Non-adiabatic KPP fronts with an arbitrary Lewis number

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Abstract

We establish existence of travelling fronts in thermo-diffusive systems of the KPP type in a shear flow with an arbitrary Lewis number and a positive heat-loss at the boundary. We then prove that the leftover concentration behind the front is positive, and that it is small when the heat-loss is small. On the other hand, when the heat-loss approaches a critical value, the temperature becomes uniformly small. Finally, in the limit of large flow amplitudes A, we show that, depending on the flow structure, the infimum of the front speeds both may become 0 or grow linearly in A for $A \gg 1$.

1 Introduction and main results

Existence and qualitative properties of thermo-diffusive fronts in flows have been a subject of active research in the last few years: see [3, 24] for extensive overviews. The majority of the results have been obtained for a single reaction-diffusion equation of the form

$$T_t + u \cdot \nabla T = \Delta T + f(T)(1-T).$$

Here $u(x)$ is a prescribed flow that is usually taken to be either spatially periodic or unidirectional (shear). In the latter case the above equation takes the form

$$T_t + u(y)T_x = \Delta T + f(T)(1-T). \tag{1.1}$$

This problem is posed in a cylinder $D = \mathbb{R}_x \times \Omega_y \subset \mathbb{R}^d$ with a regular bounded domain Ω, and with the Neumann boundary conditions along $\partial D = \mathbb{R}_x \times \partial\Omega$:

$$\frac{\partial T}{\partial n} = 0 \text{ on } \partial D. \tag{1.2}$$

The function $u(y)$ is assumed to have mean zero:

$$\int_\Omega u(y)dy = 0, \tag{1.3}$$

a non-zero mean can be taken into account by a simple change of variables. Non-planar travelling fronts are solutions of (1.1)-(1.2) of the form $T(t,x,y) = U(x-ct,y)$ with the function U that satisfies the front-like conditions at infinity:

$$U(-\infty,y) = 1, \quad U(+\infty,y) = 0, \tag{1.4}$$

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with the limits approached uniformly in $y \in \overline{\Omega}$. The scalar equation (1.1) arises as a reduction of the thermo-diffusive system

$$\begin{aligned} \frac{\partial T}{\partial t} + u(y)T_x &= \Delta T + f(T)Y \\ \frac{\partial Y}{\partial t} + u(y)Y_x &= \frac{1}{\mathrm{Le}}\Delta Y - f(T)Y \end{aligned} \tag{1.5}$$

with the Neumann boundary conditions both for the normalized temperature T and concentration Y

$$\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 \text{ on } \partial D. \tag{1.6}$$

Reduction to (1.1) is possible if the Lewis number $\mathrm{Le} = 1$ as then one may consider special solutions with the constraint $T + Y = 1$.

Existence of travelling front solutions to (1.1) has been first established in [7, 9] and later in [23] for monotonic systems. These results have been extended in [4, 8] and very recently in [14] to the case of the system (1.5)-(1.6) with either the Lewis number close to one, or a small flow $u(y)$ by means of the inverse function theorem. We also mention that interesting numerical studies of the effect of the boundary heat loss have been done in [12, 13]. The effect of the volumetric heat-loss has been studied in [15, 21].

When the boundary is not insulated, the Neumann boundary conditions should be replaced by the heat-loss conditions for the temperature:

$$\frac{\partial T}{\partial n} + qT = 0, \quad \frac{\partial Y}{\partial n} = 0, \quad y \in \partial\Omega. \tag{1.7}$$

Here $q > 0$ is the heat-loss parameter. A travelling front solution of (1.5) with the non-adiabatic boundary conditions (1.7) may not satisfy the front-like conditions as in (1.4) as the temperature behind the front vanishes because of the heat loss and shortage of available fuel. Therefore, we say that (c, T, Y) is a travelling front solution of (1.5)-(1.7) if in the moving frame $x' = x - ct$ (we drop the prime immediately) the functions T and Y satisfy

$$\left\{ \begin{array}{rcl} \Delta T + (c - u(y))T_x + f(T)Y &=& 0 \\ \mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - f(T)Y &=& 0 \end{array} \right. \quad \text{in } \mathbb{R} \times \Omega \tag{1.8}$$

together with the modified conditions at infinity

$$\left\{ \begin{array}{l} T(+\infty, \cdot) = 0, \quad Y(+\infty) = 1, \\ T_x(-\infty, \cdot) = Y_x(-\infty, \cdot) = 0 \end{array} \right. \tag{1.9}$$

and the boundary conditions (1.7). The limits in (1.9) are understood to be uniform with respect to $y \in \overline{\Omega}$. Furthermore, throughout the paper, the relative concentration Y is assumed to range in $[0, 1]$ and is not identically equal to 1. The temperature T is nonnegative and not identically equal to 0. Note that when $q \neq 0$ the thermo-diffusive system (1.8) does not reduce to a single equation even when $\mathrm{Le} = 1$ because the constraint $T + Y = 1$ is incompatible with the boundary conditions (1.7). Nevertheless, even with the heat-loss the case $\mathrm{Le} = 1$ is simpler than the case of a general Lewis number. The basic difference between the scalar equation and a system, the lack of a priori bounds on temperature, is still absent when $\mathrm{Le} = 1$. This observation has been used in [5] to construct travelling fronts solutions of (1.5) with the heat-loss boundary conditions (1.7) and with the nonlinearity $f(T)$ of the KPP type. The latter means that

$$f \in C^1([0, +\infty)),\ f(0) = 0 < f(s) \leq f'(0)s \text{ for all } s > 0,\ f' > 0 \text{ and } f(+\infty) = +\infty. \tag{1.10}$$

Let $\mu_q(\lambda)$ be the principal eigenvalue of the following elliptic problem depending on a parameter $\lambda > 0$:

$$\left\{ \begin{array}{rcll} -\Delta_y \phi_\lambda - \lambda u(y)\phi_\lambda & = & \mu_q(\lambda)\phi_\lambda & \text{in } \Omega, \\ \dfrac{\partial \phi_\lambda}{\partial n} + q\phi_\lambda & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.11}$$

That is, $\mu_q(\lambda)$ is the unique eigenvalue of (1.11) that corresponds to a positive eigenfunction $\phi(y)$.

The following theorem has been shown to hold in [5].

Theorem 1.1 [5] *(a) Assume that Le $= 1$ and $\mu_q(0) < f'(0)$. There exists $c_q^* \in \mathbb{R}$ so that for any $c > \max(0, c_q^*)$ there exists a travelling front solution of (1.8) that satisfies the boundary conditions (1.7) with the functions T and Y that are globally bounded in $L^\infty(D)$.*
(b) If Le > 0 and if there exists a solution (c, T, Y) of (1.8-1.9) and (1.7) such that $T \in L^\infty(D)$, then $\mu_q(0) < f'(0)$, $c \geq c_q^*$, $c > 0$, $T(-\infty, \cdot) = 0$, $0 < Y < 1$ and $Y(-\infty, \cdot) = Y_\infty \in [0, 1)$.

The number c_q^* is defined as follows. The function $\mu_q(\lambda)$ is concave (see [5]) in λ. When $\mu_q(0) < f'(0)$, we define

$$c_q^* = \min\{c,\ \exists\lambda > 0,\ \mu_q(\lambda) = f'(0) - c\lambda + \lambda^2\}. \tag{1.12}$$

Note that c_q^* does not depend on Le, and that c_q^* may be negative (see [5] for an example). The observation that the travelling front speed does not depend on the Lewis number for KPP reactions has been first made in [10] in the one-dimensional case without any heat-loss. It follows from the fact that the fronts are pulled by the decaying temperature profile ahead of them. In this region, the temperature equation, which does not involve the Lewis number, plays a preponderant role in the selection of speeds. This observation does not generalize to other reaction types, such as ignition or Arrhenius, for which the fronts are pushed by the whole reaction zone. We should mention that only the case $f(T) = T$ was treated in [5] but the results there extend immediately to the case where f is of the KPP type (1.10). Some of the results of [5] have been extended in [16] to non-shear periodic flows.

Part (b) of Theorem 1.1 is conditional: existence of globally bounded travelling fronts in the case of the Lewis number Le > 0 different from 1 has been left open and one of the purposes of the present paper is to close this gap. First, we show that bounded fronts, indeed, exist for all Lewis numbers.

Theorem 1.2 *Assume $\mu_q(0) < f'(0)$, with $q > 0$. For any $c > \max(c_q^*, 0)$, there exists a solution (T, Y) of (1.8-1.9) and (1.7) such that $T \in L^\infty(D)$, $T(-\infty, \cdot) = 0$, $0 < Y < 1$ and $Y(-\infty, \cdot) = Y_\infty \in (0, 1)$.*

The leftover concentration Y_∞ behind the front is not prescribed in this result – we do not know at the moment whether Y_∞ is unique for a given speed. However, we do prove that this concentration is positive. Hence, combustion is not complete and unburned fuel remains behind the front, because of the heat losses.

Next, we show that any travelling front solution of (1.7)-(1.9) such that $0 < T$ and $0 < Y < 1$ has to be uniformly bounded, with the temperature that goes to zero behind the front.

Theorem 1.3 *Let (c, T, Y) be a solution of (1.8-1.9) and (1.7), with $q > 0$, such that $0 < T$ and $0 < Y < 1$. Then $c \geq \max(0, c_q^*)$, T is bounded, $T(-\infty, \cdot) = 0$ and $Y(-\infty, \cdot) = Y_\infty \in (0, 1)$.*

We also study the dependence of the front speed on the heat-loss parameter q. Let μ^D be the first eigenvalue of the Dirichlet Laplacian in Ω. One has $\mu_q(0) \to \mu^D$ as $q \to +\infty$. There exists $q^* \in (0, +\infty]$ such that $\mu_q(0) < f'(0)$ (with $q \in [0, +\infty)$) if and only if $0 \leq q < q^*$, and q^* is finite if and only if $\mu^D > f'(0)$. We show that temperature goes to zero uniformly as the heat-loss approaches this critical value q^*. At the other extreme, we also prove that when the heat-loss is small, the leftover concentration behind the front is small.

Theorem 1.4 *(a) The function $q \mapsto c_q^*$ is a continuous decreasing function of $q \in [0, q^*)$ with $c_0^* > 0$. Furthermore, if $\mu^D > f'(0)$, that is, $q^* < +\infty$, then for any sequence $q_k \uparrow q^*$ and any sequence of solutions (c_k, T_k, Y_k) of (1.8-1.9) and (1.7) such that $0 < T_k$, $0 < Y_k < 1$ and $\sup_{k\in\mathbb{N}} c_k < +\infty$, one has $T_k \to 0$ uniformly in $\mathbb{R} \times \overline{\Omega}$ as $k \to +\infty$.*
(b) For all $\varepsilon > 0$, there exists $q_0 > 0$ so that for any $q \in (0, q_0)$ and for any solution (c, T, Y) of (1.8-1.9) and (1.7) such that $0 < T$ and $0 < Y < 1$, one has $Y_\infty < \varepsilon$.

It follows from Theorems 1.3 and 1.4 that the speeds of all travelling fronts solving (1.7)-(1.9) are bounded from below by a positive constant, provided that the heat loss $q > 0$ is small enough.

