REAL FRONTIERS OF FAKE PLANES

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ABSTRACT. In [8], we define and partially classify fake real planes, that is, minimal complex surfaces with conjugation whose real locus is diffeomorphic to the euclidean real plane $\mathbb{R}^2$. Classification results are given up to biregular isomorphisms and up to birational diffeomorphisms. In this note, we describe in an elementary way numerous examples of fake real planes and we exhibit examples of such planes of every Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$ which are birationally diffeomorphic to $\mathbb{R}^2$.

INTRODUCTION

A *complexification* of a real smooth $\mathcal{C}^\infty$-manifold M is a real algebraic manifold S (see terminological conventions at the beginning of Section 1) whose real locus is diffeomorphic to M: $S(\mathbb{R}) \approx M$. Some manifolds such as real projective spaces $\mathbb{RP}^n$ and real euclidean affine spaces $\mathbb{R}^n$ have natural algebraic complexifications, given by the complex projective and affine spaces $\mathbb{CP}^n$ and $\mathbb{C}^n$ respectively. But these also admit infinitely many other complexifications, and it is a natural problem to try to classify them up to appropriate notions of equivalence. We focus on the case when $M = \mathbb{R}^2$ which is already surprisingly involved. Consider the following two examples:

(1) Start from $S_0 = \mathbb{C}^2_{u,v}$ with the usual conjugation $(u,v) \mapsto (\overline{u}, \overline{v})$. We have $S_0(\mathbb{R}) = \mathbb{R}^2$. Blowing-up a pair of conjugated non real points of $\mathbb{C}^2$, we get a surface S_1 which is not isomorphic to $\mathbb{C}^2$, actually not even affine as it contains proper curves, but $S_1(\mathbb{R}) \approx \mathbb{R}^2$.

(2) Start from $\mathbb{CP}^2_{x:y:z}$ and choose a pair of conjugated lines $L, \overline{L}$ meeting at a point on the *line at infinity* $L_\infty := \{z = 0\}$. Let S_2 be the complement in $\mathbb{CP}^n$ of the union $L \cup \overline{L} \cup L_\infty$. Then S_2 is isomorphic to $\mathbb{C} \setminus \{$a pair of conjugated points$\} \times \mathbb{C}$ and $S_2(\mathbb{R}) \approx \mathbb{R}^2$.

In the two cases above, the complexification is *topologically far* from $\mathbb{C}^2$. So we will seek for *topologically minimal* complexifications of $\mathbb{R}^2$ which we call *fake real planes*:

Definition 1. A nonsingular quasi-projective complex algebraic surface S endowed with an anti-holomorphic involution is called a *fake real plane* if:

(1) S is a *real plane*: $S(\mathbb{R}) \approx \mathbb{R}^2$;
(2) S is *topologically minimal*: $H_k(S; \mathbb{Q}) = 0$ for all $k \geq 1$;
(3) S is *fake*: S is not biregularly isomorphic to $\mathbb{C}^2$ as a real algebraic surface.

Every fake real plane S is affine and rational over $\mathbb{R}$, see § 1.1 below. In the projective setting, we obtain complexifications of $\mathbb{RP}^2$ in the same way than described in the first example above by blowing-up pairs of conjugated non real points of $\mathbb{CP}^2$. Among the rational projective complexifications of $\mathbb{RP}^2$, the only minimal one is $\mathbb{CP}^2$ because it is the only one with Picard number $\rho = 1$. In fact, $\mathbb{CP}^2$ is the only topologically minimal complexification of $\mathbb{RP}^2$: there exists other smooth complex surfaces with the same homology as $\mathbb{CP}^2$, but none of them admits a real structure [16]. In the light of these observations, a first natural question is:

Question 2. *Is there any fake real plane at all?*

An affirmative answer to Question 2 is given in the article [8], where we introduce and partially classify fake real planes. In the present paper, as an application of general classification results in [8], we describe in an explicit way numerous examples.

Let S and S' be two complexifications of a given topological surface M. We say that S and S' are $\mathbb{R}$-*biregularly birationally equivalent* if their real loci $S(\mathbb{R})$ and $S'(\mathbb{R})$ have isomorphic Zariski open

neighborhoods in S and S', respectively. Equivalently the surfaces $S(\mathbb{R})$ and $S'(\mathbb{R})$ are *birationally diffeomorphic*, that is:

Definition 3. Let S and S' be real algebraic surfaces. Their real loci $S(\mathbb{R})$ and $S'(\mathbb{R})$ are *birationally diffeomorphic* if there is a diffeomorphism $f\colon S(\mathbb{R}) \to S'(\mathbb{R})$ which extends as a real birational map $\psi\colon S \dashrightarrow S'$ whose indeterminacy locus does not intersect $S(\mathbb{R})$, and such that the indeterminacy locus of ψ^{-1} does not intersect $S'(\mathbb{R})$.

A real algebraic surface S is *rectifiable* if its real locus $S(\mathbb{R})$ is birationally diffeomorphic to $\mathbb{R}^2$.

It is well-known that every rational projective complexification of $\mathbb{RP}^2$ is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{RP}^2$, even dropping the topological minimality condition, see [17]. Thus next natural questions are:

Question 4. *Is there any rectifiable fake real plane?*

Question 5. *Let S be a fake real plane. Is S rectifiable?*

Again the answer to Question 4 is affirmative; examples of rectifiable fake real planes of Kodaira dimensions $-\infty$ and 0 were given in [8]. In the present paper, we exhibit infinite families of rectifiable fake real planes in every Kodaira dimension. In spite of this, Question 5 remains open.

The idea of this paper emerges thanks to the conference *Frontiers of rationnality* which took place in Spitsbergen (Norway) in July 2014. The word "frontier" in our title is an attempt to give credit to this exceptional event.

1. Generalities on fake real planes

Terminological conventions.
  (1) A *real algebraic manifold* of dimension n is a quasi-projective complex algebraic manifold S of complex dimension n endowed with an anti-holomophic involution whose set of fixed points is called the *real locus* and denoted by $S(\mathbb{R})$. A *real map* is a complex map commuting with involutions. A *real algebraic surface* is a real algebraic manifold of dimension 2. The manifolds S and $S(\mathbb{R})$ are assumed to be endowed with their euclidean topology.
  (2) A *topological surface* is a real 2-dimensional $\mathcal{C}^\infty$-manifold. By our convention, a real algebraic surface S is nonsingular; as a consequence, if nonempty, the real locus $S(\mathbb{R})$ gets a natural structure of a topological surface when endowed with the euclidean topology. Furthermore $S(\mathbb{R})$ is compact if S is projective.

1.1. Generalities on topologically minimal real planes.
By virtue of results of Fujita [9] and Gurjar-Sashtri [12, 13] every smooth $\mathbb{Q}$-acyclic complex surface S is affine and rational. If S is real, then its admits a real completion $S \hookrightarrow (V, B)$ into a smooth real projective surface V with connected real boundary curve $B = V \setminus S$. In the case where S is a real plane, the connectedness of $S(\mathbb{R}) \approx \mathbb{R}^2$ implies that V is a rational complex surface with connected real locus, hence an $\mathbb{R}$-rational real projective surface, i.e. a real surface admitting a real birational map $V \dashrightarrow \mathbb{CP}^2$ to $\mathbb{CP}^2$ equipped with its standard real structure. The free abelian group $\mathbb{Z}\langle B \rangle$ generated by the irreducible components of B and the divisor class group $\mathrm{Cl}(V)$ of V both inherits a structure of G-module for the group $G = \{1, \sigma\} \simeq \mathbb{Z}_2$ generated by the real structure σ on V. The inclusion $j\colon B \hookrightarrow V$ gives rise to a homomorphism $j_*\colon \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(V)$ of G-module, hence to a induced homomorphism $H^2(j_*)\colon H^2(G, \mathbb{Z}\langle B \rangle) \to H^2(G, \mathrm{Cl}(V))$ of $\mathbb{Z}_2$-vector spaces between the Galois cohomology groups $H^2(G, M) = \mathrm{Ker}(\mathrm{id}_M - \sigma)/\mathrm{Im}(\mathrm{id}_M + \sigma)$, $M = \mathbb{Z}\langle B \rangle, \mathrm{Cl}(V)$. The next theorem will be the most useful for the constructions presented below:

Theorem 6. *Let (V, B) be a pair consisting of an $\mathbb{R}$-rational projective real surface V and a real curve $B \subset V$. Then the surface $S = V \setminus B$ is an A-acyclic, where $A = \mathbb{Z}$ or $\mathbb{Q}$, real plane if and only if the conditions are satisfied:*
  a) V and B are connected and simply connected and $j_* \otimes_{\mathbb{Z}} A\colon \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} A \to \mathrm{Cl}(V) \otimes_{\mathbb{Z}} A$ is an isomorphism.
  b) The real locus of B is nonempty and $H^2(j_*)\colon H^2(G, \mathbb{Z}\langle B \rangle) \to H^2(G, \mathrm{Cl}(V))$ is an isomorphism.

Proof. The first assertion is essentially a rephrasing of a classical criterion established first by Ramanujam [21] and Fujita [9] which asserts that the complex surface S is A-acyclic if and only if V and B are connected and the homomorphism $H_2(B; A) \to H_2(V; A)$ induced by the inclusion $B \hookrightarrow V$ is an isomorphism. Indeed, $H_2(B; A)$ is a free abelian group isomorphic to $\mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} A$ via the map which

associates to every irreducible component of B its fundamental class in homology while the isomorphism $H_2(V;A) \simeq \mathrm{Cl}(V) \otimes_{\mathbb{Z}} A$ follows from that fact that since V is rational, the cycle map $\mathrm{Cl}(V) \to H_2(V, \mathbb{Z})$ which associates to every irreducible complex curve $D \subset V$ its fundamental class is an isomorphism. The second assertion essentially follows from the combination of the well-known fact that a relatively compact topological surface $M \subset \overline{M}$ with connected boundary $\overline{M} \setminus M$ is diffeomorphic to $\mathbb{R}^2$ if and only if it is connected and $\mathbb{Z}_2$-acyclic with the cycle map construction due to Borel-Haefliger [3], see [8, Section 2]. □

Remark 7. 1) The curve B in the previous proposition need not be an SNC divisor on V, but the criterion implies in particular that if $S = V \setminus B$ is A-cyclic then its inverse image $\tau^{-1}(B)$ in a log-resolution $\tau \colon V' \to V$ of the pair (V, B) defined over $\mathbb{R}$ is a real tree of rational curves, that is a tree of rational curves with an induced action of the real structure σ on V'. It follows in particular that the real locus of B is either empty or a connected union of curves homeomorphic to a circle.

2) In the case where $j_* \otimes_{\mathbb{Z}} \mathbb{Q} \colon \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism, one infers from the long exact sequence of relative homology for the pair (V, B) that $H_1(S, \mathbb{Z})$ is a torsion group isomorphic to $\mathrm{Cl}(V)/\mathrm{Im}(j_*)$.

Example 8. The complement S of a smooth real conic B in $\mathbb{CP}^2$ is a $\mathbb{Q}$-acyclic real surface which is a not a real plane. It can be seen directly that $S(\mathbb{R})$ is either diffeomorphic to $\mathbb{RP}^2$ if $B(\mathbb{R}) = \emptyset$ or to the disjoint union of $\mathbb{R}^2$ with a Möebius band otherwise. In the setting of Theorem 6 above, the image of the generator $[B]$ of $\mathbb{Z}\langle B \rangle \simeq \mathbb{Z} \cdot [B] \simeq H_2(B; \mathbb{Z})$ by $j_* \colon \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(\mathbb{CP}^2) \simeq H_2(\mathbb{CP}^2; \mathbb{Z}) \simeq \mathbb{Z} \cdot [\ell]$ where $[\ell]$ denotes the class of a real line, is equal to $2 \cdot [\ell]$. So S is $\mathbb{Q}$-acyclic, with $H_1(S; \mathbb{Z}) \simeq \mathbb{Z}_2$, but not $\mathbb{Z}$-acyclic. Furthermore, since in this particular situation we have isomorphisms $H^2(G, \mathbb{Z}\langle B \rangle) \simeq \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and $H^2(G, \mathrm{Cl}(V)) \simeq \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $H^2(j_*) \colon H^2(G, \mathbb{Z}\langle B \rangle) \to H^2(G, \mathrm{Cl}(V))$ is then the trivial map.

2. FAKE PLANES OF NEGATIVE KODAIRA DIMENSION

It turns out that $\mathbb{C}^2$ equipped with its standard real structure is the only $\mathbb{Z}$-acyclic real plane S of negative Kodaira dimension. Indeed, a complex $\mathbb{Z}$-acyclic surface of negative Kodaira dimension is isomorphic to $\mathbb{C}^2$ by virtue of [19] while every real structure on $\mathbb{C}^2$ is isomorphic to the standard one, as a consequence of [14]. In this section, we briefly review general geometric properties of $\mathbb{Q}$-acyclic fake planes. Then as a particular instance of a large class of such planes which are known since [8] to be $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$, see Theorem 11 below, we construct a pair of non isomorphic $\mathbb{Q}$-acyclic fake planes of negative Kodaira dimension with the same homology groups, whose real loci are both birationally diffeomorphic to $\mathbb{R}^2$.

