Conical-shaped travelling fronts applied to the mathematical analysis of the shape of premixed Bunsen flames

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1 Introduction

This paper is devoted to some existence results and qualitative properties of solutions to some semilinear elliptic equations in the whole space $I\!R^N$ with asymptotic conical conditions at infinity. This work aims at analyzing the conical shape of premixed Bunsen flames.

Bunsen flames can be divided into two parts : a diffusion flame and a premixed flame inside the diffusion flame (*see* Figure 1 and the classical models of Buckmaster, Ludford [24], [25], Joulin [48], Lewis, Von Elbe [53], Liñan [56], Sivashinsky [63], [64], Williams [68]). Here we have chosen to deal with premixed flames, which are themselves divided into two zones : a fresh mixture (fuel and oxidant) and, above, a hot zone made of burnt gases. For the sake of simplicity, we assume that a simple global chemical reaction *fuel* + *oxidant* $\rightarrow$ *products* takes place in the mixture.

The level sets of the temperature have a conical shape with a curved tip and, far from its axis of symmetry, the flame is almost planar. Let us assume that the flame is stabilized and stationary in an upward flow with a uniform speed c. This uniformity assumption is reasonable at least far from the burner rim. In the classical framework of the thermodiffusive model [11], [24], [57], with unit Lewis number, the adimensionalized temperature field $u(x,y)$, which can be assumed to be defined in the whole space $I\!R^N = \{z=(x,y)\in I\!R^{N-1}\times I\!R\}$ because of the invariance of the shape of the flame with respect to the size of the Bunsen burner, satisfies the following reaction-diffusion equation :

$$\Delta u - c\frac{\partial u}{\partial y} + f(u) = 0, \quad 0\le u\le 1 \text{ in } I\!R^N. \tag{1.1}$$

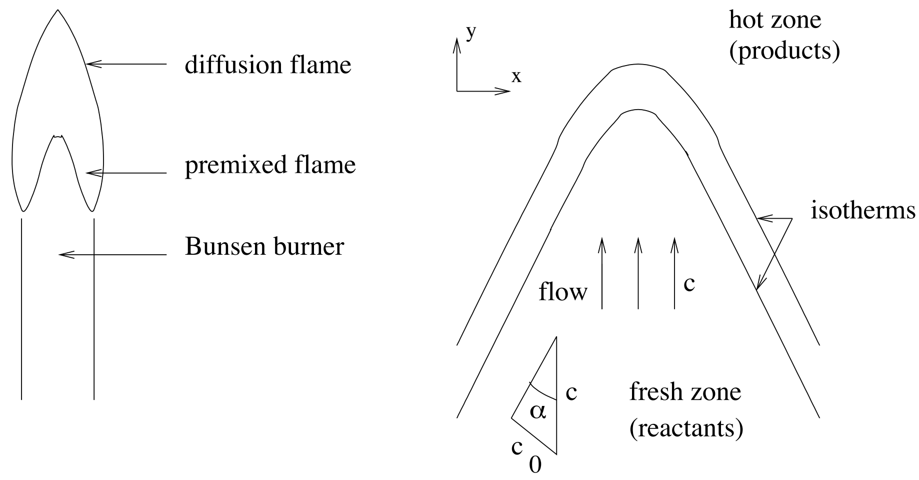

Figure 1: *Bunsen flames,* *premixed flame*

Some asymptotic conditions of the conical type have to be imposed at infinity. One can for instance require that

$$\begin{cases} \forall k \in \mathcal{C}(-e_N, \alpha), & u(\lambda k) \underset{\lambda \to +\infty}{\longrightarrow} 0 \\ \forall k \in \mathcal{C}(e_N, \pi - \alpha), & u(\lambda k) \underset{\lambda \to +\infty}{\longrightarrow} 1 \end{cases} \tag{1.2}$$

where $\alpha \in (0, \pi)$ is the angle of the flame, $e_N = (0, \cdots, 0, 1)$ is the upward unit vector and, for any vector e and any angle $\varphi \in (0, \pi)$, $\mathcal{C}(e, \varphi)$ denotes the open half-cone directed by e :

$$\mathcal{C}(e, \varphi) = \{k \in I\!\!R^N, \ k \cdot e > \|k\| \ \|e\| \cos \varphi\}.$$

Note that the solutions $u(x, y)$ of (1.1) can also be viewed as travelling fronts of the type $v(t, x, y) = u(x, y + ct)$ moving downwards with speed c in a quiescent medium and solving the parabolic reaction-diffusion equation $\partial_t v = \Delta v + f(v)$.

In practice, the speed c of the flow at the exit of the Bunsen burner is given and it determines the angle α of the flame. We here choose to solve the reverse problem, namely we assume that the angle α is given and the speed c is an unknown of the problem. We shall see later that these two formulations are actually equivalent.

The nonlinear reaction term $f(u)$ is of the "ignition temperature" type :

$$\exists \theta \in (0, 1) \text{ such that } f \equiv 0 \text{ on } [0, \theta] \cup \{1\}, \quad f > 0 \text{ on } (\theta, 1) \text{ and } f'(1) < 0. \tag{1.3}$$

Furthermore, f is assumed to be Lipschitz-continuous on $[0, 1]$ and it is extended by 0 outside the interval $[0, 1]$. Such a profile can be derived from the Arrhenius kinetics and the law of mass action (including the fact that the Lewis number is equal to 1).

