# Entire Solutions of the KPP Equation

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**Abstract**

This paper deals with the solutions defined for all time of the KPP equation

$$u_t = u_{xx} + f(u), \quad 0 < u(x,t) < 1, \; (x,t) \in \mathbb{R}^2,$$

where $f$ is a KPP-type nonlinearity defined in $[0,1]$: $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f > 0$ in $(0,1)$, and $f'(s) \leq f'(0)$ in $[0,1]$. This equation admits infinitely many traveling-wave-type solutions, increasing or decreasing in $x$. It also admits solutions that depend only on $t$. In this paper, we build four other manifolds of solutions: One is 5-dimensional, one is 4-dimensional, and two are 3-dimensional. Some of these new solutions are obtained by considering two traveling waves that come from both sides of the real axis and mix. Furthermore, the traveling-wave solutions are on the boundary of these four manifolds. © 1999 John Wiley & Sons, Inc.

## 1 Introduction

Since the pioneering paper of Kolmogorov, Petrovsky, and Piskunov [17], many works have been devoted to the so-called KPP equation

$$u_t = u_{xx} + f(u), \quad 0 < u(x,t) < 1, \; x \in \mathbb{R}, \; t \in I, \tag{1.1}$$

on a given interval $I$ of time. The nonlinearity $f$ is such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for any $0 < u < 1$. This equation arises in various biological models for gene developments or population dynamics (see, for instance, Aronson and Weinberger [1], Barenblatt and Zel'dovich [2], Fife [8], Fisher [10], Freidlin [11], Rothe [26], and Stokes [28]). Throughout this paper, we also assume that $f$ is of class $C^2$ in $[0,1]$ and that $f'(s) \leq f'(0)$ for all $s \in [0,1]$.

Our goal is to study the classical solutions that are defined for all time, namely, $I = \mathbb{R}$. We call these solutions "entire" solutions of (1.1).

Problem (1.1) admits solutions $u(x,t)$, defined for all time and not depending on $x$, that is to say, that $u(x,t) = u(t)$ is a solution of the ordinary differential equation $u'(t) = f(u)$ in $\mathbb{R}$. It is easy to see that these solutions $u(t)$ are increasing in $t$ and fulfill $u(t) \to 0$ as $t \to -\infty$, $u(t) \to 1$ as $t \to +\infty$. These solutions $u(t)$ form a 1-dimensional manifold, where the parameter can be viewed as a shift in time.

It is well-known that this problem (1.1) also has two 2-dimensional manifolds of entire solutions of traveling-wave type, namely, $u^+_{c,h}(x,t) = \phi_c(x + ct + h)$ and

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$u^-_{c,h}(x,t) = \phi_c(-x+ct+h)$, where $h$ varies in $\mathbb{R}$ and $c$ varies in $[c^*,+\infty[$ with $c^* = 2\sqrt{f'(0)} > 0$ (see, for instance, Aronson and Weinberger [1], Bramson [5], Fife [8], Freidlin [11], Hadeler and Rothe [13], Kanel' [15], Rothe [26], and Stokes [28]). For any $c \geq c^*$, the function $\phi_c$ satisfies $\phi_c'' - c\phi_c' + f(\phi_c) = 0$ in $\mathbb{R}$, $\phi_c(-\infty) = 0$ and $\phi_c(+\infty) = 1$. It is increasing and unique up to translation, and we can then assume that $\phi_c(0) = \frac{1}{2}$. Furthermore, for any $c > c^*$, there exists a positive constant $A_c$ such that

$$\phi_c(\xi) = A_c e^{\lambda_c \xi} + o(e^{\lambda_c \xi}) \quad \text{as } \xi \to -\infty \tag{1.2}$$

where $\lambda_c = (c - \sqrt{c^2 - 4f'(0)})/2 > 0$. For the minimal speed $c = c^* = 2\sqrt{f'(0)}$, it is the case that $\phi_{c^*}(\xi) = -\tilde{A}\xi e^{\sqrt{f'(0)}\xi} + O(e^{\sqrt{f'(0)}\xi})$ for some positive constant $\tilde{A}$. As far as the asymptotic behavior of $\phi_c(\xi)$ as $\xi \to +\infty$ is concerned, for any $c \geq c^*$, there exists a positive constant $B_c$ such that

$$\phi_c(\xi) = 1 - B_c e^{\mu_c \xi} + o(e^{\mu_c \xi}) \text{ as } \xi \to +\infty$$

where $\mu_c = (c - \sqrt{c^2 - 4f'(1)})/2 < 0$ (see Berestycki and Nirenberg [3], Bramson [5], Coddington and Levinson [7], Hadeler and Rothe [13], Kametaka [14], and Uchiyama [30]).

Many authors have studied the behavior for large time of the solutions of the Cauchy problem for (1.1) under a wide class of initial conditions. Special attention has been devoted to the convergence to the traveling waves and the stability of these waves (Aronson and Weinberger [1], Bramson [5, 6], Freidlin [11], Kametaka [14], Kanel' [15], Kolmogorov, Petrovsky, and Piskunov [17], Larson [18], McKean [20], Moet [21], Rothe [27], Uchiyama [30], and van Saarloos [31]). Some of these results have also been generalized in the multidimensional case in straight infinite cylinders (Berestycki and Nirenberg [3], Mallordy and Roquejoffre [19], and Roquejoffre [25]). Equation (1.1) has also been emphasized for a larger class of KPP-type equations (Biro and Kersner [4], Peletier and Troy [23, 24], van Saarloos [31], and Zhao [32]), as well as under other restrictions of the function $f$ (see Rothe [26] and Stokes [28, 29] if $c^* > 2\sqrt{f'(0)}$, or Aronson and Weinberger [1], Fife and McLeod [9], Kanel' [15, 16] if $f$ is of the "bistable" type).

The question of the existence of entire solutions of (1.1) other than the solutions independent of $x$ and the traveling-wave solutions is still open. In this paper, we construct four other manifolds of solutions: One is 5-dimensional, one is 4-dimensional, and two are 3-dimensional.

Roughly speaking, the way to build the 5- and 4-dimensional manifolds of new entire solutions is to consider two traveling waves coming from both sides of the real axis—moving in opposite directions towards each other—and mixing each other. Each of those traveling fronts is given by two parameters (a speed and a shift in $x$). Between these two fronts, when $-t$ is large, the solutions can be either almost uniform in $x$ and equal to a function $\xi(t)$ fulfilling $\xi' = f(\xi)$ (this gives a fifth parameter; see Theorem 1.1) or asymptotically small with respect to any such $\xi(t)$ (Theorem 1.3).

In order to build new entire solutions that are monotone in $x$, the idea consists of slightly perturbating a traveling wave by adding, on the side where the wave is almost 0, a function $\xi(t)$ that is a solution of the equation $\xi' = f(\xi)$. The traveling wave is given by two parameters, and the function $\xi(t)$ involves one additional parameter (Theorem 1.4 and Corollary 1.5).

We also prove in this paper that the 4- and 3-dimensional new manifolds, as well as the traveling-wave solutions and solutions $u(t)$, are on the boundary of the 5-dimensional new manifold of entire solutions of (1.1).

In the following theorems, we say that the functions $u_p(x,t)$ converge to a function $u_{p_0}(x,t)$ as $p \to p_0 \in \mathbb{R}^n$ in the sense of the topology $\mathcal{T}$ if, for any compact set $K \subset \mathbb{R}^2$, the functions $u_p$, $u_{p,x}$, $u_{p,xx}$, and $u_{p,t}$ converge uniformly in $K$ to $u_{p_0}$, $u_{p_0,x}$, $u_{p_0,xx}$, and $u_{p_0,t}$ as $p \to p_0$.

THEOREM 1.1 *For any $c, c' > c^*$, for any $h, h' \in \mathbb{R}$, and for any $K > 0$, there exists an entire solution $u(x,t) = u_{c,c',h,h',K}(x,t)$ of* (1.1) *such that*:

(i) *For any $(x,t) \in \mathbb{R}^2$,*

$$
\begin{aligned}
&\max\left(\phi_{c'}(-x+c't+h'),\ \xi(t),\ \phi_c(x+ct+h)\right) \\
&\quad \le u(x,t) \\
&\quad \le \min\left(1, \phi_{c'}(-x+c't+h') + Ke^{f'(0)t} + A_c e^{\lambda_c(x+ct+h)},\right. \\
&\qquad\qquad A_{c'}e^{\lambda_{c'}(-x+c't+h')} + Ke^{f'(0)t} + \phi_c(x+ct+h), \\
&\qquad\qquad \left. A_{c'}e^{\lambda_{c'}(-x+c't+h')} + \xi(t) + A_c e^{\lambda_c(x+ct+h)}\right)
\end{aligned}
\tag{1.3}
$$

*where $0 < \xi(t) < 1$ is a solution of $\xi'(t) = f(\xi)$, $t \in \mathbb{R}$, and $\xi(t) \sim Ke^{f'(0)t}$ as $t \to -\infty$.*

(ii) *The function $u(x,t)$ is increasing in $t$ and $u(x,t) \to 1$ as $t \to +\infty$ uniformly in $x$.*

(iii) *For any $t \in \mathbb{R}$, $u(x,t) \to 1$ as $x \to \pm\infty$, and there exists a real $x(t)$ such that $u_x(x(t),t) = 0$, $u_x(x,t) < 0$ if $x < x(t)$, and $u_x(x,t) > 0$ if $x > x(t)$; furthermore, if $c = c'$, then $x(t) \equiv x_0 = \frac{h'-h}{2}$, and for any $t \in \mathbb{R}$, $u(\cdot,t)$ is symmetric with respect to $x_0$.*

(iv) *$u(x(t),t) = \min\ u(\cdot,t) \sim Ke^{f'(0)t}$ as $t \to -\infty$.*

(v) *As $t \to -\infty$, we have*: *If $\beta > c'$, then $u(\beta t + \cdot, t) \to 1$ uniformly in any interval $]-\infty, A]$; $u(c't + \cdot, t) \to \phi_{c'}(-\cdot + h')$ uniformly in any $]-\infty, A]$. If $-c < \beta < c'$, then $u(\beta t + \cdot, t) \to 0$ uniformly in any compact subset of $\mathbb{R}$; $u(-ct + \cdot, t) \to \phi_c(\cdot + h)$ uniformly in any $[A, +\infty[$. And if $\beta < -c$, then $u(\beta t + \cdot, t) \to 1$ uniformly in any $[A, +\infty[$. All these limits also hold in the spaces $C^2_{\mathrm{loc}}$.*

