 0 \text{ on } \partial\Sigma \end{cases}$$

for some bounded function $c(x_1, y)$. From the strong maximum principle and the Hopf lemma, we infer that $z \equiv 0$ in $\overline{\Sigma}$. This is impossible since $v_\epsilon^\tau(-\infty, \cdot) = -\epsilon$ and $v \geq 0$ in $\overline{\Sigma}$.

Hence, we have $c_\epsilon \leq c$, for any ϵ small enough. By e), we have $c_\epsilon \to c_0$ as $\epsilon \to 0$. We finally conclude that

$$c_0 \leq c$$

g) Conclusion in the bistable case (3).

We can write again the parts *b*) to *f*) with a normalization of $u^n$of the type $\max_{\overline{\Sigma_-}} v^n = \max_{\overline{\omega}} v^n(0, \cdot) = \alpha$ where $v^n = u^n(x_1 + \tau'^n, y)$ and α is suitably choosen small enough. We then conclude with the same arguments to the inequality

$$c_0 \geq c$$

and at the end

$$c = c_0$$

h) Conclusion in the ignition temperature case (3).

We have to work just a little more. We assumed in this case that there exists $\mu > 0$ such that (14) holds. We can choose $\mu \leq \theta$. Let us now define τ'^n and $v^n = u^n(x_1 + \tau'^n, y)$ in such a way that

$$\max_{\overline{\Sigma_-}} v^n = \max_{\overline{\omega}} v^n(0, \cdot) = \mu$$

Up to extraction of some subsequence, we have $v^n \to v$ in $W^{2,p}_{loc}(\Sigma)$ where v is solution of $\Delta v - (c + \alpha(y))\partial_1 v + f(v) = 0$ and $\max_{\overline{\omega}} v(0, \cdot) = \max_{\overline{\Sigma_-}} v = \mu$. In Σ_-, we have $0 \leq v^n \leq \mu$, thus

$$\begin{aligned} L^n v^n :=& a^n_{ij}(x_1 + \tau'^n, v^n, \nabla v^n)\partial_{ij} v^n - (c^n + \alpha(y))\partial_1 v^n \\ &+ \vec{q}^n(x_1 + \tau'^n, v^n, \nabla v^n) \cdot \nabla v^n \geq 0 \text{ in } \Sigma_- \end{aligned}$$

Besides, $c^n \to c \geq c_0$ and we know from [9] that $\int_\omega (c_0 + \alpha(y))dy > 0$ (this comes from integration on Σ of the equation (4) satisfied by u_0). Hence, there exists $c' < c_0$ and $\epsilon > 0$ such that $\int_\omega (c' + \alpha(y))dy > 0$ and $c^n \geq c' + \epsilon$ for n large enough. By § 3 of [9], there exists a function $w' = e^{\lambda' x_1}\phi'(y)$ solution of

$$\Delta w' - (c' + \alpha(y))\partial_1 w' = 0 \text{ in } \Sigma, \quad \partial_\nu w' = 0 \text{ on } \partial\Sigma$$

with $\lambda' > 0$ and $\phi' > 0$ on $\overline{\omega}$. We can assume that $\min_{\overline{\omega}} \phi' = 1$. Let now prove that this function w' is a supersolution of L^n for n large enough. Indeed, after an easy calculation,

$$\begin{aligned} L^n w' = e^{\lambda' x_1} \, & [\Sigma_{i,j=2}^N (a_{ij}^n(x_1 + \tau'^n, v^n, \nabla v^n) - \delta_{ij})\partial_{ij}\phi'(y) \\ & + \lambda' \Sigma_{j=2}^N (a_{1j}^n(x_1 + \tau'^n, v^n, \nabla v^n) \\ & + a_{j1}^n(x_1 + \tau'^n, v^n, \nabla v^n))\partial_j \phi'(y) \\ & + \lambda'^2 (a_{11}^n(x_1 + \tau'^n, v^n, \nabla v^n) - 1)\phi'(y) + \lambda'(c' - c^n)\phi'(y) \\ & + \lambda' q_1^n(x_1 + \tau'^n, v^n, \nabla v^n)\phi'(y) \\ & + \Sigma_{j=2}^N q_j^n(x_1 + \tau'^n, v^n, \nabla v^n)\partial_j \phi'(y)] \\ = e^{\lambda' x_1} \, & [S^n(x_1, y) + \lambda'(c' - c^n)\phi'(y)] \end{aligned}$$