Before stating the mathematical results obtained for this problem (1.1-1.2), let us briefly recall some known results for travelling fronts. The first papers devoted to this question of travelling fronts were concerned with the one-dimensional case, for the equation

$$u'' - cu' + f(u) = 0, \quad u(-\infty) = 0, \ u(+\infty) = 1. \tag{1.4}$$

Under the above assumptions on f, there exists a unique solution (c_0, u_0) of (1.4), the function u_0 being increasing and unique up to translation, and the speed c_0 being positive [4], [10], [49] (other types of nonlinearities f have been dealt with in [4], [32], [33], [51], [70]). These results can be obtained by a shooting method or a study in the phase plane. The above existence, uniqueness and monotonicity results have been generalized by Berestycki, Larrouturou, Lions [12] and Berestycki, Nirenberg [15] in the multidimensional case of a straight infinite cylinder $\Sigma = \omega \times \mathbb{R} = \{z = (x, y),\ x \in \omega,\ y \in \mathbb{R}\}$, for equations of the type

$$\left\{\begin{array}{rcl} \Delta u - (c + \beta(x))\partial_y u + f(u) & = & 0 \text{ in } \Sigma = \omega \times \mathbb{R} \\ \partial_\nu u & = & 0 \text{ on } \partial\Sigma \\ u(\cdot, -\infty) = 0, \quad u(\cdot, +\infty) & = & 1, \end{array}\right. \tag{1.5}$$

where β is a given continuous function defined on the bounded and smooth section $\overline{\omega}$ of the cylinder, and $\partial_\nu u$ denotes the partial derivative of u with respect to the outward unit normal ν on $\partial\Sigma$. Under the above conditions, there exists a unique solution (c, u) of (1.5), the function $u = u(x, y)$ being increasing in y and unique up to translation in y. Variational formulas for the unique speed exist in both the one-dimensional case [40] and in the multidimensional case [41], [46].

Whereas the above problems (1.4) and (1.5) are now well understood, the questions of the existence and the qualitative properties of the solutions (c, u) of equation (1.1) with the conical conditions of the type (1.2), or with similar conical conditions detailed in the next sections, have so far remained open. Note that, although the underlying flow is here uniform unlike in (1.5), the solutions are nevertheless non-planar, because of the conical conditions (1.2) at infinity. One of our goals is to understand the influence of the conical conditions at infinity on the propagation phenomena in uniform media. We also want to establish the relationship between the speed c of the outgoing flow and the angle α of the flame. In this perspective, the results stated below are the first rigorous analysis of the conical shape of premixed Bunsen flames. Let us also mention that similar existence results of conical-shaped travelling fronts, as well as fronts with more general shapes, have recently been obtained in [45] for another nonlinearity f of the Fisher-KPP type [33], [51].

The mathematical difficulties follow on the one hand from the fact that the problem is set in the whole space $\mathbb{R}^N$ and on the other hand from the *non-standard* conical asymptotic conditions (1.2) at infinity. These conditions are rather weak and do not impose anything as far as the behavior of the function u in the directions making an angle α with respect to the unit vector $-e_N = (0, \cdots, 0, -1)$ is concerned. Note that these conditions are very different from the uniform conditions $u(z) \to 0$ as $|z| \to \infty$ which have often been considered for such nonlinear elliptic equations (*see* especially [28], [36], [37], [54], [55]).

Section 2 states an existence result of solutions in dimension $N = 2$, as well as the uniqueness of the speed c for any angle $\alpha \leq \pi/2$. Section 3 gives a non-existence result for any angle $\alpha > \pi/2$ as well as several qualitative properties of the solutions u in any dimension N with other conical conditions at infinity, slightly stronger than (1.2). Lastly, section 4 is devoted to the study of a related free boundary problem which is obtained in the classical limit of high activation energies. One also gives a complete classification of the solutions of a related overdetermined Serrin type problem. The results stated below are detailed and proved in several papers by Bonnet and Hamel [16], and Hamel and Monneau [42], [43], [44].

2 Existence result in dimension $N = 2$

This section is concerned with an existence result in dimension $N = 2$. In practice, this case corresponds to Bunsen burners with elongated rectangular outlet [68].

Remember that (c_0, u_0) denotes the unique solution of (1.4), the function $u_0 = u_0(s)$ being increasing and unique up to tranlation in s, and the speed c_0 being positive. We have obtained the existence of solutions of (1.1-1.2) for each given angle $\alpha \leq \pi/2$, as well as a characterization of the speed c :

Theorem 2.1 (Existence result in dimension $N = 2$; A. Bonnet, F. Hamel [16])

a) In dimension $N = 2$, for each function f satisfying (1.3) and for each angle $\alpha \in (0, \pi/2]$, there exists a solution (c, u) of (1.1-1.2) such that

$$c = \frac{c_0}{\sin \alpha}. \tag{2.1}$$

Moreover, $0 < u(x, y) < 1$ for all $(x, y) \in \mathbb{R}^2$, u is symmetric with respect to the variable x and it is decreasing with respect to any direction of the open cone $\mathcal{C}(-e_N, \alpha) = \mathcal{C}(-e_2, \alpha)$. The function u actually satisfies the following conical conditions, stronger than (1.2) :

$$\begin{cases} \forall\ \delta \in [0, \alpha), & \displaystyle\limsup_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}(-e_N,\delta)} u(x, y) = 0 \\ \forall\ \delta \in [0, \pi - \alpha), & \displaystyle\liminf_{|(x,y)|\to\infty,\ (x,y)\in\mathcal{C}(e_N,\delta)} u(x, y) = 1. \end{cases} \tag{2.2}$$

Lastly, for each $\lambda \in (0, 1)$, the level set $\{u(x, y) = \lambda\}$ is a curve $\{y = \phi_\lambda(x),\ x \in \mathbb{R}\}$, which has two asymptotic directions making an angle α with respect to $-e_2$. For any sequence $x_n \to \pm\infty$, the functions $u_n(x, y) = u(x + x_n, y + \phi_\lambda(x_n))$ locally converge to a translation of the planar front $u_0(y \sin \alpha \pm x \cos \alpha)$ as $x_n \to \pm\infty$.