*The functions $u_{c,c',h,h',K}(x,t)$ depend continuously on*

$$(c, c', h, h', K) \in (c^*, +\infty)^2 \times \mathbb{R}^2 \times \mathbb{R}^*_+$$

*in the sense of $\mathcal{T}$. Furthermore, they are increasing in $h$ (or in $h'$, or in $K$) and converge to* 1 *as $h \to +\infty$ (or as $h' \to +\infty$, or as $K \to +\infty$) in $\mathcal{T}$ and also uniformly for $(x,t) \in \mathbb{R} \times [A,+\infty[$ for any real $A$.*

*Remark* 1.2. Properties (iv) and (v) imply, in particular, that

$$u_1 \not\equiv u_2 \quad \text{if } (c_1, c_1', h_1, h_1', K_1) \neq (c_2, c_2', h_2, h_2', K_2).$$

THEOREM 1.3 *For any $c, c' > c^*$ and $h, h' \in \mathbb{R}$, there exists an entire solution $v(x,t) = v_{c,c',h,h'}(x,t)$ of* (1.1) *such that for any $(x,t) \in \mathbb{R}^2$*

$$\begin{aligned} &\max\left(\phi_{c'}(-x+c't+h'),\ \phi_c(x+ct+h)\right)\\ &\quad \le v(x,t)\\ &\quad \le \min\left(1,\ \phi_{c'}(-x+c't+h') + A_c e^{\lambda_c(x+ct+h)},\right.\\ &\qquad\qquad \left. A_{c'} e^{\lambda_{c'}(-x+c't+h')} + \phi_c(x+ct+h)\right) \end{aligned} \tag{1.4}$$

*and* $\min v(\cdot,t) = O(e^{(\lambda_c\lambda_{c'}+f'(0))t}) = o(e^{f'(0)t})$ *as $t \to -\infty$. Furthermore, assertions* (ii), (iii), *and* (v) *in Theorem* 1.1, *as well as the monotonicity in $h$ and $h'$, the limits $h \to +\infty$ (respectively, $h' \to +\infty$), and the continuity in $(c,c',h,h')$, are true for $v_{c,c',h,h'}$ as for $u_{c,c',h,h',K}$.*

*If $h \to -\infty$ (respectively, $h' \to -\infty$), then $v_{c,c',h,h'}(x,t) \to \phi_{c'}(-x+c't+h')$ (respectively, $\phi_c(x+ct+h)$) in $\mathcal{T}$ and uniformly for $(x,t) \in ]-\infty,A]^2$ (respectively, $(x,t) \in [A,+\infty[\times]-\infty,A]$) for any real $A$.*

*Furthermore, with the notation of Theorem* 1.1, *if $(c,c',h,h',K) \in (c^*,+\infty)^2 \times\mathbb{R}^2 \times \mathbb{R}_+^*$, then $u_{c,c',h,h',K} > v_{c,c',h,h'}$ and $u_{c,c',h,h',K} \to v_{c,c',h,h'}$ as $K \to 0^+$ in $\mathcal{T}$ and also uniformly for $(x,t) \in \mathbb{R}\times]-\infty,A]$ for any real $A$.*

THEOREM 1.4 *For any $c' > c^*$, $h' \in \mathbb{R}$, and $K > 0$, there exists a solution $w^-(x,t) = w^-_{c',h',K}(x,t)$ of* (1.1) *such that*:

(i) *For any $(x,t) \in \mathbb{R}^2$*

$$\begin{aligned} &\max\left(\phi_{c'}(-x+c't+h'), \xi(t)\right)\\ &\quad \le w^-(x,t)\\ &\quad \le \min\left(1, \phi_{c'}(-x+c't+h') + Ke^{f'(0)t},\ A_{c'}e^{\lambda_{c'}(-x+c't+h')} + \xi(t)\right) \end{aligned} \tag{1.5}$$

*where $0 < \xi(t) < 1$ is a solution of $\xi'(t) = f(\xi)$, $t \in \mathbb{R}$, and $\xi(t) \sim Ke^{f'(0)t}$ as $t \to -\infty$.*

(ii) *Assertion* (ii) *in Theorem* 1.1 *is true for $w^-$.*

(iii) *For any $t \in \mathbb{R}$ the function $x \mapsto w^-(x,t)$ is decreasing in $x$, $w^-(-\infty,t) = 1$, and $w^-(+\infty,t) = \inf w^-(\cdot,t) = \xi(t)$.*

(iv) *As $t \to -\infty$, we have*: *If $\beta > c'$, then $w^-(\beta t + \cdot, t) \to 1$ uniformly in any interval $]-\infty,A]$; $w^-(c't+\cdot,t) \to \phi_{c'}(-\cdot+h')$ uniformly in any $]-\infty,A]$; and if $\beta < c'$, then $u(\beta t+\cdot,t) \to 0$ uniformly in any $[A,+\infty[$. These limits also hold in the spaces $C^2_{\mathrm{loc}}$.*

*The functions* $w^-_{c',h',K}$ *depend continuously on* $(c',h',K)\in(c^*,+\infty)\times\mathbb{R}\times\mathbb{R}^*_+$ *in the sense of* $\mathcal{T}$. *They satisfy the same monotonicity properties with respect to* $h'$ *and* $K$ *as the functions* $u_{c,c',h,h',K}$ *in Theorem* 1.1 *and converge to* 1 *as* $h'\to+\infty$ (*respectively,* $K\to+\infty$).

*Furthermore,* $w^-_{c',h',K}(x,t)\to\phi_{c'}(-x+c't+h')$ *as* $K\to 0^+$ *in* $\mathcal{T}$ *and uniformly for* $(x,t)\in\mathbb{R}\times]-\infty,A]$ *for any* $A$. *Lastly, if* $c'>c^*$ *is fixed, then* $w^-_{c',h',K}(x,t)\to\xi(t)$ *as* $h'\to-\infty$ *in* $\mathcal{T}$ *and uniformly for* $(x,t)\in[A,+\infty[\times]-\infty,A]$ *for any real* $A$.

*For any* $c>c^*$ *and* $h\in\mathbb{R}$, *we have* $u_{c,c',h,h',K}>w^-_{c',h',K}$ *and* $u_{c,c',h,h',K}\to w^-_{c',h',K}$ *as* $h\to-\infty$ *in* $\mathcal{T}$ *and also uniformly for* $(x,t)\in]-\infty,A]\times K$ *for any real* $A$ *and any compact* $K$.

COROLLARY 1.5 *For any* $c>c^*$, $h\in\mathbb{R}$, *and* $K>0$ *the function* $w^+_{c,h,K}(x,t)=w^-_{c,h,K}(-x,t)$ *is an entire solution of* (1.1). *It is increasing in* $x$, $w^+(+\infty,t)=1$, *and* $w^+(-\infty,t)=\xi(t)$ *is a positive solution of* $\xi'=f(\xi)$. *Furthermore, the functions* $w^+_{c,h,K}$ *can also be viewed as the limits of the functions* $u_{c,c',h,h',K}$ *as* $h'\to-\infty$ *for any fixed* $c'>c^*$.

Let $\mathbb{M}_u$ (respectively, $\mathbb{M}_v$, $\mathbb{M}_{w^+}$, and $\mathbb{M}_{w^-}$) be the 5- (respectively, 4-, 3-, and 3-) dimensional manifold of the functions $u_{c,c',h,h',K}$ (respectively, $v_{c,c',h,h'}$, $w^-_{c',h',K}$, $w^+_{c,h,K}$). From Theorems 1.1, 1.3, and 1.4 and Corollary 1.5, we see that $\mathbb{M}_v$ is on the boundary of $\mathbb{M}_u$ by taking the limit $K\to 0^+$ and that both $\mathbb{M}_{w^-}$ and $\mathbb{M}_{w^+}$ are also on the boundary of $\mathbb{M}_u$ by taking the limits $h\to-\infty$ or $h'\to-\infty$. Furthermore, the two 2-dimensional manifolds of solutions of traveling-wave type $\phi_{c'}(-x+c't+h')$ and $\phi_c(x+ct+h)$ are, respectively, on the boundary of $\mathbb{M}_{w^-}$ and $\mathbb{M}_{w^+}$. Hence, the traveling waves are also boundary points of the manifold $\mathbb{M}_u$. For instance, any wave $\phi_{c'}(-x+c't+h')$ can be obtained from the $u_{c,c',h,h',K}$ by taking the limits $K\to 0^+$ and $h\to-\infty$ in any order. Similarly, the 1-dimensional manifold of the solutions $u(t)$ of $u'=f(u)$ is also on the boundary of $\mathbb{M}_u$, $\mathbb{M}_{w^-}$, and $\mathbb{M}_{w^+}$. For instance, these solutions can be obtained from the $u_{c,c',h,h',K}$ by taking the limits $h\to-\infty$ and $h'\to-\infty$ in any order.

These theorems are proved by solving sequences of Cauchy problems starting at times $-n$ with suitable initial conditions. Some a priori estimates, based on the maximum principle and on comparisons with some solutions of the linear heat equation, allow us to pass to the limit and get nontrivial solutions of (1.1).

## 2 Construction of a 5-Dimensional Manifold of Solutions: Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We build a 5-dimensional manifold of entire solutions of (1.1) that are different from the solutions depending only on $t$ and from the traveling-wave solutions. Roughly speaking, these solutions behave asymptotically as $t\to-\infty$ like two traveling waves for large $|x|$: one coming from the left and the other one coming from the right. Between these two traveling waves, the solutions are nearly uniform and equal to a positive function $\xi(t)$ solution of $\xi' = f(\xi)$.

The proof of Theorem 1.1 is divided into several steps and lemmas. In a few words, in order to get entire solutions of (1.1) fulfilling (i) through (v), the idea is to consider a countable number of functions $u_n(x,t)$ solutions of Cauchy problems starting at times $-n$ with suitable initial conditions. One of the key points will consist in getting lower and upper bounds uniform in $n$. These bounds are then sufficient to pass to the limit $n \to \infty$, and the properties fulfilled by the functions $u_n$ will hold good for the limit function $u$.