The reals ϵ, $\lambda > 0$ and the function $\phi' \geq 1$ are fixed. We have $\lambda'(c' - c^n)\phi' \leq -\epsilon\lambda' < 0$ and $S^n \to 0$ uniformly in $\overline{\Sigma}$ as $n \to \infty$. Hence, for n large enough, we have

$$L^n w' \leq 0 \text{ in } \Sigma$$

For n large enough, the operators L^n are elliptic. Since $v^n(-\infty, \cdot) = 0$ and $v^n(0, \cdot) \leq \mu$, we finally deduce from the maximum principle and the Hopf lemma that

$$v^n(x_1, y) \leq \mu e^{\lambda' x_1} \phi'(y) \text{ in } \Sigma_-$$

At the limit $n \to \infty$, we get

$$v \leq \mu e^{\lambda' x_1} \phi'(y) \text{ in } \Sigma_-$$

and we have already written

$$\max_{\overline{\omega}} \, v(0, \cdot) = \mu$$

We can now argue as in part *e*) and *f*). If we compare v to some function v^ϵ solution of $\Delta v^\epsilon - (c^\epsilon + \alpha(y))\partial_1 v^\epsilon + f^\epsilon(v^\epsilon) = 0$ in Σ, $\partial_\nu v^\epsilon = 0$ on $\partial\Sigma$, $v^\epsilon(-\infty, \cdot) = 0$, $v^\epsilon(+\infty, \cdot) = 1 + \epsilon$, where $f^\epsilon \geq f$ and $f^\epsilon > 0$ on $(\theta, 1 + \epsilon)$, $f^\epsilon = 0$ on $[0, \theta] \cup \{1 + \epsilon\}$, we get $c \leq c^\epsilon$ by a sliding method. Since $c^\epsilon \to c_0$ as $\epsilon \to 0$, we deduce $c \leq c_0$ and finally

$$c = c_0$$

Remark 4.1. – Since the limit c_0 is unique, the whole sequence c^n goes to c_0 as $n \to \infty$.

4.2. Convergence of u^n to $u_{0,\tau}$ when $\int_{\Sigma_-} u^n$ is bounded, for iginiton temperature case: proof of part *b*) of theorem 4

Let f satisfy (2) and g^n satisfy the assumption of uniform ignition temperature μ (15). Let (c^n, u^n) be solution of (13), with moreover the normalization condition (16). From the previous section, we know that $c^n \to c_0$ as $n \to \infty$.

Let x_n be the unique real such that

$$\max_{]-\infty,x_n]\times\overline{\omega}} u^n = \max_{\overline{\omega}} u^n(x_n, \cdot) = \mu$$

Let $c' = c_0 - \epsilon$ such that $\int_\omega c' + \alpha(y) > 0$ and $w' = e^{\lambda' x_1}\phi'(y)$ solution of $\Delta w' - (c' + \alpha(y))\partial_1 w' = 0$, $\partial_\nu w' = 0$ with $\lambda' > 0$ and $\phi' \geq 1$ in $\overline{\omega}$. We can even choose $\lambda' > r$ since $\lambda' \to \lambda$ as $c' \to c$ from [9] (the reals λ and r are defined in section 2). As in the previous subsection, we have, for n large enough,

$$u^n(x_1, y) \leq \mu e^{\lambda'(x_1 - x_n)}\phi'(y) \quad \forall x_1 \leq x_n,\ y \in \overline{\omega} \tag{37}$$

From standard elliptic estimates, the functions $v^n(x_1, y) := u^n(x_1 + x_n, y)$ converge (at least for some subsequence) in $W^{2,p}_{loc}(\Sigma)$ to some function v_∞ solution of

$$\Delta v_\infty - (c_0 + \alpha(y))\partial_1 v_\infty + f(v_\infty) = 0 \text{ in } \Sigma \tag{38}$$

and

$$0 \leq v_\infty(x_1, y) \leq \mu e^{\lambda' x_1}\phi'(y) \text{ in } \Sigma_-$$

$$\max_{\overline{\omega}} v_\infty(0, \cdot) = \max_{\overline{\Sigma_-}} v_\infty = \mu$$