If $\alpha = \pi/2$, the above solution u is the same as the planar front $u_0(y)$ up to translation.

b) Under the same assumptions as in a), if $\alpha \in (0, \pi/2]$ and (c, u) is a solution of (1.1), (2.2), then c is given by the formula (2.1).

It follows from this theorem that the so-built functions u are asymptotically planar along their level sets far away from the axis of symmetry $\{x = 0\}$. Furthermore, one can see from formula (2.1) that the speed c is always greater than or equal to the planar speed c_0 and that the bigger the speed c is, the smaller the angle α is and the sharper the flame is. This formula is pertinent since it can be observed in practice that an increase of the outgoing flow c makes the curvature of the flame tip increase. From this formula (2.1), it also follows that finding α when c is given is equivalent to finding c when α is given. Lastly, let us mention that this formula (2.1) is well-known [24], [53], [64] and not surprising since far away from the axis of symmetry, the flame is almost planar and, if the medium were quiescent, it would move with the speed c downwards and with the speed c_0 in the directions which are asymptotically orthogonal to the level sets of the temperature (*see* Figure 1). The speed c_0 is nothing else than the projection of the speed c on the directions $(\pm\cos\alpha, -\sin\alpha)$. Lastly, this formula (2.1) can be used to find the planar speed c_0, from the knowledge of the speed c and from a measurement of the angle α [68]. However, as far as we know, Theorem 2.1 has been the first rigorous derivation of this formula and the first rigorous analysis of the conical shape of premixed Bunsen flames.

The existence result in part a) of Theorem 2.1 can be proved by solving equivalent problems in bounded rectangles Σ_a such that the ratios between the x-length and the y-length approaches $\tan\alpha$ as the size of the rectangles goes to infinity (*see* Figure 2). One

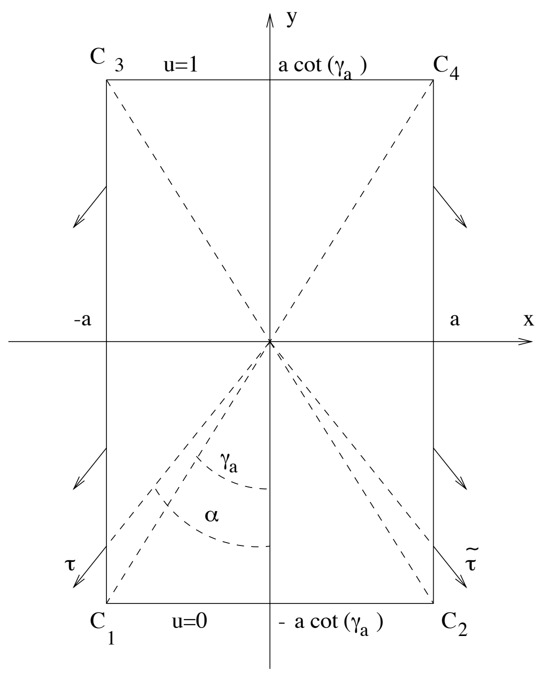

Figure 2: *The rectangle* Σ_a

imposes Dirichlet conditions 0 and 1 respectively on the lower and upper sides, and oblique Neumann boundary conditions ($\partial_\tau u = 0$ and $\partial_{\tilde\tau} u = 0$, *see* Figure 2) on the lateral sides. By imposing a normalization condition and comparing the solutions of these equivalent problems set in these rectangles Σ_a with suitable planar sub- and super-solutions, one can prove that the speeds c_a of these approximated problems are bounded independently of a. Furthermore, by using a sliding method similar to the one developped by Berestycki and Nirenberg [14], one can prove that the functions u_a are decreasing in the directions of the cone $\mathcal{C}(-e_2,\alpha)$. In order to do that, one establishes the precise asymptotic behavior of the solutions near the four corners $C_1,\cdots,C_4$ of the rectangles Σ_a, thanks to a blow-up method (*see* [29], [39], [52], [58] for similar asymptotic results). Then, one passes to the limit in the whole plane $\mathbb{R}^2$. The difficulty is to prove the asymptotic conditions at infinity of the type

(1.2) or (2.2), and to show that the level sets of the limit function u are asymptotically planar far away from the axis of symmetry $\{x = 0\}$. One especially makes several uses of the sliding method in both the directions making an angle α with respect to $-e_2$ and the orthogonal directions.

As far as part b) of Theorem 2.1 is concerned, the uniqueness of the speed can also be obtained by a sliding method and by comparing any solution with the solutions built in part a).

3 A non-existence result and qualitative properties in any dimension under various conical asymptotic conditions

This section is devoted to a non-existence result for any angle $\alpha > \pi/2$ as well as with several monotonicity and uniqueness results under various conical asymptotic conditions at infinity, slightly stronger than (1.2).

The question of the non-existence of solutions (c, u) as soon as the angle α of the flame is bigger than $\pi/2$ is physically meaningful : there cannot be any flame whose tip points downwards if the flow is going upwards. The following result agrees with this observation :

Theorem 3.1 (Non-existence result for $\alpha > \pi/2$; F. Hamel, R. Monneau [42], [43]) *In any dimension $N \geq 2$, there is no solution (c, u) of (1.1), (2.2) as soon as $\alpha > \pi/2$.*

Thus, despite its simplicity, the mathematical model which is used here to describe premixed Bunsen flames is robust enough to capture the above observation.