## 2.1 Approximating Cauchy Problems

Let $c$ and $c'$ be greater than $c^*$, let $h$ and $h'$ be two given real numbers, and let $K$ be a given positive real number. Let $n_0$ be an integer such that $Ke^{-f'(0)n_0} < 1$. For any $n \geq n_0$, let $u_n(x,t) = u_{n;c,c',h,h',K}(x,t)$ be the unique classical solution of the Cauchy problem

$$\begin{cases} (u_n)_t = (u_n)_{xx} + f(u_n), \quad x \in \mathbb{R},\ t > -n, \\ u_n(x,-n) = u_{n,0}(x) := \max\left(\phi_{c'}(-x - c'n + h'), Ke^{-f'(0)n}, \phi_c(x - cn + h)\right). \end{cases}$$

### Uniform Derivative Estimates

The above Cauchy problem is well-posed, and by the strong maximum principle, we get that $0 < u_n(x,t) < 1$ for any $n \geq n_0$, $t \geq -n$, and $x \in \mathbb{R}$. Since the functions $u_n$ are uniformly bounded, since the equation $(u_n)_t = (u_n)_{xx} + f(u_n)$ is invariant by translation in $x$ and since $f$ is of class $C^2$, the standard estimates for derivatives (see Friedman [12]) yield the existence of a constant $C$ that does not depend on $n$ or on $(c,c',h,h',K)$ such that, for all $n \in \mathbb{N}$, $t \geq -n+1$, $x \in \mathbb{R}$,

$$|(u_n)_x(x,t)|,\ |(u_n)_t(x,t)|,\ |(u_n)_{xx}(x,t)|,\ |(u_n)_{tt}(x,t)|,\ |(u_n)_{xxx}(x,t)| \leq C. \tag{2.1}$$

### Lower Bound for $u_n$

By the maximum principle for parabolic equations, it follows that for any $t \geq -n$ and any $x \in \mathbb{R}$,

$$0 < \max\left(\phi_{c'}(-x + c't + h'), \xi_n(t), \phi_c(x + ct + h)\right) \leq u_n(x,t) \leq 1 \tag{2.2}$$

where the function $\xi_n(t)$ is the solution of the Cauchy problem $\xi_n'(t) = f(\xi_n)$, $\xi_n(-n) = Ke^{-f'(0)n}$. Since $f > 0$ in $(0,1)$ and $f(1) = 0$, we have, for any $n$ large enough and for any $t > -n : 0 < Ke^{-f'(0)n} < \xi_n(t) < 1$. Hence, there exists a constant $C > 0$ such that $|\xi_n'(t)| \leq C$ for any $n$ large enough and $t > -n$. The function $f$ being of class $C^1$, $\xi_n$ is twice differentiable, and we can assume that $|\xi_n''(t)| \leq C$. By integration of the equation satisfied by $\xi_n$, we deduce that

$$\int_{Ke^{-f'(0)n}}^{\xi_n(t)} \frac{ds}{f(s)} = t + n.$$

Since $f(s) \le f'(0)s$ for any $s \in (0,1)$, we get that

$$t+n \ge \int_{Ke^{-f'(0)n}}^{\xi_n(t)} \frac{ds}{f'(0)s}$$

and then $\xi_n(t) \le Ke^{f'(0)t}$. On the other hand, $f$ being of class $C^2$ in a right neighborhood of 0 and fulfilling $f(s) \le f'(0)s$ for any $s \ge 0$, it follows that $f''(0) \le 0$. Furthermore, there exists $\varepsilon_0 \in (0,1)$ and a continuous function $\nu : [0,\varepsilon_0] \to \mathbb{R}$ such that $\nu(0) = -f''(0)/2f'(0)$ and $f(s) \ge f'(0)s(1-\nu(\varepsilon)s) > 0$ for any $\varepsilon \in [0,\varepsilon_0]$ and any $s \in [0,\varepsilon]$ (for instance, choose $\nu(s) = -(1/2f'(0))\inf_{[0,s]} f''$). Take any $\varepsilon \in ]0,\varepsilon_0]$, $n \ge n_0$, and $t > -n$. If $\xi_n(t) \le \varepsilon$, it then follows that

$$t+n \le \int_{Ke^{-f'(0)n}}^{\xi_n(t)} \frac{ds}{f'(0)s(1-\nu(\varepsilon)s)}\,.$$

After a straightforward calculation, we find that $\xi_n(t) \ge Ke^{f'(0)t}(1-\nu(\varepsilon)\varepsilon)$. Finally, we conclude that

$$\forall n \ge n_0\,,\ t > -n\,,\ \varepsilon \in [0,\varepsilon_0]\,, \quad \min(\varepsilon, Ke^{f'(0)t}(1-\nu(\varepsilon)\varepsilon)) \le \xi_n(t)\,. \tag{2.3}$$

**Monotonicity in *t***

The function $v_1(x) = \phi_{c'}(-x-c'n+h')$ satisfies

$$\begin{aligned} v_1'' + f(v_1) &= \phi_{c'}''(-x-c'n+h') + f(\phi_{c'}(-x-c'n+h')) \\ &= c'\phi_{c'}'(-x-c'n+h') > 0 \end{aligned}$$

since $c' > 0$ and $\phi_{c'}$ is increasing. Similarly, the function $v_2(x) = \phi_c(x-cn+h)$ satisfies $v_2''+f(v_2) = c\phi_c'(x-cn+h) > 0$. Lastly, the constant $Ke^{-f'(0)n}$ is such that $f(Ke^{-f'(0)n}) > 0$. Hence, the function $u_{n,0} = \sup(v_1, Ke^{-f'(0)n}, v_2)$ is a subsolution, namely, $u_{n,0}'' + f(u_{n,0}) \ge 0, \not\equiv 0$ in the distribution sense. This implies that the function $u_n(x,t)$ is increasing in $t$ for any $t > -n$, $x \in \mathbb{R}$.

**Profile of $u_n(\cdot,t)$**

Both functions $\phi_c(\xi)$ and $\phi_{c'}(\xi)$ are increasing and approach 0 (respectively, 1) as $\xi \to -\infty$ (respectively, $\xi \to +\infty$). Furthermore, set $\gamma_n = (\lambda_c - \lambda_{c'})n$. Since $-c' < \lambda_c - \lambda_{c'} < c$, both $-\gamma_n - c'n + h'$ and $\gamma_n - cn + h$ approach $-\infty$ as $n \to +\infty$. Hence, by (1.2), it follows that $\phi_{c'}(-\gamma_n - c'n + h') \sim A_{c'}e^{\lambda_{c'}(-\gamma_n - c'n+h')} = A_{c'}e^{\lambda_{c'}h'}e^{-\lambda_c\lambda_{c'}n - f'(0)n} = o(e^{f'(0)n})$ as $n \to \infty$, and, similarly,

$$\phi_c(\gamma_n - cn + h) = o(e^{f'(0)n}) \quad \text{as } n \to \infty\,.$$

Hence, for $n$ large enough, there exist two reals $y_n < z_n$ such that $y_n < \gamma_n < z_n$ and

$$u_{n,0}(x) = \begin{cases} \phi_{c'}(-x - c'n + h'),\ u'_{n,0}(x) < 0\,, & \text{if } x < y_n\,, \\ \phi_{c'}(-y_n - c'n + h') = Ke^{-f'(0)n}\,, & \text{if } x = y_n\,, \\ Ke^{-f'(0)n},\ u'_{n,0}(x) = 0\,, & \text{if } y_n < x < z_n\,, \\ Ke^{-f'(0)n} = \phi_c(z_n - cn + h)\,, & \text{if } x = z_n\,, \\ \phi_c(x - cn + h),\ u'_{n,0}(x) > 0\,, & \text{if } x > z_n\,. \end{cases} \tag{2.4}$$

Furthermore, since $(u_n)_x$ is a solution of a linear Cauchy problem, it follows that, for any $t > -n$, $(u_n)_x(\cdot,t)$ changes sign at most once (see, for instance, Nickel [22]). By (2.2), $u_n(\pm\infty,t) = 1$ whence $u_n(\cdot,t)$ cannot be monotone unless it is identically 1; the latter is ruled out by the strong maximum principle. Hence, for any $t > -n$, there exists a real $x_n(t)$ such that $(u_n)_x(x,t) < 0$ if $x < x_n(t)$, $(u_n)_x(x,t) = 0$ if $x = x_n(t)$, and $(u_n)_x(x,t) > 0$ if $x > x_n(t)$.

Let us now write down the asymptotic behavior of $y_n$ and $z_n$ (this will be useful in the sequel):

$$\begin{cases} y_n = -\lambda_{c'} n + \frac{\ln A_{c'}}{\lambda_{c'}} + h' + o(1) \\ z_n = \lambda_c n - \frac{\ln A_c}{\lambda_c} - h + o(1) \end{cases} \qquad \text{as } n \to +\infty. \tag{2.5}$$

The formula for $y_n$ comes directly from the fact that $\phi_{c'}(-y_n - c'n + h') = Ke^{-f'(0)n}$ and from the asymptotic behavior of $\phi_{c'}$ given by (1.2). The formula for $z_n$ is similar.

If $c = c'$, it is clear that $u_{n,0}$ is symmetric with respect to $(h' - h)/2$. This property holds good for $u_n(\cdot,t)$ at any time $t > -n$ because equation (1.1) is invariant by translation and reflection in $x$. This implies in particular that $x_n(t) \equiv (h' - h)/2$ for any $t > -n$.

### Upper Bound for $u_n$

The estimates (2.2) and (2.3) provide a lower bound for the functions $u_n$, which do not depend on $n$. The following lemma gives an upper bound for the $u_n$:

LEMMA 2.1 *For any couple $(x,t) \in \mathbb{R}^2$,*

$$\limsup_{n>|t|,\ n\to+\infty} u_n(x,t) \le \phi_{c'}(-x + c't + h') + Ke^{f'(0)t} + A_c e^{\lambda_c(x+ct+h)}\,, \tag{2.6}$$

$$\limsup_{n>|t|,\ n\to+\infty} u_n(x,t) \le A_{c'} e^{\lambda_{c'}(-x+c't+h')} + Ke^{f'(0)t} + \phi_c(x + ct + h)\,, \tag{2.7}$$

$$\limsup_{n>|t|,\ n\to+\infty} (u_n(x,t) - \xi_n(t)) \le A_{c'} e^{\lambda_{c'}(-x+c't+h')} + A_c e^{\lambda_c(x+ct+h)}\,. \tag{2.8}$$

PROOF: We will prove only (2.6) and (2.8), because the inequality (2.7) is similar to (2.6). To prove these inequalities, say (2.6), the key point will be to compare $u_n(x,t) - \phi_{c'}(-x + c't + h')$ with the solution of a linear heat equation for which we have an explicit formula.