2.1. **Basic properties of $\mathbb{Q}$-acyclic fake planes of negative Kodaira dimension.** By virtue of [19], the negativity of the Kodaira dimension of a smooth complex affine surface S is equivalent to the existence of an $\mathbb{A}^1$-fibration $\rho \colon S \to C$, that is, a fibration with general fibers isomorphic to $\mathbb{C}$, over a smooth curve C. One direction is clear for if $\rho \colon S \to C$ is such a fibration, then C contains a nonempty Zariski open subset C_0 over which ρ restricts to a trivial bundle $C_0 \times \mathbb{C}$, and so $\kappa(S) \le \kappa(C_0 \times \mathbb{C}) = -\infty$ by Iitaka's easy addition formula. When S is in addition real, there is no reason in general that the existing $\mathbb{A}^1$-fibration $\rho \colon S \to C$ is a real map: for instance, the complement of a smooth conic B in $\mathbb{CP}^2$ with empty real locus is a real surface of negative Kodaira dimension without any real $\mathbb{A}^1$-fibration $\rho \colon S \to C$. Indeed, if such a fibration existed then the closure in $\mathbb{CP}^2$ of its fiber over a general real point of C would be a real rational curve intersecting B in a unique point, necessarily real, which is impossible. The following theorem shows in particular that this phenomenon does not occur for real planes:

Theorem 9. ([8, Theorem 4.1]) *For a smooth affine real surface S the following are equivalent:*

1) S is a $\mathbb{Q}$-acyclic real plane of negative Kodaira dimension.

2) S admits a real $\mathbb{A}^1$-fibration $\rho \colon S \to \mathbb{C}$ whose closed fibers are all isomorphic to $\mathbb{C}$ when equipped with their reduced structure and whose fibers over the real locus of $\mathbb{C}$ have odd multiplicities.

Example 10. Let $s \ge 1$, let $[m] = (m_1, \dots, m_s)$ be a collection of integers $m_i \ge 2$, let $[p] = (p_1, \dots, p_s)$ be a collection of odd integers $p_i \ge 3$ and let $r_1, \dots, r_s \in \mathbb{R}$ be a collection of pairwise distinct real numbers. Then the real surface $S_{s,[m][p]} \subset \mathbb{C}^3$ defined by the equation

$$\prod_{i=1}^{s} (x - r_i)^{m_i} z = \sum_{i=1}^{s} \prod_{j \neq i} (x - r_j) y^{p_i} + \prod_{i=1}^{s} (x - r_i)$$

is a smooth $\mathbb{Q}$-acyclic fake plane of negative Kodaira dimension. Indeed, the smoothness of $S_{s,[m][p]}$ follows for instance from the Jacobian criterion. The restriction to $S_{s,[m][p]}$ of the projection pr_x is a real $\mathbb{A}^1$-fibration $\pi\colon S_{s,[m][p]} \to \mathbb{C}$ with irreducible fibers, restricting to a trivial bundle over $\mathbb{C} \setminus \{r_1, \ldots, r_s\}$ and whose fiber over each of the points r_i, $i = 1, \ldots, s$, has odd multiplicity p_i. So $S_{s,[m][p]}$ is a $\mathbb{Q}$-acyclic real plane of negative Kodaira dimension by the previous theorem.

The following partial result concerning the classification of $\mathbb{Q}$-acyclic real plane of negative Kodaira dimension up to $\mathbb{R}$-biregular equivalence was obtained in [8]:

Theorem 11. *Every $\mathbb{Q}$-acyclic real plane S of negative Kodaira dimension admitting a real $\mathbb{A}^1$-fibration $\rho\colon S \to \mathbb{C}$ with at most one degenerate fiber is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.*

As a consequence, the real locus of every surface S_{1,m_1,p_1} as in Example 10 is birationally diffeomorphic to $\mathbb{R}^2$. We don't know whether $\mathbb{Q}$-acyclic real plane of negative Kodaira dimension S admitting real $\mathbb{A}^1$-fibrations $\rho\colon S \to \mathbb{C}$ with more than one degenerate fiber are $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$, in particular:

Question 12. *Is the real locus of a surface $S_{s,[m][p]}$ with $s \geq 2$ as in Example 10 birationally diffeomorphic to $\mathbb{R}^2$?*

2.2. Two examples of $\mathbb{Q}$-acyclic fake planes with real loci birationally diffeomorphic to $\mathbb{R}^2$.

2.2.1. *Construction and first properties.*

1) First we let S be the complement of a real cuspidal cubic B in $V = \mathbb{CP}^2$. Since B is equivalent in $\mathrm{Cl}(V)$ to three times the class of a real line $\ell \subset \mathbb{CP}^2$, the $\mathbb{Q}$-acyclicity of S immediately follows from Theorem 6 a) applied to the pair (V, B), and we have $H_1(S; \mathbb{Z}) \simeq \mathbb{Z}_3$ by Remark 7.2. The fact that S is a real plane follows from b) in the same theorem after noting that similarly as in Example 8, $H^2(G, \mathbb{Z}\langle B \rangle) \simeq \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \mathbb{Z}_2 \cdot [B]$ and $H^2(G, \mathrm{Cl}(\mathbb{CP}^2)) \simeq \mathrm{Cl}(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \mathbb{Z}_2 \cdot [\ell]$. Alternatively, one can observe that $B(\mathbb{R})$ is homeomorphic to a simple closed curve in $\mathbb{RP}^2$ whose homotopy class is a generator of $\pi_1(\mathbb{RP}^2)$, so $\mathbb{RP}^2 \setminus B(\mathbb{R})$ is homeomorphic to an open disc, hence is diffeomorphic to $\mathbb{R}^2$. The restriction to S of the rational pencil $V \dashrightarrow \mathbb{CP}^1$ generated by B and three times its tangent T at its unique singular point restricts to a real $\mathbb{A}^1$-fibration $\rho\colon S \to \mathbb{C}$ with a unique degenerate fiber of multiplicity 3 consisting of the intersection of T with S. The real minimal resolution of the pencil $V \dashrightarrow \mathbb{CP}^1$ dominates the real minimal resolution $\tau\colon W \to V$ of the pair (V, B), and is obtained from it by blowing-up three times the intersection point of the proper transform of B with the successive total transforms of the exceptional locus of τ. We denote by $\beta\colon \tilde{V} \to V$ the so constructed surface and we denote by C the last exceptional divisor produced by this sequence of blow-ups. The dual graph of the total transform of $B \cup T$ in $\tilde{V}$ is depicted in Figure 2.1 below.
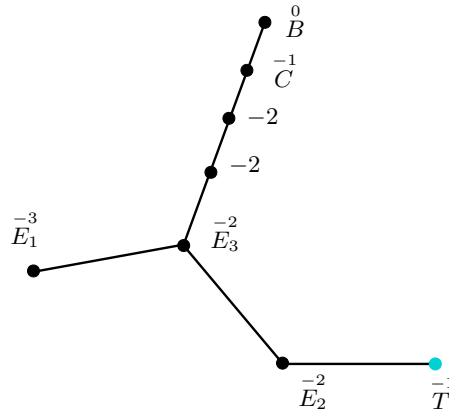

FIGURE 2.1. Dual graph of the proper transform of $B \cup T$ in $\tilde{V}$.

2) Next we let S' be the smooth real affine cubic surface in $\mathbb{C}^3$ defined by the equation $x^2 z = y^3 - x$. Taking projective closure in $\mathbb{CP}^3$ with homogeneous coordinates $[x : y : z : t]$, we view S' as the complement in the normal real cubic surface $V' = \{x^2 z - y^3 + xt^2 = 0\}$ of the real rational cuspidal hyperplane section $B' = V' \cap \{t = 0\}$. Note that V' has a unique singular point of type D_4 at $[0 : 0 : 1 : 0]$. Since the map $\mathbb{R}^2 \to S'$, $(x, z) \mapsto (x, \sqrt[3]{x^2 z + x}, z)$ is an homeomorphism between $\mathbb{R}^2$ and the real locus

of S', it follows that S' is a real plane. The divisor class group $\mathrm{Cl}(V')$ of V' is isomorphic to $\mathbb{Z}$ generated by the closure T' in V' of the real line $\{x = y = 0\} \subset S'$. The divisor class group $\mathrm{Cl}(W')$ of the real minimal log-resolution $\tau' \colon W' \to V'$ of the pair (V', B') is isomorphic to $\mathbb{Z}^6$ generated by the five exceptional divisors of τ' and the proper transform of T', and since $B' \sim 3T'$ in $\mathrm{Cl}(V')$, it follows that for an appropriate choice of bases, the map $j_* \colon \mathbb{Z}\langle \tau^{-1}(B') \rangle \to \mathrm{Cl}(W')$ is represented by a matrix M of the form

$$M = \begin{pmatrix} \mathrm{id}_5 & * \\ 0 & 3 \end{pmatrix}.$$

So by virtue of Theorem 6 a) applied to the real projective completion $(W', \tau^{-1}(B'))$ of S', S' is $\mathbb{Q}$-acyclic, with $H_1(S'; \mathbb{Z}) \simeq \mathbb{Z}_3$ by Remark 7.2. The restriction to S' of the projection pr_x is a real $\mathbb{A}^1$-fibration $\rho' \colon S' \to \mathbb{C}$ having the line $\{x = y = 0\}$ as a unique degenerate fiber of multiplicity 3. So $\kappa(S') = -\infty$. The real minimal resolution of the pencil $V' \dashrightarrow \mathbb{CP}^1$ induced by $\rho' \colon S' \to \mathbb{C}$ dominates the real minimal resolution $\tau' \colon W' \to V'$ of the pair (V', B'), and is obtained from it by blowing-up the intersection point of the proper transform of B' with the exceptional locus of τ'. We denote by $\beta' \colon \tilde{V}' \to V$ the so constructed surface and we denote by C' the last exceptional divisor produced by this sequence of blow-ups. The dual graph of the total transform of $B' \cup T'$ in $\tilde{V}'$ is depicted in Figure 2.2 below.
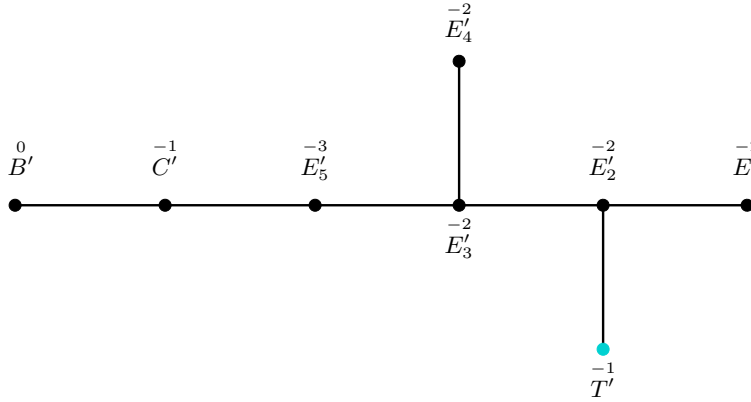

FIGURE 2.2. Dual graph of the proper transform of $B' \cup T'$ in $\tilde{V}$.

Proposition 13. *The surfaces S and S' are non isomorphic fake planes of negative Kodaira dimension.*

Proof. That S and S' are fake planes of negative Kodaira dimension follows from the construction above. The fact they are non isomorphic as complex surfaces can be deduced as follows. First since the dual graphs of the minimal resolutions of the pairs (V, B) and (V', B') are not chains, it follows from [1] that the $\mathbb{A}^1$-fibrations $\rho \colon S \to \mathbb{C}$ and $\rho' \colon S' \to \mathbb{C}$ constructed above are unique, up to composition by automorphisms of the base $\mathbb{C}$. This implies in particular that any isomorphism Φ between S and S' is an isomorphism of fibered surfaces, that is, there exists an isomorphism $\phi \colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ such that $\rho' \circ \Phi = \phi \circ \rho$. Every such isomorphism then admits a unique extension to a birational map $\overline{\Phi} \colon \tilde{V} \dashrightarrow \tilde{V}'$ with a unique proper base point, supported at the intersection of C with the proper transform of B, restricting to an isomorphism between the complements $\tilde{V} \setminus \beta_*^{-1}(B)$ and $\tilde{V}' \setminus (\beta')_*^{-1}(B')$ of the proper transforms of B and B' respectively and mapping C isomorphically onto C' (see e.g. [2] where these birational maps are called fibered modifications). But the fact that the irreducible components of $\beta^{-1}(B) \setminus \beta_*^{-1}(B)$ and $\beta^{-1}(B') \setminus \beta_*^{-1}(B')$ intersecting C and C' respectively have different self-intersections prevents the existence of any such birational map. So S and S' are not isomorphic. $\qquad\square$

Remark 14. The surfaces S and S' do not only have the same homology group but also the same fundamental group $\pi_1 \simeq \mathbb{Z}_3$. In fact, it can be shown that the complex algebraic threefolds $S \times \mathbb{A}^1$ and $S' \times \mathbb{A}^1$ are isomorphic, so that S and S' are in particular homotopically equivalent. A direct computation reveals further that the fundamental groups at infinity of S and S' (see e.g. [18, § 4.9 p. 246] for the definition and an algorithm for the computation of these groups) are both isomorphic to $\mathbb{Z}_9$. We do not know whether S and S' are homeomorphic as real 4-manifolds or not.