Finally, we look at the case when the shear flow $u(y)$ becomes fast – to highlight this problem we replace $u(y)$ by $Au(y)$ in (1.8) with $A \gg 1$ and ask how fast the fronts would propagate in such flow. We denote

$$c_q^*(A) = \min\{c,\ \exists\lambda > 0,\ \mu_q(A\lambda) = f'(0) - c\lambda + \lambda^2\}, \tag{1.13}$$

which is well-defined for all A, when $\mu_q(0) < f'(0)$. The speed up of the travelling fronts for the scalar equation (1.1) has been studied extensively: see [1, 2, 6, 11, 18, 19, 20, 22, 25] for various estimates and numerical studies. In particular, it has been shown in [3], for the unit Lewis number, that the limit of the ratio $c_0^*(A)/A$ as $A \to +\infty$ exists and is positive. We have the following generalization of this result.

Theorem 1.5 *Assume $\mu_q(0) < f'(0)$ and let ϕ_0 denote the principal eigenfunction of (1.11) with $\lambda = 0$, such that $\|\phi_0\|_{L^2(\Omega)} = 1$.*
(a) The function $A \mapsto c_q^*(A)$ is a continuous function of $A \geq 0$, and $c_q^*(0) = 2\sqrt{f'(0) - \mu_q(0)}$. If

$$\int_\Omega u(y)\phi_0(y)^2 dy\ (= -\mu_q'(0))\ \geq 0,$$

then $c_q^*(A)$ is nondecreasing with A. If

$$\int_\Omega u(y)\phi_0(y)^2 dy > 0,$$

then $c_q^*(A)$ is increasing with A and $c_q^*(A) \to +\infty$ as $A \to +\infty$.
(b) If $\max_{\lambda\geq 0} \mu_q(\lambda) < f'(0)$ and $u(y) \not\equiv 0$, then $c_q^*(A) \to +\infty$ as $A \to +\infty$. If $\max_{\lambda\geq 0} \mu_q(\lambda) = f'(0)$, then $c_q^*(A) \to 0$ as $A \to +\infty$. If $\max_{\lambda\geq 0} \mu_q(\lambda) > f'(0)$, then $c_q^*(A) \to -\infty$ as $A \to +\infty$.
(c) The ratio $c_q^*(A)/A$ is a decreasing function of $A > 0$, and

$$\lim_{A\to+\infty} \frac{c_q^*(A)}{A} \geq \int_\Omega u(y)\phi_0(y)^2 dy \quad \text{as } A \to +\infty.$$

Remark 1.1 After this paper has been submitted an anonymous referee has brought our attention to a recent paper by S. Heinze [17] which contains a variational characterization of the ratio $c^*(A)/A$, as A is large, for the adiabatic case ($q = 0$ – that is, no heat-loss). Together with (1.13) and

(3.10) below, Heinze's technique may be used to obtain the following improvement of part (c) of Theorem 1.5. Let D be the following set of functions:

$$D = \left\{ \psi \in H^1(\Omega) : \quad \int_\Omega |\nabla \psi|^2 dy + q \int_{\partial\Omega} |\psi|^2 dS_y \le f'(0), \quad \|\psi\|_{L^2(\Omega)} = 1 \right\}. \tag{1.14}$$

The set D contains ϕ_0, because one assumes $\mu_q(0) < f'(0)$, and the limit $\gamma = \lim_{A \to +\infty} \dfrac{c_q^*(A)}{A}$ satisfies the following variational characterization:

$$\gamma = \sup_{\psi \in D} \int_\Omega u(y) \psi^2(y) dy. \tag{1.15}$$

This paper is organized as follows. Theorem 1.2 is proved in Section 2. The proof of Theorem 1.3 is contained in Section 3.1, while Theorems 1.4 and 1.5 are proved in Sections 3.2 and 3.3, respectively. Section 4 contains some conclusions and a discussion of unresolved issues.

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2 Existence of travelling waves

Proof of Theorem 1.2. Let $c > \max(c_q^*, 0)$ be given. It has been shown in [5] that the system (1.8) with the boundary conditions (1.7) possesses travelling front solutions but with a possibly unbounded temperature T. The main novelty here is in the uniform bounds on T. We first recall the result of [5]. Let $\lambda > 0$ be the smallest positive solution of

$$\mu_q(\lambda) = f'(0) - c\lambda + \lambda^2 \tag{2.1}$$

and ϕ_λ be the corresponding principal eigenfunction of (1.11), normalized, say, to have $L^2(\Omega)$ norm equal to one. Furthermore, given $\delta \in (0, \lambda)$ we let ψ be the normalized first (positive) eigenfunction of the problem

$$\left\{ \begin{array}{rcll} -\mathrm{Le}^{-1} \Delta_y \psi - \delta u(y) \psi & = & \mu^N(\delta) \psi & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial n} & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{2.2}$$

corresponding to the principal eigenvalue $\mu^N(\delta)$.

Proposition 2.1 *[5] There exists a pair of solutions $(T, Y) \in L^\infty_{loc}(\overline{D})$ to (1.8) that satisfies the boundary conditions (1.7). Moreover, this solution obeys the following bounds. Let λ be defined by (2.1). There exists $\delta \in (0, \lambda)$ and $\gamma > 0$ sufficiently large so that*

$$\forall\ (x, y) \in \mathbb{R} \times \overline{\Omega}, \ \left\{ \begin{array}{l} 0 < T(x, y) < \phi_\lambda(y) e^{-\lambda x}, \\ \max\left(0, 1 - \gamma \psi(y) e^{-\delta x}\right) < Y(x, y) < 1, \end{array} \right. \tag{2.3}$$

Furthermore, T satisfies

$$\forall\ (x, y) \in \mathbb{R} \times \overline{\Omega}, \quad T(x, y) > \max\left(0, \phi_\lambda(y) e^{-\lambda x} - \kappa \phi_{\lambda+\eta}(y) e^{-(\lambda+\eta)x}\right)$$

for some small $\eta > 0$ and large $\kappa > 0$.

Our goal is to prove that T is then bounded, and even decays to 0 as $x \to -\infty$. We will argue by contradiction, assuming that T is unbounded. Let us first prove that "unbounded temperature leads to complete burning".

Lemma 2.2 *If T is not bounded, then $Y(-\infty, \cdot) = 0$.*

Proof. Let us assume that T is not bounded. then the upper bound on T in (2.3) implies that there exists a sequence of points $(x_k, y_k) \in \mathbb{R} \times \overline{\Omega}$ such that $x_k \to -\infty$ and $T(x_k, y_k) \to +\infty$ as $k \to +\infty$. On the other hand, since Y is bounded and T is positive, the Harnack inequality implies that $|\nabla T|/T$ is bounded in $\mathbb{R} \times \Omega$. Therefore, for each $R > 0$,

$$\min_{(x,y)\in[x_k-R,x_k+R]\times\overline{\Omega}} T(x,y) \quad \to +\infty \ \text{ as } k \to +\infty.$$

Set $m = f(1) > 0$ and let the function $\rho \mapsto \mu^N(\rho)$ be defined as in (2.2). The variational principle

$$\mu^N(\rho) = \min_{w\in H^1(\Omega),\ w\neq 0} \frac{\displaystyle\int_\Omega \mathrm{Le}^{-1}|\nabla w|^2 - \rho u(y) w^2}{\displaystyle\int_\Omega w^2}$$

implies that the function $\rho \mapsto \mu^N(\rho)$ is concave as a minimum of affine functions. Observe also that $\mu^N(0) = 0$. Therefore, there exist exactly two real numbers $\rho_\pm$ such that $\rho_- < 0 < \rho_+$ and

$$\mu^N(-\rho_\pm) = \mathrm{Le}^{-1}\rho_\pm^2 + c\rho_\pm - m.$$

We denote by $\psi_\pm$ the two principal eigenfunctions of the problems (2.2) with the values $\delta = -\rho_\pm$, normalized so that, say, $\min_{\overline{\Omega}} \psi_\pm = 1$. The functions $u_\pm(x,y) = e^{\rho_\pm x}\psi_\pm(y)$ then satisfy

$$\left\{ \begin{array}{rcll} \mathrm{Le}^{-1}\Delta u_\pm + (c - u(y))\partial_x u_\pm - m u_\pm & = & 0 & \text{in } \mathbb{R}\times\Omega, \\ \dfrac{\partial u_\pm}{\partial n} & = & 0 & \text{on } \mathbb{R}\times\partial\Omega. \end{array} \right.$$

Fix now any $R > 0$ and choose $N \in \mathbb{N}$ so that

$$\min_{(x,y)\in[x_k-R,x_k+R]\times\overline{\Omega}} T(x,y) \geq 1$$

for all $k \geq N$. Then, as the function $f(T)$ is increasing, we have $f(T) \geq f(1) = m$ in $[x_k - R, x_k + R] \times \overline{\Omega}$ for all $k \geq N$. Hence, the function Y satisfies

$$\mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - mY \geq 0$$

in $[x_k - R, x_k + R] \times \overline{\Omega}$ for all $k \geq N$. It also obeys the Neumann boundary conditions on $\mathbb{R} \times \partial\Omega$. Furthermore, $Y \leq 1$ in $\mathbb{R} \times \Omega$. It then follows from the maximum principle that

$$\forall (x,y) \in [x_k - R, x_k + R] \times \overline{\Omega}, \quad Y(x,y) \leq e^{\rho_+(x-x_k-R)}\psi_+(y) + e^{\rho_-(x-x_k+R)}\psi_-(y).$$

It follows that along the segment $x = x_k$ the function Y is small:

$$\limsup_{k\to+\infty} \max_{\overline{\Omega}} Y(x_k, \cdot) \leq \max\left(\max_{\overline{\Omega}} \psi_+, \max_{\overline{\Omega}} \psi_-\right) \times \left(e^{-\rho_+ R} + e^{\rho_- R}\right).$$

Now, since $Y \geq 0$, and $R > 0$ can be chosen arbitrary, one concludes that $Y(x_k, \cdot) \to 0$ uniformly in $\overline{\Omega}$ as $k \to +\infty$.