Fix any couple $(x_0,t_0)\in\mathbb{R}^2$. For $n>|t_0|$, let us consider the function $v_n(x,t)=u_n(x,t)-\phi_{c'}(-x+c't+h')$. This function is nonnegative by (2.2). Since $\phi_{c'}(-x+c't+h')$ is a solution of (1.1) and since $f'(s)\le f'(0)$ in $[0,1]$, we have that

$$(v_n)_t=(v_n)_{xx}+f(u_n)-f(\phi_{c'}(-x+c't+h'))\le(v_n)_{xx}+f'(0)v_n\,.$$

On the other hand, because of the definitions of $y_n$ and $z_n$, we have $v_n(x,-n)=0$ in $]-\infty,y_n]$, $v_n(x,-n)\le Ke^{-f'(0)n}$ in $[y_n,z_n]$, and $v_n(x,-n)\le\phi_c(x-cn+h)$ in $[z_n,+\infty[$. Therefore

$$v_n(x_0,t_0)\le \mathrm{I}+\mathrm{II}$$

where

$$\begin{cases}\mathrm{I}=\dfrac{1}{\sqrt{4\pi(t_0+n)}}e^{f'(0)(t_0+n)}\left(\displaystyle\int_{y_n}^{z_n}Ke^{-f'(0)n}e^{-\frac{(x_0-y)^2}{4(t_0+n)}}\,dy\right)\\ \mathrm{II}=\dfrac{1}{\sqrt{4\pi(t_0+n)}}e^{f'(0)(t_0+n)}\left(\displaystyle\int_{z_n}^{+\infty}\phi_c(y-cn+h)e^{-\frac{(x_0-y)^2}{4(t_0+n)}}\,dy\right).\end{cases}$$

The first term I immediately satisfies $\mathrm{I}\le Ke^{f'(0)t_0}$. Let us now emphasize the second integral II. With the change of variable $y=s+cn-h$, we get

$$\mathrm{II}=\frac{1}{\sqrt{4\pi(t_0+n)}}\,e^{f'(0)(t_0+n)-\frac{(cn-h-x_0)^2}{4(n+t_0)}}\int_{z_n-cn+h}^{+\infty}\phi_c(s)e^{-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds\,.$$

By (2.5) and since $\lambda_c<c$, it follows that $z_n-cn+h\to-\infty$ as $n\to+\infty$. Let $\varepsilon$ be a positive number. Since $c>c^*$, (1.2) implies that there exists a real $A$ such that $\phi_c(s)\le(A_c+\varepsilon)e^{\lambda_c s}$ for any $s\le A$. On the other hand, $\phi_c(s)\le1$ for any $s\in\mathbb{R}$. For $n$ large enough, we then have

$$\begin{aligned}\mathrm{II}\le{}&\frac{1}{\sqrt{4\pi(t_0+n)}}\,e^{f'(0)(t_0+n)-\frac{(cn-h-x_0)^2}{4(n+t_0)}}\\&\times\left(\int_{z_n-cn+h}^{A}(A_c+\varepsilon)e^{\lambda_c s-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds+\int_A^{+\infty}e^{-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds\right)\end{aligned}\tag{2.9}$$

We observe that

$$\int_A^{+\infty}e^{-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds\le\int_A^{+\infty}e^{-\frac{(cn-h-x_0)s}{2(n+t_0)}}ds\le\frac{2(n+t_0)}{cn-h-x_0}\,e^{-\frac{(cn-h-x_0)A}{2(n+t_0)}}\to\frac{2}{c}e^{-\frac{cA}{2}}$$

as $n\to+\infty$ and that $e^{f'(0)(t_0+n)-\frac{(cn-h-x_0)^2}{4(n+t_0)}}\to0$ since $c>c^*=2\sqrt{f'(0)}$. Hence

$$\lim_{n\to+\infty}\frac{1}{\sqrt{4\pi(t_0+n)}}e^{f'(0)(t_0+n)-\frac{(cn-h-x_0)^2}{4(n+t_0)}}\int_A^{+\infty}e^{-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds\;=0$$

On the other hand, since $\lambda_c = \frac{c-\sqrt{c^2-4f'(0)}}{2}$, it is true that

$$\lambda_c s - \frac{s^2+2(cn-h-x_0)s}{4(n+t_0)} = -\left(\frac{s+n\sqrt{c^2-4f'(0)}-2t_0\lambda_c-h-x_0}{2\sqrt{n+t_0}}\right)^2 + \frac{(n\sqrt{c^2-4f'(0)}-2t_0\lambda_c-h-x_0)^2}{4(n+t_0)}.$$

With the change of variables $s = 2\sqrt{n+t_0}\,\tau - n\sqrt{c^2-4f'(0)}+2t_0\lambda_c+h+x_0$, the first term of the right-hand side in (2.9) becomes

$$\begin{aligned}&\frac{1}{\sqrt{4\pi(t_0+n)}}e^{f'(0)(t_0+n)-\frac{(cn-h-x_0)^2}{4(n+t_0)}}\int_{z_n-cn+h}^{A}(A_c+\varepsilon)e^{\lambda_c s-\frac{s^2+2(cn-h-x_0)s}{4(n+t_0)}}ds\\ &\quad = \frac{1}{\sqrt{\pi}}(A_c+\varepsilon)e^{\alpha_n}\int_{a_n}^{b_n}e^{-\tau^2}d\tau\end{aligned}$$

where

$$\begin{aligned}\alpha_n &= f'(0)(t_0+n) - \frac{(cn-h-x_0)^2}{4(n+t_0)} + \frac{(n\sqrt{c^2-4f'(0)}-2t_0\lambda_c-h-x_0)^2}{4(n+t_0)},\\ a_n &= \frac{1}{2\sqrt{n+t_0}}(z_n-cn+h+n\sqrt{c^2-4f'(0)}-2t_0\lambda_c-h-x_0),\\ b_n &= \frac{A+n\sqrt{c^2-4f'(0)}-2t_0\lambda_c-h-x_0}{2\sqrt{n+t_0}}.\end{aligned}$$

Since $\lambda_c = \frac{c-\sqrt{c^2-4f'(0)}}{2}$, it is straightforward to check that $\alpha_n \to \lambda_c(ct_0+x_0+h)$ as $n\to+\infty$. Furthermore, the asymptotic formula for $z_n$ given in (2.5) implies that $a_n \sim -\frac{\lambda_c}{2}\sqrt{n}\to-\infty$ as $n\to+\infty$; lastly, $b_n\to+\infty$ as $n\to+\infty$. Eventually we conclude that

$$\limsup_{n\to+\infty} \mathrm{II} \le (A_c+\varepsilon)e^{\lambda_c(ct_0+x_0+h)} \quad \text{for any } \varepsilon>0.$$

Since $\varepsilon>0$ was arbitrary, this yields that

$$\limsup_{n\to+\infty}\,(u_n(x_0,t_0)-\phi_{c'}(-x_0+c't_0+h')) \le Ke^{f'(0)t_0}+A_c e^{\lambda_c(ct_0+x_0+h)}$$

and completes the proof of (2.6).

To prove (2.8), we can similarly compare the functions $u_n(x,t)-\xi_n(t)$ to the solution of the linear heat equation $v_t = v_{xx}+f'(0)v$ with initial condition at time $-n$: $v(x,-n)=\phi_{c'}(-x-c'n+h')$ if $x\le y_n$, $v(x,-n)=0$ if $y_n<x<z_n$, and $(x,-n)=\phi_c(x-cn+h)$ if $x\ge z_n$. It is then the case that

$$\begin{aligned}0 &\le u_n(x,t)-\xi_n(t)\\ &\le \frac{1}{\sqrt{4\pi(t+n)}}e^{f'(0)(t+n)}\\ &\quad\times\left(\int_{-\infty}^{y_n}\phi_{c'}(-y-c'n+h')\,e^{-\frac{(x-y)^2}{4(t+n)}}dy+\int_{z_n}^{+\infty}\phi_c(y-cn+h)\,e^{-\frac{(x-y)^2}{4(t+n)}}dy\right).\end{aligned}$$

As was done in the previous paragraphs, the limit $n \to +\infty$ gives the desired result. $\square$

## 2.2 Passage to the Limit $n \to +\infty$

From the a priori derivative estimates (2.1) and by a diagonal extraction process, there exists a subsequence $(u_{n'})$ such that $u_{n'}(\cdot,\cdot)$ converges to a function $u(\cdot,\cdot)$ in the sense of the topology $\mathcal{T}$. From the equation satisfied by $u_n$, the limit function $u(x,t)$ is an entire and classical solution of (1.1). Furthermore, since $f$ is of class $C^2$, the same kind of estimate as (2.1) holds good for $u$; that is to say, there exists a constant $C$, which does not depend on $(c,c',h,h',K)$, such that

$$\forall (x,t) \in \mathbb{R}^2, \quad |u_x|, |u_t|, |u_{xx}|, |u_{tt}|, |u_{xxx}| \leq C. \tag{2.10}$$

From the a priori estimates for $\xi_n(t)$, we can also assume that the functions $\xi_{n'}(t)$ converge to a function $\xi(t)$ in $C^1_{\mathrm{loc}}$, solution of $\xi' = f(\xi)$ in $\mathbb{R}$. By (2.3) and since $\xi_n(t) \leq Ke^{f'(0)t}$, it follows that

$$\forall t \in \mathbb{R}, \ \forall \varepsilon \in [0,\varepsilon_0], \quad \min(\varepsilon, Ke^{f'(0)t}(1-\nu(\varepsilon)\varepsilon)) \leq \xi(t) \leq Ke^{f'(0)t}. \tag{2.11}$$

In particular, for any $\delta > 0$, there exists a real $t_\delta$ such that $Ke^{f'(0)t}(1-\delta) \leq \xi(t)$ for any $t \leq t_\delta$. Finally, $\xi(t) \sim Ke^{f'(0)t}$ as $t \to -\infty$.

The estimate (1.3) is a consequence of estimates (2.2), (2.6), (2.7), and (2.8).

Since the functions $u_n$ are increasing in $t$, it follows that $u$ is nondecreasing in $t$. Since $f$ is of class $C^1$, the strong maximum principle applied to $u_t$ implies that either $u_t > 0$ or $u_t \equiv 0$ in $\mathbb{R}^2$. The latter is impossible because (1.3) implies that $u(x,t) \to 1$ as $t \to +\infty$, uniformly in $x$, whereas $u(x,t) \to 0$ as $t \to -\infty$ locally in $x$. This proves assertion (ii) in Theorem 1.1. Since $u$ is increasing in $t$, it also follows that $0 < u(x,t) < 1$ for any $(x,t) \in \mathbb{R}^2$.