2.2.2. *Birational diffeomorphisms.* Here we show that the real loci of S and S' are not only diffeomorphic to $\mathbb{R}^2$ but actually birationally diffeomorphic to it.

1) In the minimal real log-resolution $\tau\colon W \to V$ of the pair (V, B), the proper transform of any pair $(\ell, \overline{\ell})$ of general non-real complex conjugate lines in $V = \mathbb{CP}^2$ passing through the singular point p_0 of B consists of a pair of complex conjugate rational 0-curves intersecting the proper transform of B transversally in a pair of non-real complex conjugate points $(q, \overline{q})$. Choosing two distinct such general pairs $(\ell_i, \overline{\ell}_i)$, $i = 1, 2$, we let $\theta_1\colon W \dashrightarrow W_1$ be the real birational map consisting of the blow-up of the corresponding two pairs of points $(q_i, \overline{q}_i)$, $i = 1, 2$, followed by the contraction of the proper transforms of ℓ_i and $\overline{\ell}_i$, $i = 1, 2$ (see Figure 2.3 below). By construction, θ restricts to a diffeomorphism $W(\mathbb{R}) \approx W_1(\mathbb{R})$, the dual graph of the proper transform of $\tau^{-1}(B)$ by θ_1 is depicted in Figure 2.4 below.
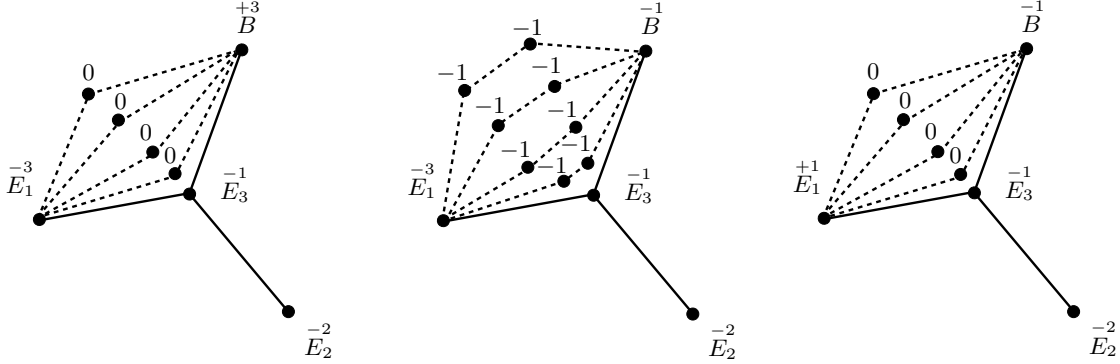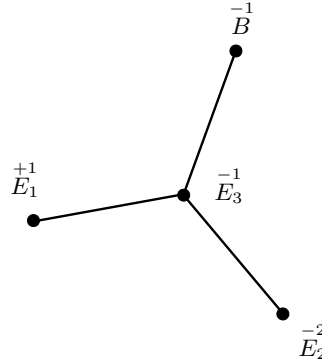

FIGURE 2.3. Elementary transformations.

FIGURE 2.4. Dual graph of the proper transform of $\tau^{-1}(B)$ by θ_1.

Now let $\theta_2\colon W_1 \dashrightarrow W_2$ be the real birational map obtained by first blowing-up the point $E_1 \cap E_3$, with exceptional divisor D_1 and then contracting successively the proper transforms of B, E_3 and E_2. By counting the number of points blown-up and the number of curves contracted during the whole process, we infer that the Picard rank of W_2 is equal to 2. So W_2 is a Hirzebruch surface in which the proper transforms of D_1 and E_1 are respectively a fiber and a section with self-intersection 1 of a $\mathbb{P}^1$-bundle structure on W_2. It follows that $W_2 \simeq \mathbb{F}_1$ and that the proper transform of D_1 is a fiber of the unique $\mathbb{P}^1$-bundle structure $\pi_1\colon \mathbb{F}_1 \to \mathbb{CP}^1$. The restriction of $\theta_2 \circ \theta_1$ to S is a real birational map $f\colon S \dashrightarrow \mathbb{F}_1 \setminus (E_1 \cup D_1) \simeq \mathbb{C}^2$ inducing a diffeomorphism $S(\mathbb{R}) \approx \mathbb{R}^2$.

2) For S', the construction is very similar although of a slightly different flavour. We first observe that the proper transform in the real minimal resolution $\tau'\colon W' \to V'$ of the pair (V', B') of a pair of general non-real complex conjugate hyperplane sections $(\ell', \overline{\ell}')$ of $V' = \{x^2 z - y^3 + xt^2 = 0\}$ of the form $V' \cap \{y - at = 0\}$ and $V' \cap \{y - \overline{a}t = 0\}$, $a \in \mathbb{C} \setminus \mathbb{R}$, consists of a pair of complex conjugate rational 0-curves intersecting the proper transform of B transversally in a pair of non-real complex conjugate points $(q', \overline{q}')$. For every such pair $(\ell', \overline{\ell}')$, the real birational map $\theta_1'\colon W' \dashrightarrow W_1'$ consisting of the blow-up of q' and $\overline{q}'$ followed followed by the contraction of the proper transforms of ℓ' and $\overline{\ell}'$ restricts to a diffeomorphism $W'(\mathbb{R}) \approx W_1'(\mathbb{R})$. The dual graph of the proper transform of $(\tau')^{-1}(B')$ by θ_1' is depicted in Figure 2.5.
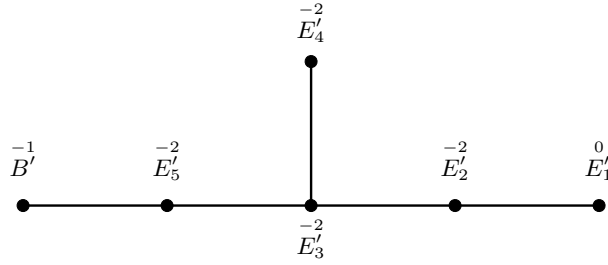
FIGURE 2.5. Dual graph of the proper transform of $(\tau')^{-1}(B')$ by θ'_1.

Letting $\theta'_2 \colon W'_1 \to W'_2$ be the real birational morphism consisting of the successive contractions of B', E'_5, E'_3 and E'_4, the resulting surface W'_2 is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ in which the proper transforms of E'_1 and E'_2 are fibers say of the first and second projection respectively. The restriction of $\theta'_2 \circ \theta'_1$ to S' is a real birational map $f' \colon S' \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus (E'_1 \cup E'_2) \simeq \mathbb{C}^2$ inducing a diffeomorphism $S'(\mathbb{R}) \approx \mathbb{R}^2$.

3. FAKE PLANES OF KODAIRA DIMENSION 0

By virtue of [18, Theorem 4.7.1 (1), p. 244], there is no smooth complex $\mathbb{Z}$-acyclic surface of Kodaira dimension 0. Smooth complex $\mathbb{Q}$-acyclic surfaces of Kodaira dimension 0 do exist, and are completely classified after the work of Fujita and Kojima (see also [18, Chapter 3, §4]). In this section, we present certain families of fake planes of Kodaira dimension 0, some of which being $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.

3.1. **Real models of Fujita's $H[-k, k]$ surfaces.** In the Hirzebruch surface $\pi_{2p} \colon \mathbb{F}_{2p} \to \mathbb{CP}^1$, $p \geq 1$, with negative section $C_0 \simeq \mathbb{CP}^1$ of self-intersection $-2p$, we choose a real section $C_1 \sim C_0 + (2p+1)f$, where f is a general real fiber of π_{2p}, and a pair $(\ell, \overline{\ell})$ of non-real complex conjugate fibers of π_{2p}. Note that since the unique intersection point of C_1 and C_0 is real, neither ℓ nor $\overline{\ell}$ passes through it. Now we let $\tau \colon V \to \mathbb{F}_{2p}$ be the smooth real projective surface obtained from $\mathbb{F}_{2p}$ by first blowing-up the pair of non-real complex conjugate points $q_1 = C_1 \cap \ell$ and $\overline{q}_1 = C_1 \cap \overline{\ell}$ with respective exceptional divisors E and $\overline{E}$ and then blowing-up the pair of non-real complex conjugate points $q'_1 = \ell \cap E$ and $\overline{q}'_1 = \overline{\ell} \cap \overline{E}$ with respective exceptional divisors F and $\overline{F}$. The dual graph of $B = C_0 \cup C_1 \cup \ell \cup \overline{\ell} \cup E \cup \overline{E}$, where we identified each curve with its proper transform in V, is a tree depicted in Figure 3.1 below.
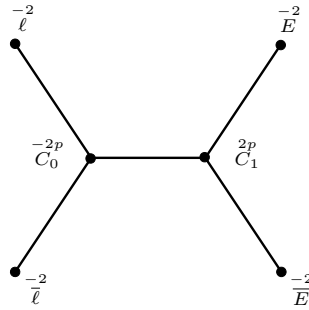
FIGURE 3.1. Dual graph of B.

Since B is a real curve, $H[-2p, 2p] = V \setminus B$ is a smooth real quasi-projective surface. The abelian group $\mathbb{Z}\langle B \rangle$ is freely generated by the classes of the irreducible curves C_0, C_1, ℓ, $\overline{\ell}$, E and $\overline{E}$ while the divisor class group $\mathrm{Cl}(V)$ of V is freely generated by $C_0, f, E, \overline{E}, F, \overline{F}$. Using the relations

$$\begin{cases} \tau^* C_1 = & C_1 + E + \overline{E} + F + \overline{F} \sim C_0 + (2p+1)f \\ \tau^* \ell = & \ell + E + 2F \sim f \\ \tau^* \overline{\ell} = & \overline{\ell} + \overline{E} + 2\overline{F} \sim f \end{cases}$$

in $\mathrm{Cl}(V)$, we find that the homomorphism $j_*\colon \mathbb{Z}\langle B\rangle \to \mathrm{Cl}(V)$ induced by the inclusion $B \hookrightarrow V$ is represented by the matrix

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2p+1 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & 1 \\
0 & -1 & -2 & 0 & 0 & 0 \\
0 & -1 & 0 & -2 & 0 & 0
\end{pmatrix}
$$

It follows from Theorem 6 a) and Remark 7 that $S = H[-2p, 2p]$ is $\mathbb{Q}$-acyclic, with $H_1(S; \mathbb{Z}) \simeq \mathbb{Z}_{8p}$. On the other hand, the real locus of B is homeomorphic to a wedge of two circles, hence is in particular nonempty. The Galois cohomology group $H^2(G, \mathbb{Z}\langle B\rangle)$ is freely generated by the classes of C_0 and C_1 while $H^2(G, \mathrm{Cl}(V))$ is freely generated by the classes of C_0 and f. The relations above imply that the matrix of the homomorphism $H^2(j_*)\colon H^2(G, \mathbb{Z}\langle B\rangle) \to H^2(G, \mathrm{Cl}(V))$ with respect to these bases is the identity. So $H[-2p, 2p]$ is a real plane by virtue of Theorem 6 b).

