REAL ALGEBRAIC CURVES WITH LARGE FINITE NUMBER OF REAL POINTS

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Abstract. We address the problem of the maximal finite number of real points of a real algebraic curve (of a given degree and, sometimes, genus) in the projective plane. We improve the known upper and lower bounds and construct close to optimal curves of small degree. Our upper bound is sharp if the genus is small as compared to the degree. Some of the results are extended to other real algebraic surfaces, most notably ruled.

1. Introduction

A *real algebraic variety* (X, c) is a complex algebraic variety equipped with an anti-holomorphic involution $c\colon X \to X$, called a *real structure*. We denote by $\mathbb{R}X$ the real part of X, *i.e.*, the fixed point set of c. With a certain abuse of language, a real algebraic variety is called *finite* if so is its real part. Note that each real point of a finite real algebraic variety of positive dimension is in the singular locus of the variety.

1.1. **Statement of the problem.** In this paper we mainly deal with the first non-trivial case, namely, finite real algebraic curves in $\mathbb{C}P^2$. (Some of the results are extended to more general surfaces.) The degree of such a curve $C \subset \mathbb{C}P^2$ is necessarily even, $\deg C = 2k$. Our primary concern is the number $|\mathbb{R}C|$ of real points of C.

Problem 1.1. For a given integer $k \geq 1$, what is the maximal number

$$\delta(k) = \max\{|\mathbb{R}C| : C \subset \mathbb{C}P^2 \text{ a finite real algebraic curve, } \deg C = 2k\}?$$

For given integers $k \geq 1$ and $g \geq 0$, what is the maximal number

$$\delta_g(k) = \max\{|\mathbb{R}C| : C \subset \mathbb{C}P^2 \text{ a finite real algebraic curve of genus } g, \deg C = 2k\}?$$

(See Section 2 for our convention for the genus of reducible curves.)

Remark 1.2. Since the curves considered are singular, we do not insist that they should be irreducible. The curves achieving the maximal possible value of $\delta(k)$ are, indeed, reducible in degrees $2k = 2, 4$ (see Section 1.2 below), whereas they are irreducible in degrees $2k = 6, 8$. It appears that in all degrees $2k \geq 6$ maximizing curves can be chosen irreducible.

The Petrovsky inequalities (see [Pet38] and Remark 2.3) result in the following upper bound:

$$|\mathbb{R}C| \leq \frac{3}{2}k(k-1) + 1.$$

Currently, this bound is the best known. Furthermore, being of topological nature, it is sharp in the realm of pseudo-holomorphic curves. Indeed, consider a rational simple Harnack curve of degree $2k$ in $\mathbb{C}P^2$ (see [Mik00, KO06, Bru15]); this curve has $(k-1)(2k-1)$ solitary real nodes (as usual, by a node we mean a non-degenerate double point, *i.e.*, an A_1-singularity) and an oval (see Remark

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2.3 for the definition) surrounding $\frac{1}{2}(k-1)(k-2)$ of them. One can erase all inner nodes, leaving the oval empty. Then, *in the pseudo-holomorphic category*, the oval can be contracted to an extra solitary node, giving rise to a finite real pseudo-holomorphic curve $C \subset \mathbb{C}P^2$ of degree $2k$ with $|\mathbb{R}C| = \frac{3}{2}k(k-1)+1$.

1.2. **Principal results.** For the moment, the exact value of $\delta(k)$ is known only for $k \le 4$. The upper (Petrovsky inequality) and lower bounds for a few small values of k are as follows:

k	1	2	3	4	5	6	7	8	9	10
$\delta(k) \le$	1	4	10	19	31	46	64	85	109	136
$\delta(k) \ge$	1	4	10	19	30	45	59	78	98	123

The cases $k = 1, 2$ are obvious (union of two complex conjugate lines or conics, respectively). The lower bound for $k = 6$ is given by Proposition 4.7, and all other cases are covered by Theorem 4.5. Asymptotically, we have

$$\frac{4}{3}k^2 \lesssim \delta(k) \lesssim \frac{3}{2}k^2,$$

where the lower bound follows from Theorem 4.5.

A finite real sextic C_6 with $|\mathbb{R}C_6| = \delta(3) = 10$ was constructed by D. Hilbert [Hil88]. We could not find in the literature a finite real octic C_8 with $|\mathbb{R}C_8| = \delta(4) = 19$; our construction given by Theorem 4.5 can easily be paraphrased without referring to patchworking. The best previously known asymptotic lower bound $\delta(k) \gtrsim \frac{10}{9}k^2$ is found in M. D. Choi, T. Y. Lam, B. Reznick [CLR80].

With the genus $g = g(C)$ fixed, the upper bound

$$\delta_g(k) \le k^2 + g + 1$$

is also given by a strengthening of the Petrovsky inequalities (see Theorem 2.5). In Theorem 4.8, we show that this bound is sharp for $g \le k - 3$.

Most results extend to curves in ruled surfaces: upper bounds are given by Theorem 2.5 (for g fixed) and Corollary 2.6; an asymptotic lower bound is given by Theorem 4.2 (which also covers arbitrary projective toric surfaces), and a few sporadic constructions are discussed in Sections 5, 6.

1.3. **Contents of the paper.** In Section 2, we obtain the upper bounds, derived essentially from the Comessatti inequalities. In Section 3, we discuss the auxiliary tools used in the constructions, namely, the patchworking techniques, hyperelliptic (*aka* bigonal) curves and *dessins d'enfants*, and deformation to the normal cone. Section 4 is dedicated to curves in $\mathbb{C}P^2$: we recast the upper bounds, describe a general construction for toric surfaces (Theorem 4.2) and a slight improvement for the projective plane (Theorem 4.5), and prove the sharpness of the bound $\delta_g(k) \le k^2 + g + 1$ for curves of small genus. In Section 5, we consider surfaces ruled over $\mathbb{R}$, proving the sharpness of the upper bounds for small bi-degrees and for small genera. Finally, Section 6 deals with finite real curves in the ellipsoid.

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2. Strengthened Comessatti inequalities

Let (X, c) be a smooth real projective surface. We denote by $\sigma^{\pm}_{\mathrm{inv}}(X, c)$ (respectively, $\sigma^{\pm}_{\mathrm{skew}}(X, c)$) the inertia indices of the invariant (respectively, skew-invariant) sublattice of the involution $c_*\colon H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ induced by c. The following statement is standard.

Proposition 2.1 (see, for example, [Wil78])**.** *One has*

$$\sigma^{-}_{\mathrm{inv}}(X, c) = \frac{1}{2}(h^{1,1}(X) + \chi(\mathbb{R}X)) - 1, \qquad \sigma^{-}_{\mathrm{skew}}(X, c) = \frac{1}{2}(h^{1,1}(X) - \chi(\mathbb{R}X)),$$

where $h^{\bullet,\bullet}$ are the Hodge numbers and χ is the topological Euler characteristic.

Corollary 2.2 (Comessatti inequalities)**.** *One has*

$$2 - h^{1,1}(X) \le \chi(\mathbb{R}X) \le h^{1,1}(X).$$

Remark 2.3. Let $C \subset \mathbb{C}P^2$ be a smooth real curve of degree $2k$. Recall that an *oval* of C is a connected component $\mathfrak{o} \subset \mathbb{R}C$ bounding a disk in $\mathbb{R}P^2$; the latter disk is called the *interior* of $\mathfrak{o}$. An oval $\mathfrak{o}$ of C is called *even* (respectively, *odd*) if $\mathfrak{o}$ is contained inside an even (respectively, odd) number of other ovals of C; the number of even (respectively, odd) ovals of a given curve C is denoted by p (respectively, n). The classical Petrovsky inequalities [Pet38] state that

$$p - n \le \frac{3}{2}k(k-1) + 1, \qquad n - p \le \frac{3}{2}k(k-1).$$

These inequalities can be obtained by applying Corollary 2.2 to the double covering of $\mathbb{C}P^2$ branched along $C \subset \mathbb{C}P^2$ (see *e.g.* [Wil78], [Man17, Th. 3.3.14]).

The Comessatti and Petrovsky inequalities, strengthened in several ways (see, *e.g.*, [Vir86]), have a variety of applications. For example, for nodal finite real rational curves in $\mathbb{C}P^2$ we immediately obtain the following statement.

Proposition 2.4. *Let $C \subset \mathbb{C}P^2$ be a nodal finite rational curve of degree $2k$. Then, $|\mathbb{R}C| \le k^2 + 1$.*

Proof. Denote by r the number of real nodes of C, and denote by s the number of pairs of complex conjugate nodes of C. We have $r + 2s = (k-1)(2k-1)$. Let, further, Y be the double covering of $\mathbb{C}P^2$ branched along the smooth real curve $C_t \subset \mathbb{C}P^2$ obtained from C by a small perturbation creating an oval from each real node of C. The union of r small discs bounded by $\mathbb{R}C_t$ is denoted by $\mathbb{R}P^2_+$; let $\bar{c}\colon Y \to Y$ be the lift of the real structure such that the real part projects onto $\mathbb{R}P^2_+$. Each pair of complex conjugate nodes of C gives rise to a pair of $\bar{c}_*$-conjugate vanishing cycles in $H_2(Y;\mathbb{Z})$; their difference is a skew-invariant class of square -4, and the s square -4 classes thus obtained are pairwise orthogonal.

