Travelling fronts for the thermodiffusive system with arbitrary Lewis numbers

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Abstract

We consider KPP-type systems in a cylinder with an arbitrary Lewis number (the ratio of thermal and material diffusivities) in the presence of a shear flow. We show that traveling fronts solutions exist for all Lewis numbers and approach uniform limits at the two ends of the cylinder.

1 Introduction and main results

KPP-type reaction-diffusion systems

Reaction-diffusion systems of the form

$$\begin{aligned} T_t &= \Delta T + f(T)Y \\ Y_t &= \mathrm{Le}^{-1}\Delta Y - f(T)Y \end{aligned} \tag{1.1}$$

describe various processes in nature, ranging from chemical and biological contexts to combustion and many-particle systems. To fix ideas we will invoke the "combustion" terminology and refer to the function T as "temperature" and to the function Y as "concentration" below. In that context the Lewis number $\mathrm{Le} > 0$ is the ratio of thermal and material diffusivities. The system (1.1) is said to be of the KPP-type if $f \in C^1([0,+\infty);\mathbb{R})$ and

$$f(0) = 0 < f(s) \le f'(0)s,\ f'(s) \ge 0 \text{ for all } s > 0 \text{ and } f(+\infty) = +\infty. \tag{1.2}$$

We will assume that (1.2) holds throughout the paper as well as that f is of class $C^{1,\alpha}$ on an interval $[0, s_0]$ for some $s_0 > 0$.

When $\mathrm{Le} = 1$ the sum $T + Y = 1$ is constant, provided that this condition holds at $t = 0$, and the system (1.1) reduces to a single equation

$$T_t = \Delta T + f(T)(1 - T), \tag{1.3}$$

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which has been extensively studied since the pioneering work of Fisher [14] and Kolmogorov, Petrovskii and Piskunov [17]. In particular, in one dimension, $x \in \mathbb{R}$, this equation admits traveling front solutions of the form $T(t,x) = U(x - ct)$ for all $c \geq c_0 = 2\sqrt{f'(0)}$ with the function $U(x)$ which has the limits at infinity:

$$U(-\infty) = 1, \quad U(+\infty) = 0.$$

Such solutions attract general solutions of the Cauchy problem with front-like initial data with the correct exponential decay at infinity.

Much less is known for the KPP-system (1.1) than for the single equation (1.3). For example, to the best of our knowledge it is not known whether solutions of the Cauchy problem for (1.1) remain uniformly bounded in time, the best L^∞-bounds we are aware of grow in time as $\log\log t$ for large times [11]. Traveling front solutions for (1.1) were constructed in [8] in one dimension using ODE techniques. The result is the same as for (1.3): for all Lewis numbers traveling wave exists for each $c \geq c_0 = 2\sqrt{f'(0)}$.

Traveling waves for a KPP equation in a shear flow

Existence of non-planar traveling fronts for a single KPP-type equation in the presence of a shear flow has been investigated in [7]:

$$T_t + u(y)T_x = \Delta T + f(T)(1 - T). \tag{1.4}$$

This problem is now posed in an infinite cylinder

$$D = \{(x,y) : \ x \in \mathbb{R}, \ y \in \omega\},$$

with a regular domain ω with the Neumann boundary conditions along $\partial\Omega$:

$$\frac{\partial T(x,y)}{\partial n} = 0 \text{ for } x \in \mathbb{R}, \, y \in \partial\Omega,$$

and limits at infinity:

$$\lim_{x \to -\infty} T(t,x,y) = 1, \quad \lim_{x \to +\infty} T(t,x,y) = 0,$$

which are uniform in $y \in \overline{\omega}$ for each time t fixed. The function $u(y)$ is assumed to be of class $C^{0,\alpha}(\overline{\omega})$ (with $\alpha > 0$) and to have mean zero:

$$\int_\omega u(y)dy = 0. \tag{1.5}$$

It has been shown in [7] that (1.4) admits non-planar traveling fronts of the form $T(t,x,y) = U(x - ct, y)$ for all speeds $c \geq c^*$. Here the function $U(x,y)$ is the solution of

$$-cU_x + u(y)U_x = \Delta U + f(U)(1 - U),$$

with the boundary conditions

$$U(-\infty, y) = 1, \quad U(+\infty, y) = 0,$$

uniformly in $y \in \overline{\omega}$.

The minimal speed c^* is determined from an auxiliary eigenvalue problem as follows. Let $\mu(\lambda)$ be the principal eigenvalue of the following elliptic problem in the cross-section ω depending on a parameter $\lambda \in \mathbb{R}$:

$$\begin{cases} -\Delta_y \phi_\lambda - \lambda u(y)\phi_\lambda &= \mu(\lambda)\phi_\lambda \quad \text{in } \omega, \\ \dfrac{\partial \phi_\lambda}{\partial n} &= 0 \qquad\qquad \text{on } \partial\omega. \end{cases} \tag{1.6}$$

That is, $\mu(\lambda)$ is the unique eigenvalue of (1.6) that corresponds to an eigenfunction ϕ_λ which is positive in $\overline{\omega}$. Up to multiplication by positive constant, one can normalize the functions ϕ_λ so that $\|\phi_\lambda\|_{L^\infty(\omega)} = 1$ which is the convention we will use throughout the paper. The function $\mu(\lambda)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$ because of (1.5) (see [3, 7] for details and further properties of the function $\mu(\lambda)$). We also have the bounds

$$-\lambda\|u\|_\infty \le \mu(\lambda) \le 0 \text{ for all } \lambda \in \mathbb{R}.$$

With this notation the minimal speed c^* is given by

$$c^* = \min\left\{c \in \mathbb{R},\ \exists\, \lambda > 0,\ \mu(\lambda) = f'(0) - c\lambda + \lambda^2\right\} = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda} > 0, \tag{1.7}$$

where

$$k(\lambda) = f'(0) - \mu(\lambda) + \lambda^2. \tag{1.8}$$

The fact that c^* is well-defined can be easily seen from elementary geometric considerations using the aforementioned properties of the function $\mu(\lambda)$. Let us just mention that the reason the eigenvalue problem (1.6) determines the minimal speed is that if ϕ_λ satisfies (1.6) with $\mu(\lambda) = f'(0) - c\lambda + \lambda^2$ then the function

$$\psi(t, x, y) = e^{-\lambda(x-ct)}\phi_\lambda(y)$$

satisfies the linearized version of (1.4)

$$\psi_t + u(y)\psi_x = -\lambda(u(y) - c)\psi = \Delta\psi + f'(0)\psi,$$

which plays a crucial role for KPP-type equations.

Note that the minimal speed c^* does not depend on the Lewis number Le. For KPP systems (1.1), the observation that the travelling front minimal speed does not depend on the Lewis number has been first made in [8] in the one-dimensional case and, as we will see below, is also true for KPP-type systems with shear flows in higher dimensions. It follows from the fact that the fronts are pulled by the decaying temperature profile ahead of them. In this region, the temperature equation, which does not involve the Lewis number, plays a preponderant role in the selection of speeds. This observation does not generalize to other reaction types, such as ignition [9] or Arrhenius [18], for which the fronts are pushed by the whole reaction zone. For instance, for nonlinearities $f(T) = T^m$ with $m \ge 2$, the minimal speed of travelling fronts of (1.1) in the one-dimensional case is known to depend on the Lewis number [8].

Traveling waves for KPP systems in a shear flow

Existence of traveling waves for the thermo-diffusive system

$$\begin{cases} \dfrac{\partial T}{\partial t} + u(y)T_x = \Delta T + f(T)Y, \\ \dfrac{\partial Y}{\partial t} + u(y)Y_x = \mathrm{Le}^{-1}\Delta Y - f(T)Y \end{cases} \tag{1.9}$$

was first investigated in [3] with the heat-loss boundary conditions along $\partial\omega$:

$$\frac{\partial Y}{\partial n} = 0, \quad \frac{\partial T}{\partial n} + \sigma T = 0 \text{ on } \partial\Omega \tag{1.10}$$

with the Lewis number Le $= 1$. Here $\sigma > 0$ is the heat-loss parameter. Note that (1.10) does not preserve the constraint $T+Y = 1$ and thus (1.9) can not be reduced to a single equation in this situation. It has been shown in [3] that (1.9)-(1.10) admits non-planar traveling front solutions of the form $T(x - ct, y)$, $Y(x - ct, y)$ for all $c > c^*_\sigma$. The limiting conditions at infinity in the presence of the heat-loss are

$$T(-\infty, y) = T(+\infty, y) = 0, \quad Y(+\infty, y) = 0, \quad Y(-\infty, y) = Y_-, \tag{1.11}$$

where Y_- is the leftover concentration. The minimal speed c^*_σ is, once again, determined by (1.6)-(1.7) but with the boundary condition

$$\frac{\partial \phi_\lambda}{\partial n} + \sigma\phi_\lambda = 0 \text{ on } \partial\omega$$

replacing the Neumann boundary condition in (1.6). This existence result for traveling waves was generalized in [16] to all Lewis numbers Le > 0 but also with a positive heat-loss $\sigma > 0$. The main technical advantage of the problem with the heat-loss is that the L^∞ bounds on temperature are relatively easy to obtain.

Traveling waves for the KPP system with the adiabatic boundary conditions

The main result of the present paper is existence of traveling waves for all Lewis numbers Le > 0 for (1.9) with the Neumann boundary conditions (also known as adiabatic in the present context) both for the temperature T and concentration Y

$$\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 \text{ on } \partial D. \tag{1.12}$$

Non-planar travelling fronts are solutions of (1.9), (1.12) of the form $T(t, x, y) = \widetilde{T}(x - ct, y)$ and $Y(t, x, y) = \widetilde{Y}(x - ct, y)$, with a speed $c \in \mathbb{R}$. Therefore, we say that (c, T, Y) is a travelling front solution of (1.9), (1.12) if in the moving frame $x' = x - ct$ (we drop the primes and tildes immediately) the functions T and Y satisfy

$$\begin{cases} \Delta T + (c - u(y))T_x + f(T)Y = 0 & \text{in } D, \\ \mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - f(T)Y = 0 & \text{in } D, \\ \dfrac{\partial T}{\partial n} = \dfrac{\partial Y}{\partial n} = 0 & \text{on } \partial D, \end{cases} \tag{1.13}$$

together with the conditions far ahead of the front:

$$T(+\infty,\cdot)=0,\quad Y(+\infty,\cdot)=1, \tag{1.14}$$

which are uniform with respect to $y\in\overline{\omega}$. Throughout the paper, the solutions T and Y are understood to be of class C^2 in $\overline{D}$. Furthermore, the relative concentration Y is assumed to range in $[0,1]$ and is not identically equal to 1. The temperature T is nonnegative and not identically equal to 0.

The main result of the present paper is existence of traveling fronts for (1.13)-(1.14) for all $c\geq c^*$ with c^* still given by (1.7).

Theorem 1.1 *Let $Le>0$, then for each $c\geq c^*$, there exists a solution (T,Y) of (1.13)-(1.14) such that $T>0$, $0<Y<1$ in $\overline{D}$ and $T\in L^\infty(D)$. Moreover, T and Y satisfy the limiting conditions far behind the front: $T(-\infty,\cdot)=1$ and $Y(-\infty,\cdot)=0$ uniformly in $\overline{\omega}$.*

A special case Le $=+\infty$ and $f(T)=T$ was considered in [1]. In this situation the time-dependent problem can be reduced to a single scalar equation for $\Phi(t,x)=\int_0^t T(s,x)ds$. This was used in [1] to construct pulsating traveling wave solutions when the coefficients (either diffusivity or advection) are spatially periodic.