Proposition 15. *For every $p \geq 1$, the surface $H[-2p, 2p]$ is a $\mathbb{Q}$-acyclic fake plane of Kodaira dimension 0, $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.*

Proof. The fact that $S = H[-2p, 2p]$ if a fake plane follows from the discussion above. Since $K_{\mathbb{F}_{2p}} \sim -2C_0 - (2p+2)f$, we deduce from the ramification formula for $\tau\colon V \to \mathbb{F}_{2p}$ and the relations in $\mathrm{Cl}(V)$ indicated above that

$$
\begin{aligned}
K_V + B &\sim -2C_0 - (2p+2)f + E + \overline{E} + 2F + 2\overline{F} + (C_0 + C_1 + E + \overline{E} + \ell + \overline{\ell}) \\
&\sim f - F - \overline{F},
\end{aligned}
$$

hence that

$$
2(K_V + B) \sim 2(f - F - \overline{F}) \sim (f - 2F) + (f - 2\overline{F}) \sim \ell + E + \overline{\ell} + \overline{E}
$$

So the linear system $|2(K_V + B)|$ is nonempty, which implies that $\kappa(S) \geq 0$, and since the intersection matrix of $\ell + E + \overline{\ell} + \overline{E}$ is negative definite it follows that $\kappa(S) = 0$. That S is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$ can be seen as follows. First since the real morphism $\tau\colon V \to \mathbb{F}_{2p}$ consists of blow-ups of non-real points only, its restriction to S is a birational morphism $S \to S' = \mathbb{F}_{2p} \setminus (C_0 \cup C_1)$ inducing a diffeomorphism $S(\mathbb{R}) \approx S'(\mathbb{R})$. The smooth real affine surface $S' = \mathbb{F}_{2p} \setminus (C_0 \cup C_1)$ admits a real $\mathbb{A}^1$-fibration $\rho\colon S' \to \mathbb{C}$ induced by the restriction of the pencil $\mathbb{F}_{2p} \dashrightarrow \mathbb{CP}^1$ generated by the linearly equivalent divisors C_1 and $C_0 + (2p+1)F_0$, where $F_0 \simeq \mathbb{CP}^1$ denotes the fiber of π_{2p} over the real point $\pi_{2p}(C_0 \cap C_1) \in \mathbb{CP}^1$. This fibration has a unique degenerate fiber of multiplicity $2p + 1$ consisting of the intersection of F_0 with S', and so S' is $\mathbb{R}$-biregularly equivalent to $\mathbb{C}^2$ by virtue of Theorem 11. So $S(\mathbb{R})$ is birationally diffeomorphic to $\mathbb{R}^2$. $\qquad\square$

3.2. **The exceptional fake plane $Y(3, 3, 3)$ (see also [8, § 5.1.1]).** Let D be the union of four general real lines $\ell_i \simeq \mathbb{CP}^1$, $i = 0, 1, 2, 3$ in $\mathbb{CP}^2$ and let $\tau\colon V \to \mathbb{CP}^2$ be the real projective surface obtained by first blowing-up the points $p_{ij} = \ell_i \cap \ell_j$ with exceptional divisors E_{ij}, $i, j = 1, 2, 3$, $i \neq j$ and then blowing-up the points $\ell_1 \cap E_{12}$, $\ell_2 \cap E_{23}$ and $\ell_3 \cap E_{13}$ with respective exceptional divisors E_1, E_2 and E_3. We let $B = \ell_0 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{12} \cup E_{23} \cup E_{13}$. The dual graphs of D, its total transform $\tau^{-1}(D)$ in V and B are depicted in Figure 3.2.
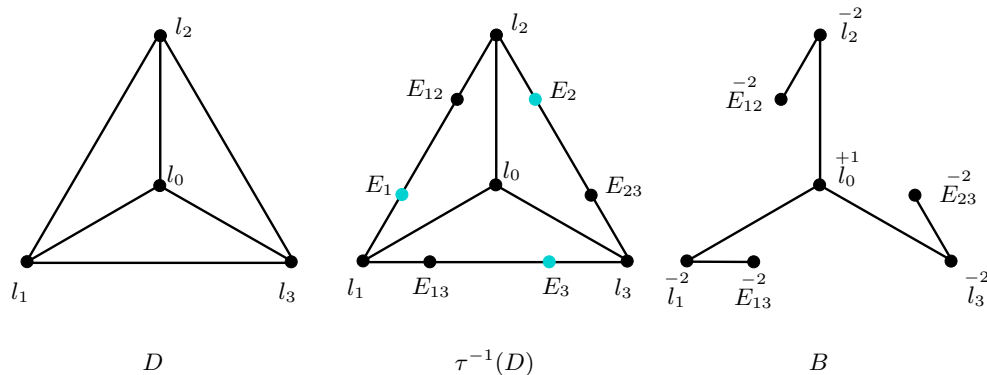

FIGURE 3.2. Construction of $Y(3, 3, 3)$

Since B is a real curve, $Y(3,3,3) := V \setminus B$ is a smooth real quasi-projective surface. The divisor class group $\mathrm{Cl}(V)$ of V is freely generated by ℓ_0, E_{12}, E_{23}, E_{13}, E_1, E_2 and E_3 and using the relations

$$\begin{cases} \tau^*\ell_1 = & \ell_1 + E_{12} + E_{13} + 2E_1 + E_3 \sim \ell_0 \\ \tau^*\ell_2 = & \ell_2 + E_{12} + E_{23} + E_1 + 2E_2 \sim \ell_0 \\ \tau^*\ell_3 = & \ell_3 + E_{13} + E_{23} + E_2 + 2E_3 \sim \ell_0 \end{cases}$$

in $\mathrm{Cl}(V)$ together with the ramification formula for τ, we find that

$$3(K_V + B) \sim 3\ell_0 - 3(E_1 + E_2 + E_3) \sim (\ell_1 + \ell_2 + \ell_3) + 2(E_{12} + E_{13} + E_{23}).$$

So $\kappa(Y(3,3,3)) \geq 0$ and since the intersection matrix of the divisor on the right hand is negative definite, it follows that $\kappa(Y(3,3,3)) = 0$. The abelian group $\mathbb{Z}\langle B \rangle$ is freely generated by the classes of the irreducible curves ℓ_i, $i = 0, \ldots, 3$, E_{12}, E_{23} and E_{13} and using the above relations, we find that the homomorphism $j_* \colon \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(V)$ induced by the inclusion $B \hookrightarrow V$ is represented by the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ \vdots & 0 & -1 & -1 & 0 & 1 & 0 \\ \vdots & -1 & 0 & -1 & \vdots & 0 & 1 \\ \vdots & -2 & -1 & 0 & \vdots & \vdots & 0 \\ \vdots & 0 & -2 & -1 & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

which has determinant $\det M = -9$. Furthermore, since $\tau \colon V \to \mathbb{CP}^2$ consists of blow-ups of real points only, $H^2(G, \mathbb{Z}\langle B \rangle) \simeq \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $H^2(G, \mathrm{Cl}(V)) \simeq \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and the homomorphism $H^2(j_*)$ is represented by the reduction modulo 2 of M. It follows from Theorem 6 that $Y(3,3,3)$ is a $\mathbb{Q}$-acyclic fake plane, with $H_1(Y(3,3,3); \mathbb{Z}) \simeq \mathbb{Z}_9$.

Question 16. *Is the real locus of $Y(3,3,3)$ birationally diffeomorphic to $\mathbb{R}^2$?*

4. Fake planes of Kodaira dimension 1

In this section, we first recall basic results on the classification of $\mathbb{Z}$-acyclic fake planes of Kodaira dimension 1. We then present examples of contractible and $\mathbb{Q}$-acyclic fake planes of Kodaira dimension 1 arising from rational real cuspidal quartics in $\mathbb{CP}^2$. We conclude this section with the construction of families of $\mathbb{Q}$-acyclic fake planes $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.

4.1. Elements of classification of $\mathbb{Z}$-acyclic fake planes of Kodaira dimension 1. Smooth complex $\mathbb{Z}$-acyclic surfaces of Kodaira dimension 1 have been classified by Gurjar and Miyanishi [11] and tom Dieck and Petrie [4]. We recall from [8] the following construction, which provides the real counter-part of this classification.

4.1.1. Let $D \subset \mathbb{CP}^2$ be the union of a collection $E_{0,0}, \ldots, E_{n,0} \simeq \mathbb{CP}^1$ of $n + 1 \geq 3$ real lines intersecting in a same point x and a general real line $C_1 \simeq \mathbb{CP}^1$. For every $i = 1, \ldots, n$, we choose a pair of coprime integers $1 \leq \mu_{i,-} < \mu_{i,+}$ in such a way that for $v_- = ^t(\mu_{1,-}, \ldots, \mu_{n,-}) \in \mathcal{M}_{n,1}(\mathbb{Z})$ and $\Delta_+ = \mathrm{diag}(\mu_{1,+}, \ldots, \mu_{n,+}) \in \mathcal{M}_{n,n}(\mathbb{Z})$, the following two conditions are satisfied:

(1)   a) $\eta = n - 1 - \displaystyle\sum_{i=1}^n \frac{1}{\mu_{i,+}} > 0$ and   b) The matrix $\mathcal{N} = \begin{pmatrix} -1 & -1 \\ v_- & \Delta_+ \end{pmatrix}$ belongs to $\mathrm{GL}_{n+1}(\mathbb{Z})$.

Then we let $\tau \colon V \to \mathbb{CP}^2$ be the smooth real projective surface obtained by the following blow-up procedure:

1) We first blow-up x with exceptional divisor $C_0 \simeq \mathbb{CP}^1$. The resulting surface is isomorphic to the Hirzebruch surface $\pi_1 \colon \mathbb{F}_1 \to \mathbb{CP}^1$ with C_0 as the negative section of π_1, the proper transforms of $E_{0,0}, \ldots, E_{n,0}$ are fibers of π_1 while the strict transform of C_1 is a section of π_1 disjoint from C_0.

2) Then for every $i = 1, \ldots, n$, we perform a sequence of blow-up of real points, starting with the blow-up of $p_i = C_1 \cap E_{i,0}$ in such a way that the following two conditions are satisfied: a) the inverse image of p_i is a chain of $\mathbb{CP}^1$ containing a unique (-1)-curve $A(p_i)$ and b) the coefficients of $A(p_i)$ in the total transform of C_1 and $E_{i,0}$ are equal to $\mu_{i,-}$ and $\mu_{i,+}$ respectively. Recall [8, Example 1.3] that

letting $x_{i,-}$ and $x_{i,+}$ be local defining equations for C_1 and $E_{i,0}$ in a Zariski open neighborhood U_i of p_i, such a blow-up sequence is obtained as the minimal resolution of the indeterminacies at p_i of the rational map $x_{i,+}^{\mu_{i,+}}/x_{i,-}^{\mu_{i,-}} : U_i \dashrightarrow \mathbb{CP}^1$. We denote by $E_{i,1}, \ldots, E_{i,r_i-1}, E_{i,r_i} = A(p_i)$ the corresponding exceptional divisors.

3) Finally, we perform a sequence of blow-ups starting with the blow-up of a real point $p_0 \in E_{0,0} \setminus (C_0 \cup C_1)$, with exceptional divisor $E_{0,1} \simeq \mathbb{CP}^1$ and continuing with a sequence of $r_0 - 1 \geq 0$ blow-ups of real points $p_{0,i} \in E_{0,i} \setminus E_{0,i-1}$, $i = 1, \ldots, r_0 - 1$, with exceptional divisors $E_{0,i+1}$. We let $A(p_0) = E_{0,r_0}$.
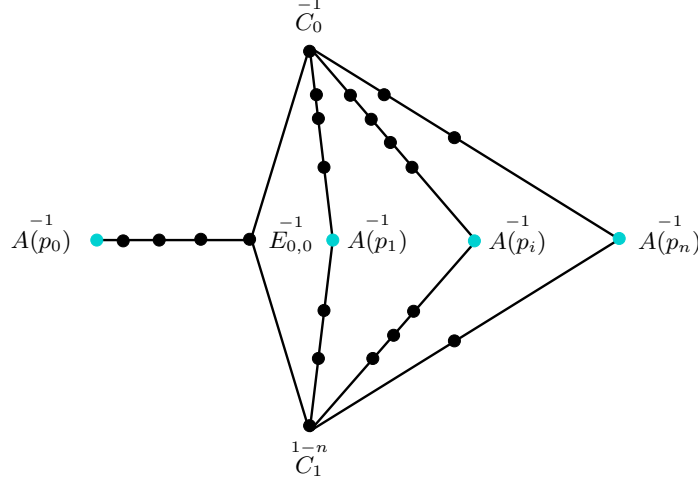

FIGURE 4.1. Dual graph of $\tau^{-1}(D)$.

The union B of the proper transforms of C_0, C_1, and the divisors $E_{i,j}$, $i = 0, \ldots, n$, $j = 0, \ldots r_{i-1}$, is a real subtree of the total transform of D by the so-constructed morphism $\tau : V \to \mathbb{CP}^2$. By virtue of [18, Lemma 4.5.3 p. 237] and the proof of Theorem 4.6.1 p. 238 in *loc. cit.* we have

$$K_V + B \quad \sim \quad (\pi_1 \circ \tau)^* K_{\mathbb{CP}^1} + \ell + \sum_{i=1}^n ((\pi_1 \circ \tau)^* E_{i,0} - A(p_i)) \sim (n-1)\ell - \sum_{i=1}^n A(p_i) \sim \eta\ell + N = P + N$$

where ℓ is the proper transform of a general real fiber of π_1, and N is an effective $\mathbb{Q}$-divisor supported on $\sum_{i=1}^n ((\pi_1 \circ \tau)^* E_{i,0})_{\mathrm{red}} - A(p_i)$. So the intersection matrix of N is negative definite and since $\eta > 0$ by hypothesis, it follows that P is nef. The surface $S = V \setminus B$ is thus a real surface of Kodaira dimension $\kappa(S) = 1$. Note further that the morphism $\pi_1 \circ \tau : V \to \mathbb{CP}^1$ coincides with that induced by the positive part of $K_V + B$ and that it restricts on S to a fibration $S \to \mathbb{CP}^1$ with general fibers isomorphic to $\mathbb{C}^*$. We have the following classification result:

Theorem 17. [8, Theorem 3.3] *Every $\mathbb{Z}$-acyclic fake plane S of Kodaira dimension 1 is isomorphic to a surface constructed by the above procedure.*

We do not know whether any of these surfaces is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.