Since $h^{1,1}(Y) = 3k^2 - 3k + 2$ (see, *e.g.* [Wil78]), Corollary 2.2 implies that

$$\chi(\mathbb{R}Y) \le h^{1,1}(Y) - 2s = 3k(k-1) + 2 - 2s = k^2 + 1 + r.$$

Thus, $r \le k^2 + 1$. □

The above statement can be generalized to the case of not necessarily nodal curves of arbitrary genus in any smooth real projective surface.

Recall that the *geometric genus* $g(C)$ of an irreducible and reduced algebraic curve C is the genus of its normalization. If C is reduced with irreducible components $C_1, \ldots, C_n$, the geometric genus of C is defined by

$$g(C) = g(C_1) + \ldots + g(C_n) + 1 - n.$$

In other words, $2 - 2g(C) = \chi(\tilde{C})$, where $\tilde{C}$ is the normalization.

Define also the *weight* $\|p\|$ of a solitary point p of a real curve C as the minimal number of blow-ups *at real points* necessary to resolve p. More precisely, $\|p\| = 1 + \sum \|p_i\|$, the summation running over all real points p_i over p of the strict transform of C blown up at p. For example, the weight of a simple node equals 1, whereas the weight of an A_{2n-1}-type point equals n. If $|\mathbb{R}C| < \infty$, we define the *weighted point count* $\|\mathbb{R}C\|$ as the sum of the weights of all real points of C.

The topology of the ambient complex surface X is present in the next statement in the form of the coefficient

$$T_{2,1}(X) = \frac{1}{6}\big(c_1^2(X) - 5c_2(X)\big) = \frac{1}{2}\big(\sigma(X) - \chi(X)\big) = b_1(X) - h^{1,1}(X)$$

of the Todd genus (see [Hir86]).

Theorem 2.5. *Let (X, c) be a simply connected smooth real projective surface with non-empty connected real part. Let $C \subset X$ be an ample reduced finite real algebraic curve such that $[C] = 2e$ in $H_2(X;\mathbb{Z})$. Then, we have*

$$|\mathbb{R}C| \le \|\mathbb{R}C\| \le e^2 + g(C) - T_{2,1}(X) + \chi(\mathbb{R}X) - 1. \tag{1}$$

Furthermore, the second inequality is strict unless all singular points of C are double.

Proof. Since $[C] \in H_2(X;\mathbb{Z})$ is divisible by 2, there exists a real double covering $\rho\colon (Y, \bar{c}) \to (X, c)$ ramified at C and such that $\rho(\mathbb{R}Y) = \mathbb{R}X$. By the embedded resolution of singularities, we can find a sequence of real blow-ups $\pi_i\colon X_i \to X_{i-1}$, $i = 1, \ldots, n$, real curves $C_i = \pi^* C_{i-1} \bmod 2 \subset X_i$, and real double coverings $\rho_i = \pi_i^* \rho_{i-1}\colon Y_i \to X_i$ ramified at C_i such that the curve C_n and surface Y_n are nonsingular. (Here, a *real blow-up* is either a blow-up at a real point or a pair of blow-ups at two conjugate points. By $\pi_i^* C_{i-1} \bmod 2$ we mean the reduced divisor obtained by retaining the *odd multiplicity components* of the divisorial pull-back $\pi_i^* C_{i-1}$.)

Using Proposition 2.1, we can rewrite (1) in the form

$$e^2 + g(C) + h^{1,1}(X) - T_{2,1}(X) + 2\chi(\mathbb{R}X) - 2\sigma^-_{\mathrm{inv}}(X, c) - \|\mathbb{R}C\| \ge 3.$$

We proceed by induction and prove a modified version of the latter inequality, namely,

$$e_i^2 + g(C_i) + h^{1,1}(X_i) - T_{2,1}(X_i) + b_1^{-1}(Y_i) + 2\chi(\mathbb{R}X_i) - 2\sigma^-_{\mathrm{inv}}(X_i, c) - \|\mathbb{R}C_i\| \ge 3, \tag{2}$$

where $[C_i] = 2e_i \in H_2(X_i;\mathbb{Z})$ and $b_1^{-1}(\cdot)$ is the dimension of the (-1)-eigenspace of ρ_* on $H_1(\cdot;\mathbb{C})$.

For the "complex" ingredients of (2), it suffices to consider a blow-up $\pi\colon \tilde{X} \to X$ at a singular point p of C, not necessarily real, of multiplicity $O \ge 2$. Denoting by C' the strict transform of C, we have $\tilde{C} = \pi^* C \bmod 2 = C' + \varepsilon E$, where $E = \pi^{-1}(p)$ is the exceptional divisor and $O = 2m + \varepsilon$, $m \in \mathbb{Z}$, $\varepsilon = 0, 1$. Then, in obvious notation,

$$e^2 = \tilde{e}^2 + m^2, \qquad g(C) = g(\tilde{C}) + \varepsilon, \qquad h^{1,1}(X) = h^{1,1}(\tilde{X}) - 1, \qquad T_{2,1}(X) = T_{2,1}(\tilde{X}) + 1.$$

Furthermore, from the isomorphisms $H_1(\tilde{Y}, \tilde{\rho}^* E) = H_1(Y, p) = H_1(Y)$ we easily conclude that

$$b_1^{-1}(Y) \ge b_1^-(\tilde{Y}) - b_1^-(\tilde{\rho}^* E) \ge b_1^{-1}(\tilde{Y}) - 2(m-1).$$

It follows that, when passing from $\tilde{X}$ to X, the increment in the first five terms of (2) is at least $(m-1)^2 + \varepsilon - 1 \ge -1$; this increment equals (-1) if and only if p is a double point of C.

For the last three terms, assume first that the singular point p above is real. Then

$$\chi(\mathbb{R}X) = \chi(\mathbb{R}\tilde{X}) + 1, \qquad \sigma^-_{\mathrm{inv}}(X_i, c) = \sigma^-_{\mathrm{inv}}(\tilde{X}_i, \tilde{c}), \qquad \|\mathbb{R}C\| = \|\mathbb{R}\tilde{C}\| + 1,$$

and the total increment in (2) is non-negative; it equals 0 if and only if p is a double point.

Now, let $\pi\colon \tilde{X} \to X$ be a pair of blow-ups at two complex conjugate singular points of C. Then

$$\chi(\mathbb{R}X) = \chi(\mathbb{R}\tilde{X}), \qquad \sigma^-_{\mathrm{inv}}(X_i, c) = \sigma^-_{\mathrm{inv}}(\tilde{X}_i, \tilde{c}) - 1, \qquad \|\mathbb{R}C\| = \|\mathbb{R}\tilde{C}\|,$$

and, again, the total increment is non-negative, equal to 0 if and only if both points are double.

To establish (2) for the last, nonsingular, curve C_n, we use the following observations:

- $\chi(Y_n) = 2\chi(X_n) - \chi(C_n)$ (the Riemann-Hurwitz formula);

- $\sigma(Y_n) = 2\sigma(X_n) - 2e_n^2$ (Hirzebruch's theorem);
- $b_1(Y_n) - b_1(X_n) = b_1^{-1}(Y_n)$, as $b_1^{+1}(Y_n) = b_1(X_n)$ *via* the transfer map;
- $\chi(\mathbb{R}Y_n) = 2\chi(\mathbb{R}X_n)$, since $\mathbb{R}C_n = \varnothing$ and $\mathbb{R}Y_n \to \mathbb{R}X_n$ is an unramified double covering.

Then, (2) takes the form

$$\sigma^-_{\mathrm{inv}}(Y_n, \bar{c}_n) \geq \sigma^-_{\mathrm{inv}}(X_n, c_n),$$

which is obvious in view of the transfer map $H_2(X_n; \mathbb{R}) \to H_2(Y_n; \mathbb{R})$: this map is equivariant and isometric up to a factor of 2.

Thus, there remains to notice that $b_1^{-1}(Y_0) = 0$. Indeed, since $C_0 = C$ is assumed ample, $X \smallsetminus C$ has homotopy type of a CW-complex of dimension 2 (as a Stein manifold). Hence, so does $Y \smallsetminus C$, and the homomorphism $H_1(C; \mathbb{R}) \to H_1(Y; \mathbb{R})$ is surjective. Clearly, $b_1^{-1}(C) = 0$. □

Corollary 2.6. *Let (X, c) and $C \subset X$ be as in Theorem 2.5. Then, we have*

$$2|\mathbb{R}C| \leq 3e^2 - e \cdot c_1(X) - T_{2,1}(X) + \chi(\mathbb{R}X),$$

the inequality being strict unless each singular point of C is a solitary real node of $\mathbb{R}C$.

Proof. By the adjunction formula we have

$$g(C) \leq 2e^2 - e \cdot c_1(X) + 1 - |\mathbb{R}C|,$$

and the result follows from Theorem 2.5. □

Remark 2.7. The assumptions $\pi_1(X) = 0$ and $b_0(\mathbb{R}X) = 1$ in Theorem 2.5 are mainly used to assure the existence of a real double covering $\rho\colon Y \to X$ ramified over a given real divisor C. In general, one should speak about the divisibility by 2 of the *real divisor class* $|C|_{\mathbb{R}}$, *i.e.*, class of real divisors modulo real linear equivalence. (If $\mathbb{R}X \neq \varnothing$, one can alternatively speak about the set of real divisors in the linear system $|C|$ or a real point of $\operatorname{Pic}(X)$.) A necessary condition is the vanishing

$$[C] = 0 \in H_{2n-2}(X; \mathbb{Z}/2\mathbb{Z}), \qquad [\mathbb{R}C] = 0 \in H_{n-1}(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}),$$

where $n = \dim_{\mathbb{C}}(X)$ and $[\mathbb{R}C]$ is the homology class of the real part of (any representative of) $|C|$ (the sufficiency of this condition in some special cases is discussed in Lemma 3.4 below). If not empty, the set of double coverings ramified over C and admitting real structure is a torsor over the space of c^*-invariant elements of $H^1(X; \mathbb{Z}/2\mathbb{Z})$.