Let now $\varepsilon > 0$ be any positive real number, and let $N \in \mathbb{N}$ be such that $Y(x_k, y) \leq \varepsilon$ for all $k \geq N$ and $y \in \overline{\Omega}$. Since the function Y actually satisfies

$$\mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x = f(T)Y \geq 0,$$

it then follows from the maximum principle that

$$Y(x, y) \leq \varepsilon$$

for all $(x, y) \in [x_k, x_N] \times \overline{\Omega}$ and $k \geq N$ such that $x_k \leq x_N$. Since $x_k \to -\infty$ as $k \to +\infty$, one concludes that $Y \leq \varepsilon$ in $(-\infty, x_N] \times \overline{\Omega}$. The conclusion of Lemma 2.2 now follows, as $Y \geq 0$. $\square$

Lemma 2.3 *The function T is bounded.*

Proof. Assume that T is not bounded. Lemma 2.2 implies that $Y(x, y)$ goes to zero as $x \to -\infty$, uniformly in $y \in \overline{\Omega}$. Hence, there exists $A \geq 0$ such that

$$\forall x \leq -A,\ \forall y \in \overline{\Omega}, \quad f'(0)\ Y(x, y) \leq \alpha := \min\left(\frac{\mu_q(0)}{2}, \frac{f'(0)}{2}\right).$$

Note that $\alpha > 0$ since both $\mu_q(0)$ (as $q > 0$) and $f'(0)$ are positive. As a consequence of the continuity of the function μ_q and (2.1) there exists $\Lambda > \lambda$ such that

$$-\mu_q(\Lambda) - c\Lambda + \Lambda^2 < -\frac{f'(0)}{2}.$$

We denote by U the positive function defined by

$$T(x, y) = U(x, y)e^{-\Lambda x}\phi_\Lambda(y),$$

where ϕ_Λ solves (1.11) with the parameter Λ and is normalized so that $\|\phi_\Lambda\|_{L^2(\Omega)} = 1$. Because of the upper bound (2.3) for T, one has $U(-\infty, \cdot) = 0$ as $\lambda < \Lambda$. The function $U(x, y)$ satisfies the homogeneous Neumann boundary condition $\partial_n U = 0$ on $\mathbb{R} \times \partial\Omega$. Furthermore, it is straightforward to check that

$$\Delta U + (c - u(y) - 2\Lambda)U_x + 2\frac{\nabla_y \phi_\Lambda}{\phi_\Lambda} \cdot \nabla_y U + (\Lambda^2 - \mu_q(\Lambda) - c\Lambda + g(x, y))U = 0 \text{ in } \mathbb{R} \times \Omega,$$

where

$$0 \leq g(x, y) = \frac{f(T(x, y))}{T(x, y)} Y(x, y) \leq f'(0)\ Y(x, y) \leq \alpha \text{ in } (-\infty, -A] \times \Omega.$$

Therefore, we have

$$\Delta U + (c - u(y) - 2\Lambda)U_x + 2\frac{\nabla_y \phi_\Lambda}{\phi_\Lambda} \cdot \nabla_y U + (\Lambda^2 - \mu_q(\Lambda) - c\Lambda + \alpha)U \geq 0 \text{ in } (-\infty, -A] \times \Omega.$$

Furthermore, $\Lambda^2 - \mu_q(\Lambda) - c\Lambda + \alpha < 0$ due to the choices of Λ and α. We shall now apply the maximum principle to the previous operator, and look for a suitable super-solution. Since $\alpha \leq \mu_q(0)/2$ and $\mu_q(0) > 0$, there exists $\beta > 0$ small such that

$$\beta^2 + c\beta - \mu_q(-\beta) + \alpha < 0.$$

One can then check that the function

$$\overline{U}(x,y) = e^{(\Lambda+\beta)x} \times \frac{\phi_{-\beta}(y)}{\phi_\Lambda(y)}$$

satisfies

$$\begin{aligned}
&\Delta\overline{U} + (c - u(y) - 2\Lambda)\overline{U}_x + 2\frac{\nabla_y \phi_\Lambda}{\phi_\Lambda} \cdot \nabla_y \overline{U} + (\Lambda^2 - \mu_q(\Lambda) - c\Lambda + \alpha)\overline{U} \\
&= (\beta^2 + c\beta - \mu_q(-\beta) + \alpha)\overline{U} \le 0 \quad \text{in } \mathbb{R}\times\Omega.
\end{aligned}$$

Furthermore, the function $\overline{U}$ obeys the Neumann boundary conditions $\partial_n \overline{U} = 0$ on $\mathbb{R}\times\partial\Omega$. It follows from the maximum principle that the difference $\bar{U} - U$ can not attain an interior negative minimum. Moreover, $\overline{U} \ge 0$ and one can normalize the function $\phi_{-\beta}$ in such a way that $U(-A,y) \le \overline{U}(-A,y)$ for all $y \in \overline{\Omega}$. Finally, both U and $\overline{U}$ tend to zero as $x \to -\infty$. We conclude that

$$\forall x \le -A,\ \forall y \in \overline{\Omega}, \quad U(x,y) \le \overline{U}(x,y).$$

In other words,

$$\forall x \le -A,\ \forall y \in \overline{\Omega}, \quad T(x,y) \le e^{\beta x}\phi_{-\beta}(y).$$

It follows that $T(-\infty,\cdot) = 0$, in addition to $T(+\infty,\cdot) = 0$ that follows directly from (2.3). That contradicts the unboundedness of T and the proof of Lemma 2.3 is complete. □

To complete the proof of Theorem 1.2, one only has to prove the following

Lemma 2.4 *The pair (T,Y) satisfies (1.9) and $T(-\infty,\cdot) = 0$, $Y(-\infty,\cdot) = Y_\infty \in [0,1)$.*

Proof. The limits of the functions T and Y as $x \to +\infty$ follow immediately from (2.3). So one only has to study the limits of T and Y as $x \to -\infty$. Moreover, since both T and Y are globally bounded, standard elliptic estimates imply that ∇T and ∇Y are globally bounded as well. We show now that the integral

$$\int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dx\ dy \tag{2.4}$$

is finite. To this end we integrate equation (1.8) satisfied by Y over the domain $(-N,N)\times\Omega$ with $N > 0$ and obtain

$$\begin{aligned}
&\int_\Omega [\mathrm{Le}^{-1}(Y_x(N,y) - Y_x(-N,y)) + (c-u(y))(Y(N,y) - Y(-N,y))]dy \\
&= \int_{(-N,N)\times\Omega} f(T(x,y))Y(x,y)dx\ dy.
\end{aligned} \tag{2.5}$$

But the left-hand side is bounded independently of N and the function $f(T)Y$ is positive in $\mathbb{R}\times\Omega$. Therefore, the integral (2.4) indeed converges.

Next, we show that the integral

$$\int_{\mathbb{R}\times\Omega} |\nabla Y|^2 dx\ dy < +\infty \tag{2.6}$$

is finite. To see this we multiply equation (1.8) satisfied by Y by Y itself, and integrate over $(-N,N)\times\Omega$ for $N > 0$:

$$\begin{aligned}
&\int_\Omega [\mathrm{Le}^{-1}(Y_x(N,y)Y(N,y) - Y_x(-N,y)Y(N,y)) + \frac{1}{2}(c-u(y))(Y^2(N,y) - Y^2(-N,y))]dy \\
&= \int_{(-N,N)\times\Omega} [f(T(x,y))Y^2(x,y) + \mathrm{Le}^{-1}|\nabla Y|^2]dx\ dy.
\end{aligned}$$

Since the left-hand side is again bounded independently of N, and since

$$0 \le \int_{(-N,N)\times\Omega} f(T(x,y))Y^2(x,y)dx\ dy \le \int_{(-N,N)\times\Omega} f(T(x,y))Y(x,y)dx\ dy$$
$$\le \int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dx\ dy < \infty,$$

one concludes that the integral (2.6) converges as well.

Choose now any sequence $(x_k)_{k\in\mathbb{N}} \to -\infty$ and define the translates

$$Y_k(x,y) = Y(x_k + x, y).$$

The functions (Y_k) are bounded in the local Sobolev norms $W^{2,p}_{loc}(\overline{D})$ for all $p < \infty$. Therefore, up to the extraction of a subsequence, the functions Y_k converge in $C^1_{loc}(\mathbb{R}\times\overline{\Omega})$ to a function $Y_\infty(x,y)$. It follows from (2.6) that Y_∞ is constant. We now show that this constant does not depend on the choice of the subsequence. Recall that $Y_x(+\infty,\cdot) = 0$ because Y converges to a constant as $x \to +\infty$ and because of standard elliptic estimates, as above. We set $N = -x_k$ in (2.5) and pass to the limit $N = -x_k \to +\infty$. This leads to

$$\int_\Omega (c - u(y))(1 - Y_\infty)dy = \int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dx\ dy.$$

Since u has zero average over Ω, one concludes that

$$c|\Omega|(1 - Y_\infty) = \int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dx\ dy.$$

Here $|\Omega|$ denotes the Lebesgue measure of the cross-section Ω. As a consequence, the limit value Y_∞ does not depend on the sequence (x_k). Thus, the limit $Y(-\infty,\cdot) = Y_\infty \in [0,1]$ exists, and $Y_x(-\infty,\cdot) = 0$. Note that the integral

$$\int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dx\ dy$$

is not zero, whence $Y_\infty < 1$.

Let us now prove that $T(-\infty,\cdot) = T_x(-\infty,\cdot) = 0$. Integrate the equation (1.8) satisfied by T over $(-N,N)\times\Omega$ for $N > 0$. One gets that

$$\int_\Omega [(T_x(N,y) - T_x(-N,y)) + (c - u(y))(T(N,y) - T(-N,y))]dy$$
$$+ \int_{(-N,N)\times\Omega} f(T(x,y))Y(x,y)dx\ dy = \int_{(-N,N)\times\partial\Omega} qT.$$

But the left-hand side is bounded independently of N and the function T is positive. Therefore, the integral

$$\int_{\mathbb{R}\times\partial\Omega} qT$$

converges. Next, we multiply by T the equation (1.8) satisfied by T over $(-N,N)\times\Omega$ and integrate by parts. One gets that the integral

$$\int_{\mathbb{R}\times\Omega} |\nabla T|^2 dx\ dy$$

converges as well. As in the above argument for the function Y, it follows that any sequence of translates $T_k(x,y) = T(x_k + x, y)$ (with $x_k \to -\infty$ as $k \to +\infty$) will converge, up to extraction of a subsequence, in $C^1_{loc}(\mathbb{R} \times \overline{\Omega})$ to a constant T_∞. The heat-loss boundary conditions (1.7) imply that $qT_\infty = 0$, whence $T_\infty = 0$. In other words, the limit value does not depend on the choice of the sequence (x_k), and then $T(-\infty, \cdot) = T_x(-\infty, \cdot) = 0$. That completes the proof of Lemma 2.4. □

The proof of Theorem 1.2 is also complete, apart from the proof of the positivity of Y_∞. This will be done in the next section, actually in the more general framework of Theorem 1.3. □

3 Qualitative properties of the fronts

3.1 Boundedness of T for any solution of (1.7-1.9)

This section is devoted to the

Proof of Theorem 1.3. Let (c, T, Y) be a locally bounded smooth solution of (1.8-1.9) with the boundary conditions (1.7) such that $0 < T$ and $0 < Y < 1$ – we do not assume here that T satisfies the exponential bounds as in Proposition 2.1. In particular, the decay rate of T as $x \to +\infty$ is not known a priori. It has been already proved in [5] that then

$$c \geq c_q^*. \tag{3.1}$$

This fact does not rely on the global boundedness of T, but simply on the study of the limiting equation as $x \to +\infty$. Here we will prove that actually T has to be globally bounded.