As a consequence, as for (5), we have

$$v_\infty(x_1, y) = \gamma e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty$$

$$u_0(x_1, y) = e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty$$

If $v_\infty(+\infty, \cdot) = 1$, then $v_\infty = u_0^\rho$ for some $\rho \in \mathbb{R}$ from the uniqueness result of [9] recalled in section 2. In the other case, we infer $\limsup_{x_1 \to +\infty} v_\infty \leq \theta$ since $\int_\Sigma f(v_\infty) < +\infty$ (from integration of (38) on Σ and the nonnegativity of f) and ∇v_∞ is bounded. Hence, we can still compare u_0 to the function

v_∞ with a sliding method and then conclude to a contradiction, using furthermore lemma 4.1 of [9].

Thus, the functions (v^n)'s converge in $W^{2,p}_{loc}(\Sigma)$ to the function u_0^ρ for some $\rho \in \mathbb{R}$. We now prove that the sequence (x_n) is bounded. Otherwise, there exists a subsequence, that we rename (x_n) such that $x_n \to +\infty$ or $x_n \to -\infty$. In the first case, we have from (37) and for n large enough such that $x_n \geq 0$:

$$u^n \leq \mu e^{-\lambda' x_n} e^{\lambda' x_1} \phi'(y) \text{ in } \Sigma_-$$

Hence,

$$\tau_n = \int_{\Sigma_-} u^n = O(e^{-\lambda' x_n}) \text{ as } n \to \infty$$

This is impossible since $\tau_n \to \tau > 0$ as $n \to \infty$.

In the case $x_n \to -\infty$, we have, for any $\gamma \in \mathbb{R}_+$,

$$\tau_n = \int_{\Sigma_-} u^n \geq \int_{[x_n-\gamma, x_n+\gamma]\times\overline{\omega}} u^n \quad \text{(for } n \text{ large enough)}$$

But, $\int_{[x_n-\gamma,x_n+\gamma]\times\overline{\omega}} u^n = \int_{[-\gamma,\gamma]\times\overline{\omega}} v^n$ go to $\int_{[-\gamma,\gamma]\times\overline{\omega}} u_0^\rho$ as $n \to \infty$. Hence $\tau \geq \int_{[-\gamma,\gamma]\times\overline{\omega}} u_0^\rho$ for any $\rho \in \mathbb{R}_+$. This is impossible since $\int_{[-\gamma,\gamma]\times\overline{\omega}} u_0^\rho \to +\infty$ as $\gamma \to \infty$.

We then deduce that the sequence (x_n) is bounded and finally that some subsequence of (u^n) converges to the function $u_0^{\rho'}$ for some $\rho' \in \mathbb{R}$. Besides, we know that $u_0^{\rho'} = O(e^{\lambda x_1})$ as $x_1 \to -\infty$ and there exists $A \geq 0$ such that $u^n \leq \mu e^{\lambda'(x_1 - A)} \phi'(y)$ for n large enough and $x_1 \leq A$. Hence, by elementary arguments, the integrals $\int_{\Sigma_-} u^n$ converge to $\int_{\Sigma_-} u_0^{\rho'}$ as $n \to \infty$. Thus $\tau = \int_{\Sigma_-} u_0^{\rho'}$. In other words, $\rho' = \rho(\tau)$ (the function ρ is defined in § 2) and u^n converge to $u_0^{\rho(\tau)} = u_{0,\tau}$ in $W^{2,p}_{loc}$. This is true for the whole sequence (u^n) by the uniqueness of the limit.

It only remains to prove that $u^n \to u_{0,\tau}$ in $D^{\rho(\tau)}$. Firstly, we have

$$\sup_{(x_1,y)\in\overline{\Sigma}} |(1+e^{-rx_1})(u^n - u_{0,\tau})| \to 0 \text{ as } n \to \infty, \tag{39}$$

otherwise there exists $\epsilon > 0$ and points $(x_n, y_n) \in \overline{\Sigma}$ such that

$$|(1+e^{-rx_n})(u^n(x_n,y_n) - u_{0,\tau}(x_n,y_n))| \geq \epsilon$$

The sequence (x_n) is not bounded because of the convergence of u^n to $u_{0,\tau}$ in $W^{2,p}_{loc}(\Sigma)$ for any $p > 1$.