4.2. **Fake planes obtained from rational real cuspidal quartic curves.**

4.2.1. First construction. Let $D \subset \mathbb{CP}^2$ be a real smooth rational quartic with a unique cusp p_0 of multiplicity 3 and a unique flex q at which D and its tangent T_q intersect with multiplicity 4. For instance, D is the zero locus of the homogeneous polynomial $z^4 - xy^3 \in \mathbb{C}[x, y, z]$, the corresponding points p_0 and q being $[1 : 0 : 0]$ and $[0 : 1 : 0]$.

Let $\tau : V \to \mathbb{CP}^2$ be the blow-up of any real point p of D different from p_0 and q, say with exceptional divisor $E_{0,1}$. Let $E_{0,0}$ and $E_{1,0}$ be the proper transforms of D and T_q respectively in V and let $B = E_{0,0} \cup E_{1,0}$. Then $S = V \setminus B$ is a contractible fake plane of Kodaira dimension 1. Indeed, the fact that S is a $\mathbb{Z}$-acyclic fake plane of Kodaira dimension 1 can be deduced directly from Theorem 17 by comparing the minimal log-resolution $\beta : V' \to V$ of the pair (V, B) depicted in Figure 4.3 below with Figure 4.1 above for $n = 2$, the curves $A(p_1)$ and $A(p_2)$ corresponding in this case to the proper transforms of the tangent line T_{p_0} to B at p_0 and of the line $\ell_{p_0 q}$ passing through p_0 and q.

Alternatively, the $\mathbb{Z}$-acyclicity follows from Theorem 7 via a computation similar to those in the previous section while the fact that $S(\mathbb{R}) \approx \mathbb{R}^2$ can be seen more directly as follows. Since $\tau : V \to \mathbb{CP}^2$

consists of the blow-up of a real point, $V(\mathbb{R})$ is a Klein bottle which we view as circle bundle $\theta\colon V(\mathbb{R}) \to S^1$ with fibers equal to the real loci of the lines through p in $\mathbb{CP}^2$. The sets $E_{0,1}(\mathbb{R})$ and $E_{1,0}(\mathbb{R})$ are two sections of θ which do no intersect each other. On the other hand $E_{0,0}(\mathbb{R})$ is a connected closed curve which intersects $E_{0,1}(\mathbb{R})$ and $E_{1,0}(\mathbb{R})$ transversally in one point and at the point q with multiplicity 4. It follows that the homology classes of $E_{0,0}(\mathbb{R})$ and $E_{1,0}(\mathbb{R})$ in $H_1(V(\mathbb{R}); \mathbb{Z}_2)$ form a basis of this group. Since $B(\mathbb{R})$ is not empty, the long exact sequence of relative homology for the pair $(V(\mathbb{R}), B(\mathbb{R}))$ with $\mathbb{Z}_2$-coefficients implies that $S(\mathbb{R})$ is connected and $\mathbb{Z}_2$-acyclic, hence diffeomorphic to $\mathbb{R}^2$.
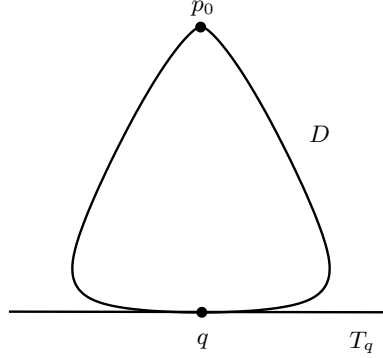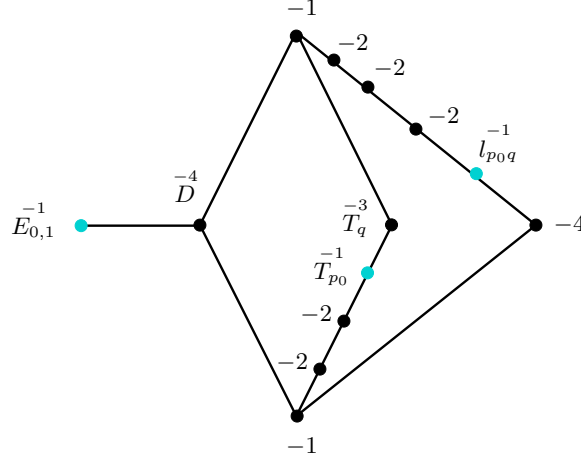

FIGURE 4.2. Real ovoid quartic.

FIGURE 4.3. Dual graph of the total transform of $B \cup T_{p_0} \cup \ell_{p_0 q} \cup E_{0,1}$ in a minimal real log-resolution $\beta\colon V' \to V$ of the pair (V, B).

The contractibility of S follows from the general classification of such surfaces given in [4]. It can also be seen directly as follows: since it is $\mathbb{Z}$-acyclic, it follows from Hurewicz theorem that S is contractible if and only if it is simply connected. Since $S \setminus E_{0,1} \simeq \mathbb{CP}^2 \setminus (D \cup T_q)$ is the complement of a closed submanifold of real codimension 2 in S, $\pi_1(S)$ is a quotient of $\pi_1(\mathbb{CP}^2 \setminus (D \cup T_q))$. Furthermore, by virtue of [24, Lemma 2.3(a)], we have an exact sequence $0 \to \mathbb{Z} \cdot \langle \mu \rangle \to \pi_1(\mathbb{CP}^2 \setminus (D \cup T_q)) \to \pi_1(\mathbb{CP}^2 \setminus D) \to 0$ where $\mathbb{Z} \cdot \langle \mu \rangle$ is the free abelian group generated by a meridian of the line T_q. Since $\pi_1(\mathbb{CP}^2 \setminus D)$ is abelian [6, Proposition 4.3 p. 130], so is $\pi_1(\mathbb{CP}^2 \setminus (D \cup T_q))$, implying in turn that $\pi_1(S)$ is abelian, hence trivial.

4.2.2. *Second construction.* Let $D \subset \mathbb{CP}^2$ be a real smooth rational quartic with a unique cusp p_0 of multiplicity 3 and a pair of real flexes q_1 and q_2 at which D and its tangent T_{q_i} intersect with multiplicity 3. For instance, D is the zero locus of the homogeneous polynomial $x^3 y - z^3 x + z^4 \in \mathbb{C}[x, y, z]$, the corresponding points p_0 and q_1 and q_2 being $[0 : 1 : 0]$, $[1 : 0 : 0]$ and $[\frac{3}{2} : \frac{4}{27} : 1]$. Note that since a complex quartic with these properties is unique up to complex projective equivalence (see e.g. [20, pp. 135, 146]), the construction below applies verbatim to any other real model of D. Let $\tau\colon V = \mathbb{F}_1 \to \mathbb{CP}^2$ be the blow-up of the intersection point p of T_{q_1} with D different from q_1, say with exceptional divisor F_1.

Since p is real, τ is a real morphism and hence, the complement in V of the proper transform B of $D \cup T_{q_1}$ is a real surface. The relations $\tau^* D = D + F_1$, $\tau^* T_{q_1} = T_{q_1} + F_1$ in $\mathrm{Cl}(V)$ and $D \sim 4T_{q_1}$ in $\mathrm{Cl}(\mathbb{CP}^2)$ imply that the homomorphism $j_*: \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(V)$ induced by the inclusion $j: B \hookrightarrow V$ is represented in appropriate bases by a matrix of the form

$$M = \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix}.$$

Furthermore, $H^2(G, \mathbb{Z}\langle B \rangle) \simeq \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $H^2(G, \mathrm{Cl}(V)) \simeq \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ as τ is the blow-up of a real point, and the homomorphism $H^2(j_*)$ is represented by the reduction modulo 2 of M. It follows from Theorem 6 and Remark 7 that S is a $\mathbb{Q}$-acyclic fake plane, with $H_1(S; \mathbb{Z}) \simeq \mathbb{Z}_3$ as $\det(M) = -3$. In contrast with the $\mathbb{Z}$-acyclic fake planes considered in § 4.1.1 which all admit a log-canonical real fibration over $\mathbb{CP}^1$ with general fibers isomorphic to $\mathbb{C}^*$, the surface S just constructed admits a fibration of this type over the complex line $\mathbb{C}$. Namely, letting $\ell_{p_0 q_1} = \{z = 0\}$ and $\ell_{p_0 p} = \{z - x = 0\}$ be the lines through p_0, q_1 and the line through p_0, p respectively, the divisors D, $3T_{p_0} + T_{q_1} = \{x^3 y = 0\}$ and $3\ell_{p_0 q_1} + \ell_{p_0 p} = \{z^3(z - x) = 0\}$ in $\mathbb{CP}^2$ generate a real pencil $V \dashrightarrow \mathbb{CP}^1$ containing $D = \{x^3 y + z^3(z - x) = 0\}$ as a member and whose lift in the minimal real log-resolution $\beta: \tilde{V} \to V$ of the pair (V, B) coincides with the $\mathbb{P}^1$-fibration $\tilde{V} \to \mathbb{CP}^1$ generated by the proper transform of D (see Figure 4.4 below). This fibration restricts on S to a real morphism $S \to \mathbb{C}$ with general fibers isomorphic to $\mathbb{C}^*$ and two degenerate fibers: one consisting of the disjoint union of $\ell_{p_0 p} \cap S \simeq \mathbb{C}$ and $\ell_{p_0 q_1} \cap S \simeq \mathbb{C}^*$ the second component having multiplicity 3, and a second one consisting of the intersection $T_{p_0} \cap S = \mathbb{C}^*$, also with multiplicity 3. The fact that $\kappa(S) = 1$ then follows from a similar computation as in § 4.1.1, see e.g. [18, Theorem 4.6.2 p. 237], which also renders the conclusion that $\tilde{V} \to \mathbb{CP}^1$ coincides with the fibration induced by the positive part of $K_{\tilde{V}} + \beta^{-1}(B)$.
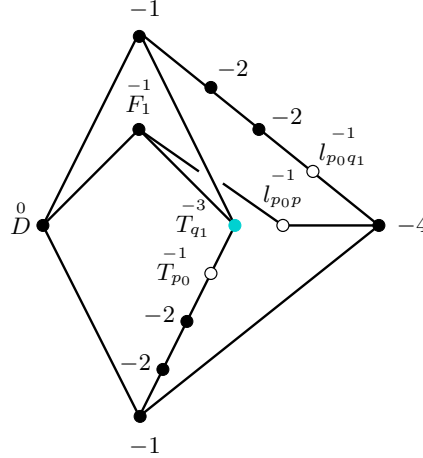

FIGURE 4.4. Dual graph of $\beta^{-1}(B \cup T_{p_0} \cup \ell_{p_0 q_1} \cup \ell_{p_0 p})$.

4.3. **$\mathbb{Q}$-acyclic fake planes with real loci birationally diffeomorphic to $\mathbb{R}^2$ (see also [8, Remark 3.2]).** To construct families of $\mathbb{Q}$-acyclic fake planes $\mathbb{R}$-biregularly birationally isomorphic to $\mathbb{C}^2$, we start similarly as in § 4.1.1 above with the union $D \subset \mathbb{CP}^2$ of a real line $E_{0,0} \simeq \mathbb{CP}^1$, a collection $(E_{1,0}, \overline{E}_{1,0}), \ldots, (E_{m,0}, \overline{E}_{m,0})$ of $m \geq 1$ distinct pairs of non-real complex conjugated lines all meeting in a same real point x, and of a general real line $C_1 \simeq \mathbb{CP}^1$. For every $i = 1, \ldots, m$, we choose a pair of coprime integers $1 \leq \nu_{i,-} < \nu_{i,+}$ in such a way that for

$$\begin{cases} v_- = {}^t(\nu_{1,-}, \ldots, \nu_{m,-}, \nu_{1,-}, \ldots, \nu_{m,-}) \in \mathcal{M}_{2m,1}(\mathbb{Z}) \\ \Delta_+ = \mathrm{diag}(\nu_{1,+}, \ldots, \nu_{m,+}, \nu_{1,+}, \ldots, \nu_{m,+}) \in \mathcal{M}_{2m,2m}(\mathbb{Z}), \end{cases}$$

the following two conditions are satisfied:

(2) a') $\eta = 2m - 1 - 2 \sum_{k=1}^{m} \dfrac{1}{\nu_{k,+}} > 0$ and b') The matrix $\mathcal{M} = \begin{pmatrix} -1 & -1 \\ v_- & \Delta_+ \end{pmatrix}$ belongs to $\mathrm{GL}_{2m+1}(\mathbb{Q})$.