Let us first prove that Y converges to a constant as $x \to -\infty$. Note that, as T is bounded for $x > 0$ and Y converges to 1 as $x \to +\infty$, it follows from the standard elliptic estimates that $Y_x(+\infty, \cdot) = 0$. Moreover, the boundary conditions (1.7) imply that $Y_x(-\infty, \cdot) = 0$. Then, as in the proof of Lemma 2.4, we conclude that the integral

$$\int_{\mathbb{R} \times \Omega} f(T(x,y))Y(x,y)dxdy \tag{3.2}$$

converges. Furthermore, the integral

$$\int_{\mathbb{R} \times \Omega} |\nabla Y(x,y)|^2 dxdy$$

converges as well. Still, in order to prove the convergence of Y to a constant as $x \to -\infty$, one cannot follow the arguments in the proof of Lemma 2.4, because the function T is not known to be a priori bounded. Therefore, the functions $Y(x_k + \cdot, \cdot)$ do not a priori satisfy the uniform $W^{2,p}_{loc}$ estimates as $x_k \to -\infty$. We shall adapt the proof to overcome this lack of a priori bounds. Fix $a \in \mathbb{R}$ and let $(x_k)_{k \in \mathbb{N}}$ be any sequence converging to $-\infty$ as $k \to +\infty$. We introduce a modified translate $Y_k(x,y) = Y(x_k + a + x, y)$. Observe that

$$\int_{(a+x_k, a+1+x_k) \times \Omega} |\nabla Y|^2 dxdy = \int_{(0,1) \times \Omega} |\nabla Y_k|^2 dxdy \to 0 \quad \text{as } k \to +\infty.$$

Hence, up to extraction of a subsequence, the functions Y_k converge in $H^1((0,1) \times \Omega)$ to a constant $Y^a_\infty \in [0,1]$. Next, we use (2.5) with $N = -x_k - a - \xi$ for $\xi \in (0,1)$ (k is chosen large enough so that $-x_k - a - 1 > 0$) and integrate over $\xi \in (0,1)$. We get

$$
\begin{aligned}
&\int_{(0,1)\times\Omega} \mathrm{Le}^{-1}(Y_x(-x_k - a - \xi, y) - Y_x(x_k + a + \xi, y))d\xi dy \\
&+\int_{(0,1)\times\Omega} (c - u(y))(Y(-x_k - a - \xi, y) - Y(x_k + a + \xi, y))dy \\
&= \int_0^1 \left(\int_{(x_k+a+\xi, -x_k-a-\xi)\times\Omega} f(T(x,y))Y(x,y)dx\ dy \right) d\xi.
\end{aligned}
$$

The first term of the left side converges to 0 as $k \to +\infty$ because $Y_x(\pm\infty, \cdot) = 0$ uniformly in $y \in \overline{\Omega}$. The second term of the left side converges to

$$
\int_\Omega (c - u(y))(1 - Y_\infty^a)dy = (1 - Y_\infty^a)c|\Omega|
$$

because $Y(+\infty, \cdot) = 1$ and $Y_k \to Y_\infty^a$ in $H^1((0,1)\times\Omega)$ as $k \to +\infty$. We used here the fact that $u(y)$ has mean zero (1.3). Lastly, the right-hand side converges to

$$
\int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dxdy
$$

by the dominated convergence theorem: this integral is finite – see (3.2). Therefore,

$$
(1 - Y_\infty^a)c|\Omega| = \int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dxdy. \tag{3.3}
$$

But the right side of (3.3) is positive because $f(T)Y$ is continuous and positive in $\mathbb{R}\times\Omega$. As a consequence, $c > 0$ and $Y_\infty^a < 1$ does not depend on a, nor on the sequence (x_k). We conclude that there exists a constant $Y_\infty \in [0,1)$ such that

$$
Y(\zeta_k + \cdot, \cdot) \to Y_\infty \ \text{ in } H^1_{loc}(\mathbb{R}\times\Omega) \ \text{ as } k \to +\infty,
$$

for any sequence $(\zeta_k)_{k\in\mathbb{N}} \to -\infty$.

Let us now prove that T is in $L^\infty(\mathbb{R}\times\Omega)$. Assume that T is unbounded. Since the right limit $T(+\infty, \cdot) = 0$ exists, there has to exist a sequence $(x_k, y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$ such that $x_k \to -\infty$ and

$$
T(x_k, y_k) \to +\infty \tag{3.4}
$$

as $k \to +\infty$. As already explained in the proof of Lemma 2.2, one deduces that

$$
\min_{(x,y)\in[x_k-1,x_k+1]\times\overline{\Omega}} T(x,y) \to +\infty
$$

as $k \to +\infty$. Hence, we also have

$$
\min_{(x,y)\in[x_k-1,x_k+1]\times\overline{\Omega}} f(T(x,y)) \to +\infty
$$

as $k \to +\infty$, because $f(+\infty) = +\infty$. But

$$
\begin{aligned}
&\int_{\mathbb{R}\times\Omega} f(T(x,y))Y(x,y)dxdy \ge \int_{(x_k-1,x_k+1)\times\Omega} f(T(x,y))Y(x,y)dxdy \\
&\ge \min_{(x,y)\in[x_k-1,x_k+1]\times\overline{\Omega}} f(T(x,y)) \ \times \ \int_{(x_k-1,x_k+1)\times\Omega} Y(x,y)dxdy,
\end{aligned}
$$

and

$$\int_{(x_k-1,x_k+1)\times\Omega} Y(x,y)dxdy \to 2|\Omega|Y_\infty$$

as $k \to +\infty$. We conclude that $Y_\infty = 0$ if T is unbounded as $x \to -\infty$. On the other hand, since $|\nabla T|/T$ is bounded, the functions

$$T_k(x,y) = \frac{T(x_k+x,y)}{T(x_k,y_k)}$$

are then locally bounded according to the Harnack inequality. They satisfy

$$\begin{cases} \Delta T_k + (c-u(y))\partial_x T_k + g_k(x,y)T_k(x,y) &= 0 \quad \text{in } \mathbb{R}\times\Omega \\ \dfrac{\partial T_k}{\partial n} + qT_k &= 0 \quad \text{on } \mathbb{R}\times\partial\Omega, \end{cases}$$

where

$$0 \le g_k(x,y) = \frac{f(T(x_k+x,y))}{T(x_k+x,y)} Y(x_k+x,y) \le f'(0)Y(x_k+x,y) \le f'(0).$$

Moreover, $g_k \to 0$ in $L^2_{loc}(\mathbb{R}\times\overline{\Omega})$ since $Y(x_k+x,y) \to 0$ in $L^2_{loc}(\mathbb{R}\times\overline{\Omega})$. Since the functions g_k are uniformly bounded in $L^\infty(\mathbb{R}\times\Omega)$, the functions T_k are then bounded in $W^{2,p}_{loc}(\mathbb{R}\times\Omega)$ for all $1 \le p < +\infty$ and they converge, up to extraction of some subsequence, weakly in $W^{2,p}_{loc}(\mathbb{R}\times\Omega)$ for all $1 \le p < +\infty$ – and then in $C^{1,\beta}_{loc}(\mathbb{R}\times\overline{\Omega})$ for all $0 \le \beta < 1$, to a solution T_∞ of

$$\begin{cases} \Delta T_\infty + (c-u(y))\partial_x T_\infty &= 0 \quad \text{in } \mathbb{R}\times\Omega \\ \dfrac{\partial T_\infty}{\partial n} + qT_\infty &= 0 \quad \text{on } \mathbb{R}\times\partial\Omega. \end{cases}$$

The elliptic regularity theory implies that the function T_∞ is actually of the class $C^{2,\alpha_0}_{loc}(\mathbb{R}\times\overline{\Omega})$. Observe that it follows from the boundary condition $T_x \to 0$ as $x \to -\infty$ in (1.7) and (3.4) that $\partial_x T_k(x,y) = T_x(x_k+x,y)/T(x_k,y_k) \to 0$ locally uniformly as $k \to +\infty$, whence $T_\infty = T_\infty(y)$. The function T_∞ is a solution of

$$\begin{cases} \Delta_y T_\infty &= 0 \quad \text{in } \Omega \\ \dfrac{\partial T_\infty}{\partial n} + qT_\infty &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Furthermore, $T_k(0,y_k) = 1$ and one can assume, up to extraction of another subsequence, that the sequence $y_k \to y_\infty \in \overline{\Omega}$ as $k \to +\infty$ – hence $T_\infty(y_\infty) = 1$. Since T_∞ is nonnegative (because the T_k are positive), the strong maximum principle and the Hopf lemma imply that T_∞ is positive in $\overline{\Omega}$. However, integrating (3.5) over Ω leads to

$$\int_{\partial\Omega} qT_\infty(y)d\sigma(y) = 0.$$

This is a contradiction. As a consequence, T belongs to $L^\infty(\mathbb{R}\times\Omega)$. It then follows from the results of [5] that $T(-\infty,\cdot) = 0$, and $Y(\zeta_k+\cdot,\cdot) \to Y_\infty$ in $C^1_{loc}(\mathbb{R}\times\overline{\Omega})$ for any sequence $\zeta_k \to -\infty$.

To complete the proof of Theorem 1.3, it only remains to prove that $Y_\infty > 0$ – one already knows that $Y_\infty \in [0,1)$. Let us argue by contraction and assume that $Y_\infty = 0$. First, since $c \ge c^*_q$ and the function $s \mapsto \mu_q(s)$ is continuous with $\mu_q(0) < f'(0)$, there exists then $\lambda > 0$ such that $\mu_q(\lambda) = f'(0) - c\lambda + \lambda^2$. Since T is bounded in $D = \mathbb{R}\times\Omega$, there exists then a constant $C_0 > 0$

such that $T(x,y) \le C_0 e^{-\lambda x}$ for all $x \le 0$ and $y \in \overline{\Omega}$. Therefore, the arguments used in the proof of Lemma 2.3 (using the fact that Y_∞ is zero) imply then that

$$T(x,y) \le \gamma e^{\beta x} \ \text{ for all } x \le 0 \text{ and } y \in \overline{\Omega}, \tag{3.6}$$

for some positive constants γ and β.