Whereas the determination of the speed c has been solved, at least in dimension $N = 2$ in Theorem 2.1, the question of the uniqueness of the possible solutions u of (1.1) with conical asymptotic conditions (1.2) or (2.2) is much more intricate, especially because of the above rather weak conditions that are imposed at infinity. That is why we shall now introduce slightly stronger conical conditions.

Before defining this new class of stronger conical conditions, let us first observe that it follows from Theorem 2.1 that in dimension $N = 2$ and for any angle $\alpha \in (0, \pi/2]$, the so-built solutions u are such that, for every $\lambda \in (0, 1)$, the level set $\{u = \lambda\} = \{y = \phi_\lambda(x)\}$ satisfies $\phi'_\lambda(x) \to \mp \cot\alpha$ as $x \to \pm\infty$ and

$$\begin{cases} \displaystyle\lim_{y_0 \to -\infty} \sup_{\Omega^-_\lambda(y_0)} u = 0 \\ \displaystyle\lim_{y_0 \to +\infty} \inf_{\Omega^+_\lambda(y_0)} u = 1 \end{cases} \tag{3.1}$$

where $\Omega^-_\lambda(y_0) = \{y < y_0 + \phi_\lambda(x)\}$ and $\Omega^+_\lambda(y_0) = \{y > y_0 + \phi_\lambda(x)\}$.

More generally speaking, in any dimension N, we can now consider the following problem with unknowns (c, u) :

$$\begin{cases} \Delta u - c\partial_y u + f(u) = 0,\ 0 \le u \le 1 \text{ in } \mathbb{R}^N \\ \displaystyle\lim_{y_0 \to -\infty} \sup_{\Omega^-(y_0)} u = 0 \\ \displaystyle\lim_{y_0 \to +\infty} \inf_{\Omega^+(y_0)} u = 1 \end{cases} \tag{3.2}$$

where, for any $y_0 \in \mathbb{R}$, $\Omega^+(y_0) = \{y > y_0 + \phi(x)\}$, $\Omega^-(y_0) = \{y < y_0 + \phi(x)\}$ and $\phi(x)$ is a non-specified, globally Lipschitz function, of class C^1 for large $|x|$, and such that

$$\lim_{|x| \to +\infty} \left(\nabla\phi(x) + \cot\alpha\, \frac{x}{|x|} \right) = 0. \tag{3.3}$$

Because of the uniformity of the limits far above or below the graph of ϕ, the problem (3.2-3.3) is clearly stronger than equation (1.1) with either conditions (1.2) or (2.2). Thus, this problem has no solution (c, u) as soon as $\alpha > \pi/2$ in any dimension $N \ge 2$. On the other hand, as already mentioned, it easily follows from Theorem 2.1 that

in dimension $N = 2$ and for any $\alpha \in (0, \pi/2]$, the solution (c, u) of Theorem 2.1 solves (3.2-3.3) where ϕ can be any level curve ϕ_λ of u.

The qualitative properties of the solutions (c, u) of (3.2-3.3) are stated in the following theorem :

Theorem 3.2 (Uniqueness of the speed c and monotonicity of u for problem (3.2-3.3); F. Hamel, R. Monneau [42], [43]) *In any dimension $N \ge 2$, if (c, u) solves (3.2-3.3) for a function ϕ satisfying (3.3), then $\alpha \le \pi/2$, $c = c_0/\sin\alpha$ and u is decreasing with respect to any direction of the open cone $\mathcal{C}(-e_N, \alpha)$. As a consequence, if $\alpha = \pi/2$, then u is planar, it only depends on the variable y and it is unique up to translation.*

Compared to problem (1.1), (2.2), the conical conditions in (3.2-3.3) then enable us to get the uniqueness of the speed in any dimension as well as the monotonicity of u in the lower cone of angle α. Nevertheless, the question of the uniqueness of u in this framework remains open.

In the problem (3.2-3.3), the graph of the non-specified Lipschitz function ϕ has asymptotic directions with angle α with respect to $-e_N$. It is then natural to consider the stronger case where ϕ satisfies $\sup_{x \in \mathbb{R}^{N-1}} |\phi(x) + |x| \cot\alpha| < +\infty$, which is the same as looking for the solutions (c, u) of

$$\begin{cases} \Delta u - c\partial_y u + f(u) = 0,\ 0 \le u \le 1 \ \text{ in } \mathbb{R}^N \\ \displaystyle\lim_{y_0 \to -\infty} \sup_{y < y_0 - |x|\cot\alpha} u = 0 \\ \displaystyle\lim_{y_0 \to +\infty} \inf_{y > y_0 - |x|\cot\alpha} u = 1. \end{cases} \tag{3.4}$$

Since this problem is stronger than the previous ones, Theorems 3.1 and 3.2 still hold. Furthermore, one has the following additional results, which clearly show a difference between the dimension $N = 2$ and the higher dimensions :

Theorem 3.3 (Uniqueness of the solutions of (3.4); F. Hamel, R. Monneau [42], [43])

a) If $N \geq 3$ and $\alpha \neq \pi/2$, or if $N = 2$ and $\alpha > \pi/2$, then problem (3.4) has no solution (c, u).

b) In dimension $N = 2$, for any angle $\alpha \leq \pi/2$, if (c, u) solves (3.4), then $c = c_0/\sin\alpha$ and the function u is unique up to translation and it is equal, up to translation, to the solution given in Theorem 2.1. Furthermore, the level sets of u are asymptotic at infinity to some translations of the two half-lines making an angle α with respect to $-e_2$: for every $\lambda \in (0, 1)$, under the notation of Theorem 2.1, there exist $a^{\pm}_{\lambda} \in I\!R$ such that $\phi_\lambda(x) + |x|\cot\alpha \to a^{\pm}_{\lambda}$ as $x \to \pm\infty$.