Then we let $\tau \colon V \to \mathbb{CP}^2$ be the real smooth projective surface obtained by the following blow-up procedure:

1) We first blow-up x with exceptional divisor $C_0 \simeq \mathbb{CP}^1$. The resulting surface is isomorphic the Hirzebruch surface $\pi_1 \colon \mathbb{F}_1 \to \mathbb{CP}^1$ with C_0 as the negative section of π_1, the proper transform of C_1 is a section of π_1 disjoint from C_0, the proper transform of $E_{0,0}$ is a real fiber of π_1 while the proper transforms of the $E_{i,0}$ and $\overline{E}_{i,0}$, $i = 1, \ldots, m$ are pairs of non-real complex conjugate fibers of π_1.

2) Then for every $i = 1, \ldots, m$, we perform a sequence of blow-ups of pairs of non-real complex conjugate points, starting with the blow-up of the points $p_i = C_1 \cap E_{i,0}$ and $\overline{p}_i = C_1 \cap \overline{E}_{i,0}$ in such a way that the following two conditions are satisfied: a) the inverse images of p_i and $\overline{p}_i$ are complex conjugate chains of $\mathbb{CP}^1$ each containing a unique (-1)-curve $A(p_i)$ and $A(\overline{p}_i)$ respectively and b) the coefficients of $A(p_i)$ (resp. of $A(\overline{p}_i)$ in the total transform of C_1 and $E_{i,0}$ (resp. $\overline{E}_{i,0}$) are equal to $\nu_{i,-}$ and $\nu_{i,+}$ respectively. We denote by $E_{i,1}, \ldots, E_{i,r_i-1}, E_{i,r_i} = A(p_i)$ and $\overline{E}_{i,1}, \ldots, \overline{E}_{i,r_i-1}, \overline{E}_{i,r_i} = A(\overline{p}_i)$ the corresponding exceptional divisors.

3) Finally, we perform a sequence of blow-ups starting with the blow-up of a real point $p_0 \in E_{0,0} \setminus (C_0 \cup C_1)$, with exceptional divisor $E_{0,1} \simeq \mathbb{CP}^1$ and continuing with a sequence of $r_0 - 1 \geq 0$ blow-ups of real points $p_{0,i} \in E_{0,i} \setminus E_{0,i-1}$, $i = 1, \ldots, r_0 - 1$, with exceptional divisors $E_{0,i+1}$. We let $A(p_0) = E_{0,r_0}$.
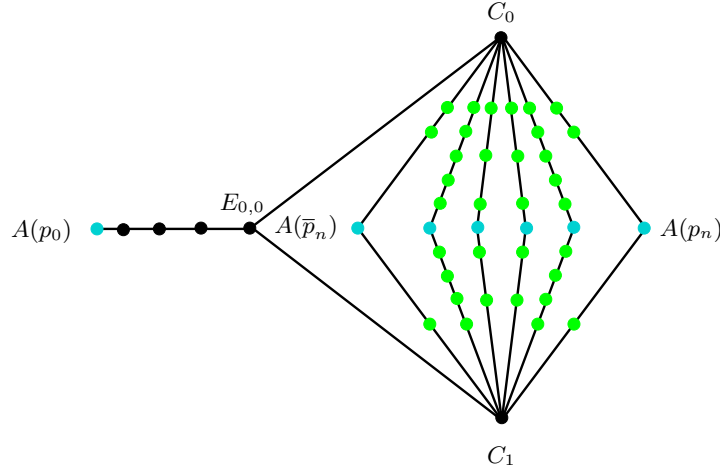

FIGURE 4.5. Dual graph of $\tau^{-1}(D)$.

The union B of the proper transforms of C_0, C_1, and the divisors $E_{i,j}$ and $\overline{E}_{i,j}$, $i = 0, \ldots, m$, $j = 0, \ldots r_{i-1}$, is a real subtree of the total transform $\tilde{B}$ of D by the so-constructed morphism $\tau \colon V \to \mathbb{CP}^2$. Condition a') guarantees by the same argument as in §4.1.1 that $S = V \setminus B$ is a real surface of Kodaira dimension 1. The classes of the curves C_0, $E_{i,j}$ and $\overline{E}_{i,j}$, $i = 1, \ldots, m$, $j = 0, \ldots r_{i-1}$, $E_{0,0}$, $C_1 - E_{0,0}$, $E_{0,i} - E_{0,0}$, $\overline{E}_{0,i} - E_{0,0}$, $i = 1, \ldots, m$, form a basis of the free abelian group $\mathbb{Z}\langle B \rangle$ generated by the irreducible components of B while $\mathrm{Cl}(V)$ is freely generated by the classes of C_0, $E_{i,j}$ and $\overline{E}_{i,j}$, $i = 1, \ldots, m$, $j = 0, \ldots r_{i-1}$, $E_{0,0}$, $A(p_0)$, $A(p_i)$ and $A(\overline{p}_i)$, $i = 1, \ldots, m$. In these bases, the matrix of the homomorphism $j_* \colon \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(V)$ induced by the inclusion takes the form

$$ M = \begin{pmatrix} \mathrm{id_n} & * \\ 0 & \mathcal{M} \end{pmatrix} $$

where $n = 2(\sum_{i=0}^m r_{i-1} + 1)$. Since $\mathcal{M} \in \mathrm{GL}_{2m+1}(\mathbb{Q})$ by hypothesis, we deduce from Theorem 6 a) that S is $\mathbb{Q}$-acyclic. But it is never $\mathbb{Z}$-acyclic as $\nu_{i,+} \geq 2$ for every $i = 1, \ldots, m$. On the other hand, the real locus of B is nonempty and the induced homomorphism $H^2(j_*) \colon H^2(G; \mathbb{Z}\langle B \rangle) \to H^2(G; \mathrm{Cl}(V))$ is an isomorphism, implying that $S(\mathbb{R}) \approx \mathbb{R}^2$ by virtue of b) in Theorem 6.

Proposition 18. *Every real surface S as above is a $\mathbb{Q}$-acyclic fake plane of Kodaira dimension 1, $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.*

Proof. It remains to show that $S(\mathbb{R})$ is birationally diffeomorphic to $\mathbb{R}^2$. The real birational morphism $\tau \colon V \to \mathbb{CP}^2$ factors through the real projective surface $\tau' \colon V' \to \mathbb{CP}^2$ obtained from $\mathbb{CP}^2$ by blowing-up x with exceptional divisor C_0 and then only performing the last sequence of blow-ups described in 3)

above, the one starting with the blow-up of a real point $p_0 \in E_{0,0} \setminus (C_0 \cup C_1)$. Since the induced real birational morphism $\beta \colon V \to V'$ consists of blow-ups of pairs of non-real complex conjugate points only, it restrict to a birational diffeomorphism between the real loci of V and V' respectively, mapping $S(\mathbb{R})$ diffeomorphically onto the real loci of the surface $S' = V' \setminus B'$ where $B' = C_0 \cup C_1 \cup \bigcup_{j=0}^{r_0-1} E_{0,j}$. A similar counting argument as in 2.2.2 above shows that the surface obtained from V' by first blowing-up the real point $C_1 \cap E_{0,0}$ with exceptional divisor C and then contracting successively the proper transforms of C_0, $E_{0,0}$, $E_{0,1}, \ldots, E_{0,r_0-1}$ has Picard rank 2, hence is a Hirzebruch surface $\pi_n \colon \mathbb{F}_n \to \mathbb{CP}^1$ for some $n \geq 0$, in which the proper transforms of C_1 and C are respectively a real fiber of π_n and a real section of it, with self-intersection r_0. The so-constructed real birational map $\theta \colon V' \dashrightarrow \mathbb{F}_n$ induces an isomorphism between the real loci of S' and that of $\mathbb{F}_n \setminus (C \cup C_1)$, and the composition $\theta \circ \beta \mid_S \colon S \to \mathbb{F}_n \setminus (C \cup C_1) \simeq \mathbb{C}^2$ is the desired $\mathbb{R}$-biregular birational isomorphism.
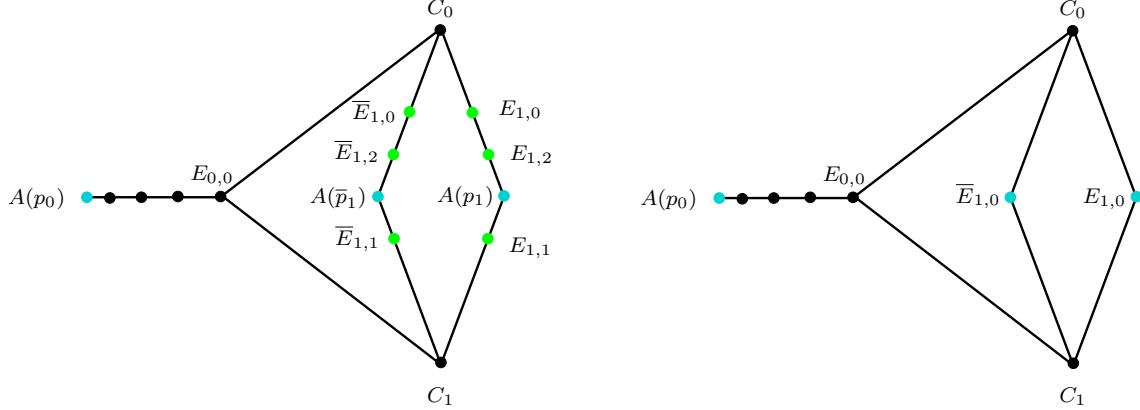

FIGURE 4.6. Dual graphs of $\tau^{-1}(D)$ and $\tau'^{-1}(D)$ in the case $m = 1$, $r_1 = 3$, $(\nu_{1,-}, \nu_{1,+}) = (2,3)$.

$\square$

5. FAKE PLANES OF GENERAL TYPE

By virtue of [23], the complement S of an irreducible rational cuspidal curve $B \subset \mathbb{CP}^2$ is a smooth affine surface of general type as soon as B has a least three cusps. Such a surface S is $\mathbb{Q}$-acyclic by virtue of Theorem 6 a), with $H_1(S; \mathbb{Z}) \simeq \mathbb{Z}_d$, where $d \geq 3$ denotes the degree of B. If in addition B is a real curve with non empty real locus, then Theorem 6 b) implies that S is a real plane if and only if d is odd. This can be seen more directly as follows: when non empty, $B(\mathbb{R})$ is homeomorphic to a simple closed curve in $\mathbb{RP}^2$, whose homotopy class is either trivial if d is even or equal to a generator of $\pi_1(\mathbb{RP}^2)$ if d is odd. So $S(\mathbb{R}) = \mathbb{RP}^2 \setminus B(\mathbb{R})$ is either diffeomorphic to the disjoint union of $\mathbb{R}^2$ with an open Möebius band in the first case or to $\mathbb{R}^2$ in the second case.

In this section, we first present different examples of contractible fake planes constructed from arrangements of lines and rational real cuspidal quartics in $\mathbb{CP}^2$ by the so-called cutting-cycle construction of tom Dieck and Petrie [5]. Then we construct an infinite family of $\mathbb{Z}$-acyclic fake planes of general type whose real loci are all birationally diffeomorphic to $\mathbb{R}^2$.

5.1. Fake planes of general type obtained from rational real cuspidal quartic curves.

5.1.1. *First construction.* We consider again as in § 4.2.2 a real smooth rational quartic $D_1 \subset \mathbb{CP}^2$ with a unique cusp $p_{1,0}$ of multiplicity 3 and a pair of real flexes $q_{1,1}$ and $q_{1,2}$ at which D_1 and its tangent $T_{q_{i,1}}(D_1)$ intersect with multiplicity 3. We let $\tau_1 \colon V_1 \to \mathbb{CP}^2$ be the real projective surface obtained from $\mathbb{CP}^2$ by first blowing-up the intersection point p_1 of $T_{q_{1,1}}(D_1)$ with D_1 different from $q_{1,1}$ with exceptional divisor $F_{1,1}$, then blowing-up the intersection point of $F_{1,1}$ with the proper transform of D_1 with exceptional divisor $F_{1,2}$ and then blowing-up the intersection point of $F_{1,2}$ with the proper transform of D_1, with exceptional divisor $A(p_1)$. The complement in V_1 of $B_1 = D_1 \cup T_{q_{1,1}}(D_1) \cup F_{1,1} \cup F_{1,2}$ is a smooth $\mathbb{Z}$-acyclic real surface S_1 with $S_1(\mathbb{R}) \approx \mathbb{R}^2$.
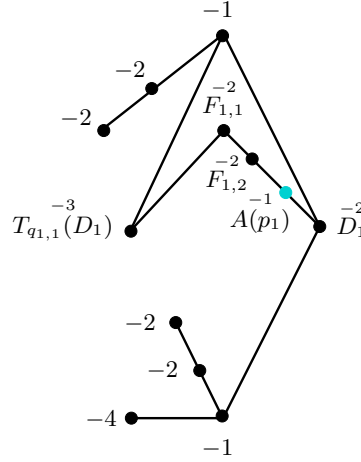
FIGURE 5.1. Dual graph of the total transform of $B_1 \cup A(p_1)$ in a minimal real log-resolution $\beta_1 \colon V_1' \to V_1$ of the pair (V_1, B_1).