We now claim that

$$\Lambda := \limsup_{x\to-\infty,\ y\in\overline{\Omega}} \frac{Y_x(x,y)}{Y(x,y)} = 0. \tag{3.7}$$

We already know that $Y > 0$ in $\mathbb{R}\times\overline{\Omega}$ from the strong maximum principle and the Hopf lemma, and that $|\nabla Y|/Y$ is globally bounded from the Harnack inequality and the fact that $f(T)$ is bounded. Therefore, Λ is finite. Furthermore, $\Lambda \ge 0$ because $Y > 0 = Y(-\infty,\cdot)$. Let now (x_k, y_k) be a sequence of points in $\mathbb{R}\times\overline{\Omega}$ such that $x_k \to -\infty$ and

$$\frac{Y_x(x_k,y_k)}{Y(x_k,y_k)} \to \Lambda \ \text{ as } k \to +\infty.$$

Up to extraction of some subsequence, one can assume that $y_k \to y_\infty \in \overline{\Omega}$ as $k \to +\infty$. Consider now the functions

$$Y_k(x,y) = \frac{Y(x+x_k,y)}{Y(x_k,y_k)}.$$

They are locally bounded in $\mathbb{R}\times\overline{\Omega}$ and satisfy

$$\mathrm{Le}^{-1}\Delta Y_k + (c-u(y))\partial_x Y_k = f(T(x+x_k,y))Y_k$$

in $\mathbb{R}\times\Omega$ and the Neumann boundary condition on $\mathbb{R}\times\partial\omega$. Moreover, $f(T(x+x_k,y)) \to 0$ locally uniformly in $\mathbb{R}\times\overline{\Omega}$ as $k\to+\infty$ because $T(-\infty,\cdot)=0$ and $f(0)=0$. From standard elliptic estimates, up to extraction of some subsequence, the functions Y_k converge in $W^{2,p}_{loc}(\mathbb{R}\times\overline{\Omega})$ weak for all $1 \le p < +\infty$, to a solution $Z(x,y)$ of

$$\mathrm{Le}^{-1}\Delta Z + (c-u(y))Z_x = 0 \ \text{ in } \mathbb{R}\times\Omega$$

satisfying the Neumann boundary condition on $\mathbb{R}\times\partial\Omega$. Furthermore, $Z(0,y_\infty)=1$, $Z \ge 0$ – and thus $Z > 0$ in $\mathbb{R}\times\overline{\Omega}$ from the strong maximum principle and the Hopf lemma, $Z_x/Z \le \Lambda$ in $\mathbb{R}\times\overline{\Omega}$ and $Z_x(0,y_\infty)/Z(0,y_\infty) = \Lambda$. However, the function $W(x,y) = Z_x(x,y)/Z(x,y)$ satisfies the equation

$$\mathrm{Le}^{-1}\Delta W + 2\mathrm{Le}^{-1}\frac{\nabla Z}{Z}\cdot\nabla W + (c-u(y))W_x = 0 \ \text{ in } \mathbb{R}\times\Omega$$

with the Neumann boundary condition on $\mathbb{R}\times\Omega$. Therefore, $W(x,y)=\Lambda$ for all $(x,y)\in\mathbb{R}\times\overline{\Omega}$ from the strong maximum principle and the Hopf lemma. In other words, $Z(x,y) = e^{\Lambda x}\phi(y)$, for some positive function ϕ in $\overline{\Omega}$ satisfying

$$\begin{cases} \mathrm{Le}^{-1}\Delta\phi + \mathrm{Le}^{-1}\Lambda^2\phi + \Lambda(c-u(y))\phi &= 0 \text{ in } \Omega \\ \dfrac{\partial\phi}{\partial n} &= 0 \text{ on } \partial\Omega. \end{cases}$$

As a consequence,

$$\mathrm{Le}^{-1}\Lambda^2 + c\Lambda = \mu^N(-\Lambda),$$

where $\mu^N(-\Lambda)$ is the first eigenvalue of problem (2.2) with $\delta = -\Lambda$. But the function μ^N is concave and $\mu^N(0) = 0$. Furthermore, as, for instance, it has been computed in [5], we have

$$(\mu^N)'(0) = -\int_\Omega u(y)\psi_0^2(y)dy,$$

where ψ_0 is the principal eigenfunction of (2.2) with $\delta = 0$ and with the unit L^2 norm. Thus ψ_0 is constant and $(\mu^N)'(0) = 0$ because u has zero average. It follows that the function μ^N is nonpositive everywhere, whence

$$\mathrm{Le}^{-1}\Lambda^2 + c\Lambda = \mu^N(-\Lambda) \le 0.$$

But $\Lambda \ge 0$ and $c > 0$. As a conclusion, $\Lambda = 0$ and (3.7) is proved. Note also that $Z(x,y)$ is then equal to $\psi(y)$, where ψ is the first eigenfunction of (2.2) with $\delta = 0$ and $\psi(y_\infty) = 1$, namely $\psi \equiv 1$ in $\overline{\Omega}$. Thus, $Z \equiv 1$ in $\mathbb{R} \times \overline{\Omega}$.

Fix now $\varepsilon > 0$ such that $\varepsilon < \beta$, where $\beta > 0$ is as in (3.6). It follows from (3.7) that there exists then $A \ge 0$ such that $Y_x(x,y)/Y(x,y) \le \varepsilon$ for all $x \le -A$ and $y \in \overline{\Omega}$. It follows immediately that

$$\forall x \le -A,\ \forall y \in \overline{\Omega},\quad Y(x,y) \ge \nu e^{\varepsilon x}, \tag{3.8}$$

where $\nu = e^{-\varepsilon A} \times \min_{y\in\overline{\Omega}} Y(-A,y) > 0$.

As we have shown in in the proof of (3.7), there exists a sequence (x_k, y_k) such that $x_k \to -\infty$ and the functions $(x,y) \mapsto Y(x+x_k,y)/Y(x_k,y_k)$ converge to the constant 1 (at least) in $C^1_{loc}(\mathbb{R}\times\overline{\Omega})$ as $k \to +\infty$. Without loss of generality, one can assume that $x_k \le -A \le 0$ for all $k \in \mathbb{N}$. Now use the fact that $Y_\infty = Y(-\infty,\cdot) = Y_x(-\infty,\cdot) = 0$ and integrate the equation (1.8) satisfied by Y, over the domain $(-\infty, x_k) \times \Omega$. One gets

$$\mathrm{Le}^{-1}\int_\Omega Y_x(x_k,y)dy + c\int_\Omega Y(x_k,y)dy - \int_\Omega u(y)Y(x_k,y)dy \le f'(0)\gamma|\Omega|\beta^{-1}e^{\beta x_k} \tag{3.9}$$

because of (3.6) and since $f(T)Y \le f'(0)T$. Furthermore, as $Y(x+x_k,y)/Y(x_k,y_k) \to 1$ in $C^1_{loc}(\mathbb{R}\times\overline{\Omega})$, and since $u(y)$ is bounded in Ω and has zero average, it follows that

$$\frac{\displaystyle\int_\Omega Y_x(x_k,y)dy}{\displaystyle\int_\Omega Y(x_k,y)dy} \to 0$$

and

$$\frac{\displaystyle\int_\Omega u(y)Y(x_k,y)dy}{\displaystyle\int_\Omega Y(x_k,y)dy} \to |\Omega|^{-1}\int_\Omega u(y)dy = 0$$

as $k \to +\infty$. Putting that together with (3.9), one gets that

$$\frac{c}{2}\int_\Omega Y(x_k,y)dy \le f'(0)\gamma|\Omega|\beta^{-1}e^{\beta x_k}$$

for k large enough, because $c > 0$. But (3.8) – and the fact that $x_k \le -A$ – then yields

$$\frac{c\nu|\Omega|}{2}e^{\varepsilon x_k} \le f'(0)\gamma|\Omega|\beta^{-1}e^{\beta x_k}$$

for k large enough. One gets a contradiction by passing to the limit $x_k \to -\infty$, because $0 < \varepsilon < \beta$.

As a conclusion, $Y_\infty = 0$ is impossible. Therefore, $Y_\infty > 0$ and the proof of Theorem 1.3 is now complete. □

3.2 Dependence on the heat loss parameter $q > 0$

This section contains the proof of Theorem 1.4.

Proof of part (a) of Theorem 1.4. Let us begin with the *continuity and monotonicity of c_q^* as a function of $q \in [0, q^*)$.* Note that c_q^* is well-defined for all $0 \le q < q^*$, because $\mu_q(0) < f'(0)$ for such q's. We first claim that, for each fixed $\lambda \in \mathbb{R}$, the principal eigenvalue $\mu_q(\lambda)$ is a continuous increasing function of $q \in [0, +\infty)$. Indeed, $\mu_q(\lambda)$ is given by the variational formula

$$\mu_q(\lambda) = \inf_{\phi \in H^1(\Omega),\ \phi \neq 0} \frac{\displaystyle\int_\Omega |\nabla \phi(y)|^2 + q \int_{\partial\Omega} \phi(y)^2 dS_y - \lambda \int_\Omega u(y)\phi(y)^2 dy}{\displaystyle\int_\Omega \phi(y)^2 dy}. \tag{3.10}$$

Therefore, $\mu_q(\lambda)$ is nondecreasing and concave in q as an infimum of nondecreasing affine functions. In order to show strict monotonicity in q we take $0 \le q < q' < +\infty$, and denote by $\phi_{\lambda,q}$ and $\phi_{\lambda,q'}$ the principal eigenfunctions of (1.11) with heat-losses q and q' respectively. Next, we multiply the equation for $\phi_{\lambda,q}$ by $\phi_{\lambda,q'}$ and that for $\phi_{\lambda,q'}$ by $\phi_{\lambda,q}$, integrate over Ω and subtract. One gets that

$$(q - q') \int_{\partial\Omega} \phi_{\lambda,q}(y)\phi_{\lambda,q'}(y) dS_y = (\mu_q(\lambda) - \mu_{q'}(\lambda)) \int_\Omega \phi_{\lambda,q}(y)\phi_{\lambda,q'}(y) dy.$$

Since both $\phi_{\lambda,q}$ and $\phi_{\lambda,q'}$ are positive in $\overline{\Omega}$ by Hopf lemma, it follows then that $\mu_q(\lambda) < \mu_{q'}(\lambda)$. Lastly, standard elliptic estimates and the uniqueness of the principal eigenvalue and eigenfunction yield the continuity of $q \mapsto \mu_q(\lambda)$.

Let us now check that c_q^* is strictly decreasing in q for $q \in [0, q^*)$. By (1.12), there exists a unique $\lambda > 0$ such that $\mu_q(\lambda) = f'(0) - c_q^*\lambda + \lambda^2$. Therefore, $\mu_{q'}(\lambda) > f'(0) - c_q^*\lambda + \lambda^2$ and there exists $c' < c_q^*$ such that $\mu_{q'}(\lambda) = f'(0) - c'\lambda + \lambda^2$. It follows that $c_{q'}^* \le c' < c_q^*$.

