A FABER-KRAHN-TYPE INEQUALITY FOR THE LAPLACIAN WITH DRIFT UNDER ROBIN BOUNDARY CONDITION

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Abstract. We prove a Faber-Krahn-type inequality for the Laplacian with drift under Robin boundary condition, provided that the β parameter in the Robin condition is large enough. The proof relies on a compactness argument, on the convergence of Robin eigenvalues to Dirichlet eigenvalues when β goes to infinity, and on a strict Faber-Krahn-type inequality under Dirichlet boundary condition. We also show the existence and uniqueness of drifts v satisfying some L^∞ constraints and minimizing or maximizing the principal eigenvalue of $-\Delta + v\cdot\nabla$ in a fixed domain and with a fixed parameter $\beta > 0$ in the Robin condition.

To Nikolai Nadirashvili, with admiration

Contents

1. Introduction

Throughout this paper, $d \geq 1$ is an integer. For all $x \in \mathbb{R}^d$, denote by $|x|$ the Euclidean norm of x and define

$$e_r(x) := \frac{x}{|x|} \text{ for all } x \in \mathbb{R}^d \setminus \{0\}\,.$$

2010 *Mathematics Subject Classification.* 35B50; 35J15; 47A75.

Key words and phrases. Laplacian with drift; Robin boundary condition; principal eigenvalue; Faber-Krahn-type inequality.

This work has been supported by the French Agence Nationale de la Recherche (ANR), in the framework of the ReaCh project ANR-23-CE40-0023-02. The authors are grateful to the anonymous reviewer for relevant suggestions which led to an improvement of the manuscript.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (connected open set) of class C^2, with outward unit normal ν on $\partial\Omega$. If $v \in L^\infty(\Omega, \mathbb{R}^d)$ is a bounded measurable vector field, set

$$\|v\|_\infty := \||v|\|_\infty .$$

We are interested in the principal eigenvalue of the operator $-\Delta + v \cdot \nabla$ in Ω under Robin boundary condition on $\partial\Omega$. More precisely, let $\beta > 0$. By [15, Theorem A.4] and Krein-Rutman theory [2], there exists a principal eigenvalue $\lambda_1^\beta(\Omega, v)$ of the problem

$$\left\{\begin{array}{ll} -\Delta\varphi^\beta_{\Omega,v} + v \cdot \nabla\varphi^\beta_{\Omega,v} = \lambda_1^\beta(\Omega, v)\varphi^\beta_{\Omega,v} & \text{in } \Omega, \\ \dfrac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.1}$$

This principal eigenvalue is simple, the corresponding eigenfunction $\varphi^\beta_{\Omega,v}$ is positive in $\overline{\Omega}$ by [2, Theorem 4.5], and none of the other eigenvalues corresponds to a positive eigenfunction (see the discussion after [15, Theorem 1.3]). By $W^{2,p}$ elliptic regularity ([15, Theorem A.29]), the function $\varphi^\beta_{\Omega,v}$ belongs to $W^{2,p}(\Omega)$ for all $p \in [1, \infty)$ and then to $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$. The first line in (1.1) is therefore understood almost everywhere in Ω, while the second line is understood in the classical sense. We usually normalize $\varphi^\beta_{\Omega,v}$ by

$$\max_{\overline{\Omega}} \varphi^\beta_{\Omega,v} = 1. \tag{1.2}$$

Moreover, there holds

$$\lambda_1^\beta(\Omega, v) > 0.$$

The C^2 smoothness of $\partial\Omega$ is used to derive the regularity of $\varphi^\beta_{\Omega,v}$. So is it in some arguments of the proofs of the following main results. We leave as an open question the derivation of Faber-Krahn-type inequalities for weaker formulations of (1.1) under weaker assumptions on $\partial\Omega$.

Fix $\tau \geq 0$ and $m > 0$. We are interested in the infimum of $\lambda_1^\beta(\Omega, v)$ when Ω and v vary under the constraints $|\Omega| = m$ (throughout the paper, $|A|$ denotes the n-dimensional Lebesgue measure of A for all measurable sets $A \subset \mathbb{R}^d$) and

$$\|v\|_\infty \leq \tau. \tag{1.3}$$

In the sequel, Ω^* stands for the Euclidean ball centered at 0 such that $|\Omega^*| = |\Omega|$.

Our main result states that, when Ω is not a ball, $\lambda_1^\beta(\Omega, v)$ is (strictly) greater than the corresponding quantity in Ω^* when $v = \tau\, e_r$, provided that β is large enough:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $\tau \geq 0$. Assume that Ω is not a ball. Then there exist $\beta_0 > 0$ and $\varepsilon > 0$, depending on τ,* *with the following property:*

$$\forall\,\beta \geq \beta_0,\ \forall\, v \in L^\infty(\Omega,\mathbb{R}^d)\ \textit{such that}\ \|v\|_\infty \leq \tau, \qquad \lambda_1^\beta(\Omega,v) \geq \lambda_1^\beta(\Omega^*,\tau e_r)+\varepsilon. \tag{1.4}$$

When $\tau=0$, *i.e.* when $-\Delta+v\cdot\nabla=-\Delta$ is merely (minus) the Laplacian, it was proved in [9, 10, 13, 14] that, for all $\beta>0$, the following Bossel-Daners inequality is satisfied:

$$\lambda_1^\beta(\Omega,0) \geq \lambda_1^\beta(\Omega^*,0), \tag{1.5}$$

and equality holds if and only if $\Omega=\Omega^*$ up to translation. When $\tau\neq 0$, Theorem 1.1 provides on the one hand a quantified strict inequality if Ω is not a ball, but the conclusion is only established above some threshold for β, contrary to [10, Theorem 1.1], and it actually can not hold for all $\beta>0$ with the same ε, since

$$\lim_{\beta\to 0}\lambda_1^\beta(\Omega,v)=\lim_{\beta\to 0}\lambda_1^\beta(\Omega^*,\tau e_r)=0$$

for each $v\in L^\infty(\Omega,\mathbb{R}^d)$ (as follows from Lemma 2.2 below). On the other hand, when $\Omega=\Omega^*$, the uniqueness part in Theorem 1.3 below ensures that, for all $v\in L^\infty(\Omega^*,\mathbb{R}^d)$ with $\|v\|_\infty\leq\tau$, if $v\neq\tau e_r$, then $\lambda_1^\beta(\Omega^*,v)>\lambda_1^\beta(\Omega^*,\tau e_r)$ for all $\beta>0$.