Let us mention that the situation is much less clear when $f(T)$ is not of the KPP type. For nonlinearities $f(T)$ of the ignition type, that is, when there exists an ignition temperature $\theta_0>0$ so that $f(T)=0$ for $T<\theta_0$ and $f(T)>0$ for $T>\theta_0$ existence of non-planar traveling waves was established in [4] and [12] for shear flows and in [13] for y-dependent nonlinearities, in both cases only for the Lewis numbers close to Le $=1$ using perturbation techniques and the inverse function theorem around the scalar case. The main difference with the KPP nonlinearities is that even if traveling waves exist for all Lewis numbers in the ignition problem one does not expect them to be stable because of the oscillatory and cellular instabilities [10, 15, 20]. For the one-dimensional system (1.1) with positive nonlinearities f of the type $f(T)=T^m$, one-dimensional instability in the temperature profiles is known to occur when Le and m are large enough [19] (extension of our existence results and qualitative bounds to this non-KPP case in the multidimensional setting remains an open problem). This instability is absent in the KPP case [21]. As the reader will see, we construct the KPP traveling wave using the by now standard procedure of restricting the problem to a finite cylinder and then passing to the limit of an infinite cylinder [5]. Heuristically, one expects this procedure to work and the a priori bounds to hold only if the traveling wave is in some sense stable, and it is the aforementioned presumed absence of cellular instabilities in the KPP case which makes our proof work from the physical point of view.

Outline of the proof of Theorem 1.1

The proof of Theorem 1.1 proceeds in several steps. The first step is to establish some a priori qualitative properties of traveling waves. As a preliminary step, we show that any travelling front solution of (1.13)-(1.14) with $T>0$ and $0<Y<1$ has its speed which is bounded from below by c^*. Furthermore, if T is bounded, then T and Y satisfy automatically the correct boundary conditions as $x\to-\infty$.

Proposition 1.2 *If (c, T, Y) is a solution of (1.13)-(1.14) such that $T > 0$ and $0 < Y < 1$ in $\overline{D}$ then $c \geq c^*$. Furthermore, if T is bounded, then $T(-\infty, \cdot) = 1$ and $Y(-\infty, \cdot) = 0$, both uniformly in $\overline{\omega}$.*

The proof of this proposition is presented in Section 2.

The second step is to establish existence of traveling waves for $c > c^*$. This is done in Section 3 using the by now standard procedure [5] of first constructing solutions in a finite cylinder $D_a = [-a, a] \times \omega$ and then passing to the limit $a \to +\infty$. This is what was also done in [3] and [16] in the KPP problem with a positive heat-loss. The main new ingredient of the present paper in this step is a uniform L^∞-bound on the temperature. It is obtained by a rather natural contradiction which comes from physical considerations: if the temperature is too large in some regions then the concentration would have to be very small there which would in turn bring down the temperature leading to a contradiction.

The last step in the proof, described in Section 5 is showing existence of traveling waves for the minimal speed $c = c^*$. We do this by taking the limit of the traveling waves (c_n, T_n, Y_n) with $c_n \downarrow c^*$. The main difficulty here is, once again, in establishing the uniform bound on T, which is obtained with the help of the following proposition.

Proposition 1.3 *Let (c_n, T_n, Y_n) be a sequence of solutions of (1.13)-(1.14) with $T_n > 0$, $0 < Y_n < 1$ in $\overline{D}$, $T_n \in L^\infty(D)$ for each $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} c_n < +\infty$. Then*

$$\sup_{n\in\mathbb{N}} \|T_n\|_{L^\infty(D)} < +\infty.$$

The proof of Proposition 1.3 is presented in Section 4.

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2 A priori qualitative properties of traveling fronts

This section is devoted to the proof of Proposition 1.2 on the a priori qualitative properties of arbitrary solutions (c, T, Y) of (1.13)-(1.14). We also prove further integral estimates which will be useful in the subsequent sections.

2.1 Proof of Proposition 1.2

A lower bound for the front speed: $c \geq c^*$

We first show that if (c, T, Y) is a solution of (1.13)-(1.14) such that $T > 0$ and $0 < Y < 1$ in $\overline{D}$ then $c \geq c^*$. As the functions Y and $f(T)/T$ are bounded in D, it follows from standard elliptic estimates and the Harnack inequality up to the boundary that the ratio $|\nabla T|/T$ is also bounded in D:

$$\frac{\nabla T}{T} \in L^\infty(\bar{D}). \tag{2.1}$$

Let Λ be defined by

$$\Lambda = -\liminf_{x\to+\infty}\left(\min_{y\in\overline{\omega}}\frac{T_x(x,y)}{T(x,y)}\right)$$

and let (x_n, y_n) be a sequence of points such that $x_n \to +\infty$ and

$$\frac{T_x(x_n,y_n)}{T(x_n,y_n)} \to \Lambda \ \text{ as } \ n\to+\infty.$$

Up to extraction of a subsequence, one can assume that $y_n \to y_\infty \in \overline{\omega}$ as $n \to +\infty$. Note that, since $T > 0$ in the cylinder $\overline{D}$, and $T(+\infty,\cdot) = 0$, the real number Λ is nonnegative:

$$\Lambda \geq 0. \tag{2.2}$$

Next, define the normalized and shifted temperature

$$T_n(x,y) = \frac{T(x+x_n,y)}{T(x_n,y_n)}$$

for all $n \in \mathbb{N}$ and $(x,y) \in \overline{D}$. Because of (2.1), the sequence of functions T_n is bounded in $L^\infty_{loc}(D)$. Each function T_n satisfies

$$\begin{cases} \Delta T_n + (c-u(y))T_{n,x} + \dfrac{f(T(x_n,y_n)T_n)\,Y_n}{T(x_n,y_n)} & = \ 0 \ \text{ in } D,\\[2mm] \dfrac{\partial T_n}{\partial n} & = \ 0 \ \text{ on } \partial D,\end{cases}$$

where

$$Y_n(x,y) = Y(x+x_n,y)$$

is the shifted concentration.

Recall that $T(x+x_n,y) \to 0$ and $Y(x+x_n,y) \to 1$ locally uniformly in $(x,y) \in \overline{D}$ as $n \to +\infty$ because of (1.14). It follows from standard elliptic estimates that, up to extraction of a subsequence, the sequence T_n converges weakly as $n \to +\infty$ in $W^{2,p}_{loc}(\overline{D})$, with $1 < p < +\infty$, to a function T_∞ which is a classical solution of

$$\begin{cases} \Delta T_\infty + (c-u(y))T_{\infty,x} + f'(0)T_\infty & = \ 0 \quad \text{in } D,\\[2mm] \dfrac{\partial T_\infty}{\partial n} & = \ 0 \quad \text{on } \partial D.\end{cases}$$

As $T_n(x,y) \geq 0$ and $T_n(0,y_n) = 1$, we have $T_\infty \geq 0$ in $\overline{D}$ and $T_\infty(0,y_\infty) = 1$, whence $T_\infty > 0$ in $\overline{D}$, as follows from the strong maximum principle and the Hopf lemma. Moreover, the function $z = T_{\infty,x}/T_\infty$ satisfies

$$z \geq -\Lambda \ \text{ in } \overline{D}$$

and $z(0,y_\infty) = -\Lambda$ owing to the definition of Λ and the choice of the sequence (x_n,y_n). However, the function z satisfies an elliptic equation

$$\begin{cases} \Delta z + 2\dfrac{\nabla T_\infty}{T_\infty}\cdot\nabla z + (c-u(y))z_x & = \ 0 \quad \text{in } D,\\[2mm] \dfrac{\partial z}{\partial n} & = \ 0 \quad \text{on } \partial D.\end{cases}$$

The strong maximum principle and Hopf lemma then yield $z(x,y) \equiv -\Lambda$ in $\overline{D}$. In other words, there exists a positive $C^2(\overline{\omega})$ function $\phi(y)$ such that $T_\infty(x,y) = e^{-\Lambda x}\phi(y)$ in $\overline{D}$. The function ϕ satisfies

$$\begin{cases} -\Delta_y \phi - \Lambda u(y)\phi &= (f'(0) - \Lambda c + \Lambda^2)\phi \quad \text{in } \omega, \\ \dfrac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial\omega. \end{cases}$$

By uniqueness of the positive solutions of (1.6), it follows that $\phi = \phi_\Lambda$ (up to multiplication by a positive constant), and

$$\mu(\Lambda) = f'(0) - \Lambda c + \Lambda^2.$$

Since $f'(0) > 0$, $\Lambda \geq 0$ and $\mu(0) = 0$, it follows that $\Lambda > 0$, whence $c \geq c^*$ from (1.7).

The left limits for temperature and concentration

Let us now assume that $T \in L^\infty(D)$ and prove that the limits

$$T(-\infty,\cdot) = 1 \text{ and } Y(-\infty,\cdot) = 0 \tag{2.3}$$

hold far behind the front, uniformly in $\overline{\omega}$. Notice first that, as both T and Y are uniformly bounded, the functions T and Y are globally $C^{2,\alpha}(\overline{D})$, from standard elliptic estimates up to the boundary.

We will obtain (2.3) from the integral bounds on the reaction rate and gradients of T and Y:

$$\int_D f(T)Y\,dxdy + \int_D |\nabla T|^2 dxdy + \int_D |\nabla Y|^2 dxdy < +\infty. \tag{2.4}$$

In order to get the bound on the reaction rate in (2.4) integrate equation (1.13) satisfied by T over a finite cylinder $D_A = (-A, A) \times \omega$, for any $A > 0$. We obtain

$$\int_{(-A,A)\times\omega} f(T(x,y))Y(x,y)\,dx\,dy = -\int_\omega \left[T_x(A,y) - T_x(-A,y)\right]\,dy$$
$$-\int_\omega \left[c - u(y)\right]\,\left[T(A,y) - T(-A,y)\right]\,dy.$$

The right-hand side is bounded independently of $A > 0$ because of the uniform bounds on T and T_x. Since $f(T)Y > 0$ in $\overline{D}$, one concludes that its integral over the whole cylinder is finite:

$$0 < \int_D f(T)Y < +\infty. \tag{2.5}$$

Next, to get the bound on ∇T in (2.4) multiply the equation for T in (1.13) by T and integrate over the same domain D_A, for any $A > 0$. One gets that

$$\int_{D_A} |\nabla T(x,y)|^2\,dx\,dy = \int_\omega \left[T(A,y)T_x(A,y) - T(-A,y)T_x(-A,y)\right]\,dy$$
$$+\frac{1}{2}\int_\omega \left[c - u(y)\right]\left[T(A,y)^2 - T(-A,y)^2\right]\,dy + \int_{D_A} f(T(x,y))Y(x,y)T(x,y)\,dx\,dy.$$

Since $0 < f(T)\,Y\,T \le f(T)\,Y\,\|T\|_{L^\infty(D)}$ in $\overline{D}$, the right-hand side is bounded independently of $A > 0$ because of (2.5). It follows that

$$\int_D |\nabla T|^2 < +\infty.$$

Similarly, by multiplying the Y-equation in (1.13) by Y and integrating over D_A for any $A > 0$, we obtain

$$\int_D |\nabla Y|^2 < +\infty.$$

Next, observe that for any sequence A_n converging to $+\infty$, the right-shifted functions $\tilde{T}(x,y) = T(x + A_n, y)$ and $\tilde{Y}(x,y) = Y(x + A_n, y)$ converge to 0 and 1 respectively, at least in $C^2_{loc}(\overline{D})$ sense, from standard elliptic estimates up to the boundary. Therefore, we also have

$$T_x(x,y) \to 0 \;\text{ and }\; Y_x(x,y) \to 0 \;\text{ as }\; x \to +\infty \text{ uniformly with respect to } \; y \in \overline{\omega}.$$