The proof of the following theorem repeats literally that of Theorem 2.5.

Theorem 2.8. *Let (X, c) be a smooth real projective surface and $C \subset X$ an ample finite reduced real algebraic curve such that the class $|C|_{\mathbb{R}}$ is divisible by 2. A choice of a real double covering $\rho\colon Y \to X$ ramified over C defines a decomposition of $\mathbb{R}X$ into two disjoint subsets $\mathbb{R}X_+ = \rho(\mathbb{R}Y)$ and $\mathbb{R}X_-$ consisting of whole components. Then, we have*

$$\|\mathbb{R}C \cap \mathbb{R}X_+\| - \|\mathbb{R}C \cap \mathbb{R}X_-\| \leq e^2 + g(C) - T_{2,1}(X) + \chi(\mathbb{R}X_+) - \chi(\mathbb{R}X_-) - 1,$$

the inequality being strict unless all singular points of C are double. □

3. Construction tools

3.1. **Patchworking.** If Δ is a convex lattice polygon contained in the non-negative quadrant $(\mathbb{R}_{\geq 0})^2 \subset \mathbb{R}^2$, we denote by $\operatorname{Tor}(\Delta)$ the toric variety associated with Δ; this variety is a surface if Δ is non-degenerate. In the latter case, the complex torus $(\mathbb{C}^*)^2$ is naturally embedded in $\operatorname{Tor}(\Delta)$. Let $V \subset (\mathbb{R}_{\geq 0})^2 \cap \mathbb{Z}^2$ be a finite set, and let $P(x, y) = \sum_{(i,j)\in V} a_{ij} x^i y^j$ be a real polynomial in two variables. The *Newton polygon* Δ_P of P is the convex hull in $\mathbb{R}^2$ of those points in V that correspond to the non-zero monomials of P. The polynomial P defines an algebraic curve in the 2-dimensional complex torus $(\mathbb{C}^*)^2$; the closure of this curve in $\operatorname{Tor}(\Delta_P)$ is an algebraic curve $C \subset \operatorname{Tor}(\Delta_P)$. If Q is a quadrant of $(\mathbb{R}^*)^2 \subset (\mathbb{C}^*)^2$ and (a,b) is a vector in $\mathbb{Z}^2$, we denote by $Q(a,b)$ the quadrant

$$\{(x,y) \in (\mathbb{R}^*)^2 \mid ((-1)^a x, (-1)^b y) \in Q\}.$$

If e is an integral segment whose direction is generated by a primitive integral vector (a,b), we abbreviate $Q(e^\perp) := Q(b,-a)$. A real algebraic curve $C \subset \operatorname{Tor}(\Delta)$ is said to be $\frac{1}{4}$-*finite* (respectively, $\frac{1}{2}$-*finite*) if the intersection of the real part $\mathbb{R}C$ with the positive quadrant $(\mathbb{R}_{>0})^2$ (respectively, the union $(\mathbb{R}_{>0})^2 \cup (\mathbb{R}_{>0})^2(1,0)$ is finite.

Given an algebraic curve $C \subset \operatorname{Tor}(\Delta)$ and an edge e of Δ, we put $T_e(C) := C \cap D(e)$, where $D(e)$ is the toric divisor corresponding to e.

Fix a subdivision $\mathcal{S} = \{\Delta_1, \ldots, \Delta_N\}$ of a convex polygon $\Delta \subset (\mathbb{R}_{\geq 0})^2$ such that there exists a piecewise-linear convex function $\nu\colon \Delta \to \mathbb{R}$ whose maximal linearity domains are precisely the non-degenerate lattice polygons $\Delta_1, \ldots, \Delta_N$. Let a_{ij}, $(i,j) \in \Delta \cap \mathbb{Z}^2$, be a collection of real numbers such that $a_{ij} \neq 0$ whenever (i,j) is a vertex of $\mathcal{S}$. This gives rise to N real algebraic curves C_k, $k = 1, \ldots, N$: each curve $C_k \subset \operatorname{Tor}(\Delta_k)$ is defined by the polynomial

$$P(x,y) = \sum_{(i,j) \in \Delta_k \cap \mathbb{Z}^2} a_{ij} x^i y^j$$

with the Newton polygon Δ_k.

Commonly, we denote by $\operatorname{Sing}(C)$ the set of singular points of an algebraic curve C.

Assume that each curve C_k is nodal and $\operatorname{Sing}(C_k)$ is disjoint from the toric divisors of $\operatorname{Tor}(\Delta_k)$ (but C_k can be tangent with arbitrary order of tangency to some toric divisors). For each inner edge $e = \Delta_i \cap \Delta_j$ of $\mathcal{S}$, the toric divisors corresponding to e in $\operatorname{Tor}(\Delta_i)$ and $\operatorname{Tor}(\Delta_j)$ are naturally identified, as they both are $\operatorname{Tor}(e)$. The intersection points of C_i and C_j with these toric divisors are also identified, and, at each such point $p \in \operatorname{Tor}(e)$, the orders of intersection of C_i and C_j with $\operatorname{Tor}(e)$ automatically coincide; this common order is denoted by $\operatorname{mult} p$ and, if $\operatorname{mult} p > 1$, the point p is called *fat*. Assume that $\operatorname{mult} p$ is even for each *fat* point p and that the local branches of C_i and C_j at each *real* fat point p are in the same quadrant $Q_p \subset (\mathbb{R}^*)^2$.

Each edge E of Δ is a union of exterior edges e of $\mathcal{S}$; denote the set of these edges by $\{E\}$ and, given $e \in \{E\}$, let $k(e)$ be the index such that $e \subset \Delta_{k(e)}$. The toric divisor $D(E) \subset \operatorname{Tor}(\Delta)$ is a smooth real rational curve whose real part $\mathbb{R}D(E)$ is divided into two halves $\mathbb{R}D_\pm(E)$ by the intersections with other toric divisors of $\operatorname{Tor}(\Delta)$; we denote by $\mathbb{R}D_+(E)$ the half adjacent to the positive quadrant of $(\mathbb{R}^*)^2$. Similarly, the toric divisor $D(e) \subset \operatorname{Tor}(\Delta_{k(e)})$ is divided into $\mathbb{R}D_\pm(e)$.

Theorem 3.1 (Patchworking construction; essentially, Theorem 2.4 in [Shu06])**.** *Under the assumptions above, there exists a family of real polynomials $P^{(t)}(x,y)$, $t \in \mathbb{R}_{>0}$, with the Newton polygon Δ, such that, for sufficiently small t, the curve $C^{(t)} \subset \operatorname{Tor}(\Delta)$ defined by $P^{(t)}$ has the following properties:*

- *the curve $C^{(t)}$ is nodal and* $\operatorname{Sing}(C^{(t)})$ *is disjoint from the toric divisors;*
- *if all curves $C_1, \ldots, C_N$ are $\frac{1}{2}$-finite (respectively, $\frac{1}{4}$-finite), then so is $C^{(t)}$;*
- *there is an injective map*

$$\Phi\colon \coprod_{k=1}^{N} \operatorname{Sing}(C_k) \to \operatorname{Sing}(C^{(t)}),$$

such that the image of each real point is a real point of the same type (solitary/non-solitary) and in the same quadrant of $(\mathbb{R}^*)^2$, and the image of each imaginary point is imaginary;

- *there is a partition*

$$\operatorname{Sing}(C^{(t)}) \smallsetminus \text{image of } \Phi = \coprod_p \Pi_p,$$

p running over all fat points, so that $|\Pi_p| = 2m-1$ if $\operatorname{mult} p = 2m$. The points in Π_p are imaginary if p is imaginary and real and solitary if p is real; in the latter case, $(m-1)$ of these points lie in Q_p and the others m points lie in $Q_p(e_p^{\perp})$, where $p \in \operatorname{Tor}(e_p)$;
- *for each edge E of Δ, there is a bijective map*

$$\Psi_E\colon \coprod_{e\in\{E\}} T_e(C_{k(e)}) \to T_E(C^{(t)})$$

preserving the intersection multiplicity and the position of points in $\mathbb{R}D_{\pm}(\cdot)$ or $D(\cdot)\smallsetminus\mathbb{R}D(\cdot)$.