The following question nevertheless remains open:

Open problem 1.2. *Let $\Omega\subset\mathbb{R}^d$ be a bounded C^2 domain, $\tau\geq 0$ and $v\in L^\infty(\Omega,\mathbb{R}^d)$ with $\|v\|_\infty\leq\tau$. Does the inequality*

$$\lambda_1^\beta(\Omega,v)\geq\lambda_1^\beta(\Omega^*,\tau e_r)$$

hold for all $\beta>0$?

Recall that, under Dirichlet boundary condition, it was proved in [17, Theorem 1.1] and [18, Remark 6.9] that, whenever (1.3) holds,

$$\lambda_1^D(\Omega,v)\geq\lambda_1^D(\Omega^*,\tau e_r), \tag{1.6}$$

where $\lambda_1^D(\Omega,v)$ stands for the principal eigenvalue of $-\Delta+v\cdot\nabla$ under Dirichlet boundary condition. Moreover, equality holds in (1.6) if and only if, up to translation, $\Omega=\Omega^*$ and $v=\tau e_r$. The inequalities (1.4)-(1.6) are called Faber-Krahn-type inequalities. This terminology originates from the results of Faber [16] and Krahn [22, 23], who proved that

$$\lambda_1^D(\Omega,0)\geq\lambda_1^D(\Omega^*,0),$$

with equality if and only if, up to translation, $\Omega=\Omega^*$. The latter inequality means that a radially symmetric membrane which is fixed at its boundary has the lowest fundamental tone among all equimeasurable membranes, answering a conjecture of Rayleigh [29] set in dimension $d=2$. Since these pioneering papers, much work has been done on various related optimization eigenvalue problems for elliptic operators, for instance on higher eigenvalues or functions of the eigenvalues of $-\Delta$ under Dirichlet boundary condition [3, 4, 6, 11, 12, 26, 27, 28, 32], under Neumann boundary condition [28, 30, 31], or for the first eigenvalue of Δ^2 under boundary conditions $\varphi = \frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ [5, 25]. We refer to the surveys [7, 20, 21] for many more references on these topics.

The second main result deals with an optimization problem when the domain Ω is fixed and v varies under the constraint (1.3). Define, for all $\beta > 0$ and $\tau \geq 0$ given:

$$\underline{\lambda}^{\beta}(\Omega,\tau) := \inf\left\{\lambda_1^{\beta}(\Omega,v) : \|v\|_{\infty} \leq \tau\right\} \tag{1.7}$$

and

$$\overline{\lambda}^{\beta}(\Omega,\tau) := \sup\left\{\lambda_1^{\beta}(\Omega,v) : \|v\|_{\infty} \leq \tau\right\}. \tag{1.8}$$

We claim that these lower and upper bounds are positive real numbers, are uniquely reached and provide an identity relating the optimizing vector fields and the corresponding eigenfunctions:

Theorem 1.3. *[Optimization in fixed domains] Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain, $\tau \geq 0$ and $\beta > 0$.*

1. *There exists a unique $\underline{v} \in L^{\infty}(\Omega,\mathbb{R}^d)$ meeting $\|\underline{v}\|_{\infty} \leq \tau$ such that $\underline{\lambda}^{\beta}(\Omega,\tau) = \lambda_1^{\beta}(\Omega,\underline{v})$. One has $|\underline{v}(x)| = \tau$ for almost every $x \in \Omega$. Moreover, if $\underline{\varphi} := \varphi^{\beta}_{\Omega,\underline{v}}$ is the corresponding eigenfunction, then*

$$|\nabla\underline{\varphi}(x)| > 0 \text{ and } \underline{v}(x) = -\tau\frac{\nabla\underline{\varphi}(x)}{|\nabla\underline{\varphi}(x)|} \text{ for almost every } x \in \Omega. \tag{1.9}$$

Lastly, if $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta\phi - \tau\,|\nabla\phi| = \lambda\phi \ \text{ and } \ \phi \geq 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} + \beta\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

and $\max_{\overline{\Omega}} \phi = 1$, then $\lambda = \underline{\lambda}^{\beta}(\Omega,\tau)$ and $\phi = \underline{\varphi}$ in $\overline{\Omega}$.

2. *Similarly, there exists a unique $\overline{v} \in L^{\infty}(\Omega,\mathbb{R}^d)$ meeting $\|\overline{v}\|_{\infty} \leq \tau$ such that $\overline{\lambda}^{\beta}(\Omega,\tau) = \lambda_1^{\beta}(\Omega,\overline{v})$. One has $|\overline{v}(x)| = \tau$ for almost every $x \in \Omega$. Moreover, if $\overline{\varphi} := \varphi^{\beta}_{\Omega,\overline{v}}$ is the corresponding eigenfunction, then*

$$|\nabla\overline{\varphi}(x)| > 0 \text{ and } \overline{v}(x) = \tau\frac{\nabla\overline{\varphi}(x)}{|\nabla\overline{\varphi}(x)|} \text{ for almost every } x \in \Omega. \tag{1.11}$$

Lastly, if $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta\phi + \tau\,|\nabla\phi| = \lambda\phi \ \text{ and } \ \phi \geq 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} + \beta\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

and $\max_{\overline{\Omega}} \phi = 1$, then $\lambda = \overline{\lambda}^{\beta}(\Omega,\tau)$ and $\phi = \overline{\varphi}$ in $\overline{\Omega}$.

3. *If $\Omega = \Omega^*$, then $\underline{v} = \tau\, e_r$, $\overline{v} = -\tau\, e_r$ in Ω^* and the functions $\underline{\varphi}$ and $\overline{\varphi}$ are radially decreasing in Ω^*.*

Remark 1.4. Notice that, in part (1) of Theorem 1.3, from elliptic regularity theory applied to (1.10), since ϕ and $|\nabla\phi|$ belong to $C^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$, the function ϕ belongs to $C^{2,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$, and the first line of (1.10) holds in the classical sense in Ω. Similarly, in part (2), the solution ϕ of (1.12) belongs to $C^{2,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$, and the first line of (1.12) holds in the classical sense in Ω.

We point out that similar properties had been derived in [17, 18] for the extremal quantities $\underline{\lambda}(\Omega,\tau)$ and $\overline{\lambda}(\Omega,\tau)$ defined like $\underline{\lambda}^{\beta}(\Omega,\tau)$ and $\overline{\lambda}^{\beta}(\Omega,\tau)$ in (1.7)-(1.8) with the Dirichlet eigenvalues $\lambda_1(\Omega,v)$ instead of the Robin ones $\lambda_1^{\beta}(\Omega,v)$. The asymptotic behavior as $\tau \to +\infty$ of the eigenfunctions associated with $\underline{\lambda}(\Omega,\tau)$ was analyzed in [19].