Indeed, by construction,

$$\tau_1^* D_1 = D_1 + F_{1,1} + 2F_{1,2} + 3A(p_1)$$
$$\tau_1^* T_{q_{1,1}}(D_1) = T_{q_{1,1}}(D_1) + F_{1,1} + F_{1,2} + A(p_1)$$

so that using the relation $\tau_1^* D_1 \sim 4\tau_1^* T_{q_{1,1}}(D_1)$ in $\mathrm{Cl}(V_1)$, we obtain that in the bases of $\mathbb{Z}\langle B_1 \rangle$ and $\mathrm{Cl}(V_1)$ given by the classes of the real curves, $T_{q_{1,1}}(D_1)$, $F_{1,1}$, $F_{1,2}$, D_1 and $T_{q_{1,1}}(D_1)$, $F_{1,1}$, $F_{1,2}$, $A(p_1)$ respectively, the homomorphism $j_* \colon \mathbb{Z}\langle B_1 \rangle \to \mathrm{Cl}(V_1)$ induced by the inclusion $B_1 \hookrightarrow V_1$ is represented by the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}).$$

So S_1 is $\mathbb{Z}$-acyclic by Theorem 6 a), and the same type of arguments as in § 4.2.1 implies that S_1 is even contractible. On the other hand, since $\tau_1 \colon V_1 \to \mathbb{CP}^2$ consists of blow-ups of real points only, $H^2(j_*)$ is represented by the reduction modulo 2 of M, which is again invertible. Since $B_1(\mathbb{R})$ is nonempty, S_1 is a real plane by Theorem 6 b).

5.1.2. *Second construction.* We let $D_2 \subset \mathbb{CP}^2$ be a real rational ramphoid quartic, i.e. with a unique cusp $p_{2,0}$ with multiplicity sequence $(2,2,2)$, and three real flexes $q_{2,1}$, $q_{2,2}$, $q_{2,3}$ at which D_2 intersects its tangent $T_{q_{2,i}}(D_2)$ with multiplicity 3. We let $\tau_2 \colon V_2 \to \mathbb{CP}^2$ be the real birational morphism obtained by blowing-up the intersection point p_2 of D_2 with $T_{q_{2,1}}(D_2)$ distinct from $q_{2,1}$, with exceptional divisor $F_{2,1}$, then blowing-up the intersection point of $F_{2,1}$ with the proper transform of D_2, with exceptional divisor $F_{2,2}$ and then blowing-up the intersection point of $F_{2,2}$ with the proper transform of D_2, with exceptional divisor $A(p_2)$. We let S_2 be the smooth real surface obtained as the complement of $B_2 = D_2 \cup T_{q_{2,1}}(D_2) \cup F_{2,1} \cup F_{2,2}$ in V_2.

The same computation as in § 5.1.1 shows that S_2 is a contractible real plane.

5.1.3. *Third construction.* Here we start with a real rational bi-cuspidal quartic $D_3 \subset \mathbb{CP}^2$, with two real cusps $p_{3,\infty}$ and $p_{3,0}$ with respective multiplicity sequences $(2,2)$ and (2) such that D_3 intersects its tangent $T_{p_{3,0}}(D_3)$ with multiplicity 3 at $p_{3,0}$ and transversally at another real point p_3. We let $\tau_3 \colon V_3 \to \mathbb{CP}^2$ be the smooth real projective surface obtained from $\mathbb{CP}^2$ by first blowing-up p_3 with exceptional divisor $F_{3,1}$, then blowing-up the intersection point of $F_{3,1}$ with the proper transform of D_3, with exceptional divisor $F_{3,2}$ and then blowing-up the intersection point of $F_{3,2}$ with the proper transform of D_3, with exceptional divisor $A(p_3)$. We let S_3 be the smooth real surface obtained as the complement of $B_3 = D_3 \cup T_{p_{3,0}}(D_3) \cup F_{3,1} \cup F_{3,2}$ in V_3.

Similar arguments as in § 5.1.1 imply that S_3 is a contractible real plane.

Proposition 19. *The surfaces S_1, S_2 and S_3 constructed in § 5.1.1, § 5.1.2 and § 5.1.3 above are pairwise non isomorphic contractible fake planes of general type.*
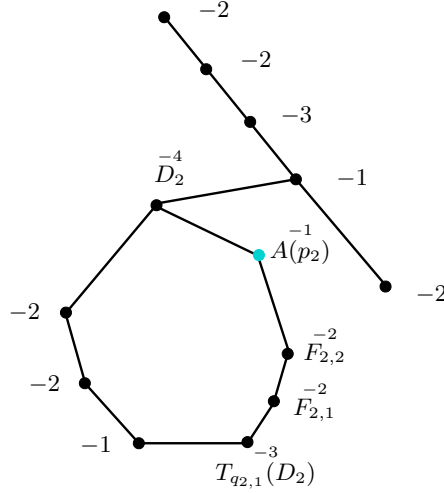
FIGURE 5.2. Dual graph of the total transform of $B_2 \cup A(p_2)$ in a minimal real log-resolution $\beta_2 \colon V_2' \to V_2$ of the pair (V_2, B_2).
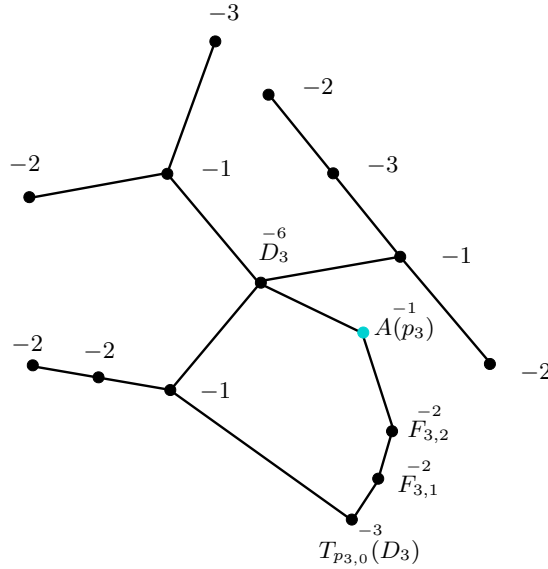

FIGURE 5.3. Dual graph of the total transform of $B_3 \cup A(p_3)$ in a minimal real log-resolution $\beta_3 \colon V_3' \to V_3$ of the pair (V_3, B_3).

Proof. Letting $\beta_i \colon V_i' \to V_i$, $i = 1, 2, 3$, be the minimal real log-resolutions of the pair (V_i, B_i) and $B_i' = \beta_i^{-1}(B_i)$, the pairs (V_i', B_i') are real minimal SNC-completion of the S_i. Note that every irreducible component of B_i' has self-intersection ≤ -1 and that every (-1)-curve in B_i' intersect three other irreducible components of B_i' (see Figure 5.1, 5.2 and 5.3 above). It follows that every birational map $\varphi \colon (V'', B'') \dashrightarrow (V_i', B_i')$ from another SNC-completion (V'', B'') of S_i restricting to an isomorphism between $V'' \setminus B''$ and $V_i' \setminus B_i'$ is a morphism. Now if S_i was isomorphic to S_j for some $j \neq i$, then the birational map $\varphi \colon (V_j', B_j') \dashrightarrow (V_i', B_i')$ extending an isomorphism $S_j \xrightarrow{\sim} S_i$ would be an isomorphism mapping B_j' isomorphically onto B_i'. In particular, the weighted dual graphs of B_j' and B_i' would be isomorphic, which is not the case. So S_1, S_2 and S_3 are pairwise non isomorphic. It remains to show that they are all of general type. Since $\mathbb{C}^2$ is the only smooth $\mathbb{Z}$-acyclic real plane and since there are no $\mathbb{Z}$-acyclic real plane of Kodaira dimension 0, to show that $\kappa(S_i) = 2$, it is enough to check that S_i is isomorphic neither to $\mathbb{C}^2$ nor to one of the surfaces described in § 4.1.1. If S_i was isomorphic to $\mathbb{C}^2$, then we would have a morphism $\varphi \colon (\mathbb{CP}^2, \ell) \to (V', B')$, where $\ell \simeq \mathbb{CP}^1$ is a real line, restricting to an isomorphism between $\mathbb{CP}^2 \setminus \ell$ and S_i, which is impossible as $\ell^2 > 0$. Now suppose that S_i is isomorphic to a surface obtained by the blow-up procedure $\tau'' \colon V'' \to \mathbb{CP}^2$ described in § 4.1.1 and let (V'', B'')

be the corresponding SNC-completion with boundary B'' consisting of the proper transforms of C_0, C_1, and the divisors $E_{i,j}$, $i = 0, \ldots, n$, $j = 0, \ldots r_{i-1}$. By construction, all irreducible components of B'' have self-intersection ≤ -1 and B'' contains at most three (-1)-curves: the proper transform of C_0 and $E_{0,0}$, and the proper transform of C_1 if $n = 2$, each of them intersecting at least three other irreducible components of B'', except in the case where $r_0 = 1$. Since the birational map $\varphi \colon (V'', B'') \dashrightarrow (V_i', B_i')$ induced by the isomorphism $V'' \setminus B'' \simeq S_i \simeq V' \setminus B$ is a morphism, the structure of B'' implies that φ is in fact an isomorphism of pairs, except possibly when $r_0 = 1$ where it can consists of the contraction of $E_{0,0}$ followed by an isomorphism of pairs. In both cases, we reach a contradiction by comparing the weighted dual graph of B_i' and the one B'' or its image by the contraction of $E_{0,0}$ in the case where $r_0 = 1$. So S_i is a fake plane of general type. $\qquad\square$

Question 20. *Is the real locus of S_1, S_2 and S_3 birationally diffeomorphic to $\mathbb{R}^2$?*

5.2. **$\mathbb{Z}$-acyclic fake planes of general type $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$.** The projective dual $D \subset \mathbb{CP}^2$ of the nodal cubic C with equation $(x - y)(x^2 + y^2) - xyz = 0$ is a real rational quartic with three ordinary cusps: a real one p_0 corresponding to the real flex $[1 : 1 : 0]$ of C and a pair of non-real complex conjugate ones q and $\bar{q}$ corresponding to the pair of non-real conjugate flexes $[i : 1 : 0]$ and $[-i : 1 : 0]$ of C. The tangent $T_{p_0}(D)$ of D at p_0 intersects D with multiplicity 3 at p_0 and transversally at another real point p. Let (μ, ν) be a pair of positive integers such that $4\nu - \mu = \pm 1$ and let $\tau \colon V \to \mathbb{CP}^2$ be the real birational morphism obtained by first blowing-up p with exceptional divisor E_1 and then, similarly as in 4.1.1 2), blowing-up a sequence of real points on the successive total transforms of E_1 in such a way that the following two conditions are satisfied: a) the inverse image of p is a chain of curves isomorphic to $\mathbb{CP}^1$ containing a unique (-1)-curve $A(p)$ and b) the coefficients of $A(p)$ in the total transform of D and $T_{p_0}(D)$ are equal to μ and ν respectively. We denote the corresponding exceptional divisors by $E_1, \ldots, E_{r-1}, E_r = A(p)$ and we let $B = D \cup T_{p_0}(D) \cup \bigcup_{i=1}^{r-1} E_i$. The weighted dual graph of the total transform of $B \cup A(p)$ in a real minimal log-resolution $\beta \colon V' \to V$ of the pair (V, B) is depicted in Figure 5.5.
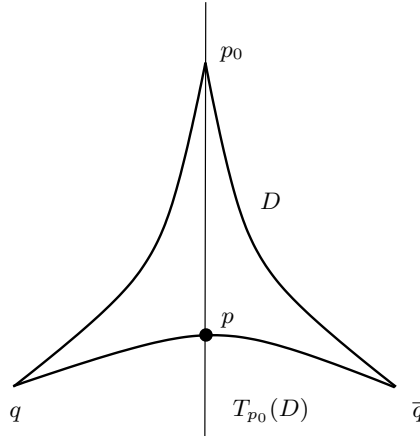

FIGURE 5.4. Real tricuspidal quartic

Proposition 21. *For every pair (μ, ν) such that $4\nu - \mu = \pm 1$, the surface $S(\mu, \nu) = V \setminus B$ is a $\mathbb{Z}$-acyclic fake plane of general type, whose real locus is birationally diffeomorphic to $\mathbb{R}^2$.*