Next, we prove the continuity of c_q^* as a function of $q \in [0, q^*)$. First, fix $q \in [0, q^*)$ and let $(q_k)_{k \in \mathbb{N}}$ be a decreasing sequence such that $q_0 < q^*$ and $q_k \to q$ as $k \to +\infty$. It follows from the monotonicity of c_q^* that

$$c_{q_k}^* \to c^* \le c_q^* \ \text{ as } k \to +\infty. \tag{3.11}$$

Furthermore, for each k, there exists a decay rate $\lambda_k > 0$ such that

$$\mu_{q_k}(\lambda_k) = f'(0) - c_{q_k}^*\lambda_k + \lambda_k^2.$$

Since, as we have shown above, $\mu_s(\lambda)$ is increasing in s, one gets that

$$\mu_q(\lambda_k) \le \mu_{q_k}(\lambda_k) = f'(0) - c_{q_k}^*\lambda_k + \lambda_k^2 \le \mu_{q_0}(\lambda_k) \le \mu_{q_0}(0) + \mu'_{q_0}(0)\lambda_k. \tag{3.12}$$

The last inequality follows from the concavity of the function $\lambda \mapsto \mu_h(\lambda)$ for each $h \ge 0$ – this follows immediately from (3.10), as $\mu_h(\lambda)$ is an infimum of affine functions in λ. We note that the derivative $\mu'_h(0)$ has been shown to exist and computed in [5]:

$$\mu'_h(0) = -\frac{\displaystyle\int_\Omega u(y)\phi_{0,h}(y)^2 dy}{\displaystyle\int_\Omega \phi_{0,h}(y)^2 dy}. \tag{3.13}$$

It follows from (3.12) that

$$f'(0) - c_{q_k}^*\lambda_k + \lambda_k^2 \le \mu_{q_0}(0) + \mu'_{q_0}(0)\lambda_k. \tag{3.14}$$

As the speeds $c^*_{q_k}$ are all uniformly bounded in k: $c^*_{q_0} \leq c^*_{q_k} \leq c^*_q$, we conclude that $\sup_{k\in\mathbb{N}} \lambda_k < +\infty$ since the left side of (3.14) is quadratic in λ_k and the right side is linear. Furthermore, since $\mu_{q_0}(0) < f'(0)$, and $c^*_{q_k}$ are uniformly bounded, (3.14) also implies that $\inf_{k\in\mathbb{N}} \lambda_k > 0$. Up to extraction of a subsequence, one can then assume that $\lambda_k \to \lambda > 0$ as $k \to +\infty$. For each $p \in \mathbb{N}$, one has

$$\mu_q(\lambda_k) \leq \mu_{q_k}(\lambda_k) = f'(0) - c^*_{q_k}\lambda_k + \lambda_k^2 \leq \mu_{q_p}(\lambda_k)$$

for all $k \geq p$. Passing to the limit $k \to +\infty$ leads to $\mu_q(\lambda) \leq f'(0) - c^*\lambda + \lambda^2 \leq \mu_{q_p}(\lambda)$ – see (3.11) – it follows that $\mu_q(\lambda) = f'(0) - c^*\lambda + \lambda^2$ by taking the limit as $p \to +\infty$. Therefore, the line $\mu = c^*\lambda$ intersects the graph of $f'(0) - \mu_q(\lambda) + \lambda^2$ and hence $c^*_q \leq c^*$ by definition of c^*_q. Taking into account (3.11) we conclude that $c^*_q = c^*$. Hence, c^*_q is a right-continuous function of q.

Next, fix $q \in (0, q^*)$ and let q_k be any increasing sequence of nonnegative numbers such that $q_k \to q$ as $k \to +\infty$. The monotonicity of c^*_q implies that the sequence $c^*_{q_k}$ is decreasing and

$$c^*_{q_k} \to \gamma^* \geq c^*_q \quad \text{as } k \to +\infty. \tag{3.15}$$

The definition (1.12) of c^*_q means that

$$\mu_{q_k}(\lambda) \leq f'(0) - c^*_{q_k}\lambda + \lambda^2$$

for all $\lambda > 0$. Passing to the limit $k \to +\infty$ leads to $\mu_q(\lambda) \leq f'(0) - \gamma^*\lambda + \lambda^2$ for all $\lambda > 0$. But since there exists $\lambda^* > 0$ such that $\mu_q(\lambda^*) = f'(0) - c^*_q\lambda^* + (\lambda^*)^2$, it follows that $\gamma^* \leq c^*_q$. Because of (3.15), one gets that $\gamma^* = c^*_q$ and hence c^*_q is a left-continuous function of q. As we have already shown that it is also right-continuous, we conclude that c^*_q is a continuous function of $q \in [0, q^*)$.

Positivity of c^*_0. The real number $\mu_0(\lambda)$, for each $\lambda \in \mathbb{R}$, is the principal eigenvalue of the operator $-\Delta - \lambda u(y)$ in Ω with the Neumann boundary condition. As in the case $q > 0$ discussed above, the function $\lambda \mapsto \mu_0(\lambda)$ is concave, and (3.13) implies that

$$\mu_0'(0) = -\frac{\displaystyle\int_\Omega u(y)\phi_{0,0}(y)^2 dy}{\displaystyle\int_\Omega \phi_{0,0}(y)^2 dy},$$

where $\phi_{0,0}$ is the principal eigenfunction for problem (1.11) with $\lambda = q = 0$, namely $\phi_{0,0}$ is a constant. Then (1.3) implies that $\mu_0'(0) = 0$, and since $\mu_0(0)$ is also equal to zero, it follows that $\mu_0(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Since $f'(0) > 0$, one concludes from formula (1.12) with $q = 0$, that $c^*_0 > 0$.

Extinction when $q \uparrow q^*$. We assume here that $\mu^D > f'(0)$. By the construction of q^*, this means that $0 < q^* < +\infty$, and $\mu_{q^*}(0) = f'(0)$. Let $q_k \uparrow q^*$ and let (c_k, T_k, Y_k) be any sequence of solutions of (1.7)-(1.9) such that $0 < T_k$, $0 < Y_k < 1$ and $\sup_{k\in\mathbb{N}} c_k < +\infty$. Let us prove that $\|T_k\|_{L^\infty(\mathbb{R}\times\Omega)} \to 0$ as $k \to +\infty$. Recall first that Theorem 1.3 implies that each function T_k is bounded and $T_k(\pm\infty, \cdot) = 0$. Furthermore, all $c_k > 0$, and $c_k \geq c^*_{q_k}$. Up to translation in the variable x_1, one can assume that

$$M_k := \max_{\mathbb{R}\times\overline{\Omega}} T_k = \max_{\overline{\Omega}} T_k(0, \cdot) = T_k(0, y_k)$$

for some $y_k \in \overline{\Omega}$.

Assume first that, up to extraction of a subsequence, $M_k \to +\infty$. Using the fact that the speeds c_k and the functions Y_k are uniformly bounded, one concludes from the standard elliptic regularity estimates that the functions T_k converge to $+\infty$ locally uniformly as $k \to +\infty$. The arguments used in Lemma 2.2 imply then that the functions Y_k converge to 0 locally uniformly in $\mathbb{R}\times\overline{\Omega}$ as $k\to+\infty$. We introduce the functions

$$\tilde{T}_k(x,y)=\frac{T_k(x,y)}{M_k}.$$

They satisfy $0<\tilde{T}_k\le 1$, $\tilde{T}_k(0,y_k)=1$. We also have

$$\left\{\begin{array}{rcl}\Delta\tilde{T}_k+(c_k-u(y))\tilde{T}_{k,x}+g_k(x,y)\tilde{T}_k & = & 0 \ \text{ in } \mathbb{R}\times\Omega\\ \dfrac{\partial\tilde{T}_k}{\partial n}+q_k\tilde{T}_k & = & 0 \ \text{ on } \mathbb{R}\times\partial\Omega,\end{array}\right.$$

where

$$0\le g_k(x,y)=\frac{f(T_k(x,y))}{T_k(x,y)}Y_k(x,y)\le f'(0)Y_k(x,y)\to 0 \ \text{ as } k\to+\infty$$

locally uniformly in (x,y). Up to extraction of a subsequence, one can then assume that $c_k\to c\ge 0$, $y_k\to y_\infty\in\overline{\Omega}$ and $\tilde{T}_k\to\tilde{T}$ in all $W^{2,p}_{loc}(\mathbb{R}\times\overline{\Omega})$ with $p<+\infty$. The function $\tilde{T}$ solves

$$\left\{\begin{array}{rcl}\Delta\tilde{T}+(c-u(y))\tilde{T}_x & = & 0 \ \text{ in } \mathbb{R}\times\Omega\\ \dfrac{\partial\tilde{T}}{\partial n}+q^*\tilde{T} & = & 0 \ \text{ on } \mathbb{R}\times\partial\Omega\end{array}\right.$$

and $0\le\tilde{T}\le 1$, $\tilde{T}(0,y_\infty)=1$. If $y_\infty\in\Omega$, the strong maximum principle implies that $\tilde{T}\equiv 1$ in $\mathbb{R}\times\overline{\Omega}$, which is impossible because of the boundary conditions on $\mathbb{R}\times\partial\Omega$. Therefore, $y_\infty\in\partial\Omega$ and $\tilde{T}<1$ in $\mathbb{R}\times\Omega$. The Hopf lemma implies then $\partial_n\tilde{T}(0,y_\infty)>0$, which is again impossible because of the boundary conditions on $\mathbb{R}\times\partial\Omega$ and because $\tilde{T}$ is positive.

As a consequence, the sequence M_k is bounded. Assume now that, up to extraction of some subsequence, $M_k\to M>0$ as $k\to+\infty$. Up to extraction of another subsequence, one can then assume that $c_k\to c\ge 0$, $y_k\to y_\infty\in\overline{\Omega}$ and that the functions T_k and Y_k converge in all $W^{2,p}_{loc}(\mathbb{R}\times\overline{\Omega})$ with $p<+\infty$ to some solutions (T,Y) of (1.7)-(1.8) with the speed c and heat loss parameter $q=q^*$. One has $0\le Y\le 1$, $0\le T\le M$ and $T(0,y_\infty)=M>0$. Using integrations by parts as in Lemma 2.4, it follows that all integrals

$$\int_{\mathbb{R}\times\Omega}f(T)Y,\quad \int_{\mathbb{R}\times\Omega}|\nabla Y|^2,\quad \int_{\mathbb{R}\times\partial\Omega}T,\quad \int_{\mathbb{R}\times\Omega}|\nabla T|^2$$

are finite, and that $T(\pm\infty,\cdot)=0$. On the other hand,

$$0=\Delta T+(c-u(y))T_x+f(T)Y\le\Delta T+(c-u(y))T_x+f'(0)T$$

and the function $\phi(x,y)=\phi_{0,q^*}(y)$ solves

$$\Delta\phi+(c-u(y))\phi_x+f'(0)\phi=\Delta_y\phi_{0,q^*}+\mu_{q^*}(0)\phi_{0,q^*}=0$$

because $\mu_{q^*}(0)=f'(0)$. The function $\phi_{0,q^*}(y)$ is uniformly positive and T is uniformly bounded from above and is not identically zero. Hence we may find γ, the smallest positive number such that $T\le\gamma\phi$ in $\mathbb{R}\times\overline{\Omega}$. Furthermore, since $T(\pm\infty,\cdot)=0$, there exists $(x^*,y^*)\in\mathbb{R}\times\overline{\Omega}$ such that $T(x^*,y^*)=\gamma\phi(x^*,y^*)$. The nonnegative function $z=\gamma\phi-T$ vanishes at (x^*,y^*) and satisfies $\Delta z+(c-u(y))z_x+f'(0)z\le 0$ in $\mathbb{R}\times\Omega$ and $\partial_n z+q^*z=0$ on $\mathbb{R}\times\partial\Omega$. The strong maximum principle and Hopf lemma imply then that $z\equiv 0$, namely $T\equiv\gamma\phi$ in $\mathbb{R}\times\overline{\Omega}$. But this is impossible because $T(\pm\infty,\cdot)=0$ and $\inf_{\mathbb{R}\times\Omega}\gamma\phi>0$.

We conclude that the sequence M_k converges to 0. This completes the proof of part (a) of Theorem 1.4.

Let us now turn to part (b) of Theorem 1.4. The proof is based on the following lemma that characterizes the linearized system as $x \to -\infty$.