The paper is organized as follows. In Section 2, we provide comparisons results between Robin, Dirichlet and Neumann eigenvalues in a fixed domain and for a given drift, and prove convergence of the Robin eigenvalues when $\beta \to +\infty$ (resp. when $\beta \to 0$) to the corresponding Dirichlet (resp. Neumann) eigenvalues. We establish Theorem 1.1 in Section 3. Finally, Section 4 is devoted to the proof of Theorem 1.3.

2. Comparisons and convergence results between Robin, Dirichlet and Neumann principal eigenvalues

This section is concerned with some comparisons and convergence results for Robin and Dirichlet principal eigenvalues in a given domain Ω. The results will be used in the proofs of the main Theorems 1.1 and 1.3.

We first start with an auxiliary comparison lemma between sub- and super-solutions.

Lemma 2.1. *Let $\mu \in \mathbb{R}$, $\beta \geq 0$, and $v \in L^\infty(\Omega,\mathbb{R}^d)$. Let $\psi,\varphi \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$, such that $\psi \geq 0$ and $\varphi \geq 0$ in Ω, $\|\psi\|_\infty = \|\varphi\|_\infty = 1$, and*

$$\begin{cases} -\Delta\psi + v\cdot\nabla\psi \geq \mu\psi & \text{a.e. in } \Omega,\\ -\Delta\varphi + v\cdot\nabla\varphi \leq \mu\varphi & \text{a.e. in } \Omega. \end{cases}$$

Assume also that

$$\frac{\partial\psi}{\partial\nu} + \beta\psi \geq 0 \geq \frac{\partial\varphi}{\partial\nu} + \beta\varphi \text{ on } \partial\Omega.$$

Then $\psi = \varphi$ in $\overline{\Omega}$.

Proof. The argument is reminiscent of the proof of [17, Lemma 2.1]. Remember first that ψ and φ belong to $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Furthermore, $\psi > 0$ in Ω from the interior strong maximum principle (otherwise, ψ would be identically 0 in Ω, contradicting $\|\psi\|_\infty = 1$). Observe now that $\psi > 0$ on $\partial\Omega$. Indeed, if there exists $x_0 \in \partial\Omega$ such that $\psi(x_0) = 0$, then the Hopf lemma shows that $\frac{\partial\psi}{\partial\nu}(x_0) < 0$, which is impossible by the boundary condition satisfied by ψ. Thus, being continuous in $\overline{\Omega}$, ψ is bounded below by a positive constant, so that there exists $\gamma > 0$ such that $\gamma\psi > \varphi$ in Ω. Define

$$\gamma^* := \inf\{\gamma > 0 : \gamma\psi > \varphi \text{ in } \Omega\}$$

and $w := \gamma^*\psi - \varphi$. Note that, since $\varphi \geq 0$ in Ω and $\|\varphi\|_\infty = 1$, $\gamma^* > 0$. The function w is nonnegative in $\overline{\Omega}$,

$$\frac{\partial w}{\partial \nu} + \beta w \geq 0 \text{ on } \partial\Omega$$

and

$$-\Delta w + v \cdot \nabla w - \mu w \geq 0 \text{ a.e. in } \Omega.$$

If $w > 0$ in Ω, then, as before, w is bounded below by a positive constant in Ω, so there exists $\delta > 0$ such that $w > \delta\varphi$ in Ω, which entails in turn

$$\frac{\gamma^*}{1+\delta}\,\psi > \varphi \text{ in } \Omega,$$

contradicting the definition of γ^*, since $\gamma^* > 0$. Therefore, there exists $x_0 \in \Omega$ such that $w(x_0) = 0$, and since $w \geq 0$ in Ω, the strong maximum principle indicates that $w(x) = 0$ everywhere in Ω and then in $\overline{\Omega}$ by continuity, meaning that $\gamma^*\psi = \varphi$ in $\overline{\Omega}$. The condition $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ finally yields $\varphi = \psi$ in $\overline{\Omega}$. □

Let now $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $v \in L^\infty(\Omega, \mathbb{R}^d)$. Denote by $\lambda_1^D(\Omega, v)$ the principal eigenvalue of $-\Delta + v\cdot\nabla$ in Ω under Dirichlet boundary condition and by $\varphi^D_{\Omega,v} \in \bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ the corresponding principal eigenfunction (which is positive in Ω) normalized by

$$\left\|\varphi^D_{\Omega,v}\right\|_\infty = 1.$$

We will show that the map $\beta \mapsto \lambda_1^\beta(\Omega, v)$ is increasing in $(0, \infty)$ and converges to $\lambda_1^D(\Omega, v)$ at infinity, and to 0 (that is, the principal eigenvalue of $-\Delta + v \cdot \nabla$ in Ω under Neumann boundary condition) as $\beta \to 0$. Although the result is natural, we are not aware of a complete proof in the literature. This is why we include a detailed proof below, making use of Lemma 2.1.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 domain and $v \in L^\infty(\Omega, \mathbb{R}^d)$. Then the map $\beta \mapsto \lambda_1^\beta(\Omega, v)$ is increasing in $(0, +\infty)$. Furthermore,*

$$\lim_{\beta\to+\infty} \lambda_1^\beta(\Omega, v) = \lambda_1^D(\Omega, v). \tag{2.1}$$

and

$$\lim_{\beta\to 0} \lambda_1^\beta(\Omega, v) = 0.$$

Proof. Let $0 < \beta_1 < \beta_2$ and assume by way of contradiction that $\lambda_1^{\beta_2}(\Omega, v) \leq \lambda_1^{\beta_1}(\Omega, v)$. Set $\varphi_1 := \varphi^{\beta_1}_{\Omega,v}$ and $\varphi_2 := \varphi^{\beta_2}_{\Omega,v}$. Both functions φ_1 and φ_2 are positive in $\overline{\Omega}$ and they satisfy

$$\begin{cases} -\Delta\varphi_1 + v \cdot \nabla\varphi_1 = \lambda_1^{\beta_1}(\Omega, v)\varphi_1 & \text{a.e. in } \Omega, \\ -\Delta\varphi_2 + v \cdot \nabla\varphi_2 = \lambda_1^{\beta_2}(\Omega, v)\varphi_2 \leq \lambda_1^{\beta_1}(\Omega, v)\varphi_2 & \text{a.e. in } \Omega, \end{cases}$$

together with

$$\frac{\partial \varphi_1}{\partial \nu} + \beta_1 \varphi_1 = 0 = \frac{\partial \varphi_2}{\partial \nu} + \beta_2 \varphi_2 > \frac{\partial \varphi_2}{\partial \nu} + \beta_1 \varphi_2 \text{ on } \partial\Omega. \tag{2.2}$$

Lemma 2.1 applied with $(\mu, \beta, \psi, \varphi) := (\lambda_1^{\beta_1}(\Omega, v), \beta_1, \varphi_1, \varphi_2)$ then entails $\varphi_1 = \varphi_2$ in $\overline{\Omega}$, contradicting the strict inequality in (2.2). Finally,

$$\lambda_1^{\beta_1}(\Omega, v) < \lambda_1^{\beta_2}(\Omega, v),$$

and the map $\beta \mapsto \lambda_1^{\beta}(\Omega, v)$ is increasing in $(0, +\infty)$.