On the other hand, for any sequence $A_n \to +\infty$, the left-shifted functions $T(x - A_n, y)$ and $Y(x - A_n, y)$ are bounded in $C^{2,\alpha}(\overline{D})$. They converge, up to extraction of a subsequence and at least in $C^2_{loc}(\overline{D})$ sense, to a pair (T_∞, Y_∞) which solves the same equation (1.13) but which might a priori depend on the sequence A_n. Since $|\nabla T|$ and $|\nabla Y|$ are uniformly bounded, and the integrals of $|\nabla T|^2$ and $|\nabla Y|^2$ and of $f(T)Y$ converge over D, the functions T_∞ and Y_∞ have to be constants, that satisfy

$$0 \le T_\infty \le \|T\|_{L^\infty(D)}, \quad 0 \le Y_\infty \le 1 \;\text{ and }\; f(T_\infty)\,Y_\infty = 0.$$

Now, integrate the equation (1.13) satisfied by T over $(-A_n, A) \times \omega$ for any $A > 0$ and pass to the limits $A \to +\infty$ and $n \to +\infty$. Since $T(+\infty, \cdot) = T_x(+\infty, \cdot) = \lim_{n \to +\infty} T_x(-A_n, \cdot) = 0$ uniformly in $\overline{\omega}$, and since u has zero mean over ω, it follows that

$$c\,|\omega|\,T_\infty = \int_D f(T)Y > 0,$$

where $|\omega|$ denotes the Lebesgue-measure of ω. Similarly, we have

$$c\,|\omega|\,(1 - Y_\infty) = \int_D f(T)Y > 0.$$

Since we have already shown that $c \ge c^* > 0$, one gets that $T_\infty > 0$ and $T_\infty = 1 - Y_\infty$. Recall that $f(T_\infty)\,Y_\infty = 0$ and $f > 0$ on $(0, +\infty)$. It follows that $Y_\infty = 0$ and thus $T_\infty = 1$. Since the limits T_∞ and Y_∞ do not depend on the sequence A_n, one concludes that $T(-\infty, \cdot) = 1$ and $Y(-\infty, \cdot) = 0$. The proof of Proposition 1.2 is now complete. □

2.2 A priori boundary conditions for positive speeds

The same arguments as above, based on integrations by parts and compactness arguments, lead to the following result, which says that if a traveling front with a positive speed $c > 0$ exists then there is no leftover concentration behind the front and the temperature ahead of the front is equal to zero. We state it as a separate proposition, since we will use it later.

Proposition 2.1 *Let (c, T, Y) be a solution of (1.13) such that $c > 0$, $T > 0$ and $Y > 0$ in $\overline{D}$, and $T \in L^\infty(D)$, $Y \in L^\infty(D)$. Then $T(+\infty, \cdot) = Y(-\infty, \cdot) = 0$ uniformly in $\overline{\omega}$. Moreover, the limits $T(-\infty, \cdot)$ and $Y(+\infty, \cdot)$ exist, are independent of $y \in \overline{\omega}$ and are equal to each other:*

$$T_- := \lim_{x \to -\infty} T(x, y) = Y_+ := \lim_{x \to +\infty} Y(x, y),$$

uniformly in $y \in \overline{\omega}$.

Proof. As above, we may deduce that the three integrals

$$\int_D f(T)\, Y, \quad \int_D |\nabla T|^2 \text{ and } \int_D |\nabla Y|^2$$

of non-negative functions converge. Therefore, as before, for any sequence $A_n \to +\infty$, there exists a subsequence such that the functions $\tilde{T}_n^\pm(x, y) = T(x \pm A_n, y)$ and $\tilde{Y}_n^\pm(x, y) = Y(x \pm A_n, y)$ converge in $C^2_{loc}(\overline{D})$ as $n \to +\infty$ to some nonnegative constants $T_\pm$ and $Y_\pm$ that satisfy

$$f(T_\pm)\, Y_\pm = 0. \tag{2.6}$$

Integrating (1.13) over the domain $(-A_n, A_n) \times \omega$ and passing to the limit $n \to +\infty$ we see that then

$$0 < \int_D f(T)\, Y = c\,(Y_+ - Y_-)\,|\omega| = c\,(T_- - T_+)\,|\omega|. \tag{2.7}$$

As $c > 0$ and $f(T)Y > 0$ everywhere it follows that $Y_+ > Y_- \geq 0$. As a consequence, we conclude from (2.6) that $f(T_+) = 0$, whence $T_+ = 0$. For the same reason it also follows from (2.7) that $T_- > 0$, so that $Y_- = 0$, again from (2.6). Thus, $Y_+ = T_-$ are given by (2.7) and since the limits $T_\pm$ and $Y_\pm$ do not depend on the sequence A_n, the conclusion of the proposition follows. □

3 Existence of fronts with non-minimal speeds

In this section, we prove existence of (bounded) solutions (T, Y) of (1.13-1.14) for each speed $c > c^*$. The case of the minimal speed c^* will be treated separately.

The decay rate ahead of the front

Throughout the present section, we fix a speed $c \in (c^*, +\infty)$, with $c^* > 0$ defined in (1.7). For each $c > c^*$ consider the equation

$$c = \frac{k(\lambda)}{\lambda}, \tag{3.1}$$

with the function $k(\lambda) = f'(0) - \mu(\lambda) + \lambda^2$ defined in (1.8). Recall that the function $\mu(\lambda)$ is concave and satisfies $-\lambda\|u\|_\infty \leq \mu(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Moreover, we have $\mu(\lambda) \to 0$ as $\lambda \to 0$ and thus $k(\lambda)/\lambda \to +\infty$ as $\lambda \to 0^+$. It follows from the above and from the definition of c^* that for all $c > c^*$ equation (3.1) has two positive solutions. We let $\lambda_c > 0$ be the smallest positive root of (3.1). In other words, it is the smallest root of

$$\mu(\lambda_c) = f'(0) - c\lambda_c + \lambda_c^2. \tag{3.2}$$

As $k(\lambda)$ is convex, $k(0) > 0$ and λ_c is the smallest root of (3.1), we have $k'(\lambda_c) \leq c$. Furthermore, if $k'(\lambda_c) = c$, then $k(\lambda) \geq c\lambda$ for all $\lambda \in \mathbb{R}$ by convexity of k, whence $c^* \geq c$, which is impossible. We conclude that $k'(\lambda_c) < c$.

Pairs of sub- and supersolutions

As we have mentioned, we will construct a traveling wave (c, T, Y) by first restricting the problem to a finite cylinder $[-a, a] \times \omega$ and then passing to the limit $a \to +\infty$. In order to ensure that we obtain a non-trivial pair (T, Y) in the limit, we will need a pair of sub- and super-solutions for T and Y (the super-solution for Y is the constant 1), that we construct now.

1. Supersolution for T. Let $\overline{T}$ be the function defined in $\overline{D}$ by

$$\overline{T}(x, y) = \phi_{\lambda_c}(y)\, e^{-\lambda_c x} > 0.$$

Here ϕ_{λ_c} is the positive principal eigenfunction of (1.6) with $\lambda = \lambda_c$, and $\|\phi_{\lambda_c}\|_{L^\infty(\omega)} = 1$. Note that $\overline{T}$ satisfies the Neumann boundary conditions on ∂D, and that $\overline{T}$ is a super-solution to(1.13) with $Y = 1$, in the sense that

$$\Delta\overline{T} + (c - u(y))\overline{T}_x + f(\overline{T}) \leq \Delta\overline{T} + (c - u(y))\overline{T}_x + f'(0)\overline{T} = 0 \;\text{ in } \overline{D}.$$

2. Sub-solution for Y. Now, since $\mu(0) = \mu'(0) = 0 < c^* < c$, one can choose $\beta > 0$ small enough so that

$$\begin{cases} 0 < \beta < \lambda_c, \\ \mu(\beta\,\mathrm{Le}) - \beta^2 + c\,\beta\,\mathrm{Le} > 0 \end{cases} \tag{3.3}$$

and then $\gamma > 0$ large enough so that

$$\begin{cases} \gamma \times \min\limits_{\overline{\omega}} \phi_{\beta\mathrm{Le}} \geq 1, \\ \gamma\,\mathrm{Le}^{-1}\,(\mu(\beta\,\mathrm{Le}) - \beta^2 + c\,\beta\,\mathrm{Le}) \times \min\limits_{\overline{\omega}} \phi_{\beta\mathrm{Le}} \geq f'(0). \end{cases} \tag{3.4}$$

Define the function

$$\underline{Y}(x, y) = \max\left(0, 1 - \gamma\,\phi_{\beta\mathrm{Le}}(y)\, e^{-\beta x}\right).$$

Note that

$$\underline{Y}(x, y) = 0 \text{ for } x < 0,$$

since $\gamma \min_{\overline{\omega}} \phi_{\beta\mathrm{Le}} \geq 1$.

Let us check that $\underline{Y}$ is a subsolution for (1.13) with $T = \overline{T}$. Note first that

$$\frac{\partial\underline{Y}}{\partial n} = 0 \text{ on } \partial D.$$

Moreover, when $\underline{Y}(x,y) > 0$, then $x > 0$ and

$$
\begin{aligned}
&\mathrm{Le}^{-1}\Delta \underline{Y} + (c-u(y))\underline{Y}_x - f(\overline{T})\underline{Y}\\
&\ge \gamma\,\mathrm{Le}^{-1}\left(\mu(\beta\,\mathrm{Le}) - \beta^2 + c\,\beta\,\mathrm{Le}\right)\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x} - f'(0)\,\phi_{\lambda_c}(y)\,e^{-\lambda_c x}\left(1-\gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x}\right)\\
&\ge \gamma\,\mathrm{Le}^{-1}\left(\mu(\beta\,\mathrm{Le}) - \beta^2 + c\,\beta\,\mathrm{Le}\right)\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x} - f'(0)\,e^{-\beta x} \ge 0
\end{aligned}
$$

because of (1.2), (3.3)-(3.4) and since $0 < \phi_{\lambda_c}(y) \le 1$ in $\overline{\omega}$.

3. Sub-solution for T. We will now use the function $\underline{Y}$ to construct a sub-solution for T. Recall that $k(\lambda_c) = c\lambda_c$ and $k'(\lambda_c) < c$. Choose first $\eta > 0$ small enough so that

$$
\begin{cases}
0 < \eta < \min(\beta, \alpha\lambda_c),\\
\varepsilon := c(\lambda_c+\eta) - k(\lambda_c+\eta) > 0,
\end{cases}
\tag{3.5}
$$

where $\alpha > 0$ is such that f is of the class $C^{1,\alpha}([0,s_0])$ for some $s_0 > 0$. Then let $M \ge 0$ be such that

$$
f(s) \ge f'(0)s - Ms^{1+\alpha} \quad \text{for all } s \in [0,s_0],
\tag{3.6}
$$

and take $x_0 \ge 0$ sufficiently large so that

$$
\underline{Y}(x,y) = 1 - \gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x} \text{ for all } (x,y) \in (x_0,+\infty)\times\overline{\omega}.
$$

Next, choose $\delta > 0$ large enough so that the following conditions hold

$$
\begin{cases}
\phi_{\lambda_c}(y)\,e^{-\lambda_c x} - \delta\,\phi_{\lambda_c+\eta}(y)\,e^{-(\lambda_c+\eta)x} \le s_0 & \text{in } \overline{D},\\
\phi_{\lambda_c}(y)\,e^{-\lambda_c x} - \delta\,\phi_{\lambda_c+\eta}(y)\,e^{-(\lambda_c+\eta)x} \le 0 & \text{in } (-\infty,x_0]\times\overline{\omega},\\
\delta\,\varepsilon \times \min\limits_{\overline{\omega}} \phi_{\lambda_c+\eta} \ge f'(0)\,\gamma + M,
\end{cases}
\tag{3.7}
$$

with $\varepsilon > 0$ defined in (3.5). Finally, we set

$$
\underline{T}(x,y) = \max\left(0, \phi_{\lambda_c}(y)\,e^{-\lambda_c x} - \delta\,\phi_{\lambda_c+\eta}(y)\,e^{-(\lambda_c+\eta)x}\right)
$$

for all $(x,y) \in \overline{D}$.