Lemma 3.1 *Let (c, T, Y) be a solution of (1.7-1.9), with $q > 0$, such that $0 < T$ and $0 < Y < 1$. Then there exists $\beta \geq 0$ such that $\mu_q(-\beta) = f'(0)Y_\infty + c\beta + \beta^2$, where $Y_\infty = Y(-\infty, \cdot) \in [0, 1)$.*

Proof. Theorem 1.3 implies that the positive function T is bounded and $T(\pm\infty, \cdot) = 0$. Furthermore, by Harnack's inequality, $|\nabla T|/T$ is globally bounded. Let

$$\beta = \limsup_{x \to -\infty} \frac{T_x}{T}$$

and let us check that β satisfies the conclusion of the lemma. First, since $T(-\infty, \cdot) = 0$ and $T > 0$, β is nonnegative. Let (x_k, y_k) be a sequence of points in $\mathbb{R} \times \overline{\Omega}$ such that $x_k \to -\infty$ and $T_x(x_k, y_k)/T(x_k, y_k) \to \beta$ as $k \to +\infty$, and set

$$T_k(x, y) = \frac{T(x_k + x, y)}{T(x_k, y_k)}.$$

The functions T_k are locally bounded in $\mathbb{R} \times \overline{\Omega}$, while the functions $(x, y) \mapsto T(x_k + x, y)$ converge to 0 locally uniformly as $k \to +\infty$. Therefore, the functions T_k are bounded in all $W^{2,p}_{loc}(\mathbb{R} \times \overline{\Omega})$ for all $1 \leq p < +\infty$ and converge, up to extraction of some subsequence, to a solution T_∞ of

$$\begin{cases} \Delta T_\infty(x, y) + (c - u(y))\dfrac{\partial T_\infty(x, y)}{\partial x} + f'(0)Y_\infty T_\infty(x, y) &= 0 \text{ in } \mathbb{R} \times \Omega \\ \dfrac{\partial T_\infty}{\partial n} + qT_\infty &= 0 \text{ on } \mathbb{R} \times \partial\Omega \end{cases}$$

One can also assume that $y_k \to y_\infty \in \overline{\Omega}$. The nonnegative function T_∞ satisfies $T_\infty(0, y_\infty) = 1$, whence $T_\infty > 0$ in $\mathbb{R} \times \overline{\Omega}$ from the strong maximum principle and Hopf lemma. Furthermore, the function

$$z(x, y) = \frac{T_{\infty,x}(x, y)}{T_\infty(x, y)}$$

satisfies $z \leq \beta$, $z(0, y_\infty) = \beta$ and

$$\begin{cases} \Delta z + 2\dfrac{\nabla T_\infty}{T_\infty} \cdot \nabla z + (c - u(y))z_x &= 0 \text{ in } \mathbb{R} \times \Omega \\ \dfrac{\partial z}{\partial n} &= 0 \text{ on } \mathbb{R} \times \partial\Omega. \end{cases}$$

The strong maximum principle and Hopf lemma yield $z \equiv \beta$ in $\mathbb{R} \times \overline{\Omega}$. In other words, there exists a positive function φ in $\overline{\Omega}$ such that $T_\infty(x, y) = e^{\beta x}\varphi(y)$. The function φ satisfies

$$\begin{cases} \Delta\varphi + \beta^2\varphi + \beta(c - u(y))\varphi + f'(0)Y_\infty\varphi &= 0 \text{ in } \Omega \\ \dfrac{\partial\varphi}{\partial n} + q\varphi &= 0 \text{ on } \partial\Omega. \end{cases}$$

By the uniqueness of the principal eigenvalue for problem (1.11), one concludes that $\mu_q(-\beta) = f'(0)Y_\infty + c\beta + \beta^2$. The proof of Lemma 3.1 is now complete. $\Box$

Proof of part (b) of Theorem 1.4. Assume that the conclusion does not hold. There exist then $\varepsilon > 0$, a sequence q_k of positive numbers converging to 0, and a sequence of solutions (c_k, T_k, Y_k) of (1.8-1.9) and (1.7), with heat losses q_k, such that $0 < T_k$, $0 < Y_k < 1$ and $l_k := Y_k(-\infty, \cdot) \geq \varepsilon$ for all k. Theorem 1.3 implies that each function T_k is bounded, $T_k(\pm\infty, \cdot) = 0$, and $c_k > 0$. By Lemma 3.1, for each k, there exists a nonnegative number $\beta_k \geq 0$ such that $\mu_{q_k}(-\beta_k) = f'(0)l_k + c_k\beta_k + \beta_k^2$. Since $\mu_q(\lambda)$ is increasing in $q \in [0, +\infty)$ for each $\lambda \in \mathbb{R}$ (see the proof of part (a) of Theorem 1.4), and since $\mu_q(\lambda)$ is concave with respect to λ for each $q \geq 0$, one gets that

$$f'(0)l_k + c_k\beta_k + \beta_k^2 = \mu_{q_k}(-\beta_k) \leq \mu_{\bar{q}}(-\beta_k) \leq \mu_{\bar{q}}(0) - \mu'_{\bar{q}}(0)\beta_k, \quad \bar{q} = \sup_k q_k < q^*,$$

for all $k \in \mathbb{N}$. Since l_k, β_k and c_k are all nonnegative, comparing the first and the last terms in the above inequality one concludes that the sequence β_k is bounded. Up to extraction of a subsequence, one can then assume that $\beta_k \to \beta \geq 0$ as $k \to +\infty$.

Fix now any positive number $q > 0$. With the same arguments as above, it follows that

$$f'(0)\varepsilon + \beta_k^2 \leq f'(0)l_k + c_k\beta_k + \beta_k^2 = \mu_{q_k}(-\beta_k) \leq \mu_q(-\beta_k) \leq \mu_q(0) - \beta_k\mu'_q(0) \tag{3.16}$$

for all k large enough so that $q_k \leq q$ – recall that $q_k \to 0$ as $k \to +\infty$. Passing to the limit $k \to +\infty$ in (3.16) yields

$$f'(0)\varepsilon + \beta^2 \leq \mu_q(0) - \beta\mu'_q(0). \tag{3.17}$$

The continuity of $\mu_q(0)$ implies that $\mu_q(0) \to \mu_0(0) = 0$ as $q \to 0$. Furthermore, (3.13) implies that

$$\mu'_q(0) = -\int_\Omega u(y)\phi_{0,q}(y)^2 dy,$$

with the first eigenfunction $\phi_{0,q}$ of (1.11) normalized to have $L^2(\Omega)$-norm equal to one. The standard elliptic estimates and the uniqueness of the first eigenfunction of (1.11) up to multiplication imply that

$$\mu'_q(0) = -\int_\Omega u(y)\phi_{0,q}(y)^2 dy \to -\int_\Omega u(y)\phi_{0,0}(y)^2 dy = 0 \quad \text{as } q \to 0$$

because $\phi_{0,0}$ is constant and u has zero mean in Ω. Passing to the limit $q \to 0$ in (3.17) yields $f'(0)\varepsilon + \beta^2 \leq 0$. This is a contradiction and the proof of Theorem 1.4 is complete. $\square$

3.3 Dependence on the amplitude of the flow: the case of fast flows

This section is concerned with the proof of Theorem 1.5.

Proof of Theorem 1.5. Throughout this subsection, we fix $0 < q < q^*$ and then we have $\mu_q(0) < f'(0)$. Since the number $\mu_q(0)$ does not depend on the underlying shear flow $Au(y)$, Theorem 1.2 implies that travelling waves exist for all A and for all speeds $c > \max(c_q^*(A), 0)$.

Let us first prove the continuity of $c_q^*(A)$ with respect to A. Observe first that formula (1.12) can be rewritten as

$$c_q^*(A) = \min_{\lambda > 0} \frac{f'(0) + \lambda^2 - \mu_q(\lambda A)}{\lambda}. \tag{3.18}$$

Since the function $s \mapsto \mu_q(s)$ is concave, one gets immediately that

$$c_q^*(A) \geq \min_{\lambda > 0} \frac{f'(0) + \lambda^2 - \mu_q(0) - \lambda A\mu'_q(0)}{\lambda} = 2\sqrt{f'(0) - \mu_q(0)} - A\mu'_q(0). \tag{3.19}$$

On the other hand, using $\lambda = 1$ in (3.18) one obtains $c_q^*(A) \le f'(0) + 1 - \mu_q(A)$. Fix now $A \in \mathbb{R}$ and let A_k be any sequence converging to A. It follows immediately from the arguments above that the sequence $c_q^*(A_k)$ is bounded. Let c^* be any limit of a subsequence, which we still call $c_q^*(A_k)$, and, for each $k \in \mathbb{N}$, let $\lambda_k > 0$ be such that

$$c_q^*(A_k)\lambda_k = f'(0) + \lambda_k^2 - \mu_q(\lambda_k A_k). \tag{3.20}$$

Therefore, we have

$$c_q^*(A_k)\lambda_k \ge f'(0) + \lambda_k^2 - \mu_q(0) - \lambda_k A_k \mu_q'(0)$$

and the sequence λ_k is bounded from above and below by two positive constants (recall that $\mu_q(0) < f'(0)$). Up to extraction of a subsequence, one can then assume that $\lambda_k \to \lambda^* > 0$ as $k \to +\infty$. Passing to the limit as $k \to +\infty$ in (3.20) gives

$$c^*\lambda^* = f'(0) + (\lambda^*)^2 - \mu_q(\lambda^* A),$$

whence $c^* \ge c_q^*(A)$ – this follows from the definition (1.12) of c_q^*. Now, using the same definition, for any fixed $\lambda > 0$, we have

$$c_q^*(A_k)\lambda \le f'(0) + \lambda^2 - \mu_q(\lambda A_k),$$

and then

$$c^*\lambda \le f'(0) + \lambda^2 - \mu_q(\lambda A)$$

in the limit $k \to +\infty$. Since this is true for any $\lambda > 0$, it follows that $c^* \le c_q^*(A)$. One concludes that $c^* = c_q^*(A)$. Therefore the limit c^* is unique and the whole sequence $c_q^*(A_k)$ converges to $c_q^*(A)$.

One immediately deduces from (3.18) that $c_q^*(0) = 2\sqrt{f'(0) - \mu_q(0)} > 0$. Assume now that

$$\int_\Omega u(y)\phi_0(y)^2 dy = -\mu_q'(0) \ge 0.$$

By concavity of the function μ_q, this function is then non-increasing in $[0, +\infty)$. Formula (3.18) implies then that $c_q^*(A)$ is non-decreasing in $A \ge 0$.

If we now assume that

$$\int_\Omega u(y)\phi_0(y)^2 dy = -\mu_q'(0) > 0, \tag{3.21}$$

then $\lim_{A\to+\infty} c_q^*(A) = +\infty$ by (3.19). Furthermore, the function $\lambda \mapsto \mu_q(\lambda)$ is then decreasing in $[0, +\infty)$. If $0 \le A' < A$ and if $\lambda > 0$ is such that $c_q^*(A)\lambda = f'(0) + \lambda^2 - \mu_q(\lambda A)$, then

$$c_q^*(A) = \frac{f'(0) + \lambda^2 - \mu_q(\lambda A)}{\lambda} > \frac{f'(0) + \lambda^2 - \mu_q(\lambda A')}{\lambda} \ge c_q^*(A').$$

Hence, $c_q^*(A)$ is increasing in A if (3.21) holds. This proves part (a) of Theorem 1.5.