Let now $\beta > 0$ and assume by way of contradiction that $\lambda_1^{\beta}(\Omega, v) \geq \lambda_1^{D}(\Omega, v)$. Both functions $\varphi_{\Omega,v}^{\beta}$ and $\varphi_{\Omega,v}^{D}$ are positive in Ω and they satisfy

$$\begin{cases} -\Delta \varphi_{\Omega,v}^{\beta} + v \cdot \nabla \varphi_{\Omega,v}^{\beta} = \lambda_1^{\beta}(\Omega, v) \varphi_{\Omega,v}^{\beta} & \text{a.e. in } \Omega, \\ -\Delta \varphi_{\Omega,v}^{D} + v \cdot \nabla \varphi_{\Omega,v}^{D} = \lambda_1^{D}(\Omega, v) \varphi_{\Omega,v}^{D} \leq \lambda_1^{\beta}(\Omega, v) \varphi_{\Omega,v}^{D} & \text{a.e. in } \Omega. \end{cases}$$

Furthermore, the Hopf lemma implies that $\frac{\partial \varphi_{\Omega,v}^{D}}{\partial \nu} < 0$ on $\partial\Omega$, whence

$$\frac{\partial \varphi_{\Omega,v}^{\beta}}{\partial \nu} + \beta \varphi_{\Omega,v}^{\beta} = 0 > \frac{\partial \varphi_{\Omega,v}^{D}}{\partial \nu} + \beta \varphi_{\Omega,v}^{D} \text{ on } \partial\Omega. \tag{2.3}$$

Lemma 2.1 applied with $(\mu, \beta, \psi, \varphi) := (\lambda_1^{\beta}(\Omega, v), \beta, \varphi_{\Omega,v}^{\beta}, \varphi_{\Omega,v}^{D})$ then entails $\varphi_{\Omega,v}^{\beta} = \varphi_{\Omega,v}^{D}$ in $\overline{\Omega}$, contradicting the strict inequality in (2.3) (or the fact that $\varphi_{\Omega,v}^{\beta} > 0 = \varphi_{\Omega,v}^{D}$ on $\partial\Omega$). Finally,

$$\lambda_1^{\beta}(\Omega, v) < \lambda_1^{D}(\Omega, v)$$

for all $\beta > 0$.

Let us now turn to the proof of (2.1). Pick up any increasing sequence $(\beta_k)_{k \in \mathbb{N}}$ of positive real numbers with $\lim_{k \to +\infty} \beta_k = +\infty$ and set $\lambda_k := \lambda_1^{\beta_k}(\Omega, v)$ for all $k \in \mathbb{N}$. The sequence $(\lambda_k)_{k \in \mathbb{N}}$ is increasing and bounded above by $\lambda_1^{D}(\Omega, v)$ and therefore converges to some $\mu \leq \lambda_1^{D}(\Omega, v)$. For all k, if φ_k is defined as $\varphi_k := \theta_k \varphi_{\Omega,v}^{\beta_k}$ with $\theta_k > 0$ such that $\|\varphi_k\|_{L^2(\Omega)} = 1$, then

$$\begin{cases} -\Delta \varphi_k + v \cdot \nabla \varphi_k = \lambda_k \varphi_k & \text{a.e. in } \Omega, \\ \dfrac{\partial \varphi_k}{\partial \nu} + \beta_k \varphi_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We claim that the sequence $(\varphi_k)_{k \in \mathbb{N}}$ is bounded in $H^1(\Omega)$. Indeed, for all $k \in \mathbb{N}$,

$$\begin{aligned} \lambda_k \int_\Omega \varphi_k^2 &= -\int_\Omega \varphi_k \Delta \varphi_k + \int_\Omega (v \cdot \nabla \varphi_k) \varphi_k \\ &= \int_\Omega |\nabla \varphi_k|^2 - \int_{\partial\Omega} \varphi_k \frac{\partial \varphi_k}{\partial \nu} + \int_\Omega (v \cdot \nabla \varphi_k) \varphi_k \\ &= \int_\Omega |\nabla \varphi_k|^2 + \beta_k \int_{\partial\Omega} \varphi_k^2 + \int_\Omega (v \cdot \nabla \varphi_k) \varphi_k. \end{aligned}$$

From this, we derive, for all $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega |\nabla\varphi_k|^2+\beta_k\int_{\partial\Omega}\varphi_k^2 &\le \lambda_k\int_\Omega \varphi_k^2+\|v\|_\infty\int_\Omega \varphi_k\,|\nabla\varphi_k| \\ &\le \Big(\lambda_k+\frac{1}{2\varepsilon}\,\|v\|_\infty\Big)\int_\Omega\varphi_k^2+\frac{\varepsilon}{2}\,\|v\|_\infty\int_\Omega|\nabla\varphi_k|^2.\end{aligned}\tag{2.5}$$

Provided $\varepsilon\,\|v\|_\infty < 2$, recalling that the sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\|\varphi_k\|_{L^2(\Omega)})_{k\in\mathbb{N}}$ are bounded, one obtains that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, that is, there is $M\in\mathbb{R}_+$ such that

$$\|\varphi_k\|_{H^1(\Omega)}=\sqrt{\|\varphi_k\|^2_{L^2(\Omega)}+\|\,|\nabla\varphi_k|\,\|^2_{L^2(\Omega)}}\le M$$

for all $k\in\mathbb{N}$. Therefore, there exists $\varphi\in H^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}\varphi_k \rightharpoonup \varphi \text{ weakly in } H^1(\Omega),\ \varphi_k\to\varphi \text{ strongly in } L^2(\Omega),\\ \varphi_k\to\varphi \text{ a.e. in } \Omega,\end{aligned}\tag{2.6}$$

as $k\to+\infty$, whence

$$\|\varphi\|_{L^2(\Omega)}=1 \ \text{ and } \ \varphi\ge 0 \text{ a.e. in } \Omega.\tag{2.7}$$