**Study of the Profile of $u(\cdot,t)$**

Let us now prove assertion (iii) in Theorem 1.1: At any time $t$, the function $u(\cdot,t)$ is decreasing in some interval $]-\infty, x(t)]$ and increasing in $[x(t),+\infty[$.

If $c = c'$, from the properties fulfilled by $u_n$, it follows that, for any $t \in \mathbb{R}$, $u(\cdot,t)$ is symmetric with respect to $x_0 = (h'-h)/2$. Furthermore, $u_x(x,t) \leq 0$ if $x \leq x_0$ and $u_x(x,t) \geq 0$ if $x \geq x_0$. For any $t \in \mathbb{R}$, since $u(x,t) < 1$ and $u(x,t) \to 1$ as $x \to \pm\infty$ by (1.3), there exist two sequences $\alpha_n \to -\infty$ and $\beta_n \to +\infty$ such that $u_x(\alpha_n,t) < 0$ and $u_x(\beta_n,t) > 0$. Finally, for any $t' > t$, $u_x(\cdot,t')$ can change sign at most once in $\mathbb{R}$, this change of sign occurring then at the point $x_0$; that is to say, for any $t' > t$, $u_x(x,t') < 0$ if $x < x_0$ and $u_x(x,t')$ if $x > x_0$. Since $t$ is arbitrary, this gives assertion (iii) in Theorem 1.1 if $c = c'$.

Let us now consider the general case where $c$ and $c'$ may be equal or not equal. At any time $t = -k$, $k \in \mathbb{N}$, we know that, for any $n > k$, there exists a real $x_n(-k)$ such that $u_n(\cdot,-k)$ is decreasing in $]-\infty, x_n(-k)]$ and increasing in $[x_n(-k),+\infty[$. If the sequence $(x_n(-k))_{n>k}$ were not bounded, then, say, there exists a subsequence $n'$ such that $x_{n'}(-k) \to +\infty$ as $n' \to +\infty$. In particular, for any $x_1 \leq x_2$, it follows that $u_{n'}(x_1,-k) \geq u_{n'}(x_2,-k)$ for $n'$ large enough. The limit $n' \to +\infty$ would imply that $u(x_1,-k) \geq u(x_2,-k)$. Since $x_1 \leq x_2$ are arbitrary, this means that $u(\cdot,-k)$ is nonincreasing. Since $u(\pm\infty,-k) = 1$, we would get that $u(\cdot,-k) \equiv 1$, which is impossible because $0 < u < 1$. Hence, the sequence $(x_n(-k))_{n>k}$ is bounded for any $k \in \mathbb{N}$.

By the diagonal extraction process, there exists then a subsequence $n' \to +\infty$ such that, for any $k \in \mathbb{N}$, $x_{n'}(-k) \to x(-k) \in \mathbb{R}$. For any $k \in \mathbb{N}$ and for any $x_1 \leq x_2 < x(-k)$, we deduce that $u_{n'}(x_1,-k) \geq u_{n'}(x_2,-k)$, whence $u(x_1,-k) \geq u(x_2,-k)$. The function $u(\cdot,-k)$ is nonincreasing in $]-\infty, x(-k)]$, and similarly, it is nondecreasing in $[x(-k),+\infty[$. Hence, for any time $t > -k$, the function $u_x(\cdot,t)$ changes sign at most once, and as above, we conclude that there exists a unique real $x(t)$ such that $u_x(\cdot,t)$ is negative in $(-\infty, x(t))$ and positive in $(x(t),+\infty)$. Since $k \in \mathbb{N}$ and $t > -k$ are arbitrary, this gives the desired assertion (iii) of Theorem 1.1.

By (1.3), we have $u(x(t),t) = \min u(\cdot,t) \geq \xi(t)$. Furthermore, remember that $\xi(t) \sim Ke^{f'(0)t}$ as $t \to -\infty$. On the other hand, for any $A \in \mathbb{R}$, there exists a real $T$ such that, for any $t \leq T$,

$$\phi_{c'}(-x+c't+h') \leq 2A_{c'}e^{\lambda_{c'}(-x+c't+h')} = o(e^{f'(0)t}) \quad \text{as } t \to -\infty,$$

uniformly in $x \in [-A,A]$ (since $\lambda_{c'}c' = \lambda_{c'}^2 + f'(0) > f'(0)$). This also holds good for $\phi_c(x+ct+h)$. Finally, we deduce from the upper bound in (1.3) that

$$\sup_{[-A,A]} \left(u(\cdot,t) - Ke^{f'(0)t}\right) = o(e^{f'(0)t}) \quad \text{as } t \to -\infty.$$

We conclude that $u(x(t),t) \sim Ke^{f'(0)t}$ and even that $u(x,t) \sim Ke^{f'(0)t}$ uniformly in any $[-A,A]$ as $t \to -\infty$.

**Behavior of $u(\beta t + \cdot, t)$ as $t \to -\infty$**

Let us now emphasize part (v) of Theorem 1.1. First of all, by the lower bound in (1.3), it is clear that, if $\beta > c'$, then $u(\beta t + x, t) \to 1$ as $t \to -\infty$ uniformly in any interval $]-\infty, A]$. Furthermore, we have already seen that the lower bound in (1.3) implies that $\lim_{t\to-\infty} u(0,t) = 0$. In particular, $\lim_{t\to-\infty}\inf_{x\in\mathbb{R}} u(\beta t + x, t) = 0$ and the convergence of $u(\beta t + x, t)$ to 1 as $t \to -\infty$ cannot be uniform in $x \in \mathbb{R}$. Similarly, if $\beta < -c$, then $u(\beta t + x, t) \to 1$ as $t \to -\infty$ uniformly in any interval $[A, +\infty[$.

Now, if $-c < \beta < c'$, then the lower bound in (1.3) immediately yields that $u(\beta t + \cdot, t) \to 0$ as $t \to -\infty$ uniformly in any compact subset of $\mathbb{R}$. Notice that this last convergence can only be local in $x$ because $u(\pm\infty, t) = 1$ for any $t \in \mathbb{R}$.

Consider now the case where $\beta = c'$. Let $t_n$ be any sequence converging to $-\infty$ and define the functions $v_n(x) = u(c't_n + x, t_n)$. For any $x \in \mathbb{R}$, (1.3) implies that

$$\begin{aligned} &\max(\phi_{c'}(-x+h'), \xi(t_n), \phi_c(x+(c+c')t_n+h)) \\ &\quad \leq v_n(x) \end{aligned}$$

$$\leq \min\Big(1,\ \phi_{c'}(-x+h')+Ke^{f'(0)t_n}+A_c e^{\lambda_c((c+c')t_n+x+h)}\,,$$
$$A_{c'}e^{\lambda_{c'}(-x+h')}+Ke^{f'(0)t_n}+\phi_c((c+c')t_n+x+h)\Big)\,.$$

On the other hand, since the functions

$$v_n(x)\,,\qquad\qquad v_n'(x)=u_x(t_n,c't_n+x)\,,$$
$$v_n''(x)=u_{xx}(t_n,c't_n+x)\,,\qquad\qquad v_n'''(x)=u_{xxx}(t_n,c't_n+x)\,,$$

are uniformly bounded in $n$ and $x$, there exists a function $\psi(x)$ such that, up to extraction of some subsequence, $v_n(x)\to\psi(x)$ in $C^2_{\mathrm{loc}}$ as $n\to+\infty$. Passing to the limit $t_n\to-\infty$ in the above inequality for $v_n(x)$, we get that, for any $x\in\mathbb{R}$,

$$\phi_{c'}(-x+h')\leq\psi(x)\leq\min\Big(1,\phi_{c'}(-x+h'),A_{c'}e^{\lambda_{c'}(-x+h')}\Big)\leq\phi_{c'}(-x+h')\,.\tag{2.12}$$

Hence $\psi\equiv\phi_{c'}(-\cdot+h')$: This means that, up to extraction of some subsequence, the functions $v_n(x)=u(c't_n+x,t_n)$ approach $\phi_{c'}(-x+h')$ as $t_n\to-\infty$ in $C^2_{\mathrm{loc}}$ norms. The limit does not depend on the sequence $t_n$ whence the convergence holds good for the functions $x\mapsto u(c't+x,t)$ as $t\to-\infty$. Furthermore, since $1\geq u(c't+x,t)\geq\phi_{c'}(-x+h')$ for any $x\in\mathbb{R}$ and since $\phi(+\infty)=1$, the functions $x\mapsto u(c't+x,t)$ converge to $\phi_{c'}(-x+h')$ as $t\to-\infty$ uniformly in any interval $]-\infty,A]$. The convergence cannot occur uniformly in $x\in\mathbb{R}$ because at any time $t$, $u(\xi,t)\to1$ whereas $\phi_{c'}(-\xi)\to0$ as $\xi\to+\infty$.

Similarly, we would get that the functions $u(-ct+x,t)$ approach the function $\phi_c(x+h)$ as $t\to-\infty$ in $C^2_{\mathrm{loc}}$ norms, and also uniformly in any interval $[A,+\infty[$.

**Continuity in $(\boldsymbol{c},\boldsymbol{c}',\boldsymbol{h},\boldsymbol{h}',\boldsymbol{K})$**

Consider a sequence

$$(c_k,c_k',h_k,h_k',K_k)\to(c,c',h,h',K)\in(c^*,+\infty)^2\times\mathbb{R}^2\times(0,+\infty)\,.$$

Set $u_k(x,t)=u_{c_k,c_k',h_k,h_k',K_k}(x,t)$ and $u(x,t)=u_{c,c',h,h',K}(x,t)$. Call $\xi_k(t)$ the function solution of $\xi_k'(t)=f(\xi_k)$ in $\mathbb{R}$ and appearing in the bounds (1.3) for the function $u_k$.

From the a priori estimates (2.10) for the functions $u_k(x,t)$, there exists a function $\tilde u(x,t)$ such that $u_k\to\tilde u$ as $k\to+\infty$ (up to extraction of some subsequence) in the sense of $\mathcal{T}$. In particular, the function $\tilde u$ is an entire solution of (1.1). Since the functions $\xi_k$ are uniformly bounded in $C^2(\mathbb{R})$, we can assume that they converge in $C^1_{\mathrm{loc}}(\mathbb{R})$ to a function $\xi(t)$ solution of $\xi'=f(\xi)$ in $\mathbb{R}$. Furthermore, (2.11) holds good for $\xi$ as well as for $\xi_k$ (remember that the real $\varepsilon_0$ appearing in (2.11) only depends on $f$). As a consequence, $\xi(t)\sim Ke^{f'(0)t}$ as $t\to-\infty$.