Proof. To deduce the statement from [Shu06, Theorem 2.4], one can use Lemma 5.4(ii) in [Shu05] and the deformation patterns described in [IKS15], Lemmas 3.10 and 3.11 (*cf.* also the curves $C_{*,0,0}$ in Lemma 3.2 below). □

3.2. **Bigonal curves *via dessins d'enfants*.** We denote by Σ_n, $n \ge 0$, the Hirzebruch surface of degree n, *i.e.*, $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1})$. Recall that $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and Σ_1 is the blow-up of $\mathbb{C}P^2$ at a point. The bundle projection induces a map $\pi\colon \Sigma_n \to \mathbb{C}P^1$, and we denote by F a fiber of π; it is isomorphic to $\mathbb{C}P^1$. The images of $\mathcal{O}_{\mathbb{C}P^1}$ and $\mathcal{O}_{\mathbb{C}P^1}(n)$ are denoted by B_0 and B_∞, respectively; these curves are sections of π. The group $H^2(\Sigma_n;\mathbb{C}) = H^{1,1}(X;\mathbb{C})$ is generated by the classes of B_0 and F, and we have

$$[B_0]^2 = n,\quad [B_\infty]^2 = -n,\quad [F]^2 = 0,\quad B_\infty \sim B_0 - nF,\quad c_1(\Sigma_n) = 2[B_0] + (2-n)[F].$$

(If $n > 0$, the *exceptional section* B_∞ is the only irreducible curve of negative self-intersection.) In other words, we have $D \sim aB_0 + bF$ for each divisor $D \subset \Sigma_n$, and the pair $(a,b) \in \mathbb{Z}^2$ is called the *bidegree* of D. The cone of effective divisors is generated by B_∞ and F, and the cone of ample divisors is $\{aB_0 + bF \mid a, b > 0\}$.

In this section, we equip $\mathbb{C}P^1$ with the standard complex conjugation, and the surface Σ_n with the real structure c induced by the standard complex conjugation on $\mathcal{O}_{\mathbb{C}P^1}(n)$. Unless $n = 0$, this is the only real structure on Σ_n with nonempty real part. In particular c acts on $H^2(\Sigma_n;\mathbb{C})$ as $-\operatorname{Id}$, and so $\sigma^-_{\mathrm{inv}}(X,c) = 0$. The real part of Σ_n is a torus if n is even, and a Klein bottle if n is odd. In the former case, the complement $\mathbb{R}\Sigma_n \smallsetminus (\mathbb{R}B_0 \cup \mathbb{R}B_\infty)$ has two connected components, which we denote by $\mathbb{R}\Sigma_{n,\pm}$.

Lemma 3.2. *Given integers $n > 0$, $b \ge 0$, and $0 \le q \le n+b-1$, there exists a real algebraic rational curve $C = C_{n,b,q}$ in Σ_{2n} of bidegree $(2,2b)$ such that (see Figure 1):*

(1) all singular points of C are $2n+2b-1$ solitary nodes; $n+b+q$ of them lie in $\mathbb{R}\Sigma_{2n,+}$, and the other $n+b-q-1$ lie in $\mathbb{R}\Sigma_{2n,-}$;

(2) the real part $\mathbb{R}C$ has a single extra oval $\mathfrak{o}$, which is contained in $\mathbb{R}\Sigma_{2n,-} \cup \mathbb{R}B_0 \cup \mathbb{R}B_\infty$ and does not contain any of the nodes in its interior;

(3) each intersection $p_\infty := \mathfrak{o} \cap B_\infty$ and $p_0 := \mathfrak{o} \cap B_0$ consists of a single point, the multiplicity being $2b$ and $4n+2b-2q$, respectively; the points p_0 and p_∞ are on the same fiber F.

This curve can be perturbed to a curve $\widetilde{C}_{n,b,q} \subset \Sigma_{2n}$ satisfying conditions (1) and (2) and the following modified version of condition (3):

($\tilde{3}$) the oval $\mathfrak{o}$ intersects B_∞ and B_0 at, respectively, b and $2n+b-q$ simple tangency points.

Note that $C_{n,b,q}$ intersects B_0 in q additional pairs of complex conjugate points.

Proof. Up to elementary transformations of Σ_{2n} (blowing up the point of intersection $C \cap B_\infty$ and blowing down the strict transforms of the corresponding fibers) we may assume that $b = 0$ and, hence, C is disjoint from B_∞. Then, C is given by $P(x,y) = 0$, where

$$P(x,y) = y^2 + a_1(x)y + a_2(x),\qquad \deg a_i(x) = 2in. \tag{3}$$

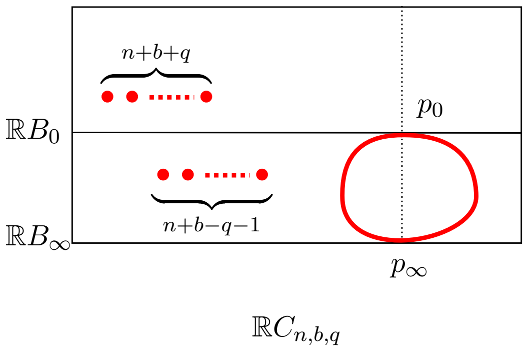


FIGURE 1.

(Strictly speaking, a_i are sections of appropriate line bundles, but we pass to affine coordinates and regard a_i as polynomials.) We will construct the curves using the techniques of *dessins d'enfants*, *cf.* [Ore03, DIK08, Deg12]. Consider the rational function $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ given by

$$f(x) = \frac{a_1^2(x) - 4a_2(x)}{a_1^2(x)}.$$

(This function differs from the j-invariant of the trigonal curve $C + B_0$ by a few irrelevant factors.) The *dessin* of C is the graph $\mathcal{D} := f^{-1}(\mathbb{R}P^1)$ decorated as shown in Figure 2. In addition to $\times$-, $\circ$-,

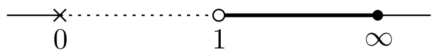


FIGURE 2. Decoration of a dessin

and $\bullet$-vertices, it may also have *monochrome* vertices, which are the pull-backs of the real critical values of f other that 0, 1, or ∞. This graph is real, and we depict only its projection to the disk $D := \mathbb{C}P^1/x \sim \bar{x}$, showing the boundary ∂D by a wide grey curve: this boundary corresponds to the real parts $\mathbb{R}C \subset \mathbb{R}\Sigma_{2n} \to \mathbb{R}P^1$. Assuming that a_1, a_2 have no common roots, the real special vertices and edges of $\mathcal{D}$ have the following geometric interpretation:

- a $\times$-vertex x_0 corresponds to a double root of the polynomial $P(x_0, y)$; the curve is tangent to a fiber if $\operatorname{val} x_0 = 2$ and has a double point of type A_{p-1}, $p = \frac{1}{2}\operatorname{val} x_0$, otherwise;
- a $\circ$-vertex x_0 corresponds to an intersection $\mathbb{R}C \cap \mathbb{R}B_0$ of multiplicity $\frac{1}{2}\operatorname{val} x_0$;
- the real part $\mathbb{R}C$ is empty over each point of a solid edge and consists of two points over each point of any other edge;
- the points of $\mathbb{R}C$ over two $\times$-vertices x_1, x_2 are in the same half $\mathbb{R}\Sigma_{2n,\pm}$ if and only if one has $\sum \operatorname{val} z_i = 0 \bmod 8$, the summation running over all $\bullet$-vertices z_i in any of the two arcs of ∂D bounded by x_1, x_2.

(For the last item, observe that the valency of each $\bullet$-vertex is 0 mod 4 and the sum of all valencies equals $2\deg f = 8n$; hence, the sum in the statement is independent of the choice of the arc.)

Now, to construct the curves in the statements, we start with the dessin $\widetilde{\mathcal{D}}_{n,0,0}$ shown in Figure 3, left: it has $2n$ $\bullet$-vertices, $2n$ $\circ$-vertices, and $(2n+1)$ $\times$-vertices, two bivalent and $(2n-1)$ four-valent, numbered consecutively along ∂D. To obtain $\widetilde{\mathcal{D}}_{n,0,q}$, we replace q disjoint embraced fragments with copies of the fragment shown in Figure 3, right; by choosing the fragments replaced around *even-numbered* $\times$-vertices, we ensure that the solitary nodes would migrate from $\mathbb{R}\Sigma_{2n,-}$ to $\mathbb{R}\Sigma_{2n,+}$. Finally, $\mathcal{D}_{n,0,q}$ is obtained from $\widetilde{\mathcal{D}}_{n,0,q}$ by contracting the dotted real segments connecting the real $\circ$-vertices, so that the said vertices collide to a single $(8n-4q)$-valent one. Each of these dessins $\mathcal{D}$

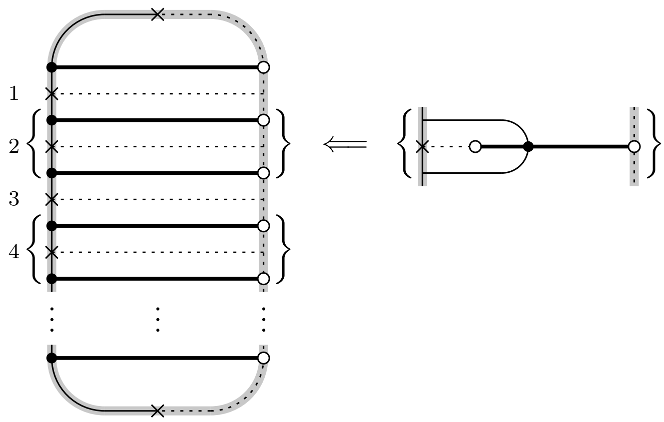


FIGURE 3. The dessin $\widetilde{\mathcal{D}}_{n,0,0}$ and its modifications

gives rise to a (not unique) equivariant topological branched covering $f\colon S^2 \to \mathbb{C}P^1$ (*cf.* [Ore03, DIK08, Deg12]), and the Riemann existence theorem gives us an analytic structure on the sphere S^2 making f a real rational function $\mathbb{C}P^1 \to \mathbb{C}P^1$. There remains to take for a_1 a real polynomial with a simple zero at each (double) pole of f and let $a_2 := \frac{1}{4}a_1^2(1-f)$. □