Moreover, since $\lim_{k\to+\infty}\beta_k=+\infty$, (2.5) shows that

$$\lim_{k\to+\infty}\mathrm{tr}(\varphi_k)=0 \text{ strongly in } L^2(\partial\Omega),\tag{2.8}$$

where $\mathrm{tr}\ : H^1(\Omega)\to L^2(\partial\Omega)$ denotes the trace operator. Since this trace operator is compact from $H^1(\Omega)$ to $L^2(\partial\Omega)$ with the topologies induced by the norms (see [24, Corollary 18.4]), and since $\varphi_k\rightharpoonup\varphi$ weakly in $H^1(\Omega)$ as $k\to+\infty$, it follows that $\mathrm{tr}(\varphi_k)\to\mathrm{tr}(\varphi)$ strongly in $L^2(\partial\Omega)$ as $k\to+\infty$, whence $\mathrm{tr}(\varphi)=0$ by (2.8), meaning that $\varphi\in H^1_0(\Omega)$. Consider now $\psi\in C^\infty_c(\Omega)$. One has

$$\begin{aligned}\int_\Omega\nabla\varphi\cdot\nabla\psi+\int_\Omega(v\cdot\nabla\varphi)\psi &= \lim_{k\to+\infty}\int_\Omega\nabla\varphi_k\cdot\nabla\psi+\int_\Omega(v\cdot\nabla\varphi_k)\psi\\ &= \lim_{k\to+\infty}\lambda_k\int_\Omega\varphi_k\psi\\ &= \mu\int_\Omega\varphi\psi,\end{aligned}$$

which means that φ is an $H^1_0(\Omega)$ weak solution of

$$\begin{cases}-\Delta\varphi+v\cdot\nabla\varphi=\mu\varphi & \text{in } \Omega,\\ \mathrm{tr}(\varphi)=0 & \text{on } \partial\Omega.\end{cases}$$

Elliptic H^2 and $W^{2,p}$ estimates show that $\varphi\in W^{2,p}(\Omega)$ for all $1\le p<\infty$, and, since $\varphi\ge 0$ in Ω and $\|\varphi\|_{L^2(\Omega)}=1$, the strong maximum principle entails that $\varphi>0$ in Ω. Thus, by uniqueness of the principal eigenvalue of $-\Delta+v\cdot\nabla$ under Dirichlet boundary condition, one gets that

$$\mu=\lambda_1^D(\Omega,v),$$

which ends the proof.

Lastly, let us investigate the limit of $\lambda_1^\beta(\Omega,v)$ as $\beta\to 0$. Pick up any decreasing sequence $(\beta_k)_{k\in\mathbb{N}}$ of positive real numbers with $\lim_{k\to+\infty}\beta_k=0$ and set $\lambda_k:=\lambda_1^{\beta_k}(\Omega,v)$ for all $k\in\mathbb{N}$. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is decreasing and bounded below by 0, and therefore converges to some $\lambda\geq 0$. For all k, if φ_k is defined as $\varphi_k:=\theta_k\varphi_{\Omega,v}^{\beta_k}$ with $\theta_k>0$ such that $\|\varphi_k\|_{L^2(\Omega)}=1$, then as above (2.4)-(2.5) still hold and there exists $\varphi\in H^1(\Omega)$ satisfying (2.6)-(2.7), up to a subsequence. Pick now any $\psi\in H^1(\Omega)$. For all $k\in\mathbb{N}$, one has

$$\lambda_k\int_\Omega\varphi_k\psi=\int_\Omega\nabla\varphi_k\cdot\nabla\psi+\beta_k\int_{\partial\Omega}\varphi_k\psi+\int_\Omega(v\cdot\nabla\varphi_k)\psi.$$

But $\beta_k\to 0$ as $k\to+\infty$ and the sequence $(\mathrm{tr}(\varphi_k))_{k\in\mathbb{N}}$ is bounded in $L^2(\partial\Omega)$ (since so is $(\varphi_k)_{k\in\mathbb{N}}$ in $H^1(\Omega)$). Hence, by (2.6), the passage to the limit as $k\to+\infty$ in the above formula leads to

$$\lambda\int_\Omega\varphi\psi=\int_\Omega\nabla\varphi\cdot\nabla\psi+\int_\Omega(v\cdot\nabla\varphi)\psi.$$

In other words, φ is an $H^1(\Omega)$ weak solution of

$$\begin{cases}-\Delta\varphi+v\cdot\nabla\varphi=\lambda\varphi & \text{in }\Omega,\\ \dfrac{\partial\varphi}{\partial\nu}=0 & \text{on }\partial\Omega.\end{cases}$$

Elliptic H^2 and $W^{2,p}$ estimates show that $\varphi\in W^{2,p}(\Omega)$ for all $1\leq p<\infty$, and by (2.7) the strong maximum principle and Hopf lemma entail that $\varphi>0$ in $\overline{\Omega}$. Thus, by uniqueness of the principal eigenvalue of $-\Delta+v\cdot\nabla$ under Neumann boundary condition, one gets that $\lambda=0$. The proof of Lemma 2.2 is thereby complete. □

3. Proof of the minimization result

Let us now prove Theorem 1.1. Arguing by contradiction, assume that the conclusion does not hold. There exist then a sequence $(\beta_k)_{k\in\mathbb{N}}$ of positive numbers such that $\lim_{k\to+\infty}\beta_k=+\infty$ and a sequence of vector fields $(v_k)_{k\in\mathbb{N}}$ such that, for all $k\in\mathbb{N}$, $\|v_k\|_\infty\leq\tau$ and

$$\lambda_1^{\beta_k}(\Omega,v_k)<\lambda_1^{\beta_k}(\Omega^*,\tau e_r)+\frac{1}{k+1}. \tag{3.1}$$

For all $k\in\mathbb{N}$, write

$$\varphi_k:=\varphi_{\Omega,v_k}^{\beta_k}\ \text{ and }\ \lambda_k:=\lambda_1^{\beta_k}(\Omega,v_k).$$

One has

$$\begin{cases}-\Delta\varphi_k+v_k\cdot\nabla\varphi_k=\lambda_k\varphi_k & \text{a.e. in }\Omega,\\ \dfrac{1}{\beta_k}\dfrac{\partial\varphi_k}{\partial\nu}=-\varphi_k & \text{on }\partial\Omega.\end{cases} \tag{3.2}$$