Furthermore, the functions $\phi_c(z)$ are continuous with respect to $c\in[c^*,+\infty[$ in the norms $C^2_{\mathrm{loc}}(\mathbb{R})$. Indeed, if $c_l\to c\in[c^*,+\infty[$, then by the standard elliptic estimates and by an diagonal extraction process, there exists a subsequence $l'$ such that $\phi_{c_{l'}} \to \phi$ in $C^2_{\mathrm{loc}}(\mathbb{R})$, where $\phi$ is a solution of

$$\phi'' - c\phi' + f(\phi) = 0 \quad \text{in } \mathbb{R}.$$

By passing to the limit $c_{l'} \to c$, the function $\phi$ is nondecreasing and, since the functions $\phi_{c_l}$ are normalized in 0, it follows that $\phi(0) = \frac{1}{2}$. Since $f$ is positive on $(0,1)$, this yields that $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Finally, $\phi \equiv \phi_c$ and the whole sequence $\phi_{c_l}$ converges to $\phi_c$ in $C^2_{\mathrm{loc}}(\mathbb{R})$.

The coefficients $\lambda_c$ are continuous in $c \in [c^*, +\infty[$, because of their definition. Lastly, we claim that

$$c \mapsto A_c \text{ is continuous in } c \in (c^*, +\infty). \tag{2.13}$$

Assume this claim temporarily. By passage to the limit $k \to +\infty$ in (1.3), the function $\tilde{u}(x,t)$ fulfills the estimates

$$\begin{aligned} &\max\left(\phi_{c'}(-x + c't + h'),\ \xi(t),\ \phi_c(x + ct + h)\right) \\ &\quad \le \tilde{u}(x,t) \\ &\quad \le \min\Big(1, \phi_{c'}(-x + c't + h') + Ke^{f'(0)t} + A_c e^{\lambda_c(x+ct+h)}, \\ &\qquad\qquad A_{c'} e^{\lambda_{c'}(-x+c't+h')} + Ke^{f'(0)t} + \phi_c(x + ct + h), \\ &\qquad\qquad A_{c'} e^{\lambda_{c'}(-x+c't+h')} + \xi(t) + A_c e^{\lambda_c(x+ct+h)}\Big). \end{aligned} \tag{2.14}$$

Let us now prove that $\tilde{u} \equiv u = u_{c,c',h,h',K}$. Remember that the functions $u_n(x,t)$, which are solutions of the Cauchy problems $(u_n)_t = (u_n)_{xx} + f(u_n)$, $t > -n$, $x \in \mathbb{R}$, with the initial conditions

$$u_n(x, -n) = u_{n,0}(x) = \max(\phi_{c'}(-x - c'n + h'), Ke^{-f'(0)n}, \phi_c(x - cn + h)),$$

converge to the function $u(x,t)$ in the sense of $\mathcal{T}$. Let us now compare the functions $\tilde{u}(\cdot, -n)$ to the functions $u_{n,0}(\cdot)$. Notice first that, from (2.12), for any $c > c^*$ and for any $z \in \mathbb{R}$, $\phi_c(z) \le A_c e^{\lambda_c z}$. By (2.14) and from the definition of $(y_n, z_n)$ in (2.4), we get that

$$|\tilde{u}(x, -n) - u_{n,0}(x)| \le \begin{cases} Ke^{-f'(0)n} + A_c e^{\lambda_c(x-cn+h)} & \text{if } x \le y_n, \\ |\xi(-n) - Ke^{-f'(0)n}| + A_{c'} e^{\lambda_{c'}(-x-c'n+h')} \\ \quad + A_c e^{\lambda_c(x-cn+h)} & \text{if } x \in [y_n, z_n], \\ Ke^{-f'(0)n} + A_{c'} e^{\lambda_{c'}(-x-c'n+h')} & \text{if } x \ge z_n. \end{cases} \tag{2.15}$$

Fix a couple $(x_0, t_0) \in \mathbb{R}^2$. For $n > |t_0|$, as we did in the proof of Lemma 2.1, we can compare $\tilde{u} - u_n$ with a solution of the linear heat equation $v_t = v_{xx} + f'(0)v$, which has as initial condition at time $-n$ the right-hand side of (2.15). We deduce that

$$
\begin{aligned}
&|\tilde{u}(x_0,t_0)-u_n(x_0,t_0)|\\
&\quad\le \frac{1}{\sqrt{4\pi(t_0+n)}}e^{f'(0)(t_0+n)}\\
&\qquad\times\Bigg(\int_{-\infty}^{y_n}\Big(Ke^{-f'(0)n}+A_ce^{\lambda_c(y-cn+h)}\Big)\ e^{-\frac{(x_0-y)^2}{4(t_0+n)}}dy\\
&\qquad\quad+\int_{y_n}^{z_n}\Big(|\xi(-n)-Ke^{-f'(0)n}|+A_{c'}e^{\lambda_{c'}(-y-c'n+h')}+A_ce^{\lambda_c(y-cn+h)}\Big)\\
&\qquad\qquad\qquad\times e^{-\frac{(x_0-y)^2}{4(t_0+n)}}dy\\
&\qquad\quad+\int_{z_n}^{+\infty}\Big(Ke^{-f'(0)n}+A_{c'}e^{\lambda_{c'}(-y-c'n+h')}\Big)e^{-\frac{(x_0-y)^2}{4(t_0+n)}}dy\Bigg).
\end{aligned}
$$

Call I, II, and III the three terms in the right-hand side of this last inequality. Consider the first integral I and write it $\mathrm{I}=\mathrm{I}_1+\mathrm{I}_2$ with obvious notation. With the change of variables $y=x_0+2\sqrt{t_0+n}\,s$, it follows that

$$
0\le \mathrm{I}_1=\frac{K}{\sqrt{\pi}}e^{f'(0)t_0}\int_{-\infty}^{\frac{y_n-x_0}{2\sqrt{t_0+n}}}e^{-s^2}ds\to 0\quad \text{as } n\to+\infty,
$$

since $y_n\sim-\lambda_{c'}n$ by (2.5). With the same change of variables and since $\lambda_c^2-\lambda_c c+f'(0)=0$, we get that

$$
\begin{aligned}
0\le \mathrm{I}_2&\le \frac{A_c}{\sqrt{\pi}}e^{(f'(0)-\lambda_c c)n+f'(0)t_0+\lambda_c(x_0+h)}\int_{-\infty}^{\frac{y_n-x_0}{2\sqrt{t_0+n}}}e^{-s^2+2\lambda_c\sqrt{t_0+n}\,s}\,ds\\
&\le \frac{A_c}{\sqrt{\pi}}e^{-\lambda_c^2n+f'(0)t_0+\lambda_c(x_0+h)}\frac{e^{\lambda_c(y_n-x_0)}}{2\lambda_c\sqrt{t_0+n}}\ \to 0\quad \text{as } n\to+\infty.
\end{aligned}
$$

Similarly, we have $\mathrm{III}\to 0$ as $n\to+\infty$.

Lastly, the integral II can be divided into three terms $\mathrm{II}_1$, $\mathrm{II}_2$, and $\mathrm{II}_3$ with obvious notation. First of all, $0\le \mathrm{II}_1\le e^{f'(0)(t_0+n)}|\xi(-n)-Ke^{-f'(0)n}|\to 0$ as $n\to+\infty$ since $\xi(t)\sim Ke^{f'(0)t}$ as $t\to-\infty$. Let us now deal with term $\mathrm{II}_3$. With the successive change of variables $y=x_0+2\sqrt{t_0+n}\,s$ and $s=\tau+\lambda_c\sqrt{t_0+n}$, we get that

$$
\begin{aligned}
0\le \mathrm{II}_3&=\frac{A_c}{\sqrt{\pi}}e^{-\lambda_c^2n+f'(0)t_0+\lambda_c(x_0+h)}\int_{\frac{y_n-x_0}{2\sqrt{t_0+n}}}^{\frac{z_n-x_0}{2\sqrt{t_0+n}}}e^{-s^2+2\lambda_c\sqrt{t_0+n}\,s}\,ds\\
&=\frac{A_c}{\sqrt{\pi}}e^{f'(0)t_0+\lambda_c(x_0+h)+\lambda_c^2t_0}\int_{a_n}^{b_n}e^{-\tau^2}\,d\tau
\end{aligned}
$$

where $a_n=\frac{y_n-x_0}{2\sqrt{t_0+n}}-\lambda_c\sqrt{t_0+n}$ and $b_n=\frac{z_n-x_0}{2\sqrt{t_0+n}}-\lambda_c\sqrt{t_0+n}$. Since $y_n\sim-\lambda_{c'}n$ and $z_n\sim\lambda_c n$, we deduce that $a_n,b_n\to-\infty$ and that $\mathrm{II}_3\to 0$ as $n\to+\infty$. Similarly, $\mathrm{II}_2\to 0$ as $n\to+\infty$.

Eventually, $|\tilde{u}(x_0,t_0)-u_n(x_0,t_0)|\to 0$ as $n\to+\infty$. Since $u_n(x_0,t_0)\to u(x_0,t_0)$ and $(x_0,t_0)\in\mathbb{R}^2$ is arbitrary, we get that $\tilde{u}\equiv u$. The limit function being unique, the whole sequence $(u_k)$ converges to $u$.

PROOF OF (2.13): Notice first that by (1.2), for any $c>c^*$, the function $\tilde{\phi}_c=\phi_c(\cdot-\frac{\ln A_c}{\lambda_c})$ is the only solution of $\tilde{\phi_c}''-c\tilde{\phi_c}'+f(\tilde{\phi_c})=0$, $\tilde{\phi_c}(-\infty)=0$, $\tilde{\phi_c}(+\infty)=1$ fulfilling $\tilde{\phi_c}(\xi)\sim e^{\lambda_c\xi}$ as $\xi\to-\infty$.