If $x_n \to -\infty$ (or at least some subsequence), then we have $u_{0,\tau}(x_n, y_n) = O(e^{\lambda x_n})$ and $u^n(x_n, y_n) = O(e^{\lambda' x_1})$ as $n \to \infty$ from the remarks of the previous subsection. Since $\lambda > r$ and we have choosen $\lambda' > r$, we obtain a contradiction.

In the last case, we have $x_n \to +\infty$ (at least for some subsequence). Since $u_{0,\tau}(+\infty, \cdot) = 1$, we have $u^n(x_n, y_n) \leq 1 - \epsilon/2$ for n large enough. We now define $x'_n \geq x_n$ in such a way that $\min_{[x'_n, +\infty[\times\overline{\omega}} u^n = \min_{\overline{\omega}} u^n(x'_n, \cdot) = 1 - \epsilon/2$. We can choose ϵ such that $f(s) \geq -f'(1)/2\ (1 - s)$ for $1 - \epsilon/2 \leq s \leq 1$. As in the previous subsection, we have, for $x_1 \geq x'_n$, $u^n \geq 1 - \epsilon/2\, e^{\mu'(x_1 - x'_n)}\psi'(y)$ where $\mu' < 0$, $\psi'(y) \geq 1$ and $w' = e^{\mu' x_1}\psi'(y)$ is solution of $\Delta w' - (c' + \alpha(y))\partial_1 w' + f'(1)/2\ w' = 0$, $\partial_\nu w' = 0$ for some fixed $c' > c$. The functions $v^n = u^n(x'_n + x_1, y)$ go to some function v solution of $\Delta v - (c_0 + \alpha(y))\partial_1 v + f(v) = 0$, $\partial_\nu v = 0$ and $v \geq 1 - \epsilon/2\ e^{\mu' x_1}\psi'(y)$ in Σ_+, $\min_{\overline{\omega}} v(0, \cdot) = 1 - \epsilon/2$. If $\liminf_{x_1 \to -\infty} v(x_1, \cdot) > 0$, then we conclude by a sliding method as in [9] (with lemma 4.1 of [9]) that $v = u_0^\mu$ for some $\mu \in \mathbb{R}$. Hence, we always have $\liminf_{x_1 \to -\infty} v(x_1, \cdot) = 0$. Then there exists $\gamma \geq 0$ and $y_0 \in \overline{\omega}$ such that $v(-\gamma, y_0) \leq 1/2 \min(\mu, \theta)$. Let A such that $u_{0,\tau} \geq (1 + \theta)/2$ in $[A, +\infty[\times\overline{\omega}$. For n large enough, we have then $u^n \geq \theta$ in $[A, A + 1] \times \overline{\omega}$, $x'_n - \gamma > A + 1$ and $u^n(x'_n - \gamma, y_0) = v^n(-\gamma, y_0) < \min(\mu, \theta)$. The set $\Omega = (A + 1, x'_n) \times \omega \cap \{u^n(x_1, y) < \min(\mu, \theta)\}$ is not empty and on Ω, we have $M^n(u^n - \min(\mu, \theta)) \equiv 0$ where M^n is an elliptic operator with no zero-order term. Besides, from the values of u^n at $x_1 = x'_n$ or $A + 1$, $u^n = \min(\mu, \theta)$ on $\partial\Omega$. This is in contradiction with the maximum principle and ends the proof of (39).

The sequence $z^n - u_{0,\tau}$ goes to 0 in B^0 and satisfies the equations

$$
\begin{aligned}
&a^n_{ij}\partial_{ij}z^n - (c^n + \alpha(y))\partial_1 z^n + \vec{q}^n \cdot \nabla z^n \\
&\quad = (\delta_{ij} - a^n_{ij})\partial_{ij}u_{0,\tau} + (c^n - c_0)\partial_1 u_{0,\tau} \\
&\qquad - \vec{q}^n \cdot \nabla u_{0,\tau} + f(u_{0,\tau}) - f(u^n) - g^n(x_1, y, u^n, \nabla u^n)
\end{aligned}
$$

From (15) and since $u^n(x_1, y) \leq \mu e^{\lambda' x_1 - A}\phi'(y)$ where $\lambda' > r$ for some $A \geq 0$ and for any n large enough (by the arguments above), the terms $g^n(x_1, y, u^n, \nabla u^n)$ go to 0 in B^0. Since f is lipschitz-continuous and $\partial_i u_{0,\tau}$, $\partial_{ij}u_{0,\tau} \in B^0$, it finally comes from the standard elliptic estimates that $\partial_i z^n$, $\partial_{ij}z^n$ to 0 in B^0. Hence $u^n \to u_{0,\tau}$ in D^0 and even in $D^{\rho(\tau)}$ from the inequalities (8). This achieves the proof of part $b)$ of theorem 4.