Lemma 2.2 shows that $\lambda_k\leq\lambda_1^D(\Omega,v_k)$ for all $k\in\mathbb{N}$, while [8, Proposition 5.1] ensures that the sequence $(\lambda_1^D(\Omega,v_k))_{k\in\mathbb{N}}$ is bounded (recall that

$\|v_k\|_\infty \le \tau$ for all $k \in \mathbb{N}$). Furthermore, each λ_k is a positive real number. Therefore, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded. Arguing as in the proof of Lemma 2.2, one concludes that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is then bounded in $H^1(\Omega)$, which entails that the sequence $(\mathrm{tr}(\varphi_k))_{k\in\mathbb{N}}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$. Therefore, together with the boundedness of the sequence $(1/\beta_k)_{k\in\mathbb{N}}$, [1, Theorem 15.2] implies that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,2}(\Omega)$, and a bootstrap argument therefore shows that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega)$ for all $1 \le p < \infty$. Thus, there exist $\mu \in \mathbb{R}$, $\varphi \in \bigcap_{1\le p<\infty} W^{2,p}(\Omega)$ and $f \in L^\infty(\Omega)$ such that, up to a subsequence,

$$\lim_{k\to+\infty} \lambda_k = \mu,$$

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \varphi \text{ weakly in } W^{2,p}(\Omega) \text{ and } \varphi_k \underset{k\to+\infty}{\to} \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all $1 \le p < \infty$ and all $\alpha \in (0,1)$, and

$$v_k \cdot \nabla\varphi_k \rightharpoonup f \text{ weakly-}* \text{ in } L^\infty(\Omega).$$

Furthermore, since $\varphi_k \to \varphi$ in (at least) $C^1(\overline{\Omega})$ and $\beta_k \to +\infty$ as $k \to +\infty$, it follows from (3.2) that $\varphi = 0$ in $\partial\Omega$. One therefore has

$$\begin{cases} -\Delta\varphi + f = \mu\varphi & \text{a.e. in } \Omega, \\ \varphi \ge 0 & \text{in } \overline{\Omega}, \\ \varphi = 0 & \text{on } \partial\Omega, \\ \max\limits_{\overline{\Omega}} \varphi = 1 & \end{cases}$$

and $f \ge -\tau\,|\nabla\varphi|$ a.e. in Ω, so that

$$-\Delta\varphi - \tau\,|\nabla\varphi| \le \mu\varphi \ \text{ a.e. in } \Omega.$$

Define

$$v(x) := \begin{cases} -\tau \dfrac{\nabla\varphi(x)}{|\nabla\varphi(x)|} & \text{if } \nabla\varphi(x) \ne 0, \\ 0 & \text{if } \nabla\varphi(x) = 0, \end{cases}$$

so that $\|v\|_\infty \le \tau$ and

$$-\Delta\varphi + v\cdot\nabla\varphi \le \mu\varphi \ \text{ a.e. in } \Omega.$$

Let now $\psi := \varphi^D_{\Omega,v}$, so that $\psi > 0$ in Ω and

$$-\Delta\psi + v\cdot\nabla\psi = \lambda_1^D(\Omega, v)\psi \ \text{ a.e. in } \Omega.$$

If $\mu < \lambda_1^D(\Omega, v)$, then

$$-\Delta\varphi + v\cdot\nabla\varphi \le \mu\varphi \le \lambda_1^D(\Omega, v)\varphi \ \text{ a.e. in } \Omega,$$

and [17, Lemma 2.1] implies that $\varphi = \psi$ in $\overline{\Omega}$, therefore $\mu = \lambda_1^D(\Omega, v)$, a contradiction. Finally,

$$\lambda_1^D(\Omega, v) \le \mu.$$

But (3.1) and Lemma 2.2 imply that

$$\mu \le \lambda_1^D(\Omega^*, \tau e_r),$$

and one therefore obtains

$$\lambda_1^D(\Omega, v) \leq \lambda_1^D(\Omega^*, \tau e_r),$$

which contradicts the "equality" statement in [17, Theorem 1.1] since Ω is not a ball. This concludes the proof of Theorem 1.1. □

4. Optimization of the principal eigenvalue in a fixed domain

This section is devoted to the proof of Theorem 1.3.

Proof of Theorem 1.3. *Part 1.* We first focus on the infimum problem and begin with the existence part. First of all, since $\lambda_1^\beta(\Omega, v) > 0$ for every $v \in L^\infty(\Omega, \mathbb{R}^d)$, we already know that $\underline{\lambda}^\beta(\Omega, \tau)$ is a nonnegative real number. Let then $(v_k)_{k\in\mathbb{N}}$ be a sequence of vector fields in $L^\infty(\Omega, \mathbb{R}^d)$ such that $\|v_k\|_\infty \leq \tau$ for all k and

$$\lim_{k\to+\infty} \lambda_1^\beta(\Omega, v_k) = \underline{\lambda}^\beta(\Omega, \tau).$$

For all $k \in \mathbb{N}$, define $\lambda_k := \lambda_1^\beta(\Omega, v_k)$ and let $\varphi_k := \varphi^\beta_{\Omega, v_k}$ be the corresponding eigenfunction, normalized with $\max_{\overline{\Omega}} \varphi_k = 1$. Since $(v_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega, \mathbb{R}^d)$ and the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded, $W^{2,p}$ elliptic estimates ([15, Theorem A.29]) show that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. Up to a subsequence, there exist $\varphi \in \bigcap_{1\leq p<\infty} W^{2,p}(\Omega)$ and $f \in L^\infty(\Omega)$ such that, as $k \to +\infty$,

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } W^{2,p}(\Omega)$$

for all $1 \leq p < \infty$,

$$\varphi_k \to \varphi \text{ strongly in } C^{1,\alpha}(\overline{\Omega})$$

for all $\alpha \in (0, 1)$, and

$$v_k \cdot \nabla\varphi_k \overset{*}{\rightharpoonup} f \text{ weak-}* \text{ in } L^\infty(\Omega).$$

As a consequence, after integrating the equation satisfied by φ_k against any function in $C_c^\infty(\Omega)$, passing to the limit as $k \to +\infty$ and recalling that both $\Delta\varphi$, f and φ are (at least) in $L^1_{loc}(\Omega)$, we finally obtain that

$$-\Delta\varphi + f = \underline{\lambda}^\beta(\Omega, \tau)\varphi \text{ a.e. in } \Omega$$

and

$$-\Delta\varphi - \tau\,|\nabla\varphi| \leq \underline{\lambda}^\beta(\Omega, \tau)\varphi \text{ a.e. in } \Omega.$$