Remark 3.4 In the case $\alpha = \pi/2$, problem (3.4) is similar to a version of a conjecture of De Giorgi. This strong version says that any solution u of $\Delta u + f(u) = 0$ such that $u(x, y) \to \pm 1$ as $y \to \pm\infty$ uniformly in $x \in I\!R^{N-1}$, where the nonlinearity f is of the bistable type, is actually planar. The latter has been proved in any dimension N [5], [9], [31]. On the other hand, the original conjecture of De Giorgi [30] requires that the function u be increasing in y but that it not necessarily satisfy the uniform limits $u(x, y) \to \pm 1$ as $y \to \pm\infty$ uniformly in x, or at least that it satisfy these limits for each x. The De Giorgi conjecture says that u is planar. This conjecture has been proved in dimension $N = 2$ [8], [35] and in dimension $N = 3$ [2].

The price to pay for the uniqueness result in Theorem 3.3 is that, unlike problems (1.1-1.2) or (3.2-3.3), one does not know in full generality if problem (3.4) has a solution, even in dimension $N = 2$ and for an angle $\alpha < \pi/2$.

However, one can give two sufficient conditions for problem (3.4) to have a solution, in dimension $N = 2$ and for $\alpha \leq \pi/2$. Namely, one has two sufficient conditions for the solution of (1.1-1.2) given in Theorem 2.1 to be a solution of (3.4) :

Theorem 3.5 (Sufficient conditions for the existence of a solution of (3.4) in dimension $N = 2$; F. Hamel, R. Monneau [42], [43]) *Assume $N = 2$ and let f be a function satisfying (1.3).*

a) If the function f is C^1 and is such that $c_0^2 > (4/9)\sup_{[0,1]} f'$, where $c_0 = c_0(f)$ is the unique planar speed solving (1.4), then there exists $\alpha_0 \in (0, \pi/2)$ such that problem (3.4) has a solution for any angle $\alpha \in (0, \alpha_0)$. Furthermore, for any $\varepsilon > 0$, there exist some functions f such that $\alpha_0 \geq \pi/2 - \varepsilon$.

b) If the solution $(c, u) = (c_0/\sin\alpha, u)$ of (1.1), (2.2) given in Theorem 2.1 satisfies

$$\exists \underline{y} \in I\!R,\ \exists \underline{\xi} \in (0,1) \text{ such that } u(x, -|x|\cot\alpha + \underline{y}) \geq \underline{\xi} \text{ for all } x \in I\!R,$$

then (c, u) is a solution of (3.4) as well.

The result a) on the existence of solutions of (3.4) for some functions f in dimension $N = 2$ contrasts widely with the non-existence result in dimension $N \geq 3$ for $\alpha \neq \pi/2$. This existence result is based on the explicit construction of a solution from a sub-solution and integral estimates. Part b) can be proved thanks to the construction of suitable sub- and super-solutions defined in domains rotating around a fixed point.

The proofs of Theorems 3.1, 3.2 and 3.3 make an intensive use of sliding methods in various directions, as well as a version of the maximum principle in $\mathbb{R}^N$ with conical conditions at infinity. We state this principle as a comparison principle between sub- and super-solutions of problems of the type (3.2-3.3) :

Theorem 3.6 (Comparison principle in $\mathbb{R}^N$; F. Hamel, R. Monneau, [42], [43]) *Let $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a uniformly continuous function and, for any $y_0 \in \mathbb{R}$, let*

$$\begin{cases} \Omega^+(y_0) & = \{y > y_0 + \phi(x)\} \\ \Omega^-(y_0) & = \{y < y_0 + \phi(x)\} \\ \Gamma(y_0) & = \{y = y_0 + \phi(x)\}. \end{cases}$$

Let $a < b$ be two real numbers and let us consider the semilinear elliptic equation, set in $\mathbb{R}^N = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}\}$:

$$I(u) := \sum_{1\le i,j\le N} a_{ij}(x)\partial_{ij}u + \sum_{1\le i\le N} b_i(x)\partial_i u + f(x,u) = 0 \text{ in } \mathbb{R}^N \tag{3.5}$$

where the functions u are such that $a \le u \le b$ and

$$\begin{cases} \lim\limits_{y_0\to-\infty} \sup\limits_{\Omega^-(y_0)} u = a \\ \lim\limits_{y_0\to+\infty} \inf\limits_{\Omega^+(y_0)} u = b \end{cases} \tag{3.6}$$

Suppose there exist $0 < \gamma \le \nu$ such that $\gamma|\xi|^2 \le \sum_{1\le i,j\le N} a_{ij}(x)\xi_i\xi_j \le \nu|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and $x \in \mathbb{R}^{N-1}$. Assume moreover that $a_{ij} \in C^{2,\delta}$, $b_i \in C^{1,\delta}$ with $\delta > 0$, that f is continuous and bounded in $\mathbb{R}^{N-1}\times[a,b]$ and satisfies $|f(\tilde{x}',\tilde{u})-f(x,u)| \le C(|\tilde{x}'-x|^\delta+|\tilde{u}-u|)$ for some positive constant $C > 0$. Suppose there exist $a < a' < b' < b$ such that f is nonincreasing with respect to u for all u in $[a,a']$ or $[b',b]$ and for all $x \in \mathbb{R}^{N-1}$. For all $x \in \mathbb{R}^{N-1}$, the function $f(x,\cdot)$ is extended in $\mathbb{R}$ by $f(x,u) = f(x,a)$ for $u < a$ and $f(x,u) = f(x,b)$ for $u > b$.