Proof. By construction $\tau^* D = D + \mu A(p) + R$ and $\tau^* T_{p_0}(D) = T_{p_0}(D) + \nu A(p) + R'$ where R and R' are effective divisors supported on the union of $E_1, \ldots, E_{r-1}$. The free abelian group $\mathbb{Z}\langle B \rangle$ is generated by the classes of $T_{p_0}(D)$, $E_i, \ldots, E_{r-1}$ and $D - 4T_{p_0}(D)$ while $\mathrm{Cl}(V)$ is freely generated by the classes of $T_{p_0}(D)$, $E_1, \ldots, E_{r-1}$ and $A(p)$. Since $\tau^* D \sim 4\tau^* T_{p_0}(D)$ in $\mathrm{Cl}(V)$, the matrix of the homomorphism $j_* \colon \mathbb{Z}\langle B \rangle \to \mathrm{Cl}(V)$ with respect to these bases has the form

$$M = \begin{pmatrix} \mathrm{id}_{r+1} & * \\ 0 & 4\nu - \mu \end{pmatrix} \in \mathcal{M}_{r+2}(\mathbb{Z}).$$

The hypothesis that $4\nu - \mu = \pm 1$ implies that $M \in \mathrm{GL}_{r+2}(\mathbb{Z})$ so that S is $\mathbb{Z}$-acyclic by Theorem 6 a). Furthermore, $H^2(G, \mathbb{Z}\langle B \rangle) \simeq \mathbb{Z}\langle B \rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $H^2(G, \mathrm{Cl}(V)) \simeq \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ as τ consists of the blow-ups of
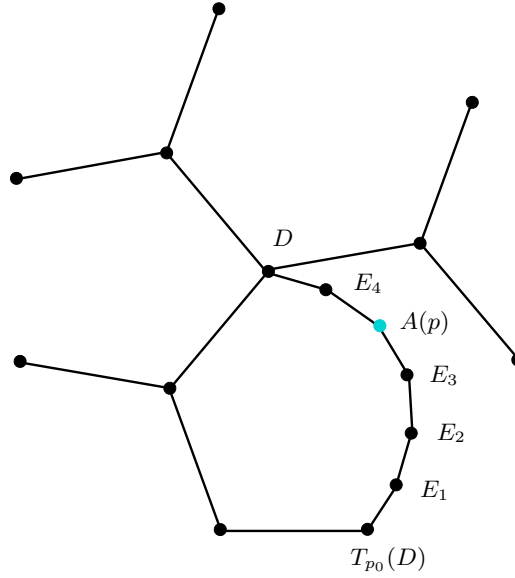
FIGURE 5.5. Dual graph of the total transform of $B \cup A(p)$ in a minimal real log-resolution $\beta\colon V' \to V$ of the pair (V, B) in the case $(\mu, \nu) = (7, 2)$.

real points only, and the homomorphism $H^2(j_*)$ is represented by the reduction modulo 2 of M. Since μ is necessarily odd, $H^2(j_*)$ is thus an isomorphism and since $B(\mathbb{R})$ is not empty, we conclude from Theorem 6 b) that S is a real plane. The fact that $\kappa(S) = 2$ follows from the same argument as in the proof of Proposition 19 above using the structure of the weighted dual graph of the total transform of B in a real minimal log-resolution $\beta\colon V' \to V$ of the pair (V, B) (see Figure 5.5).

It remains to show that S is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{C}^2$. Let $\theta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ be the real birational involution consisting of the blow-up of the points p_0 with exceptional divisor $L \simeq \mathbb{P}^1_\mathbb{R}$, q and $\bar{q}$ followed by the contraction of the proper transforms of the real line $L_1 \simeq \mathbb{P}^1_\mathbb{R}$ passing through q and $\bar{q}$ and of the pair of non-real conjugate lines passing through p_0 and q and p_0 and $\bar{q}$ respectively. The images of D, $T_{p_0}(D)$ and L by θ are respectively a smooth conic $\tilde{D}$, a real line $\tilde{T}$ intersecting $\tilde{D}$ at p and another real point p_∞, and the tangent line $\tilde{L}$ of $\tilde{D}$ at p_∞, see Figure 5.6 below.
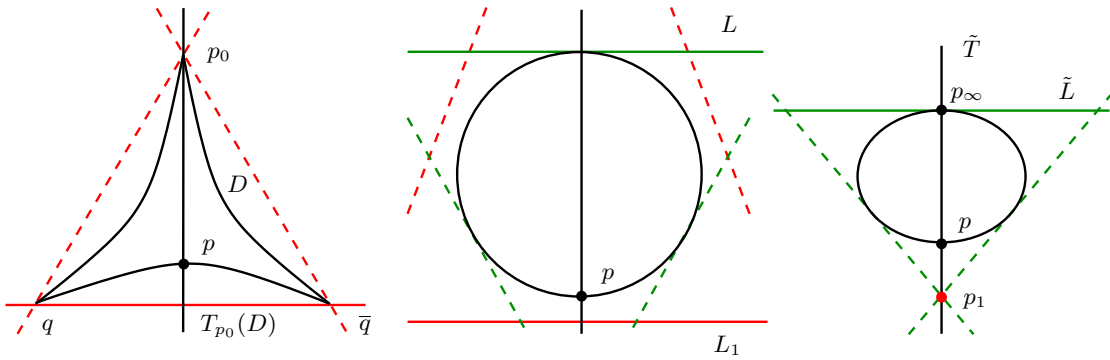
FIGURE 5.6. Behavior of the tricuspidal quartic under the standard Cremona transformation.

This map θ lifts to a real birational map $\tilde{\theta}\colon V \dashrightarrow \tilde{V}$ to the real projective surface $\tilde{\tau} : \tilde{V} \to \mathbb{CP}^2$ obtained from $\mathbb{CP}^2$ by blowing-up the real point $p_1 \in \tilde{L} \setminus (\tilde{D} \cap \tilde{L})$ on which L_1 was contracted and performing the same sequence of blow-ups as for the construction of V over the point $p \in \tilde{D} \cap \tilde{T}$. Let $\tilde{L}_1$ and $\tilde{E}_1, \ldots, \tilde{E}_{r-1}, \tilde{E}_r = \tilde{A}(p)$ be the corresponding exceptional divisors of $\tilde{\tau}$ over p_1 and p respectively and let $\tilde{B} = \tilde{D} \cup \tilde{T} \cup \tilde{L} \cup \bigcup_{i=1}^{r-1} \tilde{E}_i$. Then $\tilde{\theta}$ restricts to a diffeomorphism between the real loci of the

surfaces S and $\tilde{S} = \tilde{V} \setminus \tilde{B}$. Using the relations

$$\begin{cases} \tilde{\tau}^* \tilde{T} = \tilde{\tau}_*^{-1} \tilde{T} + \tilde{L}_1 + \nu \tilde{A}(p) + R_1 & \sim \tilde{\tau}^* \tilde{L} \\ \tilde{\tau}^* \tilde{D} = \tilde{\tau}_*^{-1} \tilde{D} + \mu \tilde{A}(p) + R_2 & \sim 2\tilde{\tau}^* \tilde{L} \end{cases}$$

in the divisor class group of $\tilde{V}$, where R_1 and R_2 are effective divisors supported on $\bigcup_{i=1}^{r-1} \tilde{E}_i$, we conclude that the homomorphism $\tilde{j}_* : \mathbb{Z}\langle \tilde{B} \rangle \to \mathrm{Cl}(\tilde{V})$ induced by the inclusion $\tilde{B} \hookrightarrow \tilde{V}$ is represented in appropriate bases by a matrix of the form

$$\tilde{M} = \begin{pmatrix} \mathrm{id}_{r+1} & * & * \\ 0 & 0 & -1 \\ 0 & -\mu & -\nu \end{pmatrix}.$$

Since $4\nu - \mu = \pm 1$, μ is odd, $\tilde{M} \in \mathrm{GL}_{r+3}(\mathbb{Q})$, and we deduce from Theorem 6 a) and Remark 7 that $\tilde{S}$ is $\mathbb{Q}$-acyclic, with $H_1(\tilde{S}, \mathbb{Z}) \simeq \mathbb{Z}_\mu$. So $\tilde{S}$ is in particular affine. Furthermore, the proper transform of $\tilde{L}$ in a minimal real log-resolution $\tilde{\beta} : \tilde{V}' \to \tilde{V}$ of the pair $(\tilde{V}, \tilde{B})$ is a (-1)-curve. Contracting it, we obtain a completion of $\tilde{S}$ into a smooth real projective surface $(\tilde{V}'', \tilde{B}'')$ whose boundary $\tilde{B}''$ is a chain of rational curves. It follows from [10] that $\tilde{S}$ admits an $\mathbb{A}^1$-fibration over $\mathbb{C}$. So $\kappa(\tilde{S}) = -\infty$, and we then deduce from Theorem 9 that $\tilde{S}$ admits a real $\mathbb{A}^1$-fibration $\tilde{\rho} : \tilde{S} \to \mathbb{C}$. By [7, Proposition 2.15], $\tilde{\rho}$ has at most one degenerate fiber, and so, $\tilde{S}$ is $\mathbb{R}$-biregularly equivalent to $\mathbb{C}^2$ by virtue of Theorem 11. This completes the proof. $\qquad\square$

Example 22. In the simplest case $(\mu, \nu) = (3, 1)$, the dual graph of the total transform of $\tilde{L}_1 \cup \tilde{B} \cup \tilde{A}(p)$ in the minimal log-resolution $\tilde{\beta} : \tilde{V}' \to \tilde{V}$ of the pair $(\tilde{V}, \tilde{B} = \tilde{D} \cup \tilde{T} \cup \tilde{L} \cup \tilde{E}_1 \cup \tilde{E}_2)$ is depicted on the right hand side of Figure 5.7, where $E_{p_\infty,1}$ and $E_{p_\infty,2}$ denote the two exceptional divisors of $\tilde{\beta}$

FIGURE 5.7. Dual graphs of the total transforms of $B \cup A(p)$ and $\tilde{B} \cup \tilde{A}(p) \cup \tilde{L}_1$ in a minimal log-resolution of the pairs (V, B) and $(\tilde{V}, \tilde{B})$ respectively for $(\mu, \nu) = (3, 1)$.
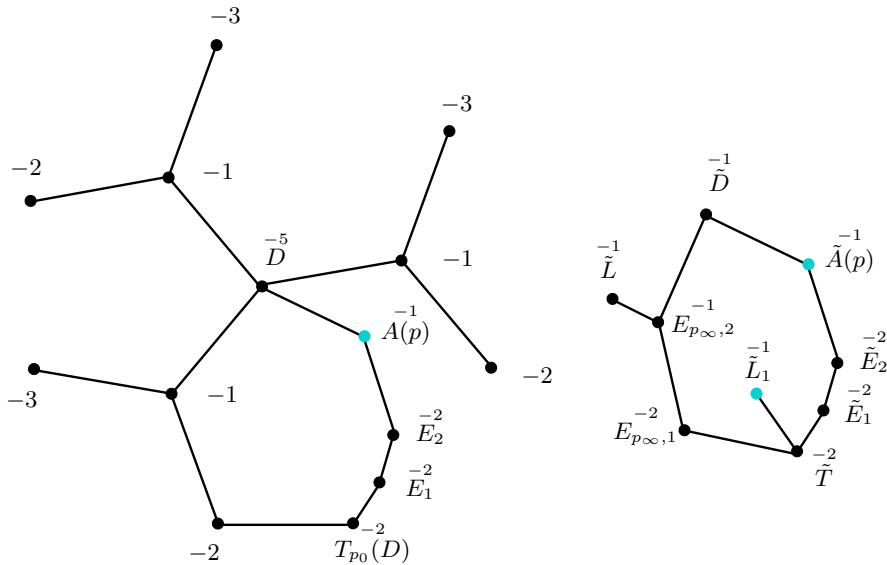
over p_∞. Let $\gamma : \tilde{V}' \dashrightarrow \tilde{V}''$ be the real birational map consisting of the blow-up of the real point $E_{p_\infty,1} \cap E_{p_\infty,2}$ with exceptional divisor C, followed by the contraction of $\tilde{L}$ and $\tilde{D}$. The total image $\tilde{B}'' = E_{p_\infty,2} \cup C \cup E_{p_\infty,1} \cup \tilde{T} \cup \tilde{E}_1 \cup \tilde{E}_2$ of $\tilde{B}'$ by γ is a chain of rational curves with self-intersections $(0, -1, -3, -2, -2, -2)$. The complete linear system $|E_{p_\infty,2}|$ on $\tilde{V}''$ defines a real $\mathbb{P}^1$-fibration $\overline{\rho} : \tilde{V}'' \to \mathbb{CP}^1$ having C has a section and $E_{p_\infty,1} + 3\tilde{T} + 3\tilde{L}_1 + 2\tilde{E}_1 + \tilde{E}_2$ as a unique degenerate fiber. The restriction of $\overline{\rho}$ to $\tilde{S} \simeq \tilde{V}'' \setminus \tilde{B}''$ is a real $\mathbb{A}^1$-fibration $\tilde{\rho} : \tilde{S} \to \mathbb{C}$ having $\tilde{L}_1 \cap \tilde{S} \simeq \mathbb{C}$ has unique degenerate fiber, of multiplicity 3.

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