Moreover, $\varphi \geq 0$ in Ω, $\|\varphi\|_\infty = 1$ and

$$\frac{\partial\varphi}{\partial\nu} + \beta\varphi = 0 \text{ on } \partial\Omega.$$

Define now $v \in L^\infty(\Omega, \mathbb{R}^d)$ by

$$v(x) := \begin{cases} -\tau\dfrac{\nabla\varphi(x)}{|\nabla\varphi(x)|} & \text{if } \nabla\varphi(x) \neq 0, \\ 0 & \text{if } \nabla\varphi(x) = 0. \end{cases} \tag{4.1}$$

Notice that $\|v\|_\infty \le \tau$, which entails that $\underline{\lambda}^\beta(\Omega,\tau) \le \lambda_1^\beta(\Omega,v)$. On the one hand,

$$(4.2)\quad -\Delta\varphi + v\cdot\nabla\varphi = -\Delta\varphi - \tau\,|\nabla\varphi| \le \underline{\lambda}^\beta(\Omega,\tau)\varphi \le \lambda_1^\beta(\Omega,v)\varphi \ \text{ a.e. in } \Omega.$$

On the other hand, $\varphi^\beta_{\Omega,v} > 0$ in $\overline{\Omega}$,

$$\frac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} = 0 = \frac{\partial\varphi}{\partial\nu} + \beta\varphi \ \text{ on } \partial\Omega$$

and $\|\varphi^\beta_{\Omega,v}\|_\infty = 1$. Lemma 2.1 applied with $(\mu,\beta,\psi,\varphi) := (\lambda_1^\beta(\Omega,v),\beta,\varphi^\beta_{\Omega,\tau},\varphi)$ yields

$$\varphi^\beta_{\Omega,v} = \varphi \ \text{ in } \overline{\Omega}.$$

As a consequence, all inequalities in (4.2) are equalities and

$$\underline{\lambda}^\beta(\Omega,\tau) = \lambda_1^\beta(\Omega,v).$$

Furthermore, since $\varphi \in W^{2,p}(\Omega)$ for each $1 \le p < \infty$, it follows that, for each $1 \le i \le d$, $\partial_{x_i}\varphi := \frac{\partial\varphi}{\partial x_i} \in W^{1,p}(\Omega)$ and then

$$|\nabla(\partial_{x_i}\varphi)| \times \mathbb{1}_{\{\partial_{x_i}\varphi = 0\}} = 0 \ \text{ a.e. in } \Omega$$

by Stampacchia's lemma, whence

$$\Delta\varphi \times \mathbb{1}_{\{\nabla\varphi = 0\}} = 0 \ \text{ a.e. in } \Omega.$$

Since $-\Delta\varphi + v\cdot\nabla\varphi = \lambda_1^\beta(\Omega,v)\varphi > 0$ a.e. in Ω, one gets that the set $\{x \in \Omega : \nabla\varphi(x) = 0\}$ is negligible. Therefore, in addition to $v\cdot\nabla\varphi = -\tau|\nabla\varphi|$ a.e. in Ω, (4.1) also entails that $|v(x)| = \tau$ for almost every $x \in \Omega$. The vector field $\underline{v} := v$ and the function

$$\underline{\varphi} := \varphi = \varphi^\beta_{\Omega,v}$$

then fulfill the required conclusions of part 1 of Theorem 1.3.

Let us now turn to the uniqueness result in part 1 of Theorem 1.3. Assume that $w \in L^\infty(\Omega,\mathbb{R}^d)$ is such that $\|w\|_\infty \le \tau$ and $\lambda_1^\beta(\Omega,w) = \underline{\lambda}^\beta(\Omega,\tau)$. One has

$$(4.3)\quad \begin{cases} -\Delta\varphi^\beta_{\Omega,v} + w\cdot\nabla\varphi^\beta_{\Omega,v} \ge -\Delta\varphi^\beta_{\Omega,v} - \tau|\nabla\varphi^\beta_{\Omega,v}| = \underline{\lambda}^\beta(\Omega,\tau)\varphi^\beta_{\Omega,v}, \\ -\Delta\varphi^\beta_{\Omega,w} + w\cdot\nabla\varphi^\beta_{\Omega,w} = \underline{\lambda}^\beta(\Omega,\tau)\varphi^\beta_{\Omega,w}, \end{cases}$$

a.e. in Ω, together with

$$\frac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} = 0 = \frac{\partial\varphi^\beta_{\Omega,w}}{\partial\nu} + \beta\varphi^\beta_{\Omega,w} \ \text{ on } \partial\Omega.$$

Furthermore, both functions $\varphi^\beta_{\Omega,v}$ and $\varphi^\beta_{\Omega,w}$ are positive (in $\overline{\Omega}$), with L^∞ norms equal to 1. Lemma 2.1 applied with

$$(\mu,\beta,\psi,\varphi) := (\underline{\lambda}^\beta(\Omega,\tau),\beta,\varphi^\beta_{\Omega,v},\varphi^\beta_{\Omega,w}),$$

and the vector field w instead of v, then entails

$$\varphi^\beta_{\Omega,v} = \varphi^\beta_{\Omega,w} \ \text{ in } \overline{\Omega}.$$

Consequently, the first line in (4.3) then yields

$$w \cdot \nabla \varphi^\beta_{\Omega,v} = -\tau |\nabla \varphi^\beta_{\Omega,v}| \ \text{ a.e. in } \Omega,$$

that is, $w \cdot \nabla\varphi = -\tau|\nabla\varphi|$ a.e. in Ω. Since $\nabla\varphi \neq 0$ a.e. in Ω and $\|w\|_\infty \le \tau$, one concludes that

$$w = -\tau \frac{\nabla\varphi}{|\nabla\varphi|} \ \text{ a.e. in } \Omega,$$

that is, $w = v$ a.e. in Ω.