Generalizing, one can consider a geometrically ruled surface $\pi\colon \Sigma_n(\mathcal{O}) := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_B) \to B$, where B is a smooth compact real curve of genus $\mathfrak{g} \geq 1$ and $\mathcal{O}$ is a line bundle, $\deg \mathcal{O} = n \geq 0$. If $\mathcal{O}$ is also real, the surface $\Sigma_n(\mathcal{O})$ acquires a real structure; the sections B_0 and B_∞ are also real and we can speak about $\mathbb{R}B_0$, $\mathbb{R}B_\infty$. The real line bundle $\mathcal{O}$ is said to be *even* if the $GL(1,\mathbb{R})$-bundle $\mathbb{R}\mathcal{O}$ over $\mathbb{R}B$ is trivial (*cf.* Remark 2.7). In this case, the real part $\mathbb{R}\Sigma_n(\mathcal{O})$ is a disjoint union of tori, one torus T_i over each real component $\mathbb{R}_iB$ of B, and each complement $T_i^\circ := T_i \smallsetminus (\mathbb{R}B_0 \cup \mathbb{R}B_\infty)$ is made of two connected components (open annuli).

A smooth compact real curve B of genus $\mathfrak{g}$ is called *maximal* if it has the maximal possible number of real connected components: $b_0(\mathbb{R}B) = \mathfrak{g} + 1$.

Lemma 3.3. *Let n, $\mathfrak{g}$ be two integers, $n \geq \mathfrak{g} - 1 \geq 0$. Then there exists an even real line bundle $\mathcal{O}$ of degree* $\deg \mathcal{O} = 2n$ *over a maximal real algebraic curve B of genus $\mathfrak{g}$, and a nodal real algebraic curve $C_n(\mathfrak{g}) \subset \Sigma_{2n}(\mathcal{O})$ realizing the class $2[B_0] \in H_2(\Sigma_{2n}(\mathcal{O}); \mathbb{Z})$ such that*

(1) $\mathbb{R}C_n(\mathfrak{g}) \cap T_1$ consists of $2n$ solitary nodes, all in the same connected component of T_1°;
(2) $\mathbb{R}C_n(\mathfrak{g}) \cap T_2$ is a smooth connected curve, contained in a single connected component of T_2° except for n real points of simple tangency of C_n and B_0;
(3) $\mathbb{R}C_n(\mathfrak{g}) \cap T_i$, $i \geq 3$, is a smooth connected curve, contained in a single connected component of T_2° except for one real point of simple tangency of C_n and B_0.

Note that we can only assert the existence of a ruled surface $\Sigma_{2n}(\mathcal{O})$: the analytic structure on B and line bundle $\mathcal{O}$ are given by the construction and cannot be fixed in advance.

Proof. We proceed as in the proof of Lemma 3.2, with the "polynomials" a_i sections of $\mathcal{O}^{\otimes i}$ in (3) and half-dessin $\mathcal{D}_n(\mathfrak{g})/c_B$ in the surface $D := B/c_B$, which, in the case of maximal B, is a disk with $\mathfrak{g}$ holes; as above, we have $\partial D = \mathbb{R}B$. The following technical requirements are necessary and sufficient for the existence of a topological ramified covering $f\colon B \to \mathbb{C}P^1$ (see [DIK08, Deg12]) with B the orientable double of D:

- each *region* (connected component of $D \smallsetminus \mathcal{D}$) should admit an orientation inducing on the boundary the orientation inherited from $\mathbb{R}P^1$ (the order on $\mathbb{R}$), and

- each *triangular* region (*i.e.*, one with a single vertex of each of the three special types $\times$, $\circ$, and $\bullet$ in the boundary) should be a topological disk.

(For example, in the dessins $\widetilde{\mathcal{D}}_{n,0,q}$ in Figure 3 the orientations are given by a chessboard coloring and all regions are triangles.)

The curve $C_n(\mathfrak{g})$ as in the statement is obtained from the dessin $\mathcal{D}_n(\mathfrak{g})$ constructed as follows. If $\mathfrak{g} = 1$, then $\mathcal{D}_n(1)$ is the dessin in the annulus shown in Figure 4, left (which is a slight modification

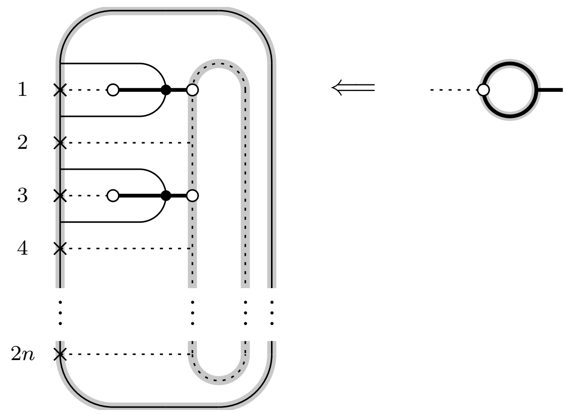


FIGURE 4. The dessin $\mathcal{D}_n(1)$ and its modifications

of $\widetilde{\mathcal{D}}_{n,0,n-1}$ in Figure 3): it has $2n$ real four-valent $\times$-vertices, n inner four-valent $\bullet$-vertices, and $2n$ $\circ$-vertices, n real four-valent and n inner bivalent. (Recall that each inner vertex in D doubles in B, so that the total valency of the vertices of each kind sums up to $8n = 2\deg f$, as expected.) This dessin is maximal in the sense that all its regions are triangles. To pass from $\mathcal{D}_n(1)$ to $\mathcal{D}_n(1+q)$, $q \le n$, we replace small neighbourhoods of q inner $\circ$-vertices with the fragments shown in Figure 4, right, creating q extra boundary components.

Each dessin $\mathcal{D}_n(\mathfrak{g})$ satisfies the two conditions above and, thus, gives rise to a ramified covering $f\colon B \to \mathbb{C}P^1$. The analytic structure on B is given by the Riemann existence theorem, and $\mathcal{O}$ is the line bundle $\mathcal{O}_B(\frac{1}{2}P(f))$, where $P(f)$ is the divisor of poles of f. (All poles are even.) Then, the curve in question is given by "equation" (3), with the sections $a_i \in H^0(B; \mathcal{O}^{\otimes i})$ almost determined by their zeroes: $Z(a_1) = \frac{1}{2}P(f)$ and $Z(a_2) = Z(1-f)$. Further details of this construction (in the more elaborate trigonal case) can be found in [DIK08, Deg12]. $\square$

Next few lemmas deal with the real lifts of the curves constructed in Lemma 3.3 under a ramified double covering of $\Sigma_{2n}(\mathcal{O})$. First, we discuss the existence of such coverings, *cf.* Remark 2.7.

Lemma 3.4. *Let $\Sigma_n(\mathcal{O})$ be a real ruled surface over a real algebraic curve B such that $\mathbb{R}B \neq \varnothing$, and let D be a real divisor on X. Then there exists a real divisor E on X such that $|D|_{\mathbb{R}} = 2|E|_{\mathbb{R}}$ if and only if $[\mathbb{R}D] = 0 \in H_1(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$.*

Proof. By [Har77, Proposition 2.3], we have

$$\operatorname{Pic}(\Sigma_n(\mathcal{O})) \simeq \mathbb{Z}B_0 \oplus \operatorname{Pic}(B),$$

and this isomorphism respects the action induced by the real structures. Let

$$|D| = m|B_0| + |D_0|.$$

Then $m = [\mathbb{R}D] \circ [\mathbb{R}F] \bmod 2$, where F is the fiber of the ruling over a real point $p \in \mathbb{R}B$, and $D_0 = D \circ B_\infty$, so that $[\mathbb{R}D_0] = [\mathbb{R}D] \circ [\mathbb{R}B_\infty]$. There remains to observe that $|D_0|_{\mathbb{R}}$ is divisible by 2 in $\mathbb{R}\,\mathrm{Pic}(B)$ if and only if $[\mathbb{R}D_0] = 0 \in H_0(B;\mathbb{Z}/2\mathbb{Z})$. The "only if" part is clear, and the "if" part follows from the fact that D_0 can be deformed, through real divisors, to $(\deg D_0)p$. □

Lemma 3.5. *Let $X := \Sigma_n(\mathcal{O})$ be a real ruled surface over a real algebraic curve B such that $\mathbb{R}B \neq \varnothing$, and let C be a reduced real divisor on X such that $[\mathbb{R}C] = 0 \in H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$. Then, for any surface $S \subset \mathbb{R}X$ such that $\partial S = \mathbb{R}C$, there exists a real double covering $Y \to X$ ramified over C such that $\mathbb{R}Y$ projects onto S.*