The function $\underline{T}$ satisfies the Neumann boundary conditions:

$$
\frac{\partial \underline{T}}{\partial n} = 0 \text{ on } \partial D.
$$

Let us check that $\underline{T}$ is a sub-solution to (1.13) with $Y = \underline{Y}$. Note first that $0 \le \underline{T}(x,y) \le s_0$ for all $(x,y) \in \overline{D}$ and that if $\underline{T}(x,y) > 0$ then $x > x_0 \ge 0$, whence

$$
0 \le \underline{Y}(x,y) = 1 - \gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x} \qquad \text{if } \underline{T}(x,y) > 0.
$$

Therefore, if $\underline{T}(x,y) > 0$ then

$$
\begin{aligned}
&\Delta \underline{T} + (c-u(y))\underline{T}_x + f(\underline{T})\,\underline{Y}\\
&\ge \Delta \underline{T} + (c-u(y))\underline{T}_x + \left(f'(0)\underline{T} - M\underline{T}^{1+\alpha}\right)\left(1-\gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x}\right)\\
&\ge -\delta\,\left(k[\lambda_c+\eta] - c(\lambda_c+\eta)\right)\,\phi_{\lambda_c+\eta}(y)\,e^{-(\lambda_c+\eta)x} - f'(0)\,\gamma\,\underline{T}\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta x} - M\,\underline{T}^{1+\alpha}\\
&\ge \delta\,\varepsilon\,\phi_{\lambda_c+\eta}(y)\,e^{-(\lambda_c+\eta)x} - f'(0)\,\gamma\,e^{-(\lambda_c+\beta)x} - M\,e^{-\lambda_c(1+\alpha)x}\\
&\ge \left(\delta\,\varepsilon\,\phi_{\lambda_c+\eta}(y) - f'(0)\,\gamma - M\right)e^{-(\lambda_c+\eta)x} \ge 0
\end{aligned}
\tag{3.8}
$$

because of (3.2), (3.5)-(3.7) and since $0 < \phi_{\lambda_c+\eta}(y),\ \phi_{\beta\,\mathrm{Le}}(y) \le 1$ in $\overline{\omega}$.

The finite cylinder problem

This step follows a part of Section 4 of [3]: here we construct a solution of (1.13) in a finite cylinder $D_a = (-a, a) \times \omega$. In the last step we will pass to the limit $a \to +\infty$. For a given $a > 0$, let $C(\overline{D_a})$ denote the space of continuous functions in $\overline{D_a}$, with the usual sup-norm. Observe that $0 \le \underline{T} < \overline{T}$ and $0 \le \underline{Y} < 1$ in $\overline{D}$, and denote by E_a the set

$$E_a = \{(T, Y) \in C(\overline{D_a}; \mathbb{R}^2),\ \underline{T} \le T \le \overline{T} \text{ and } \underline{Y} \le Y \le 1 \text{ in } \overline{D_a}\}.$$

The set E_a is a convex closed bounded subset of the Banach space $C(\overline{D_a}; \mathbb{R}^2)$.

We now set up a fixed point problem for an approximation of the traveling wave in D_a. For any pair $(T_0, Y_0) \in E_a$, let $(T, Y) = \Phi_a(T_0, Y_0)$ be the unique solution of

$$\begin{cases} \Delta T + (c - u(y))T_x &= -f(T_0)Y_0 \quad \text{in } D_a, \\ \mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - f(T_0)Y &= 0 \quad\quad\quad\ \ \text{in } D_a, \end{cases}$$

with the boundary conditions

$$\begin{cases} T(\pm a, y) = \underline{T}(\pm a, y),\ Y(\pm a, y) = \underline{Y}(\pm a, y) & \text{for } y \in \overline{\omega}, \\ \dfrac{\partial T}{\partial n} = \dfrac{\partial Y}{\partial n} = 0 & \text{on } [-a, a] \times \partial\omega. \end{cases}$$

Such a solution (T, Y) exists, it belongs to $C(\overline{D_a}; \mathbb{R}^2)$ and it is unique (see [2, 6]). Our next goal is to show that the map Φ_a has a fixed point. To this end we will show that Φ_a leaves the set E_a invariant and that the map Φ_a is compact.

1. The set E_a is invariant. Let us now check that the mapping Φ_a leaves the set E_a invariant:

$$\Phi_a(E_a) \subset E_a. \tag{3.9}$$

To do so, choose any $(T_0, Y_0) \in E_a$ and denote $(T, Y) = \Phi_a(T_0, Y_0)$. Given any $(T_0, Y_0) \in E_a$, using (3.8), monotonicity of $f(s)$ in s (see (1.2)) and the definition of the set E_a, it is immediate to verify that the function $\underline{T}$ satisfies the inequality

$$\Delta\underline{T} + (c - u(y))\underline{T}_x \ge -f(\underline{T})\underline{Y} \ge -f(T_0)Y_0,$$

in the sense of distributions in D_a. Furthermore, $\underline{T}$ satisfies the same boundary conditions as T on the boundary of D_a. The weak maximum principle implies that $\underline{T} \le T$ in $\overline{D_a}$. The inequalities $T \le \overline{T}$, $\underline{Y} \le Y$ and $Y \le 1$ in $\overline{D_a}$ can be checked similarly. We conclude that (3.9) holds.

2. The map Φ_a is compact. This is a rather standard fact. We introduce $(h_1, k_1) = \Phi_a(\overline{T}, 1)$ and $(h_2, k_2) = \Phi_a(\underline{T}, 1)$. For any pair $(T_0, Y_0) \in E_a$ and $(T, Y) = \Phi_a(T_0, Y_0)$, one has

$$\Delta h_1 + (c - u(y))h_{1,x} = -f(\overline{T}) \le -f(T_0)Y_0 \ \text{ in } D_a,$$

and thus $T \le h_1$ in $\overline{D_a}$ (recall that h_1 satisfies the same boundary conditions as T). Similarly, using monotonicity of f one checks that

$$\mathrm{Le}^{-1}\Delta k_2 + (c - u(y))k_{2,x} - f(T_0)k_2 = (f(\underline{T}) - f(T_0))k_2 \le 0 \ \text{ in } D_a,$$

so that $Y \leq k_2$ in $\overline{D_a}$. Thus we obtain

$$\begin{cases} \underline{T} \leq T \leq h_1 \leq \overline{T} \\ \underline{Y} \leq Y \leq k_2 \leq 1 \end{cases} \text{ in } \overline{D_a} \tag{3.10}$$

for all $(T_0, Y_0) \in E_a$ and $(T, Y) = \Phi_a(T_0, Y_0)$.

Let (T_0^n, Y_0^n) be a sequence in E_a, and set

$$(T^n, Y^n) = \Phi_a(T_0^n, Y_0^n).$$

As follows from the standard elliptic estimates up to the boundary, the sequence (T^n, Y^n) is bounded in $C^1(K; \mathbb{R}^2)$ norm, for any compact subset

$$K \subset \Sigma_a = \overline{D_a} \setminus \{\pm a\} \times \partial\omega.$$

Therefore, using the diagonal extraction process, there exists a subsequence, still denoted by (T^n, Y^n), which converges locally uniformly in Σ_a to a pair (T, Y) of continuous functions in Σ_a. Since each (T^n, Y^n) satisfies (3.10) in $\overline{D_a}$, it follows that (T, Y) satisfies (3.10) in Σ_a. As we have

$$\begin{cases} h_1(\pm a, y) = \underline{T}(\pm a, y), \\ k_2(\pm a, y) = \underline{Y}(\pm a, y) \end{cases}$$

for all $y \in \omega$, and both $\underline{T}$, $\underline{Y}$, h_1 and k_2 are continuous in $\overline{D_a}$, the functions (T, Y) can be extended in $\overline{D_a}$ to two continuous functions, still denoted by (T, Y), satisfying (3.10) in $\overline{D_a}$. For any $\varepsilon > 0$, there exists $\kappa > 0$ such that

$$\begin{cases} 0 \leq h_1 - \underline{T} \leq \varepsilon \\ 0 \leq k_2 - \underline{Y} \leq \varepsilon \end{cases} \text{ in } [-a, -a+\kappa] \times \overline{\omega} \,\cup\, [a-\kappa, a] \times \overline{\omega},$$

and thus $|T^n - T| \leq \varepsilon$ and $|Y^n - Y| \leq \varepsilon$ in the same sets, for all n. On the other hand, (T^n, Y^n) converges uniformly in $[-a+\kappa, a-\kappa] \times \overline{\omega}$ to (T, Y). Therefore, (T^n, Y^n) converges uniformly to (T, Y) in $[-a, a] \times \overline{\omega}$ and thus the map Φ_a is compact.

3. A fixed point of Φ_a. As a consequence, the set $\overline{\Phi(E_a)}$ is compact in E_a. One concludes from the Schauder fixed point theorem that Φ_a has a fixed point in E_a. In other words, there exists a classical solution $(T_a, Y_a) \in E_a$ of problem

$$\begin{cases} \Delta T_a + (c - u(y))T_{a,x} + f(T_a)Y_a = 0 & \text{in } D_a, \\ \mathrm{Le}^{-1}\Delta Y_a + (c - u(y))Y_{a,x} - f(T_a)Y_a = 0 & \text{in } D_a \end{cases} \tag{3.11}$$

with the boundary conditions

$$T_a(\pm a, y) = \underline{T}(\pm a, y),\ \ Y_a(\pm a, y) = \underline{Y}(\pm a, y) \ \text{ for } y \in \overline{\omega}, \tag{3.12}$$

and

$$\frac{\partial T_a}{\partial n} = \frac{\partial Y_a}{\partial n} = 0 \ \text{ on } [-a, a] \times \partial\omega. \tag{3.13}$$

Furthermore, we have $0 \leq \underline{T} \leq T_a \leq \overline{T}$ and $0 \leq \underline{Y} \leq Y_a \leq 1$ in $[-a, a] \times \overline{\omega}$.