Next, we prove part (b) of this theorem. Note that if $u(y) \equiv 0$, then $\mu_q(s) = \mu_q(0)$ for all s. If $u(y) \not\equiv 0$, then, since u is continuous and has zero mean in Ω, there exists $\delta > 0$ and a ball $B \subset \Omega$ so that $u(y) \ge \delta$ for all $y \in B$. Therefore, for all $\lambda \ge 0$,

$$\mu_q(\lambda) \le \mu_D(-\Delta - \lambda u(y), B) \le \mu_D(-\Delta, B) - \lambda\delta \to -\infty \text{ as } \lambda \to +\infty,$$

where $\mu_D(-\Delta - \lambda u(y), B)$ and $\mu_D(-\Delta, B)$ denote, respectively, the principal eigenvalue of the operators $-\Delta - \lambda u(y)$ and $-\Delta$ in B with the Dirichlet boundary conditions on ∂B. Therefore, in all cases, the (continuous) function $\lambda \mapsto \mu_q(\lambda)$ achieves its maximum over $[0, +\infty)$, at some value $\Lambda \ge 0$ and $\mu_q(+\infty) = -\infty$.

Assume now that

$$u(y) \not\equiv 0 \text{ and } \max_{\lambda \geq 0} \mu_q(\lambda) < f'(0). \tag{3.22}$$

This happens, for instance, if $\mu_q'(0) < 0$. Set $\eta = f'(0) - \mu_q(\Lambda) > 0$. For each $A \geq 0$, let $\lambda_A > 0$ be such that

$$c_q^*(A)\lambda_A = f'(0) + \lambda_A^2 - \mu_q(\lambda_A A) \geq \eta + \lambda_A^2. \tag{3.23}$$

It follows that $c_q^*(A) > 0$. In order to show that $c_q^*(A) \to +\infty$ as $A \to +\infty$ we assume for contradiction that there exists a sequence A_k such that $A_k \to +\infty$ but $c_q^*(A_k) \to c^* \in [0, +\infty)$ as $k \to +\infty$. We see from (3.23) that then the numbers λ_{A_k} are bounded from above and below by two positive constants and one can assume up to extraction of a subsequence, that $\lambda_{A_k} \to \lambda > 0$ as $k \to +\infty$. However, then the left side in (3.23) is bounded as $A = A_k \to +\infty$, while the middle term converges to $+\infty$ because $u \not\equiv 0$, and thus $\mu_q(+\infty) = -\infty$. This is a contradiction, and thus $c_q^*(A) \to +\infty$ as $A \to +\infty$ if (3.22) holds.

Next, we assume that

$$\max_{\lambda \geq 0} \mu_q(\lambda) = f'(0). \tag{3.24}$$

This means that $\Lambda > 0$, that is, the maximum of the function $\mu_q(\lambda)$ over $\lambda \in [0, +\infty)$ is not achieved at $\lambda = 0$ – recall that $\mu_q(0) < f'(0)$. With the same notation as above, one has

$$c_q^*(A)\lambda_A = f'(0) + \lambda_A^2 - \mu_q(\lambda_A A) \geq \lambda_A^2,$$

whence $c_q^*(A) > 0$. However, using $\lambda = \Lambda/A$ in (3.18), we obtain

$$c_q^*(A) \leq \frac{A}{\Lambda}\left[f'(0) + \frac{\Lambda^2}{A^2} - \mu_q(\Lambda)\right] = \frac{\Lambda}{A},$$

hence $c_q^*(A) \to 0$ as $A \to +\infty$ if (3.24) holds.

Lastly, if

$$\max_{\lambda \geq 0} \mu_q(\lambda) = \mu_q(\Lambda) > f'(0), \tag{3.25}$$

the same calculations as above imply that

$$c_q^*(A) \leq \frac{A}{\Lambda}\left[f'(0) + \frac{\Lambda^2}{A^2} - \mu_q(\Lambda)\right] = \frac{(f'(0) - \mu_q(\Lambda))A}{\Lambda} + \frac{\Lambda}{A} \to -\infty \text{ as } A \to +\infty.$$

This finishes the proof of part (b) of Theorem 1.5.

It remains only to prove part (c) of Theorem 1.5. For $A > 0$, one can write $c_q^*(A)/A$ as

$$\gamma_A^* := \frac{c_q^*(A)}{A} = \min_{\lambda > 0} \frac{f'(0) + \lambda^2 - \mu_q(\lambda A)}{\lambda A} = \min_{\lambda > 0} \frac{f'(0) + \dfrac{\lambda^2}{A^2} - \mu_q(\lambda)}{\lambda}. \tag{3.26}$$

This immediately implies that γ_A^* is non-increasing in $A > 0$. Furthermore, if $0 < A < A'$ and $\lambda_A > 0$ is such that $\gamma_A^* = (f'(0) + \lambda_A^2/A^2 - \mu_q(\lambda_A))/\lambda_A$, then

$$\frac{c_q^*(A')}{A'} \leq \frac{f'(0) + \dfrac{\lambda_A^2}{A'^2} - \mu_q(\lambda_A)}{\lambda_A} < \frac{f'(0) + \dfrac{\lambda_A^2}{A^2} - \mu_q(\lambda_A)}{\lambda_A} = \frac{c_q^*(A)}{A}.$$

Thus, the function $A \mapsto c_q^*(A)/A$ is actually decreasing in A. Finally, since

$$\mu_q(\lambda) \le \mu_q(0) + \mu_q'(0)\lambda = \mu_q(0) - \lambda \int_\Omega u(y)\phi_0(y)^2 dy,$$

expression (3.26) yields

$$\frac{c_q^*(A)}{A} \ge \inf_{\lambda>0} \frac{f'(0) - \mu_q(0) - \lambda\mu_q'(0)}{\lambda} = -\mu_q'(0) = \int_\Omega u(y)\phi_0(y)^2 dy.$$

This completes the proof of Theorem 1.5. □

4 Conclusions and discussion

We have shown that the KPP type thermo-diffusive systems in a shear flow and with a boundary heat loss possess travelling front solutions even when the Lewis number is different from one. The temperature profile is a "bump" as the temperature of the burnt material drops due to the heat loss. We also prove that there is always a leftover concentration of unburnt fuel behind the front – the limiting value Y_∞ is strictly positive. The range of possible speeds is a semi-axis, as in the case of a scalar KPP equation. As it has been already shown in [5], it is possible that the infimum of all speeds is not a travelling front speed itself — this may happen, for instance, when the range of speeds is $\{c > 0\}$. A natural question is existence of a travelling front moving with the speed equal to the infimum c_q^* of all possible speeds provided that c_q^* is positive – this issue remains open. It would also be interesting to further study the passage to the adiabatic limit $q \to 0$ – the problem of existence of adiabatic fronts for a general Lewis number remains indeed unresolved. Finally, another open problem is that of the existence of travelling fronts with an ignition or Arrhenius nonlinearity. In particular, the fronts would then be pushed and the travelling front speed would certainly depend on the Lewis number unlike in the present KPP case.

References

[1] M. Abel, A. Celani, D. Vergni and A. Vulpiani, Front propagation in laminar flows, Physical Review E, **64**, 2001, 046307.

[2] B. Audoly, H. Berestycki and Y. Pomeau, Réaction-diffusion en écoulement stationnaire rapide, C. R. Acad. Sci. Paris, **328 II**, 2000, 255–262.

[3] H. Berestycki, The influence of advection on the propagation of fronts in reaction-diffusion equations, in *Nonlinear PDEs in Condensed Matter and Reactive Flows*, NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[4] H. Berestycki, Some nonlinear PDE's in the theory of flame propagation. ICIAM 99 (Edinburgh), 13–22, Oxford Univ. Press, Oxford, 2000.

[5] H. Berestycki, F. Hamel, A. Kiselev and L. Ryzhik, Quenching and propagation in KPP reaction-diffusion equations with a heat loss, Arch. Rational Mech. Anal., **178**, 2005, 57–80.

[6] H. Berestycki, F. Hamel and N. Nadirashvili, Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena, Comm. Math. Phys., **253**, (2005), 451–480.

[7] H. Berestycki, B. Larrouturou and P.-L. Lions, Multi-dimensional traveling wave solutions of a flame propagation model, Arch. Rational Mech. Anal., **111**, 1990, 33–49.

[8] H. Berestycki, B. Larrouturou, P.-L. Lions and J.-M. Roquejoffre, An elliptic system modelling the propagation of a multidimensional flame, Unpublished manuscript.

[9] H. Berestycki and L. Nirenberg, Traveling fronts in cylinders, Annales de l'IHP, Analyse non linéaire, **9**, 1992, 497–572.

[10] J. Billingham and D. Needham, The development of traveling waves in a quadratic and cubic autocatalysis with unequal diffusion. I. Permanent form traveling waves, Phil. Trans. R. Soc. Lond. A **334**, 1991, 1–24.

[11] P. Constantin, A. Kiselev, A. Oberman and L. Ryzhik, Bulk burning rate in passive-reactive diffusion, Arch. Rat. Mech. Anal. **154**, 2000, 53–91.

[12] J. Daou and M. Matalon, Flame propagation in Poiseuille flow under adiabatic conditions, Combustion and Flame, **124**, 2001, 337–349.

[13] J. Daou and M. Matalon, Influence of conductive heat-losses on the propagation of premixed flames in channels, Combustion and Flame **128**, 2001, 321–339.

[14] A. Ducrot, M. Marion and V. Volpert, Reaction-diffusion problems with non Fredholm operators, Preprint, 2004.

[15] V. Giovangigli, Nonadiabatic plane laminar flames and their singular limits. SIAM J. Math. Anal., **21**, 1990, 1305–1325.

[16] P. Gordon, L. Ryzhik and N. Vladimirova, The KPP system in a periodic flow with a heat loss, Nonlinearity, **18**, 2005, 571–589.

[17] S. Heinze, Large convection limits for KPP fronts, Preprint, 2005.

[18] S. Heinze, G. Papanicolaou and A. Stevens, Variational principles for propagation speeds in inhomogeneous media, SIAM J. Appl. Math. **62**, 2001, 129–148.

[19] A. Kiselev and L. Ryzhik, An upper bound for the bulk burning rate for systems, Nonlinearity, **14**, 2001, 1297–1310.

[20] J. Nolen and J. Xin, Reaction-diffusion front speeds in spatially-temporally periodic shear flows, SIAM Jour. MMS, **1**, 2003, 554–570.

[21] L. Roques, Existence de deux solutions du type front progressif pour un modèle de combustion avec pertes de chaleur, C. R. Acad. Sci. Paris Ser. I, **340**, 2005, 493–497

[22] N. Vladimirova, P. Constantin, A. Kiselev, O. Ruchayskiy and L. Ryzhik, Flame Enhancement and Quenching in Fluid Flows, Combust. Theory Model., **7**, 2003, 487–508

[23] V.A. Volpert and A.I. Volpert, Existence and stability of multidimensional travelling waves in the monostable case, Israel Jour. Math., **110**, 1999, 269–292.

[24] J. Xin, Analysis and modelling of front propagation in heterogeneous media, SIAM Rev., **42**, 2000, 161–230.

[25] J. Xin, KPP front speeds in random shears and the parabolic Anderson model, Meth. Appl. Anal., **10**, 2003, 191–198.