4.3. Convergence of u^n to $u_0^{\mu(h)}$ when $\max_{\overline{\Sigma_-}} u^n$ is bounded, for ignition temperature case: proof of part *c*) of theorem 4

We now assume that $\max_{\overline{\Sigma_-}} u^n = h^n \to h \in (0, 1)$. We argue exactly as in the previous subsection and define in the same way x_n such that

$$\max_{]-\infty, x_n] \times \overline{\omega}} u^n = \max_{\overline{\omega}} u^n(x_n, \cdot) = \mu$$

The functions $v^n = u^n(x_1 + x_n, y)$ go to u_0^ρ for some $\rho \in \mathbb{R}$.

If $x_n \to -\infty$ (at least for some subsequence), then, for any $\gamma \geq 0$, we have

$$h^n = \max_{\overline{\Sigma_-}} u^n \geq \max_{[x_n - \gamma, x_n + \gamma] \times \overline{\omega}} u^n \text{ for } n \text{ large enough}$$

$$\to \max_{[-\gamma, \gamma] \times \overline{\omega}} u_0^\rho$$

Since $u_0^\rho(+\infty, \cdot) = 1$ and $h^n \to h < 1$, we obtain a contradiction. On the other side, if $x_n \to +\infty$, then $u^n \leq \mu e^{\lambda'(x_1 - x_n)} \phi'(y)$ in Σ_- with $\lambda' > 0$, and this yields a contradiction.

Finally, the sequence (x_n) is bounded and we conclude in the same way as in the previous subsection.

5. CONCLUDING REMARKS AND OPEN QUESTIONS

All the results presented in this paper also hold if the term $c + \alpha(y)$ is replaced by $\beta(y, c)$, where β is of class $C^{2,\delta}$ with respect to y, increasing in c and $\beta(y, c) \to \pm\infty$ as $c \to \pm\infty$ (*cf.* [9]). For instance, in some models, $\beta(y, c) = c\alpha(y)$ with $\alpha > 0$ on $\overline{\omega}$.

We remarked that the functions u constructed in theorem 1-3 are not necessarily increasing in x_1 although they are close to some function u_0^ρ increasing in x_1. Nevertheless, in dimension 1, if $f + g$ has a constant sign, then u is increasing from the maximum principle. In higher dimension, we cannot apply a sliding method as in [9] for the invariant case by translation, or in [14] when the coefficients are monotone in x_1. As a consequence, the question of stability of the solutions constructed for small $(a - 1, \vec{q}, g)$ seems to be intricate.

If $(a, \vec{q}, g) = (1, \vec{0}, 0)$, the set of solutions of (10) is $\{(c_0, u_{0,\tau}),\ \tau \in \mathbb{R}_+^*\}$ which is a C^2 manifold in Y^0. For $I \subset \mathbb{R}_+^*$, $\inf I > 0$, if $(a - 1, \vec{q}, g)$ is small enough, we proved that the set of solutions (c, u) of (10) contains a C^1 manifold $\{(c(\tau), u(\tau),\ \tau \in I\}$ (theorem 1), each $u(\tau)$ being close to $u_{0,\tau}$ and such that $\int_{\Sigma_-} u(\tau) = \int_{\Sigma_-} u_{0,\tau} = \tau$. In theorem 3, we proved that for $(a-1, \vec{q}, g)$ small enough and for any $\tau > 0$, there exists a solution (c, u) of (10) such that $\int_{\Sigma_-} u = \tau$. The final result would be to prove that the set of solutions of (10) is actually a C^1 manifold, for small $(a-1, \vec{q}, g)$.