Passage to the infinite cylinder

Finally, let a_n be an increasing sequence of positive numbers such that $a_n \to +\infty$ as $n \to +\infty$. Let (T_{a_n}, Y_{a_n}) be a sequence of solutions of (3.11)-(3.13) with $a = a_n$. We know from the standard elliptic estimates up to the boundary that the sequence of functions (T_{a_n}, Y_{a_n}) is then bounded in, say, $C^{2,\alpha}_{loc}(\overline{D})$ (remember that the flow u is of class $C^{0,\alpha}(\overline{\omega})$ and f is locally Lipschitz-continuous). Up to extraction of some subsequence, the functions (T_{a_n}, Y_{a_n}) converge in $C^2_{loc}(\overline{D})$ to a pair (T, Y) of $C^2(\mathbb{R} \times \overline{\omega})$ solutions (T, Y) of

$$\begin{cases} \Delta T + (c - u(y))T_x + f(T)Y = 0 & \text{in } \mathbb{R} \times \overline{\omega}, \\ \mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - f(T)Y = 0 & \text{in } \mathbb{R} \times \overline{\omega}, \\ \dfrac{\partial T}{\partial n} = \dfrac{\partial Y}{\partial n} = 0 & \text{on } \mathbb{R} \times \partial\omega, \\ 0 \le \underline{T} \le T \le \overline{T}, \quad 0 \le \underline{Y} \le Y \le 1 & \text{in } \mathbb{R} \times \overline{\omega}. \end{cases}$$

In particular, T and Y approach their limits $T(+\infty, y) = 0$ and $Y(+\infty, y) = 1$ uniformly in $y \in \overline{\omega}$, and the pair (T, Y) solves (1.13)-(1.14). Furthermore, the strong maximum principle implies that $Y > 0$ and $T > 0$ in $\overline{D}$ (note that $\underline{Y}(x, y)$ and $\underline{T}(x, y)$ are positive for large x and thus $T \not\equiv 0$ and $Y \not\equiv 0$). Since $f(T) > 0$, the function Y cannot be identically equal to 1, whence $Y < 1$ in $\overline{D}$ from the strong maximum principle.

Boundedness of T

The last step in the proof of Theorem 1.1 in the case $c > c^*$ is to show that the solution (T, Y) that we just have constructed has uniformly bounded temperature: $T \in L^\infty(D)$. Assume for the sake of a contradiction that $T \notin L^\infty(D)$.

1. The function T blows up on the left. Since $0 \le \underline{T} \le T \le \overline{T}$ in $\overline{D}$, the only possibility for the function T to grow is on the left. Thus there exists then a sequence (x_n, y_n) of points in $\overline{D}$ such that

$$T(x_n, y_n) \to +\infty \quad \text{and} \quad x_n \to -\infty \quad \text{as } n \to +\infty.$$

One can assume without loss of generality that the sequence x_n is decreasing. Since the function $|\nabla T|/T$ is globally bounded (from the Schauder and Harnack estimates up to the boundary), it follows that

$$m_n := \min_{y \in \overline{\omega}} T(x_n, y) \to +\infty$$

as $n \to +\infty$. Furthermore, the function T satisfies

$$\Delta T + (c - u(y))T_x = -f(T)Y < 0 \quad \text{in } \overline{D},$$

together with Neumann boundary conditions on ∂D. Hence, the function T can not attain a local minimum inside $\overline{D}$, and we have, for all $n < p$:

$$T \ge \min(m_n, m_p) \quad \text{in } [-x_p, -x_n] \times \overline{\omega}.$$

This implies that

$$T(x,y) \to +\infty \text{ as } x \to -\infty, \tag{3.14}$$

uniformly in $y \in \overline{\omega}$.

2. An upper bound for $Y(x,y)$ on the left. Define now

$$M = \max_{y\in\overline{\omega}} |c - u(y)| \geq c > 0,$$

and

$$m = \min_{(x,y)\in(-\infty,0]\times\overline{\omega}} f(T(x,y)).$$

It follows from (3.14) that $m > 0$. We also set

$$\overline{\lambda} = \frac{-M + \sqrt{M^2 + 4\,\mathrm{Le}^{-1} m}}{2\,\mathrm{Le}^{-1}} > 0.$$

For an arbitrary $x_0 < 0$, define

$$\overline{Y}(x,y) = e^{\overline{\lambda}(x_0 - x)} + e^{\overline{\lambda} x}.$$

The functions Y and $\overline{Y}$ satisfy the same Neumann boundary conditions on ∂D, together with $Y(x,y) \leq 1 \leq \overline{Y}(x,y)$ for $x = x_0$ and $x = 0$ and all $y \in \overline{\omega}$. Furthermore, there holds

$$\mathrm{Le}^{-1}\Delta Y + (c - u(y))Y_x - mY \geq 0 \ \text{ in } (-\infty, 0] \times \overline{\omega},$$

and

$$\mathrm{Le}^{-1}\Delta\overline{Y} + (c - u(y))\overline{Y}_x - m\overline{Y} \leq (\mathrm{Le}^{-1}\overline{\lambda}^2 + M\overline{\lambda} - m)\overline{Y} = 0 \ \text{ in } \overline{D}.$$

It follows from the maximum principle that

$$0 \leq Y(x,y) \leq \overline{Y}(x,y) = e^{\overline{\lambda}(x_0-x)} + e^{\overline{\lambda}x}$$

for all $(x,y) \in [x_0, 0] \times \overline{\omega}$. Since this is true for all $x_0 < 0$, the passage to the limit as $x_0 \to -\infty$ yields

$$0 \leq Y(x,y) \leq e^{\overline{\lambda}x} \ \text{ for all } (x,y) \in (-\infty, 0] \times \overline{\omega}. \tag{3.15}$$

3. The function T is actually bounded. Now, choose $\lambda \in \mathbb{R}$ so that

$$0 < \lambda < \overline{\lambda} \ \text{ and } \ \mu(-\lambda) - \lambda^2 - c\lambda < 0. \tag{3.16}$$

This is possible since μ is nonpositive and $c > 0$. On the other hand, we know from (3.2) that the positive real number λ_c satisfies

$$-\mu(\lambda_c) + \lambda_c^2 - c\lambda_c = -f'(0) < 0.$$

Thus, there exists $\rho > \lambda_c$ such that

$$-\mu(\rho) + \rho^2 - c\rho < 0.$$

Because of (1.2) and (3.15)-(3.16), there exists $x_1 \le 0$ such that all of the following conditions hold:

$$\begin{cases} -\mu(\rho) + \rho^2 - c\rho + \dfrac{f(T)Y}{T} \le 0 & \text{in } (-\infty, x_1] \times \overline{\omega}, \\ [\mu(-\lambda) - \lambda^2 - c\lambda] \times \min\limits_{\overline{\omega}} \phi_{-\lambda} + f'(0)\, e^{(\overline{\lambda}-\lambda)x} \le 0 & \text{in } (-\infty, x_1] \times \overline{\omega}, \\ e^{\lambda x_1} \times \max\limits_{\overline{\omega}} \phi_{-\lambda} \le \dfrac{1}{2}. \end{cases} \tag{3.17}$$

Then, set

$$A = 2 \times \max_{y \in \overline{\omega}} T(x_1, y) > 0. \tag{3.18}$$

Let now U and $\overline{U}$ be the functions defined in $\overline{D}$ by

$$U(x,y) = \frac{e^{\rho x}\, T(x,y)}{\phi_\rho(y)} \quad \text{and} \quad \overline{U}(x,y) = \frac{A\, e^{\rho x}\, (1 - \phi_{-\lambda}(y)\, e^{\lambda x})}{\phi_\rho(y)}.$$

Our goal is to show that $U \le \overline{U}$ in $(-\infty, x_1] \times \overline{\omega}$, which would finally imply that T is bounded. The function U is positive in $\overline{D}$, while $\overline{U}$ is positive in $(-\infty, x_1] \times \overline{\omega}$, from the third condition on x_1 in (3.17). Since

$$0 < T(x,y) \le \overline{T}(x,y) = \phi_{\lambda_c}(y)\, e^{-\lambda_c x} \quad \text{in } \overline{D},$$

and $\rho > \lambda_c$, we have $U(-\infty, \cdot) = 0$. It is also true that $\overline{U}(-\infty, \cdot) = 0$. Furthermore,

$$U(x_1, y) \le \overline{U}(x_1, y) \quad \text{for all } y \in \overline{\omega},$$

again, from the third assertion in (3.17) and the choice of A in (3.18). Notice also that

$$\frac{\partial U}{\partial n} = \frac{\partial \overline{U}}{\partial n} = 0 \text{ on } \partial D.$$

It is straightforward to check that

$$\Delta U + B(x,y) \cdot \nabla_{x,y} U + C(x,y) U = 0 \quad \text{in } \overline{D},$$

where

$$B(x,y) = (c - u(y) - 2\,\rho, 2\,\phi_\rho(y)^{-1} \nabla_y \phi_\rho(y))$$

and

$$C(x,y) = -\mu(\rho) + \rho^2 - c\rho + \frac{f(T)Y}{T} \le 0 \quad \text{in } (-\infty, x_1] \times \overline{\omega},$$

from the first assertion in (3.17). Set now

$$\overline{V}(x,y) = 1 - \phi_{-\lambda}(y)\, e^{\lambda x}.$$

As can be verified directly, in $(-\infty,x_1]\times\overline{\omega}$, the function $\overline{U}$ satisfies

$$\Delta\overline{U}+B(x,y)\cdot\nabla_{x,y}\overline{U}+C(x,y)\overline{U}=\frac{A\,e^{\rho x}}{\phi_\rho(y)}\times\left[\Delta\overline{V}+(c-u(y))\overline{V}_x+\frac{f(T)\,Y\,\overline{V}}{T}\right]$$
$$\le\frac{A\,e^{\rho x}}{\phi_\rho(y)}\times\left[(\mu(-\lambda)-\lambda^2-c\lambda)\,\phi_{-\lambda}(y)\,e^{\lambda x}+f'(0)\,e^{\overline{\lambda}x}\right]\le 0,$$

from (3.16) and the second assertion in (3.17). The weak maximum principle then implies that

$$U(x,y)\le\overline{U}(x,y)\ \text{ in }(-\infty,x_1]\times\overline{\omega}.$$

Therefore, we have

$$T(x,y)\le A\,(1-\phi_{-\lambda}(y)\,e^{\lambda x})\text{ in }(-\infty,x_1]\times\overline{\omega},$$

which contradicts (3.14). This contradiction shows that the function T is bounded in D and the proof of Theorem 1.1 in the case $c>c^*$ is complete.

4 Proof of Proposition 1.3

Let (c_n,T_n,Y_n) be a sequence of solutions of (1.13)-(1.14) such that $T_n>0$, $0<Y_n<1$ in $\overline{D}$ and $T_n\in L^\infty(D)$ for each $n\in\mathbb{N}$. Assume in addition that

$$\sup_{n\in\mathbb{N}}c_n<+\infty.$$

This implies that the sequence c_n is bounded, since $c_n\ge c^*>0$ for each $n\in\mathbb{N}$ according to Proposition 1.2. Up to extraction of a subsequence, one can assume that $c_n\to c_\infty\in[c^*,+\infty)$ as $n\to+\infty$.

Assume now, for the sake of a contradiction, that the sequence $\|T_n\|_{L^\infty(D)}$ is not bounded. Up to extraction of another subsequence, one can assume without loss of generality that $\|T_n\|_{L^\infty(D)}\to+\infty$ as $n\to+\infty$ and $\|T_n\|_{L^\infty(D)}>1$ for each $n\in\mathbb{N}$. Once again, from Proposition 1.2, we know that each pair (T_n,Y_n) satisfies $T_n(-\infty,\cdot)=1$ and $Y_n(-\infty,\cdot)=0$. Then the boundary conditions (1.14) imply that each T_n attains a maximum inside the cylinder $\overline{D}$, and there exists a sequence of points (x_n,y_n) in $\overline{D}$ such that

$$T_n(x_n,y_n)=\max_{\overline{D}}T_n\to+\infty\ \text{ as }\ n\to+\infty.$$

After yet another extraction of a subsequence we may assume that $y_n\to y_\infty\in\overline{\omega}$ as $n\to+\infty$.