Proof. Pick one covering $Y_0 \to X$, which exists by Lemma 3.4, and let S_0 be the projection of $\mathbb{R}Y_0$. We can assume that $S_0 \cap T_1 = S \cap T_1$ for one of the components T_1 of $\mathbb{R}X$. Given another component T_i, consider a path γ_i connecting a point in T_i to one on T_1, and let $\tilde{\gamma}_i = \gamma_i + c_*\gamma_i$; in view of the obvious equivariant isomorphism $H_1(Y;\mathbb{Z}/2\mathbb{Z}) \simeq H_1(B;\mathbb{Z}/2\mathbb{Z})$, these loops form a partial basis for the space of c_*-invariant classes in $H_1(X;\mathbb{Z}/2\mathbb{Z})$. Now, it suffices to twist Y_0 (*cf.* Remark 2.7) by a cohomology class sending $\tilde{\gamma}_i$ to 0 or 1 if $S \cap T_i$ coincides with $S_0 \cap T_i$ or with the closure of its complement, respectively. □

Lemma 3.6. *Let n, $\mathfrak{g}$ be two integers, $n \geq \mathfrak{g} - 1 \geq 0$, and let B, $\mathcal{O}$, and $C_n(\mathfrak{g}) \subset \Sigma_{2n}(\mathcal{O})$ be as in Lemma 3.3. Then there exists a real double covering $\Sigma_n(\mathcal{O}') \to \Sigma_{2n}(\mathcal{O})$ ramified along $B_0 \cup B_\infty$ and such that the pullback of $C_n(\mathfrak{g})$ is a finite real algebraic curve $C'_n(\mathfrak{g}) \subset \Sigma_n(\mathcal{O}')$ with*

$$|\mathbb{R}C'_n(\mathfrak{g})| = 5n - 1 + \mathfrak{g}.$$

Proof. By Lemma 3.5, there exists a real double covering $\Sigma_n(\mathcal{O}') \to \Sigma_{2n}(\mathcal{O})$ ramified along the curve $B_0 \cup B_\infty$, such that the pull back in $\Sigma_n(\mathcal{O}')$ of the curve $C_n(\mathfrak{g})$ from Lemma 3.3 is a finite real algebraic curve $C'_n(\mathfrak{g})$. Each node of $C_n(\mathfrak{g})$ gives rise to two solitary real nodes of $C'_n(\mathfrak{g})$, and each tangency point of $C_n(\mathfrak{g})$ and $\mathbb{R}B_0$ gives rise to an extra solitary node of $C'_n(\mathfrak{g})$. □

3.3. **Deformation to the normal cone.** We briefly recall the deformation to normal cone construction in the setting we need here, and refer for example to [Ful84] for more details. Given X a non-singular algebraic surface, and $B \subset X$ a non-singular algebraic curve, we denote by $N_{B/X}$ the normal bundle of B in X, its projective completion by $E_B = \mathbb{P}(N_{B/X} \oplus \mathcal{O}_B)$, and we define $B_\infty = E_B \setminus N_{B/X}$. Note that if both X and B are real, then so are E_B and B_∞.

Let $\mathcal{X}$ be the blow up of $X \times \mathbb{C}$ along $B \times \{0\}$. The projection $X \times \mathbb{C} \to \mathbb{C}$ induces a flat projection $\sigma\colon \mathcal{X} \to \mathbb{C}$, and one has $\sigma^{-1}(t) = X$ if $t \neq 0$, and $\sigma^{-1}(0) = X \cup E_B$. Furthermore, in this latter case $X \cap E_B$ is the curve B in X, and the curve B_∞ in E_B. Note that if both X and B are real, and if we equip $\mathbb{C}$ with the standard complex conjugation, then the map σ is a real map.

Let $C_0 = C_X \cup C_B$ be an algebraic curve in $X \cup E_B$ such that:

(1) $C_X \subset X$ is nodal and intersects B transversely;
(2) $C_B \subset E_B$ is nodal and intersects B_∞ transversely; let $a = [C_B] \circ [F]$ in $H_2(E_B;\mathbb{Z})$;
(3) $C_X \cap B = C_B \cap B_\infty = C_X \cap C_B$.

In the following two propositions, we use [ST06, Theorem 2.8] to ensure the existence of a deformation C_t in $\sigma^{-1}(t)$ within the linear system $|C_X + aB|$ of the curve C_0 in some particular instances. We denote by $\mathcal{P}$ the set of nodes of $C_0 \setminus (X \cap E_B)$, and by $\mathcal{I}_X$ (resp. $\mathcal{I}_B$) the sheaf of ideals of $\mathcal{P} \cap X$ (resp. $\mathcal{P} \cap E_B$).

Proposition 3.7. *In the notation above, suppose that $X \subset \mathbb{C}P^3$ is a quadric ellipsoid, and that B is a real hyperplane section. If C_0 is a finite real algebraic curve, then there exists a finite real algebraic curve C_1 in X in the linear system $|C_X + aB|$ such that*

$$|\mathbb{R}C_1| = |\mathbb{R}C_0|.$$

Proof. One has the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(C_X) \otimes \mathcal{I}_X \longrightarrow \mathcal{O}(C_X) \longrightarrow \mathcal{O}_{\mathcal{P}\cap X} \longrightarrow 0.$$

(To shorten the notation, we abbreviate $\mathcal{O}(D) = \mathcal{O}_X(D)$ for a divisor $D \subset X$ when the ambient variety X is understood.) Since $H^1(X, \mathcal{O}(C_X)) = 0$, one obtains the following exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}(C_X)\otimes\mathcal{I}_X) \longrightarrow H^0(X, \mathcal{O}(C_X)) \longrightarrow H^0(\mathcal{P}\cap X, \mathcal{O}_{\mathcal{P}\cap X}) \longrightarrow H^1(X, \mathcal{O}(C_X)\otimes\mathcal{I}_X) \longrightarrow 0.$$

The surface $\mathbb{C}P^1 \times \mathbb{C}P^1$ is toric and it is a classical application of Riemann-Roch Theorem that $H^0(X, \mathcal{O}(C_X) \otimes \mathcal{I}_X)$ has codimension $|\mathcal{P} \cap X|$ in $H^0(X, \mathcal{O}(C_X))$ (see for example [Shu99, Lemma 8 and Corollary 2]). Since $h^0(\mathcal{P} \cap X, \mathcal{O}_{\mathcal{P}\cap X}) = |\mathcal{P} \cap X|$, we deduce that

$$H^1(X, \mathcal{O}(C_X) \otimes \mathcal{I}_X) = 0.$$

The curve B is rational, and the surface E_B is the surface Σ_2. In particular, E_B is a toric surface and B_∞ is an irreducible component of its toric boundary. Hence we analogously obtain

$$H^1(E_B, \mathcal{O}(C_B - B_\infty) \otimes \mathcal{I}_B) = 0.$$

Hence by [ST06, Theorem 3.1], the proposition is now a consequence of [ST06, Theorem 2.8]. □

Recall that $H^0(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B)$ is the set of elements of $H^0(E_B, \mathcal{O}(C_B))$ vanishing on $\mathcal{P} \cap E_B$.

Proposition 3.8. *Suppose that $X = \mathbb{C}P^2$, that B is a non-singular real cubic curve, and that $C_X = \varnothing$. If C_B is a finite real algebraic curve and if $H^0(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B)$ is of codimension $|\mathcal{P}|$ in $H^0(E_B, \mathcal{O}(C_B))$, then there exists a finite real algebraic curve C_1 in $\mathbb{C}P^2$ of degree $3a$ such that*

$$|\mathbb{R}C_1| = |\mathbb{R}C_B|.$$

Proof. Recall that E_B is a ruled surface over B, *i.e.*, is equipped with a $\mathbb{C}P^1$-bundle $\pi\colon E_B \to B$. By [Har77, Lemma 2.4], we have

$$H^i(E_B, \mathcal{O}(C_B)) \simeq H^i(B_\infty, \pi_*\mathcal{O}(C_B)), \qquad i \in \{0, 1, 2\}.$$

In particular the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(C_B - B_\infty) \longrightarrow \mathcal{O}(C_B) \longrightarrow \mathcal{O}_{B_\infty} \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow H^0(E_B, \mathcal{O}(C_B - B_\infty)) \longrightarrow H^0(E_B, \mathcal{O}(C_B)) \longrightarrow H^0(B_\infty, \mathcal{O}_{B_\infty}) \longrightarrow$$
$$\longrightarrow H^1(E_B, \mathcal{O}(C_B - B_\infty)) \longrightarrow H^1(E_B, \mathcal{O}(C_B)) \xrightarrow{\iota_1} H^1(B_\infty, \mathcal{O}_{B_\infty}) \longrightarrow 0.$$

Furthermore, by [GP96, Proposition 3.1] we have $H^1(E_B, \mathcal{O}(C_B - B_\infty)) = 0$, hence the map ι_1 is an isomorphism.

On the other hand, the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(C_B) \otimes \mathcal{I}_B \longrightarrow \mathcal{O}(C_B) \longrightarrow \mathcal{O}_{\mathcal{P}} \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow H^0(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B) \longrightarrow H^0(E_B, \mathcal{O}(C_B)) \xrightarrow{r_1} H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}}) \longrightarrow$$
$$\longrightarrow H^1(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B) \xrightarrow{\iota_2} H^1(E_B, \mathcal{O}(C_B)) \longrightarrow 0.$$

By assumption, the map r_1 is surjective, so we deduce that the map ι_2 is an isomorphism.