Lastly, let $\lambda \in \mathbb{R}$ and $\phi \in \bigcap_{1\le p<\infty} W^{2,p}(\Omega)$ satisfy

$$\begin{cases} -\Delta\phi - \tau\,|\nabla\phi| = \lambda\phi \ \text{ and } \ \phi \ge 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} + \beta\phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\max_{\overline{\Omega}} \phi = 1$. Define $q \in L^\infty(\Omega, \mathbb{R}^d)$ by

$$q(x) := \begin{cases} -\tau \dfrac{\nabla\phi(x)}{|\nabla\phi(x)|} & \text{if } \nabla\phi(x) \neq 0, \\ 0 & \text{if } \nabla\phi(x) = 0. \end{cases}$$

Notice that $\|q\|_\infty \le \tau$. Since $-\tau|\nabla\phi| = q \cdot \nabla\phi$ a.e. in Ω, the nonnegativity of ϕ and the uniqueness of the pair of principal eigenvalue and principal normalized eigenfunction imply that

$$\lambda = \lambda^\beta_1(\Omega, q) \ge \underline{\lambda}^\beta(\Omega,\tau), \ \text{ and } \ \phi = \varphi^\beta_{\Omega,q} \text{ in } \overline{\Omega}.$$

Both functions $\phi = \varphi^\beta_{\Omega,q}$ and $\varphi = \varphi^\beta_{\Omega,v}$ are positive in $\overline{\Omega}$ with L^∞ norms equal to 1, and they satisfy

$$\begin{cases} -\Delta\varphi^\beta_{\Omega,q} + v\cdot\nabla\varphi^\beta_{\Omega,q} \ge -\Delta\varphi^\beta_{\Omega,q} - \tau|\nabla\varphi^\beta_{\Omega,q}| = \lambda^\beta_1(\Omega,q)\varphi^\beta_{\Omega,q}, \\ -\Delta\varphi^\beta_{\Omega,v} + v\cdot\nabla\varphi^\beta_{\Omega,v} = \underline{\lambda}^\beta(\Omega,\tau)\varphi^\beta_{\Omega,v} \le \lambda^\beta_1(\Omega,q)\varphi^\beta_{\Omega,v}, \end{cases} \tag{4.4}$$

a.e. in Ω, together with

$$\frac{\partial\varphi^\beta_{\Omega,q}}{\partial\nu} + \beta\varphi^\beta_{\Omega,q} = 0 = \frac{\partial\varphi^\beta_{\Omega,v}}{\partial\nu} + \beta\varphi^\beta_{\Omega,v} \ \text{ on } \partial\Omega.$$

Lemma 2.1 applied with $(\mu,\beta,\psi,\varphi) = (\lambda^\beta_1(\Omega,q), \beta, \varphi^\beta_{\Omega,q}, \varphi^\beta_{\Omega,v})$ then entails

$$\varphi^\beta_{\Omega,q} = \varphi^\beta_{\Omega,v} \ \text{ in } \overline{\Omega},$$

that is, $\phi = \varphi = \underline{\varphi}$ in $\overline{\Omega}$. Furthermore, all inequalities in (4.4) are equalities and

$$\underline{\lambda}^\beta(\Omega,\tau) = \lambda^\beta_1(\Omega,q),$$

whence $\lambda = \underline{\lambda}^\beta(\Omega,\tau)$. All properties in part 1 of Theorem 1.3 have now been proved.

Part 2. Notice that, for all $v \in L^\infty(\Omega)$, $\lambda^\beta_1(\Omega,v) \le \lambda^D_1(\Omega,v)$ by Lemma 2.2. Since

$$\sup_{v\in L^\infty(\Omega,\mathbb{R}^d),\, \|v\|_\infty\le\tau} \lambda^D_1(\Omega,v) < +\infty$$

by [8, Proposition 5.1], [17, Theorem 1.5] or [18, Theorem 6.6], it follows that the quantity $\overline{\lambda}^{\beta}(\Omega,\tau)$ defined in (1.8) is a real number. Then, arguments similar to those in part 1 above yield the conclusions of part 2.

Part 3. Consider now the case $\Omega=\Omega^*$ and denote

$$\phi:=\varphi^{\beta}_{\Omega^*,\tau e_r}.$$

This function ϕ is positive in $\overline{\Omega^*}$, it is of class $W^{2,p}(\Omega^*)$ for all $1\le p<\infty$, and $\max_{\overline{\Omega^*}}\phi=1$. For any $\mathcal{R}\in O(d)$ (the group of orthogonal transformations in $\mathbb{R}^d$), the function $\phi\circ\mathcal{R}$ satisfies the same equation as ϕ in Ω^* and the same boundary condition on $\partial\Omega^*$. The uniqueness of the pair of eigenvalue and principal normalized eigenfunction then entails that $\phi\circ\mathcal{R}=\phi$ in $\overline{\Omega^*}$ for any $\mathcal{R}\in O(d)$, that is, ϕ is radially symmetric in $\overline{\Omega^*}$. Let R denote the radius of Ω^*. For any $\sigma\in(0,R]$, there holds

$$-\Delta\phi+\tau e_r\cdot\nabla\phi=\lambda_1^{\beta}(\Omega^*,\tau e_r)\phi>0$$

almost everywhere in $\{x:|x|\le\sigma\}$ and ϕ is constant on the sphere $\{y:|y|=\sigma\}$. The weak maximum principle then implies that $\phi(x)\ge\phi(y)$ for all $|x|\le|y|=\sigma$, and the Hopf lemma even yields $e_r\cdot\nabla\phi(y)<0$ for all $|y|=\sigma$. As a conclusion, ϕ is radially decreasing and

$$\tau e_r\cdot\nabla\phi=-\tau|\nabla\phi|$$

everywhere in $\overline{\Omega^*}\setminus\{0\}$ and the function ϕ then fulfills (1.10) in Ω^* with $\lambda:=\lambda_1^{\beta}(\Omega^*,\tau e_r)$. It then follows from the last result of part 1 of the present theorem that

$$\underline{\lambda}^{\beta}(\Omega^*,\tau)=\lambda_1^{\beta}(\Omega^*,\tau e_r),$$

and the uniqueness of the vector field minimizing $\lambda_1^{\beta}(\Omega^*,v)$ implies that $\underline{v}=\tau e_r$.

By denoting $\psi:=\varphi^{\beta}_{\Omega^*,-\tau e_r}$, one proves similarly that ψ is radially decreasing and one still has $\tau e_r\cdot\nabla\psi=-\tau|\nabla\psi|$, that is, $-\tau e_r\cdot\nabla\psi=\tau|\nabla\psi|$, everywhere in $\overline{\Omega^*}\setminus\{0\}$. Part 2 of the present theorem then implies that

$$\overline{\lambda}^{\beta}(\Omega^*,\tau)=\lambda_1^{\beta}(\Omega^*,-\tau e_r)$$

and $\overline{v}=-\tau e_r$. The proof of Theorem 1.3 is thereby complete. □

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