In [30], Xin proved the existence of solutions (c, U) of

$$(\nabla_y + \vec{k}\partial_s)(A(y)(\nabla_y + \vec{k}\partial_s)U) \\ + \vec{b}(y) \cdot (\nabla_y + \vec{k}\partial_s)U + c\partial_s U + f(U) = 0 \text{ in } \mathbb{R} \times\ T$$

for $A(y)$, $\vec{b}(y)$ not necessarly close to $(Id, \vec{0})$. By a method of continuation, he solved the same problem for $(A^t(y), \vec{b}^t(y)) = (1-t)(Id, \vec{0}) + t(A(y), \vec{b}(y))$. At any step $t_0 \in [0,1]$, there is a continuation because the linearized operator is invertible. This method do not work in our case because we have no a priori properties for the linearized operator around a solution of (10), due to the dependance of the coefficients of (10) on x_1 (the equations investigated by Xin in [30] were invariant by translation in s).

We can nevertheless ask the question of the existence of solutions (c, u) of (10) when $||(a-1, \vec{q}, g)||$ increases. Are there any bifurcation phenomena, any non-existence results as for similar problems treated in [32] or [10] in periodic media? or transition between existence and non-existence according to the intensity of $||\vec{q}||$ like in counterflow flames models in [21]?

We mentionned in the introduction the existence of solutions of a system of two reaction-diffusion equations for Lewis numbers close to 1. Under additionnal assumptions, monotone solutions can be constructed for systems of reaction-diffusion ordinary differential equations. Because of the monotonicity, it could be interesting to investigate small perturbations of such systems.

ACKNOWLEDGEMENTS

The author would like to thank Professors Berestycki and Roquejoffre for their helpful support and suggestions in this work.

REFERENCES

[1] S. Agmon and L. Nirenberg, Properties of solutions of ordinary differential equations in Banach space, *Comm. Pure Appl. Math.*, Vol. **16**, 1963, pp. 121-239.