We denote by $\widetilde{\mathcal{L}}_0$ the invertible sheaf on the disjoint union of E_B and $\mathbb{C}P^2$ and restricting to $\mathcal{O}(C_B)$ and $\mathcal{O}_{\mathbb{C}P^2}$ on E_B and $\mathbb{C}P^2$ respectively. Finally, we denote by $\mathcal{L}_0$ the invertible sheaf on $\sigma^{-1}(0)$ for which C_0 is the zero set of a section. The natural short exact sequence

$$0 \longrightarrow \mathcal{L}_0 \otimes \mathcal{I}_B \longrightarrow \widetilde{\mathcal{L}}_0 \otimes \mathcal{I}_B \longrightarrow \mathcal{O}_B \longrightarrow 0$$

gives rise to the long exact sequence

$$0 \longrightarrow H^0(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) \longrightarrow H^0(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B) \oplus H^0(\mathbb{C}P^2, \mathcal{O}_{\mathbb{C}P^2}) \xrightarrow{r_2} H^0(B, \mathcal{O}_B) \longrightarrow$$

$$\longrightarrow H^1(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) \longrightarrow H^1(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B) \xrightarrow{\iota} H^1(B, \mathcal{O}_B) \longrightarrow H^2(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) \longrightarrow 0.$$

The restriction of the map r_2 to the second factor $H^0(\mathbb{C}P^2, \mathcal{O}_{\mathbb{C}P^2})$ is clearly an isomorphism, hence we obtain the exact sequence

$$0 \longrightarrow H^1(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) \longrightarrow H^1(E_B, \mathcal{O}(C_B) \otimes \mathcal{I}_B) \xrightarrow{\iota} H^1(B, \mathcal{O}_B) \longrightarrow H^2(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) \longrightarrow 0.$$

Since $\iota = \iota_1 \circ \iota_2$ is an isomorphism, we deduce that $H^1(\sigma^{-1}(0), \mathcal{L}_0 \otimes \mathcal{I}_B) = 0$. Now the proposition follows from [ST06, Theorem 2.8]. □

4. Finite curves in $\mathbb{C}P^2$

In the case $X = \mathbb{C}P^2$, Theorem 2.5 and Corollary 2.6 specialize as follows.

Theorem 4.1. *Let $C \subset \mathbb{C}P^2$ be a finite real algebraic curve of degree $2k$. Then,*

$$|\mathbb{R}C| \leq k^2 + g(C) + 1, \tag{4}$$

$$|\mathbb{R}C| \leq \frac{3}{2}k(k-1) + 1. \tag{5}$$

In the rest of this section, we discuss the sharpness of these bounds.

4.1. **Asymptotic constructions.** The following asymptotic lower bound holds for any projective toric surface with the standard real structure.

Theorem 4.2. *Let $\Delta \subset \mathbb{R}^2$ be a convex lattice polygon, and let X_Δ be the associated toric surface. Then, there exists a sequence of finite real algebraic curves $C_k \subset X_\Delta$ with the Newton polygon $\Delta(C_k) = 2k\Delta$, such that*

$$\lim_{k\to\infty} \frac{1}{k^2}|\mathbb{R}C_k| = \frac{4}{3}\mathrm{Area}(\Delta),$$

where $\mathrm{Area}(\Delta)$ *is the lattice area of* Δ.

Remark 4.3. In the settings of Theorem 4.2, assuming X_Δ smooth, the asymptotic upper bound for finite real algebraic curves $C \subset X_\Delta$ with $\Delta(C) = 2k\Delta$ is given by Theorem 2.5:

$$|\mathbb{R}C| \lesssim \frac{3}{2}\mathrm{Area}(\Delta).$$

Proof of Theorem 4.2. There exists a (unique) real rational cubic $C \subset (\mathbb{C}^*)^2$ such that

- $\Delta(C)$ is the triangle with the vertices $(0,0)$, $(2,1)$, and $(1,2)$;
- the coefficient of the defining polynomial f of C at each corner of $\Delta(C)$ equals 1;
- $\mathbb{R}C \cap \mathbb{R}^2_{>0}$ is a single solitary node.

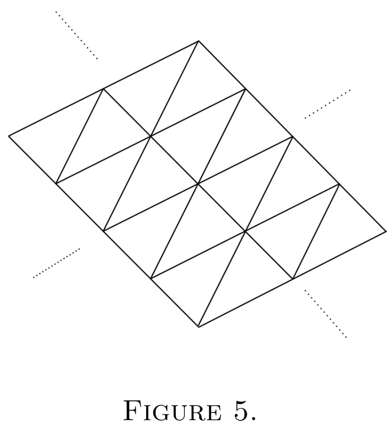


Figure 5.

Figure 5 shows a tilling of $\mathbb{R}^2$ by lattice congruent copies of $\Delta(C)$. Intersecting this tilling with $k\Delta$ and making an appropriate adjustment in the vicinity of the boundary, we obtain a convex subdivision of $k\Delta$ containing $\frac{1}{3}k^2\text{Area}(\Delta)+O(k)$ copies of $\Delta(C)$. Now, to each of these copies, we associate the curve given by an appropriate monomial multiple of either $f(x,y)$ or $f(1/x,1/y)$. Applying Theorem 3.1, we obtain a real polynomial f_k whose zero locus in $\mathbb{R}^2_{>0}$ consists of $\frac{1}{3}k^2\text{Area}(\Delta)+O(k)$ solitary nodes. There remains to let $C_k = \{f_k(x^2,y^2)=0\}$. □

Corollary 4.4. *There exists a sequence of finite real algebraic curves $C_k \subset \mathbb{C}P^2$, $\deg C_k = 2k$, such that*

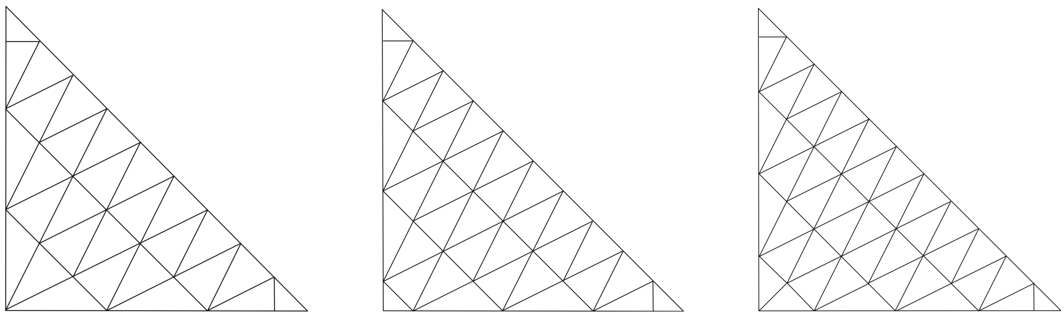
$$\lim_{k\to+\infty} \frac{1}{k^2}|\mathbb{R}C_k| = \frac{4}{3}.$$

In the next theorem, we tweak the "adjustment in the vicinity of the boundary" in the proof of Theorem 4.2 in the case $X_\Delta = \mathbb{C}P^2$.

Theorem 4.5. *For any integer $k \geq 3$, there exists a finite real algebraic curve $C \subset \mathbb{C}P^2$ of degree $2k$ such that*

$$|\mathbb{R}C| = \begin{cases} 12l^2 - 4l + 2 & \text{if } k = 3l, \\ 12l^2 + 4l + 3 & \text{if } k = 3l+1, \\ 12l^2 + 12l + 6 & \text{if } k = 3l+2. \end{cases}$$

Proof. Following the proof of Theorem 4.2, we use the subdivision of the triangle $k\Delta$ (with the vertices $(0,0)$, $(k,0)$, and $(0,k)$) shown in Figure 6. In the t-axis ($t = x$ or y), the missing coefficients are chosen so that the truncation of the resulting polynomial to each segment of length 1, 2 or 3 is an appropriate monomial multiple of 1, $(t-1)^2$ or $(t-1)^2(t+1)$, respectively. Thus, each segment ℓ of length 2 or 3 gives rise to a point of tangency of the t-axis and the curve $\{f_k = 0\}$, resulting in two extra solitary nodes of C_k. Similarly, each vertex of $k\Delta$ contained in a segment of length 1 gives rise to an extra solitary node of C_k. □



a) $k = 3l$ b) $k = 3l+1$ c) $k = 3l+2$

FIGURE 6.

Remark 4.6. The construction of Theorem 4.5 for $k = 3, 4$ can easily be performed without using the patchworking technique.

4.2. **A curve of degree** 12**.** The construction given by Theorem 4.5 is the best known if $k \leq 5$. If $k = 6$, it can be improved from 43 to 45.

Proposition 4.7. *There exists a finite real algebraic curve $C \subset \mathbb{C}P^2$ of degree* 12 *such that*

$$|\mathbb{R}C| = 45.$$

Proof. Let $C' = C'_9(1)$ be a finite real algebraic curve in $\Sigma_9(\mathcal{O}')$ as in Lemma 3.6. Let us denote by $\mathcal{P}$ the set of nodes of C', and by $\mathcal{I}$ the sheave of ideals on $\Sigma_9(\mathcal{O}')$ defining $\mathcal{P}$. Since $\mathbb{R}B \neq \varnothing$, there exists a real line bundle $\mathcal{L}_0$ of degree 3 over B such that $\mathcal{O}' = \mathcal{L}_0^{\otimes 3}$. This bundle $\mathcal{L}_0$ embeds B into $\mathbb{C}P^2$ as a real cubic curve for which $\mathcal{L}_0^{\otimes 3} = \mathcal{O}$ is the normal bundle. The proposition will then follow from Proposition 3.8 once we prove that $H^0(\Sigma_9(\mathcal{O}'), \mathcal{O}(4B_0) \otimes \mathcal{I})$ is of codimension 45 in $H^0(\Sigma_9(\mathcal{O}'), \mathcal{O}(4B_0))$. Let us show that this is indeed the case, *i.e.*, let us show that given any node p of C', there exists an algebraic curve in $\mathcal{O}(4B_0)$ on $\Sigma_9(\mathcal{O}')$ passing through all nodes of C' but p. Recall that there exists a real double covering $\rho\colon \Sigma_9(\mathcal{O}') \to \Sigma_9(\mathcal{O}'^{\otimes 2})$ ramified along $B_0 \cup B_\infty$ with respect to which C' is symmetric, and that C' has 18 pairs of symmetric nodes and 9 nodes on B_0.