Fix a real $c_0>c^*$. In order to prove that the $A_c$ are continuous in $c$ at $c_0$, it is enough to prove that the $\tilde{\phi}_c(0)$ are continuous in $c$ at $c_0$. Indeed, suppose that $\tilde{\phi}_{c_n}(0)\to\tilde{\phi}_{c_0}(0)$ for a sequence $c_n\to c_0$. Assume that $A_{c_n}\not\to A_{c_0}$ and that, without loss of generality, there exist a real $\varepsilon>0$ and a subsequence $c_{n'}\to c_0$ such that $A_{c_{n'}}\le A_{c_0}-\varepsilon$. Then

$$\tilde{\phi}_{c_{n'}}(0)=\phi_{c_{n'}}\left(-\frac{\ln A_{c_{n'}}}{\lambda_{c_{n'}}}\right)\ge\phi_{c_{n'}}\left(-\frac{\ln(A_{c_0}-\varepsilon)}{\lambda_{c_{n'}}}\right)$$

since the $\phi_c$ are increasing. On the other hand, we have

$$\tilde{\phi}_{c_n}(0)\to\tilde{\phi}_{c_0}(0)=\phi_{c_0}\left(-\frac{\ln A_{c_0}}{\lambda_{c_0}}\right)$$

and

$$\phi_{c_{n'}}\left(-\frac{\ln(A_{c_0}-\varepsilon)}{\lambda_{c_{n'}}}\right)\to\phi_{c_0}\left(-\frac{\ln(A_{c_0}-\varepsilon)}{\lambda_{c_0}}\right)$$

since we have proved that the functions $\phi_c(\cdot)$ are continuous in $C^2_{\mathrm{loc}}$ with respect to $c$. We deduce that

$$\phi_{c_0}\left(-\frac{\ln A_{c_0}}{\lambda_{c_0}}\right)\ge\phi_{c_0}\left(-\frac{\ln(A_{c_0}-\varepsilon)}{\lambda_{c_0}}\right).$$

This is impossible because $\phi_{c_0}$ is increasing.

Let $\Gamma(x)$ be a given smooth function such that $\Gamma(x)=1$ if $x\le 0$ and $\Gamma(x)=0$ if $x\ge 1$. Let us define $w_c(x)=\tilde{\phi_c}(x)-e^{\lambda_c x}\Gamma(x)$ and prove that $w_c(0)$ is continuous in $c$ at the point $c_0$. The functions $w_c$ satisfy

$$\begin{aligned}F(c,w_c):=\;&w_c''-cw_c'+f\big(e^{\lambda_c x}\Gamma(x)+w_c(x)\big)-f'(0)e^{\lambda_c x}\Gamma(x)\\&+\big(\Gamma''(x)+(2\lambda_c-c)\Gamma'(x)\big)e^{\lambda_c x}\\=\;&0\end{aligned}$$

Let $UC$ be the set of uniformly continuous and bounded functions on $\mathbb{R}$. Let $r>0$ and $X=\{w\in UC:(1+e^{-(r+\lambda_{c_0})x})w\in UC\}$ embedded with the norm $\|w\|=\|(1+e^{-(r+\lambda_{c_0})x})w\|_\infty$. Let $L$ be the operator defined by

$$Lv=v''-c_0v'+f'(\tilde{\phi}_{c_0})v$$

on its domain $D(L)=\{v\in X\cap\bigcap_{p\ge1}W^{2,p}_{\mathrm{loc}}(\mathbb{R}):v''\in X\}$, embedded with the norm $\|w\|_{D(L)}=\|w\|+\|Lw\|$. From proposition 5.5 in the paper by Mallordy and Roquejoffre [19], it is the case that $L$ is an isomorphism for $r=r_0$ small enough.

On the other hand, it is straightforward to check that the function $F(c,w)$ is of class $C^1$ on $(c_0-\delta, c_0+\delta)\times D(L)$ for some $\delta > 0$ small enough and that $\partial_w F(c_0, w_{c_0}) = L$. The implicit function theorem implies that $w_c$ is in $D(L)$ and is continuous with respect to $c$ (and even of class $C^1$) in the space $D(L)$ in a neighborhood of the point $c_0$. In particular, $w_c(0)$ is continuous in $c$. This gives the desired result.

We are grateful to J.-M. Roquejoffre for the proof of (2.13). □

### Monotonicity with Respect to $h$, $h'$, and $K$

Since the functions $\phi_c$ and $\phi_{c'}$ are increasing, it follows that the functions $u_{n,0}$ are nondecreasing in $h$ (respectively, $h'$)—the other parameters being fixed. Hence, the functions $u_n(x,t)$, and then the functions $u_{c,c',h,h',K}(x,t)$, are nondecreasing in $h$ (respectively, $h'$). They are even increasing in $h$ (respectively, $h'$) from the strong maximum principle. Similarly, the functions $u_{c,c',h,h',K}(x,t)$ are increasing in $K$.

Notice that we cannot hope for any monotonicity in $c$ or $c'$ because the traveling wave $\phi_c$ is neither decreasing or increasing in $c$: This can be seen from the asymptotic behavior of the $\phi_c$ at $\pm\infty$.

Let us now prove the convergence of $u_{c,c',h,h',K}$ to the function 1 as $h \to +\infty$, the parameters $c$, $c'$, $h'$, and $K$ being fixed. Since $\phi_c(+\infty) = \phi_{c'}(+\infty) = 1$, it is clear from the lower bound in (1.3) that

$$\inf_{(x,t)\in\mathbb{R}\times[A,+\infty[} u(x,t) \to 1 \quad \text{as } h\to+\infty \text{ for any } A\in\mathbb{R}\,.$$

Furthermore, from the estimates (2.10) for the derivatives, this convergence also takes place in the sense of $\mathcal{T}$. Similarly, the functions $u_{c,c',h,h',K}(\cdot,\cdot)$ approach 1 as $h' \to +\infty$.

Finally, let $c$, $c'$, $h$, and $h'$ be fixed and let a sequence $K_n \to +\infty$. Set $u_{K_n} = u_{c,c',h,h',K_n}$. For any $n$ and for any $(x,t)\in\mathbb{R}^2$, we know that $u_{K_n}(x,t) \geq \xi_{K_n}(t)$, where $0 < \xi_{K_n}(t) < 1$ is a solution of $\xi'_{K_n} = f(\xi_{K_n})$ in $\mathbb{R}$. Furthermore, there exists a real $\varepsilon_0 \in (0,1)$, which depends only on the function $f$ such that for any $\varepsilon\in[0,\varepsilon_0]$ and any $t\in\mathbb{R}$,

$$\xi_{K_n}(t) \geq \min(\varepsilon, K_n e^{f'(0)t}(1-\nu(\varepsilon)\varepsilon)) \quad \text{where } 1-\nu(\varepsilon)\varepsilon > 0\,.$$

On the other hand, since $\xi_{K_n}(t)$, $\xi'_{K_n}(t)$, and $\xi''_{K_n}(t)$ are uniformly bounded in $t$ and $K_n$, up to extraction of some subsequence, we can assume that $\xi_{K_n}(t)$ converges in $C^1_{\text{loc}}(\mathbb{R})$ to a function $0\leq\xi(t)\leq 1$ solution of $\xi' = f(\xi)$ in $\mathbb{R}$. By taking the limit in the above lower bound for $\xi_{K_n}(t)$ applied to $\varepsilon = \varepsilon_0$, it follows that $\xi(t)\geq\varepsilon_0 > 0$ for any $t\in\mathbb{R}$. Since $0\leq\xi(t)\leq 1$ for any $t$ and $f>0$ in $(0,1)$, the function $\xi$ is nondecreasing in $\mathbb{R}$ and then converges to a limit as $t\to-\infty$. This limit is a zero of $f$ and is bigger than $\varepsilon_0 > 0$. Hence, $\xi(-\infty) = 1$ and then $\xi(t)\equiv 1$ in $\mathbb{R}$. Eventually, this implies that $u_{K_n}(x,t)$ approaches the constant 1 locally in $t$—and then uniformly in any interval $[A,+\infty[$ by monotonicity—and uniformly in $x$. Because this limit is independent of $K_n$, the functions $u_{c,c',h,h',K}(x,t)$ converge to 1, uniformly in $t \in [A, +\infty[$ and uniformly in $x$, as $K \to +\infty$. As in the case mentioned above, this convergence is also true in the sense of $\mathcal{T}$.

## 3 Construction of the 4-Dimensional Manifold $\mathbb{M}_v$: Proof of Theorem 1.3

In this section, our aim is to construct a 4-dimensional manifold of entire solutions of (1.1) and to prove that this new manifold of solutions is on the boundary of the 5-dimensional manifold $\mathbb{M}_u$ given in Section 2. Roughly speaking, this will be done by considering the limit $K \to 0$ for the solutions $u_{c,c',h,h',K}$ given in Theorem 1.1.

The construction of the functions $v_{c,c',h,h'}$ defined in Theorem 1.3 proceeds almost exactly the same way as that of the functions $u_{c,c',h,h',K}$ in Theorem 1.1. For any $n \in \mathbb{N}$, let $v_n(x,t) = v_{n;c,c',h,h'}(x,t)$ be the solution of the Cauchy problem

$$\begin{cases} (v_n)_t = (v_n)_{xx} + f(v_n), \quad x \in \mathbb{R},\ t > -n, \\ v_n(x,-n) = v_{n,0}(x) := \max\left(\phi_{c'}(-x - c'n + h'), \phi_c(x - cn + h)\right). \end{cases}$$

We observe that there exists a real $x_n$ such that $v_{n,0}(x) = \phi_{c'}(-x - c'n + h')$, $v'_{n,0}(x) < 0$ if $x < x_n$, $v_{n,0}(x_n) = \phi_{c'}(-x_n - c'n + h') = \phi_c(x_n - cn + h)$, and $v_{n,0}(x) = \phi_c(x - cn + h)$, $v'_{n,0}(x) > 0$ if $x > x_n$. Furthermore, $v_{n,0}(x_n) \to 0$ as $n \to +\infty$. Since $v_{n,0}(x_n) = \phi_{c'}(-x_n - c'n + h') = \phi_c(x_n - cn + h)$, it is easy to check that

$$x_n = (\lambda_c - \lambda_{c'})n + B + o(1) \quad \text{as } n \to +\infty,$$
$$B = \frac{\ln A_{c'} - \ln A_c + \lambda_{c'}h' - \lambda_c h}{\lambda_c + \lambda_{c'}}.$$

The lower bounds (2.2) work for $v_n$ with $\xi_n \equiv 0$. Furthermore, since $y_n < x_n < z_n$ for $n$ large enough, where $y_n$ and $z_n$ satisfy (2.5), we deduce from the proof of Lemma 2.1 that the upper estimates (2.6) and (2.7) work for the functions $v_n$ with $K = 0$.