[2] D. G. ARONSON and H. F. WEINBERGER, Multidimensional nonlinear diffusions arising in population genetics, *Adv. in Math.*, Vol. **30**, 1978, pp. 33-76.
[3] D. L. BARROW and P. W. BATES, Bifurcation and stability of periodic travelling wave for a reaction-diffusion system, *J. Diff. Eqns.*, Vol. **50**, 1983 , pp. 218-233.
[4] H. BERESTYCKI and B. LARROUTUROU, Quelques aspects mathématiques de la propagation des flammes prémélangées, Nonlinear p.d.e. and their applications, Collège de France seminar, Vol. **10**, Brézis and Lions eds, Pitman Longman, Harbow, UK, 1990.
[5] H. BERESTYCKI and B. LARROUTUROU, A semilinear elliptic equation in a strip arising in a two-dimensional flame propagation model, *J. Reine Angew. Math.*, Vol. **396**, 1989, pp. 14-40.
[6] H. BERESTYCKI, B. LARROUTUROU and P. L. LIONS, Multidimensional traveling-wave solutions of a flame propagation model, *Arch. Rat. Mech. Anal.*, Vol. **111**, 1990, pp. 33-49.
[7] H. BERESTYCKI, B. LARROUTUROU, P. L. LIONS and J.-M. ROQUEJOFFRE, *An elliptic system modelling the propagation of a multidimensional flame*, preprint.
[8] H. BERESTYCKI, B. LARROUTUROU and J. M. ROQUEJOFFRE, Stability of traveling fronts in a curved flame model, Part I, Linear Analysis, *Arch. Rat. Mech. Anal.*, Vol. **117**, 1992, pp. 97-117.
[9] H. BERESTYCKI and L. NIRENBERG, Travelling fronts in cylinders, *Ann. Inst. H.Poincaré, Analyse Non Linéaire*, Vol. **9**, 5, 1992, pp. 497-572.
[10] H. BERESTYCKI and G. I. SIVASHINSKY, Flame extinction by a periodic flow field, *SIAM, J. Appl. Math.*, Vol. **51**, 1991, pp. 344-350.
[11] J. D. BUCKMASTER and G. S. S. LUDFORD, Lectures on Mathematical Combustion, *CBMS-NSF Conf. Series in Applied Math.*, Vol. **43**, SIAM, 1983.
[12] P. C. FIFE and J. B. MCLEOD, The approach of solutions of non-linear diffusion equations to traveling front solutions, *Arch. Rat. Mech. Anal.*, Vol. **65**, 1977, pp. 335-361.
[13] P. S. HAGAN, Travelling waves and multiple travelling waves solutions of parabolic equations, *SIAM J. Math. Anal.*, Vol. **13**, 1982, pp. 717-738.
[14] F. HAMEL, Reaction-diffusion problems in cylinders with no invariance by translation. Part II: Monotone perturbations, *Ann. Inst. H. Poincaré*, Vol. **14**, 1997.
[15] F. HAMEL, Problèmes de réaction-diffusion sans invariance par translation dans des cylindres infinis, *C. R. Acad. Sci. Paris*, Vol. **321**, I, 1995, pp. 855-860.
[16] F. HAMEL, Quelques problèmes d'ondes progressives dans les équations aux dérivées partielles et applications à la théorie de la combustion, *Ph. D. Thesis*, Univ. Paris VI, France, 1996.
[17] D. HENRY, Geometric theory of semilinear parabolic equations, *Lectures Notes in Math.*, Springer Verlag, New York, 1981.
[18] Ja. I. KANEL, On the stability of solutions of the equations of combustion theory for finite initial functions, *Mat Sbornik*, Vol. **65**, (107), 1964, pp. 398-413.
[19] A. N. KOLMOGOROV, I. G. PETROVSKY and N. S. PISKUNOV, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.), Série internationale, section A.1, 1937, pp. 1-26.
[20] M. G. KREIN and M. A. RUTMAN, Linear operators leaving invariant a cone in a Banach space, *AMS Translation*, Vol. **26**, 1950.
[21] A. LIÑAN, *Fluid Dynamic Aspects of Combustion Theory*, course of IAC, Mauro Picone, Rome, 1989.
[22] G. PAPANICOLAOU and X. XIN, Reaction-diffusion Fronts in Periodically Layered media, *J. Statist. Phys.*, Vol. **63**, 1991, pp. 915-931.
[23] A. PAZY, Asymptotic expansions of solutions of ordinary differential equations in Hilbert space, *Arch. Rat. Mech. Anal.*, Vol. **24**, 1967, pp. 193-218.
[24] J. M. ROQUEJOFFRE, Stability of traveling fronts in a curved flame model, Part II: Non-linear Orbital Stability, *Arch. Rat. Mech. Anal.*, Vol. **117**, 1992, pp. 119-153.
[25] D. H. SATTINGER, Stability of waves of nonlinear parabolic systems, *Adv. Math.*, Vol. **22**, 1976, pp. 312-355.

[26] G. I. Sivashinsky, Instabilities, pattern formation and turbulence in flames, *Ann. Rev. Fluid Mech.*, Vol. **15**, 1983, pp. 179-199.

[27] J. M. Vega, Multidimensional travelling fronts in a model from combustion theory and related problems, *Diff. and Int. Eq.*, Vol. **6**, 1993, pp. 131-155.

[28] F. Williams, *Combustion Theory*, Addison-Wesley, Reading MA, 1983.

[29] X. Xin, Existence and Uniqueness of Travelling Waves in a Reaction-Diffusion Equation with Combustion Nonlinearity, *Idiana Univ. Math. J.*, Vol. **40**, No 3, 1991.

[30] X. Xin, Existence of planar flame fronts in convective-diffusive periodic media, *Arch. Rat. Mech. Analysis*, Vol. **121**, 1992, pp. 205-233.

[31] X. Xin, Existence and stability of travelling waves in periodic media governed by a bistable nonlinearity, *J. Dyn. Diff. Eq.*, Vol. **3**, 1991, pp. 541-573.

[32] X. Xin, Existence and non existence of travelling waves and reaction, diffusion front propagation in periodic media, *J. Statist. Phys.*, Vol. **73**, 1993, pp. 893-926.

[33] J. B. Zeldovic and D. A. Frank-Kamenetskii, A theory of thermal propagation of flame, *Acta physiochimica URSS*, Vol. **9**, 1938, pp. 341-350, English translation in *Dynamics of curved fronts*, R. Pelcé ed., Perspectives in Physics Series, Academic Press, New York 1988, pp. 131-140.

(Manuscript received June 7, 1995;
revised October 2, 1996.)