Define now the normalized shifts

$$U_n(x,y)=\frac{T_n(x+x_n,y)}{T_n(x_n,y_n)}.$$

Each function U_n satisfies $0<U_n\le 1$ in $\overline{D}$ and solves

$$\begin{cases}\Delta U_n+(c_n-u(y))U_{n,x}+\dfrac{f(T_n(x_n,y_n)U_n)}{T_n(x_n,y_n)}\,Z_n &=\ 0\ \text{ in }D,\\[2mm] \dfrac{\partial U_n}{\partial n} &=\ 0\ \text{ on }\partial D.\end{cases}$$

where

$$Z_n(x,y) = Y_n(x + x_n, y)$$

is the shifted concentration.

In order to pass to the limit as $n \to +\infty$, we shall use the following lemma that says that a very high temperature may be achieved only at the expense of a small concentration:

Lemma 4.1 *Let* $(\widetilde{c}_n, \widetilde{T}_n, \widetilde{Y}_n)$ *be a sequence of solutions of* (1.13) *such that* $\sup_{n\in\mathbb{N}} |\widetilde{c}_n| < +\infty$, $\widetilde{T}_n > 0$, $0 < \widetilde{Y}_n < 1$ *in* $\overline{D}$ *and assume that there exists a sequence of points* $(\widetilde{x}_n, \widetilde{y}_n)$ *in* $\overline{D}$ *such that*

$$\widetilde{T}_n(\widetilde{x}_n, \widetilde{y}_n) \to +\infty \quad \textit{as } n \to +\infty.$$

Then

$$\max_{(x,y)\in K} \widetilde{Y}_n(x + \widetilde{x}_n, y) \to 0 \quad \textit{as} \quad n \to +\infty$$

for any compact $K \subset \overline{D}$.

We postpone the proof of this lemma until the end of this section. Let us now complete the proof of Proposition 1.3. It follows from Lemma 4.1 that the functions Z_n converge to 0 locally uniformly in $\overline{D}$. Since the functions U_n are uniformly bounded (by 1) in $L^\infty(D)$, and since

$$0 < \frac{f(T_n(x_n, y_n)U_n)}{T_n(x_n, y_n)} \le f'(0)\, U_n \le f'(0) \ \text{ in } D,$$

by the KPP property (1.2) of the function $f(s)$, the functions U_n converge as $n \to +\infty$, up to extraction of a subsequence and in all $W^{2,p}_{loc}(\overline{D})$ weak (with $1 < p < +\infty$) to a function U_∞ which satisfies

$$\left\{ \begin{array}{rcl} \Delta U_\infty + (c_\infty - u(y))U_{\infty,x} & = & 0 \ \text{ in } D, \\ \dfrac{\partial U_\infty}{\partial n} & = & 0 \ \text{ on } \partial D. \end{array} \right.$$

Furthermore, $0 \le U_\infty \le 1$ and $U_\infty(0, y_\infty) = 1$. The strong maximum principle and the Hopf lemma imply that $U_\infty = 1$ in $\overline{D}$. As a consequence, $\nabla U_n \to 0$ locally uniformly in $\overline{D}$ as $n \to +\infty$.

Integrate now the equation (1.13) satisfied by T_n over a finite cylinder $(x_n, A) \times \omega$ and pass to the limit as $A \to +\infty$. As in the proof of Proposition 1.2, the contribution of the boundary terms at $x = A$ vanishes as $A \to +\infty$ and we get

$$\int_{(x_n,+\infty)\times\omega} f(T_n(x,y))\, Y_n(x,y)\, dx\, dy = \int_\omega [T_{n,x}(x_n, y) + c_n T_n(x_n, y) - u(y) T_n(x_n, y)]\, dy. \quad (4.1)$$

On the other hand, we know that $f(T_n)\, Y_n > 0$ in $\overline{D}$, and, moreover, integrating the equation for T_n in (1.13) we get

$$\int_D f(T_n)\, Y_n = c_n\, |\omega|$$

from Proposition 1.2, as $T_n(-\infty, \cdot) = 1$. After dividing (4.1) by $T_n(x_n, y_n)$, it follows that

$$\frac{c_n\, |\omega|}{T_n(x_n, y_n)} \ge \int_\omega [U_{n,x}(0, y) + c_n U_n(0, y) - u(y) U_n(0, y)]\, dy.$$

As we have shown that $|U_n(0,y)-1|+|U_{n,x}(0,y)|\to 0$ uniformly with respect to $y\in\overline{\omega}$ and $T_n(x_n,y_n)\to+\infty$ as $n\to+\infty$, and since u has zero mean over ω, one concludes that $0\geq c_\infty|\omega|\geq c^*|\omega|>0$, which is impossible. Therefore, the sequence $\|T_n\|_{L^\infty(D)}$ is bounded and the proof of Proposition 1.3 is complete. □

Proof of Lemma 4.1

Since the functions $\widetilde{Y}_n$ and $f(\widetilde{T}_n)/\widetilde{T}_n$ are bounded in D uniformly with respect to $n\in\mathbb{N}$, it follows from Harnack inequality up to the boundary that

$$\widetilde{T}_n(x+\widetilde{x}_n,y)\to+\infty \ \text{ as } \ n\to+\infty \ \text{ locally uniformly in } \ (x,y)\in\overline{D}. \tag{4.2}$$

Let K be any compact set in $\overline{D}$. Take $a\geq 1$ such that $K\subset[-a+1,a-1]\times\overline{\omega}$. Define also, for each $n\in\mathbb{N}$:

$$M=\sup_{n\in\mathbb{N},\ y\in\overline{\omega}}|\widetilde{c}_n-u(y)|<+\infty,$$

and

$$m_n=\min_{(x,y)\in[-a,a]\times\overline{\omega}} f(\widetilde{T}_n(x+\widetilde{x}_n,y))\in(0,+\infty).$$

Observe that (4.2) and the fact that $f(+\infty)=+\infty$ imply that $m_n\to+\infty$ as $n\to+\infty$. For each $n\in\mathbb{N}$, define

$$\lambda_n=\frac{-M+\sqrt{M^2+4\,\mathrm{Le}^{-1}m_n}}{2\,\mathrm{Le}^{-1}}>0,$$

the positive solution of

$$\mathrm{Le}^{-1}\lambda_n^2+M\lambda_n-m_n=0. \tag{4.3}$$

Note that $\lambda_n\to+\infty$ as $n\to+\infty$. Define the shift $\psi_n(x,y)=\widetilde{Y}_n(x+\widetilde{x}_n,y)$, and set

$$\overline{Y}_n(x,y)=e^{-\lambda_n(x+a)}+e^{-\lambda_n(-x+a)}.$$

We show now that $\overline{Y}_n$ is a super-solution for ψ_n in the domain $D_a=(-a,a)\times\omega$. Both functions ψ_n and $\overline{Y}_n$ satisfy the Neumann boundary conditions on ∂D while at the horizontal boundaries of D_a we have

$$\psi_n(\pm a,\cdot)\leq 1\leq\overline{Y}_n(\pm a,\cdot) \ \text{ in } \overline{\omega}.$$

Inside the domain D_a the function ψ_n is a solution of

$$0=\mathrm{Le}^{-1}\Delta\psi_n+(\widetilde{c}_n-u(y))\psi_{n,x}-f(\widetilde{T}_n(x+\widetilde{x}_n,y))\psi_n\leq\mathrm{Le}^{-1}\Delta\psi_n+(\widetilde{c}_n-u(y))\psi_{n,x}-m_n\psi_n,$$

while $\overline{Y}_n$ satisfies

$$\mathrm{Le}^{-1}\Delta\overline{Y}_n+(\widetilde{c}_n-u(y))\overline{Y}_{n,x}-m_n\overline{Y}_n\leq\left(\mathrm{Le}^{-1}\lambda_n^2+M\lambda_n-m_n\right)\overline{Y}_n=0 \ \text{ in } D_a,$$

owing to the definition of λ_n. The weak maximum principle then yields

$$0\leq\ \psi_n\leq\overline{Y}_n \ \text{ in } D_a,$$

for each $n\in\mathbb{N}$. Since $K\subset[-a+1,a-1]\times\overline{\omega}$ and $\lambda_n\to+\infty$ as $n\to+\infty$, it follows from the definition of $\overline{Y}_n$ that

$$\max_{(x,y)\in K}\widetilde{Y}_n(x+\widetilde{x}_n,y)=\max_{(x,y)\in K}\psi_n(x,y)\leq\max_{(x,y)\in K}\overline{Y}_n(x,y)\to 0 \ \text{ as } n\to+\infty,$$

which is the desired result. □

5 Existence of fronts with minimal speed c^*

In this section we prove the last part of Theorem 1.1, that is, the existence of bounded nontrivial solutions of (1.13)-(1.14) with the minimal speed c^*. We will do this using an approximation by a sequence of fronts with speeds larger than c^* that we have already constructed. To do this, let c_n be a sequence of speeds such that $c_n > c^*$ for all n, and such that

$$c_n \to c^* \ \text{ as } \ n \to +\infty.$$

It follows from the results of Section 3 that for each n, there exists a bounded solution (T_n, Y_n) of (1.13)-(1.14) with the speed $c = c_n$, such that $T_n > 0$, $0 < Y_n < 1$ in $\overline{D}$ and $T_n \in L^\infty(D)$. According to (1.14), we have the correct limits on the right:

$$T_n(+\infty, \cdot) = 0 \text{ and } Y_n(+\infty, \cdot) = 1.$$

It also follows from Propositions 1.2 and 1.3 that

$$T_n(-\infty, \cdot) = 1, \quad Y_n(-\infty, \cdot) = 0$$

and that there exists a constant $M > 0$ such that

$$\forall\ n \in \mathbb{N},\ \forall\ (x, y) \in \overline{D}, \quad 0 < T_n(x, y) \le M. \tag{5.1}$$

As we have mentioned, our strategy is to pass to the limit as $n \to +\infty$, in order to get a solution of (1.13)-(1.14) with the speed $c = c^*$ and $T \in L^\infty(D)$. Any shift of the traveling wave (T_n, Y_n) in the variable x along the cylinder is, of course, also a traveling wave, and the main technical difficulty here is to shift suitably the functions (T_n, Y_n) so that the limit pair is non-trivial and satisfies the correct limiting conditions at infinity. For that we have to identify a region where both T_n and Y_n are uniformly not very flat.

Locating the interface

For each $a \in (0, 1)$ and $n \in \mathbb{N}$, define

$$x_n^a = \min \{x \in \mathbb{R},\ Y_n \ge a \text{ in } [x, +\infty) \times \overline{\omega}\}.$$

Since the functions Y_n are continuous in $\overline{D}$ and satisfy $Y_n(+\infty, \cdot) = 1$ and $Y_n(-\infty, \cdot) = 0$, x_n^a are well-defined. Moreover, x_n^a is nondecreasing in $a \in (0, 1)$ for each $n \in \mathbb{N}$ fixed. Observe that, also,

$$\begin{cases} Y_n \ge a \ \text{ in } [x_n^a, +\infty) \times \overline{\omega}, \\ \min\limits_{\overline{\omega}} Y_n(x_n^a, \cdot) = a. \end{cases}$$

Since Y_n is "flat at $+\infty$", that is, $Y_n(+\infty, \cdot) = 1$, we have

$$\|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \overline{\omega})} := \max_{(x,y) \in [x_n^a, +\infty) \times \overline{\omega}} |\nabla Y_n(x, y)| > 0.$$

Furthermore, since $|\nabla Y_n(x, y)| \to 0$ as $x \to +\infty$ uniformly in $y \in \overline{\omega}$, the points

$$\widetilde{x}_n^a = \min \left\{x \in [x_n^a, +\infty),\ \exists\ y \in \overline{\omega},\ |\nabla Y_n(x, y)| = \|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \overline{\omega})}\right\}$$

are well-defined.