Let $\overline{u}$ and $\underline{u}$ be two globally Lipschitz-continuous super- and sub-solutions of (3.5-3.6) in the sense that

$$\begin{cases} I(\overline{u}) \le 0 \text{ in } \mathcal{D}'(\mathbb{R}^N),\ a \le \overline{u} \le b \text{ in } \mathbb{R}^N \text{ and } \lim\limits_{y_0\to+\infty} \inf\limits_{\Omega^+(y_0)} \overline{u} = b \\ I(\underline{u}) \ge 0 \text{ in } \mathcal{D}'(\mathbb{R}^N),\ a \le \underline{u} \le b \text{ in } \mathbb{R}^N \text{ and } \lim\limits_{y_0\to-\infty} \sup\limits_{\Omega^-(y_0)} \underline{u} = a. \end{cases} \tag{3.7}$$

For all $t \in \mathbb{R}$, call $\overline{u}^t(x,y) = \overline{u}(x,y+t)$ and $\overline{u}^{-\infty}(x) = \liminf_{t\to-\infty} \overline{u}(x,t)$.

Then the set $I = \{t \in \mathbb{R},\ \forall s \ge t,\ \overline{u}^s \ge \underline{u} \text{ in } \mathbb{R}^N\}$ is not empty. Let $t^* = \inf\ I$. One has $\overline{u}^{t^*} \ge \underline{u}$ in $\mathbb{R}^N$ and, if $t^* > -\infty$, then either $\overline{u}^{t^*} \equiv \underline{u}$ in $\mathbb{R}^N$, or $\overline{u}^{t^*} > \underline{u}$ in $\mathbb{R}^N$ and $\inf\limits_{\Gamma(y_0)} (\overline{u}^{t^*} - \underline{u}) = 0$ for every $y_0 \in \mathbb{R}$.

In other words, two sub- and super-solutions of (3.7) can be compared up to translation in the variable y, until a critical shift where they are either identically equal or asymptotically equal along the graphs of ϕ.

Note that this comparison principle does not work anymore if $a = -\infty$ or $b = +\infty$: for instance, the functions y and $2y$ solve $u'' = 0$ in $\mathbb{R}$ but cannot be compared up to translation.

Theorem 3.6 can be proved by applying the maximum principle for the functions $\overline{u}^t + \varepsilon$ and $\underline{u}$ in the domains $\Omega^+(y_0)$ and $\Omega^-(y_0)$ (*see* also [54], [67]). One concludes by passing to the limit $\varepsilon \to 0$ and by sliding $\overline{u}$ in the direction y.

A consequence of Theorem 3.6 is the following monotonicity result for the *solutions* of (3.5-3.6) :

Corollary 3.7 ([42], [43]) *Under the assumptions of Theorem 3.6, if a function u solves (3.5-3.6), then u is increasing with respect to the variable y.*

The monotonicity result stated in Theorem 3.2 then follows immediately from Corollary 3.7 even if it means rotating the frame.

Remark 3.8 Some results similar to Theorems 3.6 and 3.7 work in straight infinite cylinders $\Sigma = \omega \times \mathbb{R}$ where ω is a connected smooth open subset of $\mathbb{R}^{N-1}$ - which may be unbounded - with Dirichlet or Neumann type boundary conditions on $\partial\Sigma$.

Open questions. The main open problem is the question of the existence of solutions, for the angles $\alpha < \pi/2$, to the problem (3.2-3.3). Even the existence of axisymmetric solutions of the type $u(|x|, y)$ is not known. Note that other shapes, polyhedral for instance, have been obtained for other asymptotic models [23].

The question of the uniqueness of the functions u for problem (3.4) has been solved but this framework is too strong for solutions to exist in dimension $N \geq 3$ as soon as $\alpha \neq \pi/2$. An interesting open question is that of the uniqueness (up to translation) of the functions u for problem (3.2-3.3). Provided the existence of such solutions can be proved, that would mean that this problem (3.2-3.3) is well-posed in any dimension $N \geq 2$ and for any angle $\alpha \leq \pi/2$.

Another open question consists in wondering if the so-built solutions are stable for the associated evolution equations, which seems reasonable at least for compactly supported perturbations. It would be interesting to define a general class of perturbations for which the solutions would be stable up to translation.

Other open problems concern the models of premixed Bunsen flames with non unit Lewis numbers (*see* [63], [64]), with heat losses or with nonconstant density.

4 A free boundary problem and a Serrin type problem arising in the limit of high activation energies

This section is devoted to the analysis of the model described in the introduction for the dimension $N = 2$, in the limit of high activation energies. In this limit, the source term

$f(u)$ vanishes as soon as the temperature is below that of the brunt gases and the zone where the chemical reaction takes place becomes infinitely thin. Below this flame, the gases are fresh and the reaction cannot happen, and above the flame, the gases are burnt and the reaction does not happen either because at least one of the reactants has a zero concentration.