Up to extraction of some subsequence, the functions $v_n(x,t)$ converge in the sense of $\mathcal{T}$ to a function $v(x,t)$, satisfying (1.4) and increasing in $t$. Assertions (iii) and (v) in Theorem 1.1 work for $v_{c,c',h,h'}(x,t)$ exactly the same way as for $u_{c,c',h,h',K}(x,t)$. The monotonicity in $h$ and $h'$ as well as the convergence to 1 as $h \to +\infty$ (respectively, $h' \to +\infty$) are also true.

The only change deals with the minimum point $x(t)$ of the function $v(\cdot,t)$ and with the value of $v(x(t),t)$. If $c = c'$, the minimum point $x(t)$ of $v(\cdot,t)$ is still constant and equal to $x_0 = \frac{h'-h}{2}$. From the upper bound in (1.4), we get that

$$\phi_c\left(ct + \frac{h+h'}{2}\right) \le v(x_0,t) \le \phi_c\left(ct + \frac{h+h'}{2}\right) + A_c e^{\lambda_c(ct + \frac{h+h'}{2})}.$$

By (1.2) and since $\lambda_c c = \lambda_c^2 + f'(0)$, it follows that

$$A_c e^{\lambda_c \frac{h+h'}{2}} \leq \liminf_{t\to-\infty} (e^{-(\lambda_c^2+f'(0))t} \min v(\cdot,t))$$
$$\leq \limsup_{t\to-\infty} (e^{-(\lambda_c^2+f'(0))t} \min v(\cdot,t)) \leq 2A_c e^{\lambda_c \frac{h+h'}{2}}.$$

This gives the required result in assertion (iv) of Theorem 1.3.

If $c \neq c'$, thanks to (1.4), it is easy to see that $x(t) \sim (\lambda_{c'} - \lambda_c)t$ as $t \to -\infty$. In particular, if $c > c'$, then $\lambda_c < \lambda_{c'}$, whence $x(t) \to -\infty$ as $t \to -\infty$, and the converse is true if $c < c'$. Furthermore, both $(\lambda_{c'} - \lambda_c)t + ct$ and $-(\lambda_{c'} - \lambda_c)t + c't$ approach $-\infty$ as $t \to -\infty$. By using the lower and upper bounds in (1.4) and the asymptotic behavior of $\phi_c$ and $\phi_{c'}$ in (1.2), a straightforward calculation yields that, for any $\varepsilon > 0$,

$$\begin{cases} v((\lambda_{c'} - \lambda_c)t + B, t) < v((\lambda_{c'} - \lambda_c)t + B - \frac{\ln 2}{\lambda_{c'}} - \varepsilon, t) \\ v((\lambda_{c'} - \lambda_c)t + B, t) < v((\lambda_{c'} - \lambda_c)t + B + \frac{\ln 2}{\lambda_c} + \varepsilon, t) \end{cases}$$

for $-t$ large enough. Hence,

$$B - \frac{\ln 2}{\lambda_{c'}} \leq \liminf_{t\to-\infty} (x(t) - (\lambda_{c'} - \lambda_c)t) \leq \limsup_{t\to-\infty} (x(t) - (\lambda_{c'} - \lambda_c)t) \leq B + \frac{\ln 2}{\lambda_c}.$$

By using again (1.4) and the above inequalities, it then follows that

$$C \leq \liminf_{t\to-\infty} (e^{-(\lambda_c\lambda_{c'}+f'(0))t} v(x(t),t)) \leq \limsup_{t\to-\infty} (e^{-(\lambda_c\lambda_{c'}+f'(0))t} v(x(t),t)) \leq 3C$$

where $C = A_c^{\frac{\lambda_{c'}}{\lambda_c+\lambda_{c'}}} A_{c'}^{\frac{\lambda_c}{\lambda_c+\lambda_{c'}}} e^{\frac{\lambda_c\lambda_{c'}}{\lambda_c+\lambda_{c'}}(h+h')}$. This gives the desired result.

**Continuity in $(c, c', h, h')$**

As we did for the functions $u$, let a sequence $(c_k, c'_k, h_k, h'_k)$ converge to $(c, c', h, h') \in (c^*, +\infty)^2 \times \mathbb{R}^2$ and set $v_k = v_{c_k, c'_k, h_k, h'_k}$ and $v = v_{c,c',h,h'}$. Up to extraction of some subsequence, the functions $v_k$ approach a solution $\tilde{v}$ of (1.1) that satisfies (1.4). Hence, we can easily compare the functions $\tilde{v}(x, -n)$ to the functions $v_{n,0}(x)$ on both sides of the point $x_n$. Arguing as in the proof of the continuity of the functions $u$ in Section 2, we deduce that $|\tilde{v}(x_0, t_0) - v_n(x_0, t_0)| \to 0$ as $n \to +\infty$. Therefore, $\tilde{v} \equiv v$.

**Limit $K \to 0^+$ in $u_{c,c',h,h',K}$**

Fix a quadruple $(c, c', h, h') \in (c^*, +\infty)^2 \times \mathbb{R}^2$. For any $K > 0$ and any $n \in \mathbb{N}$, we see that $u_{n,0}(x) \geq, \not\equiv v_{n,0}(x)$ in $\mathbb{R}$ for $n$ large enough. The limit $n \to +\infty$ yields that $u(x,t) \geq v(x,t)$ in $\mathbb{R}^2$. On the other hand, $\min u(\cdot,t)$ and $\min v(\cdot,t)$ have two different asymptotic behaviors as $t \to -\infty$ by assertions (iv) in Theorems 1.1 and 1.3. The strong maximum principle then implies that $u > v$ in $\mathbb{R}^2$.

Furthermore, we have $0 \leq u_{n,0}(x) - v_{n,0}(x) \leq Ke^{-f'(0)n}$ in $\mathbb{R}$. Hence, by comparing the function $u_n(x,t) - v_n(x,t)$ to the solution of the linear heat equation $w_t = w_{xx} + f'(0)w$ with initial condition $Ke^{-f'(0)n}$ at time $-n$, we get that $0 \leq$

$u_n(x,t) - v_n(x,t) \le Ke^{f'(0)t}$ for any $(x,t) \in \mathbb{R}^2$ and for any $n \ge |t|$. The limit $n \to +\infty$ gives that $0 \le u(x,t) - v(x,t) \le Ke^{f'(0)t}$. Therefore, $u(x,t) \to v(x,t)$ as $K \to 0^+$ uniformly in $\mathbb{R}\times]-\infty, A]$ for any $A \in \mathbb{R}$. Lastly, for any sequence $K_n \to 0^+$, the functions $u_{c,c',h,h',K_n}$ satisfy the a priori estimates (2.10) (which do not depend on $K_n$). Up to extraction of some subsequence, they converge in the sense of $\mathcal{T}$ to a solution of (1.1), which turns out to be $v$. The limit does not depend on the sequence $K_n$, whence all the functions $u_{c,c',h'h,K}$ converge in $\mathcal{T}$ to the function $v_{c,c',h,h'}$ as $K \to 0^+$.

**Limits $\boldsymbol{h, h'} \to -\infty$**

Let us finally prove that, say, $v_{c,c',h,h'}(x,t) \to \phi_{c'}(-x + c't + h')$ as $h \to -\infty$. Let $h_n \to -\infty$. Since the estimates (2.10) do not depend on $h$, the functions $v_{c,c',h_n,h'}(x,t)$ converge, up to extraction of some subsequence, in the sense of $\mathcal{T}$, to a function $\psi(x,t)$, solution of $\psi_t = \psi_{xx} + f(\psi)$. By (1.4), it also follows that, for any $(x,t) \in \mathbb{R}^2$,

$$\phi_{c'}(-x + c't + h') \le \psi(x,t) \le \min(1, \phi_{c'}(c't - x + h')).$$

Eventually, $\psi(x,t) = \phi_{c'}(-x + c't + h')$ for any $(x,t) \in \mathbb{R}^2$. As usual, we can also add that the convergence of $v_{c,c',h,h'}(x,t)$ to $\phi_{c'}(-x + c't + h')$ is true as $h \to -\infty$ (and not only for some sequence). Furthermore, from (1.4), this convergence occurs uniformly in $(x,t) \in ]-\infty, A]^2$ for any real $A$.

## 4 Two 3-Dimensional Manifolds of Solutions Monotone in $x$: Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. It deals with the construction of two 3-dimensional manifolds of solutions of (1.1) that are monotone in $x$, unlike the solutions $u$ and $v$ given in Theorems 1.1 and 1.3. These new solutions can also be viewed as boundary points of the 5-dimensional manifold $\mathbb{M}_u$.

The proof of Theorem 1.4 is very similar to those of Theorems 1.1 and 1.3. We only outline it. Consider the functions $w_n^-(x,t)$ solutions of the Cauchy problems

$$\begin{cases} (w_n^-)_t = (w_n^-)_{xx} + f(w_n^-) \\ w_n^-(x,-n) = w_{n,0}^-(x) := \max\left(\phi_{c'}(-x - c'n + h'), Ke^{-f'(0)n}\right). \end{cases}$$

These functions converge as $n \to +\infty$ to a function $w^-(x,t) = w^-_{c',h',K}(x,t)$ fulfilling all the requirements of Theorem 1.4.

The only fact that we point out is the convergence of $u_{c,c',h,h',K}$ to $w^-_{c',h',K}$ as $h \to -\infty$ for any fixed $c > c^*$. Indeed, consider a sequence $h_k \to -\infty$. Up to extraction of some subsequence, by (2.10) and (1.3), the functions $u_k = u_{c,c',h_k,h',K}$ converge in the sense of $\mathcal{T}$ to a function $\tilde{w}^-$ solution of (1.1) and fulfilling (1.5). By estimating the difference between $\tilde{w}^-(-n,\cdot)$ and $w^-_{n,0}(\cdot)$, we get that $\limsup_{n\to+\infty} |\tilde{w}^-(x,t) - w_n^-(x,t)| = 0$ (with the same arguments as in the proof of the continuity of the functions $u_{c,c',h,h',K}$ in Theorem 1.1). This implies that $\tilde{w}^- \equiv w^-$.

Lastly, the convergences of $w^-_{c',h',K}$ to $\xi(t)$ as $h' \to -\infty$ and to $\phi_{c'}(-x+c't+h')$ as $K \to 0^+$ come directly from (1.5).

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