The key step is the following lemma that shows that to the right of x_n^a there are regions where Y_n are uniformly "non-flat".

Lemma 5.1 *For all $a \in (0,1)$, we have*

$$\inf_{n \in \mathbb{N}} \|\nabla Y_n\|_{L^\infty([x_n^a,+\infty)\times\overline{\omega})} > 0.$$

The proof of this lemma is postponed until the end of the section.

Normalization of (T_n, Y_n) and passage to the limit

Let us now complete the proof of the existence of a non-trivial bounded solution (T, Y) of (1.13)-(1.14) with the speed $c = c^*$. Choose any $a \in (0,1)$ and let $\widetilde{y}_n^a$ be a sequence of points in the cross-section $\overline{\omega}$ such that

$$|\nabla Y_n(\widetilde{x}_n^a, \widetilde{y}_n^a)| = \|\nabla Y_n\|_{L^\infty([x_n^a,+\infty)\times\overline{\omega})} \text{ for all } n \in \mathbb{N}.$$

Lemma 5.1 implies that

$$\inf_{n \in \mathbb{N}} |\nabla Y_n(\widetilde{x}_n^a, \widetilde{y}_n^a)| > 0. \tag{5.2}$$

For each $n \in \mathbb{N}$ and $(x,y) \in \overline{D}$, define the shifted functions

$$T_n^a(x,y) = T_n(x + \widetilde{x}_n^a, y), \qquad Y_n^a(x,y) = Y_n(x + \widetilde{x}_n^a, y). \tag{5.3}$$

Proposition 1.3 implies that both T_n and Y_n are uniformly bounded in $\overline{D}$, independently of n, that is (5.1). Then the standard elliptic estimates up to the boundary imply that these functions, as well as the shifts T_n^a and Y_n^a, are also bounded in $C^{2,\alpha}(\overline{D})$, also uniformly in n. Up to extraction of a subsequence, one can assume that the sequence $\widetilde{y}_n^a$ converges: $\widetilde{y}_n^a \to \widetilde{y}^a \in \overline{\omega}$, and that $(T_n^a, Y_n^a) \to (T^a, Y^a)$ in $C^2_{loc}(\overline{D})$ as $n \to +\infty$. Passing to the limit, we conclude that the pair (T^a, Y^a) satisfies

$$\left\{\begin{array}{rcll} \Delta T^a + (c^* - u(y))T_x^a + f(T^a)Y^a & = & 0 & \text{in } D, \\ \mathrm{Le}^{-1}\Delta Y^a + (c^* - u(y))Y_x^a - f(T^a)Y^a & = & 0 & \text{in } D, \\ \dfrac{\partial T^a}{\partial n} = \dfrac{\partial Y^a}{\partial n} & = & 0 & \text{on } \partial D, \end{array}\right. \tag{5.4}$$

and they obey the uniform bounds $0 \le T^a \le M$ and $0 \le Y^a \le 1$ in $\overline{D}$. Furthermore, (5.2) and normalization (5.3) imply that

$$|\nabla Y^a(0, \widetilde{y}^a)| > 0. \tag{5.5}$$

Since $\tilde{Y} \equiv 1$ is a supersolution of the Y^a-equation, the strong maximum principle and Hopf lemma imply that $Y^a < 1$ in $\overline{D}$, – otherwise, we would have $Y^a \equiv 1$, contradicting (5.5). For the same reason we have $Y^a > 0$ in $\overline{D}$. Therefore, the function Y^a is non-trivial.

If T^a vanishes somewhere in $\overline{D}$, then it is identically equal to 0, from the same arguments. Let us rule out this possibility. Assume that $T^a \equiv 0$. Then, the function Y^a would satisfy

$$\left\{ \begin{array}{rcll} \mathrm{Le}^{-1}\Delta Y^a + (c^* - u(y))Y^a_x & = & 0 & \text{in } D, \\ \dfrac{\partial Y^a}{\partial n} & = & 0 & \text{on } \partial D. \end{array} \right. \tag{5.6}$$

We apply now the same method as in the second part of the proof of Proposition 1.2 in Section 2. If we multiply (5.6) by Y^a, integrate over a finite cylinder $(-A, A) \times \omega$ and pass to the limit as $A \to +\infty$, we would obtain that the integral

$$\int_D |\nabla Y^a|^2 < +\infty$$

converges. Then, for a sequence $A_n \to +\infty$, the shifted functions $Y^a(\pm A_n + x, y)$ would converge in $C^2_{loc}(\overline{D})$ to two constants $Y^a_\pm \in [0, 1]$. Integrating (5.6) over $(-A_n, A_n) \times \omega$ and passing to the limit as $n \to +\infty$ yields that $c^*(Y^a_+ - Y^a_-) = 0$, that is $Y^a_+ = Y^a_-$. Finally, once again, multiplying (5.6) by Y^a, integrating over $(-A_n, A_n) \times \omega$ and passing to the limit as $n \to +\infty$, but now with the above information in hand, finally implies that

$$\int_D |\nabla Y^a|^2 = 0,$$

which contradicts (5.5). As a consequence, we conclude that

$$T^a > 0 \ \text{ in } \overline{D},$$

so that, in particular, T^a is not a constant, since the forcing term $f(T^a)\,Y^a$ is positive in $\overline{D}$.

The limits at infinity

It remains only to show that T^a and Y^a attain the correct limits at infinity. Observe that, since $Y_n \geq a$ in $[x^a_n, +\infty) \times \overline{\omega}$ and $\widetilde{x}^a_n \geq x^a_n$, we have $Y^a_n \geq a$ in $[0, +\infty) \times \overline{\omega}$, and thus $Y^a(x, y) \geq a > 0$ for all $x \geq 0$ and $y \in \overline{\omega}$. Since $c^* > 0$, it follows immediately from Proposition 2.1 that

$$T^a(+\infty, \cdot) = Y^a(-\infty, \cdot) = 0, \tag{5.7}$$

uniformly in $\overline{\omega}$. The second part of this proposition implies that the limits $T^a(-\infty, y)$ and $Y^a(+\infty, y)$ exist, are independent of $y \in \overline{\omega}$ and are equal:

$$T^a(-\infty, y) = Y^a(+\infty, y) = (c^*|\omega|)^{-1} \int_D f(T^a)\,Y^a > 0. \tag{5.8}$$

We now claim that the sequence $z^a_n = \widetilde{x}^a_n - x^a_n \geq 0$ is bounded. Otherwise, up to extraction of another subsequence, we would have $z_n \to +\infty$ as $n \to +\infty$. Thus, for each $(x, y) \in \overline{D}$, we would have $x + \widetilde{x}^a_n \geq x^a_n$ for sufficiently large n, and so

$$Y^a_n(x, y) = Y_n(x + \widetilde{x}^a_n, y) \geq a$$

for n large enough, which would imply that $Y^a(x,y) \geq a$ in $\overline{D}$. In particular, it would follow that $Y^a(-\infty,\cdot) \geq a > 0$, which contradicts (5.7).

Let now b be any real number in $(a,1)$. As in the previous argument, the shifted functions

$$T_n^b(x,y) = T_n(x+\widetilde{x}_n^b,y), \qquad Y_n^b(x,y) = Y_n(x+\widetilde{x}_n^b,y)$$

converge in $C^2_{loc}(\overline{D})$ as $n \to +\infty$, up to extraction of another subsequence, to a pair (T^b,Y^b) of solutions of (5.4) (with b instead of a), such that $Y^b(-\infty,\cdot) = 0$. We claim that the sequence $x_n^b - x_n^a \geq 0$ is bounded. Indeed, as we know that the sequence $\tilde{z}_n^b = \tilde{x}_n^b - x_n^b$ is bounded, if the sequence $(x_n^b - x_n^a)$ is unbounded then the sequence of nonnegative numbers $(\widetilde{x}_n^b - x_n^a)$ would be unbounded, which, in turn, would imply that $Y^b(-\infty,\cdot) \geq a > 0$, contradicting (5.7) for Y^b. As a consequence, the sequence $x_n^b - \widetilde{x}_n^a$ is also bounded and there exists $A_a^b \geq 0$, which depends on a and b but not n such that $x_n^b - \widetilde{x}_n^a \leq A_a^b$ for all $n \in \mathbb{N}$. However, for each $(x,y) \in [A_a^b,+\infty) \times \overline{\omega}$, we have then $x + \widetilde{x}_n^a \geq x_n^b$, and thus

$$Y_n^a(x,y) = Y_n(x+\widetilde{x}_n^a,y) \geq b,$$

for all $n \in \mathbb{N}$. As a consequence, we have $Y^a(x,y) \geq b$ for all $x \geq A_a^b$ and $y \in \overline{\omega}$, and, in particular, $Y^a(+\infty,y) \geq b$.

Since b was arbitrarily chosen in $(a,1)$ and since $Y^a(+\infty,\cdot) \leq 1$, we deduce that

$$Y^a(+\infty,y) = 1.$$

Now, (5.8) implies that, in addition, $T^a(-\infty,y) = 1$. As a conclusion, the pair (T^a,Y^a) solves (1.13)-(1.14) with $c = c^*$, together with $0 < T^a \leq M$ and $0 < Y^a < 1$ in $\overline{D}$. This completes the proof of Theorem 1.1. □

Proof of Lemma 5.1

We now prove Lemma 5.1. Assume that the conclusion of lemma does not hold for a real number $a \in (0,1)$. As $\|\nabla Y_n\|_{L^\infty([x_n^a,+\infty)\times\overline{\omega})}$ is positive for each $n \in \mathbb{N}$, up to extraction of a subsequence, one can then assume without loss of generality that

$$\|\nabla Y_n\|_{L^\infty([x_n^a,+\infty)\times\overline{\omega})} \to 0 \;\text{ as } n \to +\infty. \tag{5.9}$$

Temperature is small on the right

We first claim that in this case the ”temperature interface” is located far to the left of the ”concentration interface”, that is, we have

$$\|T_n\|_{L^\infty([x_n^a,+\infty)\times\overline{\omega})} \to 0 \;\text{ as } n \to +\infty. \tag{5.10}$$

Indeed, assume now that (5.9) holds and (5.10) does not. Then there exist $\varepsilon > 0$ and a sequence (x_n,y_n) in $\overline{D}$ such that

$$x_n \geq x_n^a \;\text{ and } T_n(x_n,y_n) \geq \varepsilon \;\text{ for all } n \in \mathbb{N}.$$

Up to extraction of a subsequence, one can assume that $y_n \to y_\infty \in \overline{\omega}$ as $n \to +\infty$. The standard elliptic estimates imply that the sequences of shifted functions $T_n(x + x_n, y)$ and $Y_n(x + x_n, y)$ converge in $C^2_{loc}(\overline{D})$, up to extraction of another subsequence, to a pair (T, Y) solving (1.13) with $c = c^*$. Furthermore, T and Y satisfy

$$\begin{cases} 0 \le Y \le 1, \quad 0 \le T \le M & \text{in } \overline{D}, \\ Y \ge a\ (>0), \quad |\nabla Y| = 0 & \text{in } [0, +\infty) \times \overline{\omega}, \end{cases}$$

and $T(0, y_\infty) \ge \varepsilon$. The strong maximum principle and Hopf lemma imply that $T > 0$ and $Y > 0$ in $\overline{D}$. On the other hand, Proposition 2.1 yields

$$T(+\infty, \cdot) = Y(-\infty, \cdot) = 0.$$

Finally, the positive maximum m of Y in $\overline{D}$ is reached. But since m is a supersolution for the Y equation, the strong maximum principle and Hopf lemma imply that $Y = m$ in $\overline{D}$, which leads to a contradiction since $Y(-\infty, \cdot) = 0$. As a consequence, (5.10) has to hold if assumption (5.9) is true.