One is then led to the following model for the adimensionalized temperature $u(z) = u(x, y)$ (this model will be rigorously derived in Theorem 4.1) :

$$\left\{\begin{array}{rl} \Delta u - c\partial_y u = 0 & \text{in } \Omega = \{0 < u < 1\} \subset I\!R^2 \\ u = 1 & \text{in } I\!R^2 \backslash \Omega \\ \partial_\nu u = c_0 & \text{on } \Gamma := \partial\Omega \\ \limsup\limits_{d(z,\Gamma)\to+\infty,\ z\in\Omega} u(z) = 0 & \\ u \text{ is continuous} & \text{across } \Gamma \end{array}\right. \tag{4.1}$$

where the curve Γ represents the flame front, $d(z, \Gamma)$ denotes the distance of a point $z \in I\!R^2$ to Γ, and $\partial_\nu u$ denotes the normal derivative on Γ of the restriction of the function u to $\overline{\Omega}$. The continuity of u across Γ means that there is no jump of temperature across the flame. In the above equations (4.1), the reaction term $f(u)$ has disappeared. Nevertheless, the chemical reaction takes place on the flame Γ. The condition $\partial_\nu u = c_0$ on Γ is a memory of this reaction term and simply means that the normal burning velocity is constant along the flame front and equal to some planar speed c_0 (*see* [18], [19], [20], [21], [22], [24], [25], [27] for other occurences of this type of jump condition in related problems). Note that this condition will also be obtained *ex nihilo* from a passage to the limit of smooth solutions of some reaction-diffusion equations with particular reaction terms (*see* Theorem 4.1 below).

Furthermore, the "fresh" zone (where the temperature is below that of the burnt gases) is located below the flame and the latter has a conical shape. Assuming this flame front can be written as a graph $\Gamma = \{y = \phi(x),\ x \in I\!R\}$ and the "fresh" zone is the set $\Omega = \{y < \phi(x)\}$, it is then natural to impose the following asymptotic condition

$$\phi'(x) \pm \cot\alpha \to 0 \text{ as } x \to \pm\infty, \tag{4.2}$$

where $\alpha \in (0, \pi)$ is a given angle (as in the previous two sections). Note that the function ϕ is itself unknown and Γ can be viewed as a free boundary for problem (4.1).

Problem (4.1) had been studied in various asymptotic formal limits : the case of very sharp flames $\alpha \to 0^+$ with Lewis number close to 1 has been considered by Buckmaster and Ludford (this limit is reduced to a parabolic free boundary problem after a blow-down in the direction y [22], [24], [25]). Multiscale asymptotic expansions have been carried out by Sivashinsky, leading to different shapes of the flame fronts according to the position of the Lewis number with respect to 1 [64] (*see* also [53], [63] for the three-dimensional case). Another approach has been used by Michelson [59], in the case of a unit Lewis number; namely, Michelson has used the fourth-order Kuramoto-Sivashinsky equation ([17], [34], [65], [66]) for the description of the graph of the flame front and he has obtained the existence and the uniqueness of such graphs for angles α close to 0 (*see* also [60] for three-dimensional results).

However, the study of the solutions (c, u, Ω) of (4.1) satisfying the conical condition (4.2) at infinity, where c_0 and α are given, has so far remained open. We here state the existence of solutions, as well as their uniqueness. The uniqueness is even obtained under very general assumptions and gives a complete classification of the solutions of a class of more general overdetermined Serrin type problems.

The existence result can be stated as follows :

Theorem 4.1 (Existence of a solution of the free boundary problem (4.1); F. Hamel, R. Monneau [44]) *Let $\alpha \in (0, \pi/2]$. There exists a solution $(c^\alpha, u^\alpha, \Omega^\alpha)$ of (4.1-4.2). The function u^α is obtained as the limit (locally uniform in $\mathbb{R}^2$) of the sequence of functions u_ε solving (1.1), (2.2)* (see *Theorem 2.1) for nonlinearities f_ε of the type $f_\varepsilon = \frac{1}{\varepsilon} f(1 - \frac{1-s}{\varepsilon})$ where f is a given function satisfying (1.3) with some ignition temperature $\theta \in (0, 1)$. The unique speed c^α_ε solving (1.1), (2.2) for the nonlinearity f_ε approaches $c^\alpha = c_0 / \sin\alpha$ as $\varepsilon \to 0^+$, where $c_0 = \sqrt{2\int_0^1 f} > 0$.*

The function u^α is globally Lipschitz-continuous in $\mathbb{R}^2$, symmetric with respect to the variable x, and it is nonincreasing in the directions of the lower cone $\mathcal{C}(-e_2, \alpha)$. The free boundary $\Gamma^\alpha = \partial\Omega^\alpha$ is analytic, it has a bounded curvature, and the restriction of u^α to $\overline{\Omega^\alpha}$ is analytic.

Lastly, the set Ω^α is a sub-graph $\Omega^\alpha = \{y < \phi^\alpha(x)\}$. The (even) function ϕ^α satisfies $(\phi^\alpha)' \geq 0$ in $\mathbb{R}_-$ and it has two asymptots in both directions making an angle α with respect to $-e_2$, namely :

$$\exists L \in \mathbb{R}, \quad \phi^\alpha(x) + |x| \cot\alpha \to L \text{ as } x \to \pm\infty.$$

Remark 4.2 This result especially enables us to solve the flame tip problem, which has been set by Buckmaster and Ludford [25] (the original problem in [25] has been set in a half-plane, say $\{x \geq 0\}$, with Neumann boundary conditions on $\{x = 0\}$).

Remark 4.3 Once again, the speed c^α is given by the natural formula $c^\alpha = c_0 / \sin\alpha$ as in sections 2 et 3.

The method to prove this result consists in working with regularizing approximations and passing to the limit. Namely, one considers the solutions $(c^\alpha_\varepsilon, u^\alpha_\varepsilon)$ of (1.1), (1.2) with the functions $f_\varepsilon(s) = \frac{1}{\varepsilon} f(1 - \frac{1-s}{\varepsilon})$ as in Theorem 4.1 (note here that $1/\varepsilon$ is proportional to the activation energy in this model). This limiting procedure comes back to Zeldovich and Frank-Kamenetskii [70] and had already been used in dimension $N = 1$ for systems of two ordinary differential equations [13] (*see* also [38] for more general classes of functions f_ε, [6] for similar problems in infinite cylinders with heterogeneous velocity fields, and [27] for parabolic related problems). The above functions f are chosen so that the condition $\partial_\nu u = c_0$ on $\partial\Omega$ holds at the limit $\varepsilon \to 0^+$. As far as the regularity of Γ and u is concerned, one uses some general results in [26] and [50]. The main difficulty here is to get the limiting conical conditions (4.2) after the passage to the limit. That is done thanks to some uniform exponential upper bounds of the functions u^α_ε in the lower cone $\mathcal{C}(-e_2, \alpha)$.

Let us now investigate the question of the uniqueness of the solutions (c, u, Ω) of (4.1-4.2). We here state a uniqueness result, in a framework actually more general than (4.1-4.2).

Namely, we deal with the classification of all the solutions of an overdetermined Serrin type problem. Actually, even without assuming that the conical condition (4.2) holds or that the domain Ω is a sub-graph, problem (4.1) can be viewed as an overdetermined Serrin type problem, for which the unknown function u satisfies an elliptic equation $\Delta u - c\partial_y u = 0$ in an *unknown* domain $\Omega = \{u < 1\}$ with two boundary conditions $u = 1$ and $\partial_\nu u = c_0$ on $\partial\Omega$. Problems of that type have first been considered by Serrin [62] in bounded domains for equations of the type $\Delta u + f(u) = 0$, which are invariant by rotation. For such problems it has been proved that, under some conditions on f and u, the domain Ω is necessarily a ball (*see* also [1], [7], [47], [61] for similar problems in other types of geometries).

For problem (4.1), one cannot expect any radial symmetry because of the first-order term $c\partial_y u$. However, under some smoothness assumptions for Γ and without assuming (4.2), we have proved that, besides some trivial planar solutions, the solutions given in Theorem 4.1 are the only solutions of (4.1) :

Theorem 4.4 (Classification of the solutions of (4.1); F. Hamel, R. Monneau, [44]) *Let (c, u, Ω) be a solution of the free boundary problem (4.1), where Ω is an open set such that both Ω and $\mathbb{R}^2\backslash\Omega$ are not empty, and the restriction of u to $\overline{\Omega}$ is C^1 in $\overline{\Omega}$. Assume that the free boundary $\Gamma = \partial\Omega$ is globally $C^{1,1}$ with bounded curvature. Assume moreover that $\mathbb{R}^2\backslash\Omega$ has no bounded connected components.*

Then, even if it means changing (c, u, Ω) into $(-c, u(-x, -y), -\Omega)$, one has $c \geq c_0$ and, if $\alpha \in (0, \pi/2]$ denotes the only solution of $c = c_0/\sin\alpha$, the following three and only three cases (up to translation) may occur:

– either Ω is the half-space $\{y < x\cot\alpha\}$ and $u(x, y) = U_0(y\sin\alpha - x\cos\alpha)$, where U_0 is equal to : $U_0(s) = e^{c_0 s}$ for $s \leq 0$ and $U_0(s) = 1$ for $s \geq 0$.

– or the same conclusion holds up to symmetry in x: $\Omega = \{y < -x\cot\alpha\}$ and $u(x, y) = U_0(y\sin\alpha + x\cos\alpha)$,

– or $\Omega = \Omega^\alpha$ and $u = u^\alpha$ where $(u^\alpha, \Omega^\alpha)$ is the solution of (4.1)-(4.2) given in Theorem 4.1.

Remark 4.5 In the particular case $c = c_0$, then, under the above regularity and connexity assumptions on Ω, any solution (u, Ω) of (4.1) is planar : Ω is a half-space of the type $\Omega = \{\pm(y - h) < 0\}$ for some $h \in \mathbb{R}$ and u only depends on the variable y, namely, $u(x, y) = e^{\pm c_0(y-h)}$ in Ω.

Remark 4.6 The method we use to prove this theorem allows us for additional *a priori* estimates in $\mathbb{R}^N = \{(x, y),\ x \in \mathbb{R}^{N-1}, y \in \mathbb{R}\}$ for dimensions $N \geq 3$. Nevertheless, the question of the existence of solutions for the angles $\alpha < \pi/2$ is still open in dimensions 3 or higher.

It follows from Theorems 4.1 and 4.4 that the free boundary problem (4.1)-(4.2) is well-posed, in dimension $N = 2$, for any angle $\alpha \in (0, \pi/2]$ whereas no solution exists whenever α is larger than $\pi/2$ or whenever c is smaller than c_0, as for the case with a source term $f(u)$ in Theorem 3.1. Note that despite its simplicity the model we have used is robust enough to capture that the tip of the flame cannot point downwards.

Theorem 4.4 is proved in several steps. The first step consists in proving that, up to a change of (c, u, Ω) into $(-c, u(-x, -y), -\Omega)$, the domain Ω is a Lipschitz sub-graph. The second step is based on a method of rotation of the domain up to a critical angle, for which the function in the rotated frame is asymptotically planar in a vertical direction. One also uses various versions of the sliding method as well as comparison principles and monotonicity results similar to Theorem 3.6 and Corollary 3.7.

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