Temperature decays exponentially on the right

We then claim that under assumptions (5.9) (and hence (5.10)) T_n decays exponentially uniformly to the right of x_n^a: there exist a positive number $\lambda > 0$, an integer $n \in \mathbb{N}$ and $A \ge 0$ so that for all $n \ge N$ and all $(x, y) \in [x_n^a + A, +\infty) \times \overline{\omega}$ we have

$$\frac{T_{n,x}(x, y)}{T_n(x, y)} \le -\lambda. \tag{5.11}$$

As $T_n > 0$, while Y_n and $f(T_n)/T_n$ are bounded independently of n and satisfy (1.13) with the speeds c_n which are uniformly bounded (since $\lim_{n\to+\infty} c_n = c^*$), it follows from standard elliptic estimates and the Harnack inequality that the functions $|\nabla T_n|/T_n$ are bounded in D independently of n. Assume now that the claim (5.11) does not hold. Then, after extraction of a subsequence, there exists a sequence of points (x_n, y_n) in $[x_n^a, +\infty) \times \overline{\omega}$ such that

$$\lim_{n\to+\infty} (x_n - x_n^a) = +\infty \tag{5.12}$$

and

$$\liminf_{n\to+\infty} \frac{T_{n,x}(x_n, y_n)}{T_n(x_n, y_n)} \ge 0. \tag{5.13}$$

Set the normalized and shifted temperature

$$U_n(x, y) = \frac{T_n(x + x_n, y)}{T_n(x_n, y_n)}$$

for all $n \in \mathbb{N}$ and $(x, y) \in \overline{D}$. Again, up to another extraction of a subsequence, one can assume that $y_n \to y_\infty \in \overline{\omega}$ as $n \to +\infty$. The functions U_n satisfy

$$\begin{cases} \Delta U_n + (c_n - u(y))U_{n,x} + \dfrac{f(T_n(x_n, y_n)U_n)}{T_n(x_n, y_n)} Z_n & = 0 \text{ in } D, \\ \dfrac{\partial U_n}{\partial n} & = 0 \text{ on } \partial D, \end{cases}$$

where

$$Z_n(x,y) = Y_n(x+x_n,y)$$

is the shifted concentration. The sequence U_n is bounded in all $W^{2,p}_{loc}(\overline{D})$ (for all $1 \le p < +\infty$), while $T_n(x_n,y_n) \to 0$ as $n \to +\infty$, as can be seen from (5.10) because $x_n \ge x_n^a$. On the other hand, the sequence of functions Z_n is globally bounded in $C^{2,\alpha}(\overline{D})$. Hence, up to extraction of a subsequence, the functions Z_n converge to a function Z in $C^2_{loc}(\overline{D})$ as $n \to +\infty$. But (5.9) and (5.12) imply that Z is a constant: $Z \equiv Z_0$. Furthermore, the constant Z_0 is such that

$$0 < a \le Z_0 \le 1, \tag{5.14}$$

since $a \le Y_n \le 1$ in $[x_n^a, +\infty) \times \overline{\omega}$. As a consequence, up to extraction of another subsequence, the positive functions U_n converge in all $W^{2,p}_{loc}(\overline{D})$ weak (for $1 < p < +\infty$), to a classical nonnegative solution U of

$$\begin{cases} \Delta U + (c^* - u(y))U_x + f'(0)\, Z_0\, U & = & 0 \text{ in } D, \\ \dfrac{\partial U}{\partial n} & = & 0 \text{ on } \partial D. \end{cases} \tag{5.15}$$

Furthermore, we have $U(0,y_\infty) = 1$, while (5.13) implies

$$\frac{U_x(0,y_\infty)}{U(0,y_\infty)} \ge 0. \tag{5.16}$$

It follows from the strong maximum principle and the Hopf lemma that $U > 0$ in $\overline{D}$ and from standard elliptic estimates and the Harnack inequality that the function $|\nabla U|/U$ is bounded in D. Let (x_n', y_n') be a sequence a points in $\overline{D}$ such that

$$\frac{U_x(x_n',y_n')}{U(x_n',y_n')} \to \sup_{\overline{D}} \frac{U_x}{U} =: \overline{M} \ge 0 \quad \text{as } n \to +\infty. \tag{5.17}$$

Next, with the same arguments as above, the functions

$$V_n(x,y) = \frac{U(x+x_n',y)}{U(x_n',y_n')}$$

are bounded in $C^{2,\alpha}_{loc}(\overline{D})$ independently of n and converge in $C^2_{loc}(\overline{D})$, up to extraction of a subsequence, to a nonnegative function V solving the same linear equation (5.15) as U, and such that $V(0,y_\infty') = 1$, where $y_\infty' = \lim_{n\to+\infty} y_n'$ (after extraction of a subsequence). Therefore, V is positive in $\overline{D}$. Moreover, at the point $(0,y_\infty')$ we have

$$\frac{V_x}{V} \le \overline{M} \ \text{ in } \overline{D} \ \text{ and } \frac{V_x(0,y_\infty')}{V(0,y_\infty')} = \overline{M}.$$

However, the function V_x/V satisfies a linear elliptic equation in $\overline{D}$ without the zeroth-order term, together with the Neumann boundary condition on ∂D and attains its maximum at the point $(0,y_\infty')$. The maximum principle implies that $V_x/V \equiv \overline{M}$ in $\overline{D}$. In other words, there exists a positive function $\phi(y)$ such that $V(x,y) = e^{\overline{M}x}\phi(y)$ in $\overline{D}$. It follows that $\phi(y)$ satisfies

$$\begin{cases} \Delta_y \phi + \left[\overline{M}\,(c^* - u(y)) + \overline{M}^2 + f'(0)\,Z_0\right]\phi &= 0 \text{ in } \overline{\omega}, \\ \dfrac{\partial \phi}{\partial n} &= 0 \text{ on } \partial\omega. \end{cases}$$

In other words, $\phi(y)$ is the unique positive eigenfunction of (1.6) and, moreover,

$$\mu(-\overline{M}) = c^*\overline{M} + \overline{M}^2 + f'(0)\,Z_0. \tag{5.18}$$

Recall that $\overline{M} \geq 0$ (see (5.17)), while $c^* > 0$, $f'(0) > 0$ and $Z_0 \geq a > 0$ from (5.14). Hence, the right side of (5.18) is positive. However, as we have mentioned in the introduction, the function μ is nonpositive, since it is concave and $\mu(0) = \mu'(0) = 0$. One has then reached a contradiction which shows that (5.11) must hold.

A sub-solution for Y_n

The last step in the proof of Lemma 5.1 is to use the exponential decay bound on T_n in order to find a suitable sub-solution for the function Y_n in $[x_n^a, +\infty) \times \overline{\omega}$, which will contradict our assumption (5.9). We have just shown that, for all $n \geq N$ and $(x,y) \in [x_n^a + A, +\infty) \times \overline{\omega}$ we have

$$0 < T_n(x,y) \leq T_n(x_n^a + A, y)\, e^{-\lambda(x - x_n^a - A)} \leq M\, e^{-\lambda(x - x_n^a - A)}.$$

The last inequality above follows from (5.1). On the other hand, for all $x \in [x_n^a, x_n^a + A]$, one has $e^{-\lambda(x - x_n^a - A)} \geq 1$. We conclude that the above bound holds in the whole half-strip $x \geq x_n^a$:

$$\forall\, n \geq N,\ \forall\, (x,y) \in [x_n^a, +\infty) \times \overline{\omega}, \quad 0 < T_n(x,y) \leq M\, e^{-\lambda(x - x_n^a - A)}. \tag{5.19}$$

We apply the same strategy as in Step 1 of the proof of Theorem 1.1 for $c > c^*$: use the above exponential bound for temperature to create a sub-solution for Y_n. First, since $\mu(0) = \mu'(0) = 0 < c^*$, one can choose $\beta > 0$ small enough so that

$$\begin{cases} 0 < \beta < \lambda, \\ \mu(\beta\,\mathrm{Le}) - \beta^2 + c^*\beta\,\mathrm{Le} > 0. \end{cases} \tag{5.20}$$

Then pick $\gamma > 0$ large enough so that

$$\begin{cases} \gamma \times \min\limits_{\overline{\omega}} \phi_{\beta\,\mathrm{Le}} \geq 1, \\ \gamma\,\mathrm{Le}^{-1}\,(\mu(\beta\,\mathrm{Le}) - \beta^2 + c^*\beta\,\mathrm{Le}) \times \min\limits_{\overline{\omega}} \phi_{\beta\,\mathrm{Le}} \geq f'(0)\,M\,e^{\lambda A}, \end{cases} \tag{5.21}$$

where $\phi_{\beta\,\mathrm{Le}}$ denotes the positive principal eigenfunction of (1.6) with parameter $\beta\,\mathrm{Le}$. For each $n \geq N$, define

$$\underline{Y}_n(x,y) = \max\left(0, 1 - \gamma\,\phi_{\beta\,\mathrm{Le}}(y)\, e^{-\beta(x - x_n^a)}\right),$$

for all $(x,y) \in \overline{D}$. Each function $\underline{Y}_n$ satisfies

$$\frac{\partial \underline{Y}_n}{\partial n} = 0 \text{ on } \partial D,$$

while $0 \le \underline{Y}_n \le 1$ and $\underline{Y}_n(+\infty,\cdot) = Y_n(+\infty,\cdot) = 1$ uniformly in $\overline{\omega}$. In addition, we have

$$\underline{Y}_n = 0 \text{ in } (-\infty, x_n^a] \times \overline{\omega},$$

as follows from the first property in (5.21). Hence, in the region where $\underline{Y}_n(x,y) > 0$ we have $x > x_n^a$ and thus there $\underline{Y}_n$ satisfies

$$\begin{aligned} &\mathrm{Le}^{-1}\Delta\underline{Y}_n + (c_n - u(y))\underline{Y}_{n,x} - f(T_n)\underline{Y}_n \ge \gamma\,\mathrm{Le}^{-1}\left(\mu(\beta\,\mathrm{Le}) - \beta^2 + c_n\,\beta\,\mathrm{Le}\right)\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta(x-x_n^a)} \\ &\quad -f'(0)\,M\,e^{-\lambda(x-x_n^a-A)}\left(1-\gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta(x-x_n^a)}\right) \\ &\quad \ge \gamma\,\mathrm{Le}^{-1}\left(\mu(\beta\,\mathrm{Le}) - \beta^2 + c^*\,\beta\,\mathrm{Le}\right)\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta(x-x_n^a)} - f'(0)\,M\,e^{\lambda A}\,e^{-\beta(x-x_n^a)} \ge 0 \end{aligned}$$

because of (1.2), (5.19)–(5.21) and since $c_n > c^*$. As $f(T_n) \ge 0$, it then follows from the weak maximum principle that we have a lower bound for Y_n:

$$\forall\ n \ge N,\ \forall\ (x,y) \in [x_n^a, +\infty) \times \overline{\omega}, \quad Y_n(x,y) \ge \underline{Y}_n(x,y) \ge 1 - \gamma\,\phi_{\beta\,\mathrm{Le}}(y)\,e^{-\beta(x-x_n^a)}.$$

In particular, it follows that there exists $L_0 > 0$ which is independent of n so that we have $Y_n(x_n^a + L_0, y) \ge (1+a)/2$ for all $y \in \overline{\omega}$. However, since $\min_{y\in\overline{\omega}} Y_n(x_n^a, y) = a < 1$ for all n, we finally reach a contradiction to our assumption (5.9). This completes the proof of Lemma 5.1. □

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