By Riemann-Roch Theorem, for any line bundle $\mathcal{O}$ over B_0 of degree $n \geq 1$, and given any set $\mathcal{P}$ of $n-2$ points on distinct fibers of $\Sigma_n(\mathcal{O})$ and any disjoint finite subset $\overline{\mathcal{P}}$ of $\Sigma_n(\mathcal{O})$, there exists an algebraic curve in $\mathcal{O}(B_0)$ containing $\mathcal{P}$ and avoiding $\overline{\mathcal{P}}$. As a consequence, there exists a symmetric curve in $\mathcal{O}(2B_0)$ on $\Sigma_9(\mathcal{O}')$ passing through any 16 pairs of symmetric nodes of C' and avoiding all other nodes of C'. Altogether, we see that given any node p of C', there exists a reducible curve in $\mathcal{O}(4B_0)$ on $\Sigma_9(\mathcal{O}')$, consisting in the union of a symmetric curve in $\mathcal{O}(2B_0)$ and two curves in $\mathcal{O}(B_0)$, and passing through all nodes of C' but p. □

4.3. **Curves of low genus.** Here we show that inequality (4) of Theorem 4.1 is sharp when the degree is large compared to the genus.

Theorem 4.8. *Given integers $k \geq 3$ and $0 \leq g \leq k-3$, there exists a finite real algebraic curve $C \subset \mathbb{C}P^2$ of degree $2k$ and genus g such that*

$$|\mathbb{R}C| = k^2 + g + 1.$$

Proof. Consider a real rational curve $C_1 \subset \mathbb{C}^2$ with the following properties:

- the Newton polygon of C_1 is the triangle with the vertices $(0,0)$, $(0,k-2)$ and $(2k-4,0)$,
- C_1 intersects the axis $y=0$ in a single point with multiplicity $2k-4$,
- $\mathbb{R}C_1 \cap \{y>0\}$ consists of $\frac{1}{2}(k-2)(k-3)$ solitary nodes.

Such a curve exists: for example, one can take a rational simple Harnack curve with the prescribed Newton polygon (see [Mik00, KO06, Bru15]). Shift the Newton polygon $\Delta(C_1)$ by 2 units up and place in the trapezoid with the vertices $(0,0)$, $(2k,0)$, $(2k-4,2)$, $(0,2)$ a defining polynomial of the curve $\widetilde{C}_{1,k-2,g+1}$ given by Lemma 3.2. Applying Theorem 3.1, we obtain a real rational curve $C_2 \subset \mathbb{C}^2$ such that

- $\mathbb{R}C_2 \cap \{y>0\}$ consists of $\frac{1}{2}(k-2)(k-3) + 2k + g - 2$ solitary nodes,
- C_2 intersects the line $y=0$ in $k-g-1$ real points of multiplicity 2, and in $g+1$ additional pairs of complex conjugated points.

If C_2 is given by an equation $f(x,y)=0$ positive on $y>0$, we define C as the curve $f(x,y^2)=0$. Each node $p \in \{y>0\}$ of C_2 gives rise to two solitary real nodes of C, and each tangency point of C_2 and the axis $y=0$ gives rise to an extra solitary node of C. The genus $g(C)=g$ is given by the Riemann–Hurwitz formula applied to the double covering $C \to C_2$: its normalization is branched at the $2(g+1)$ points of transverse intersection of C_2 and the axis $y=0$. □

5. Finite curves in real ruled surfaces

We use the notation $B, \mathcal{O}, B_0, F, \Sigma_n(\mathcal{O})$ introduced in Section 3.2. A real algebraic curve C in $\Sigma_n(\mathcal{O})$ realizing the class $u[B_0] + v[F] \in H_2(\Sigma_n(\mathcal{O}); \mathbb{Z})$ may be finite only if both $u = 2a$ and $v = 2b$ are even. General results of the previous sections specialize as follows.

Theorem 5.1. *Let $C \subset \Sigma_n(\mathcal{O})$ be a finite real algebraic curve, $[C] = 2a[B_0]+2b[F] \in H_2(\Sigma_n(\mathcal{O}); \mathbb{Z})$, $a > 0$, $b > 0$. Then,*

$$|\mathbb{R}C| \le na^2 + 2ab + g(C) + 1 - 2g(B), \tag{6}$$

$$|\mathbb{R}C| \le \frac{1}{2}na(3a-1) + 3ab - (a+b) + 1 + (a-1)g(B). \tag{7}$$

Proof. The statement is an immediate consequence of Theorem 2.8 and Corollary 2.6: due to Lemma 3.5, we can choose $\mathbb{R}X_+ = \mathbb{R}X$ and $\mathbb{R}X_- = \varnothing$. □

As in the case of $\mathbb{C}P^2$, we do not know whether the upper bounds (6) and (7) are sharp in general. In the rest of the section, we discuss the special cases of small a or small genus. The two next propositions easily generalize to ruled surfaces over a base of any genus (in the same sense as explained after Lemma 3.3). For simplicity, we confine ourselves to the case of a rational base.

Proposition 5.2 ($a = 1$)**.** *Given integers $b, n \ge 0$, there exists a finite real algebraic curve $C \subset \Sigma_n$ of bidegree $(2, 2b)$ such that $|\mathbb{R}C| = n + 2b$.*

Proof. A collection of $n + 2b$ generic real points in Σ_n determines a real pencil of curves of bidegree $(1, b)$, and one can take for C the union of two complex conjugate members of this pencil. □

Proposition 5.3 ($a = 2$)**.** *Given integers $b, n \ge 0$, and $-1 \le g \le n + b - 2$, there exists a finite real algebraic curve $C \subset \Sigma_n$ of bidegree $(4, 2b)$ and genus g such that*

$$|\mathbb{R}C| = 4n + 4b + g + 1.$$

In particular, if $b + n \ge 1$, then there exists a finite real algebraic curve $C \subset \Sigma_n$ of bidegree $(4, 2b)$ such that

$$|\mathbb{R}C| = 5n + 5b - 1.$$

Proof. We argue as in the proof of Lemma 3.6, starting from the curve $\widetilde{C}_{n,b,g+1}$ given by Lemma 3.2. The genus $g(C)$ is computed by the Riemann–Hurwitz formula. □

All rational ruled surfaces are toric, and Theorem 4.2 takes the following form.

Theorem 5.4. *Given integers $a > 0$ and $b \ge 0$, there exists a sequence of finite real algebraic curves $C_k \subset \Sigma_n$ of bidegree (ka, kb) such that*

$$\lim_{k \to +\infty} \frac{1}{k^2} |\mathbb{R}C_k| = \frac{4}{3}(na^2 + 2ab). \qquad \square$$

Theorem 4.8 extends to curves in Σ_n as follows.

Theorem 5.5 (low genus)**.** *Given integers $a > 0$, $b, n \ge 0$, and $-1 \le g \le n(a-1) + b - 2$, there exists a finite real algebraic curve $C \subset \Sigma_n$ of bidegree $(2a, 2b)$ and genus g such that*

$$|\mathbb{R}C| = na^2 + 2ab + g + 1. \qquad \square$$

Proof. The proof is a literal repetition of that of Theorem 4.8, choosing for C_1 and C_2 curves with the Newton polygons with the vertices $(0, 0), (0, a-2), (2b, a-2), (2n(a-2) + 2b, 0)$ and $(0, 0), (0, 2), (2n(a-2) + 2b, 2), (2na + 2b, 0)$, respectively. □

6. Finite curves in the ellipsoid

The algebraic surface $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ has two real structures with non-empty real part, namely $c_h(z, w) = (\bar{z}, \bar{w})$ and $c_e(z, w) = (\bar{w}, \bar{z})$. The first one was considered in Section 5. In this section, Σ_0 is assumed equipped with the real structure c_e, and we have $\mathbb{R}\Sigma_0 = S^2$.

6.1. **General bounds.** Let e_1 and e_2 be the classes in $H_2(\Sigma_0;\mathbb{Z})$ represented by the two rulings. The action of c_e on $H_2(\Sigma_0;\mathbb{Z})$ is given by $c_e(e_i) = -e_{3-i}$, and so $\sigma^-_{\text{inv}}(\Sigma_0, c_e) = 1$.

The classes in $H_2(\Sigma_0;\mathbb{Z})$ realized by real algebraic curves are those of the form $m(e_1 + e_2)$. For any $m \geq 1$, a real algebraic curve of bidegree (m, m) may have finite real part.

Theorem 6.1. *Let C be a reduced finite real algebraic curve in (Σ_0, c_e) of bidegree (m, m), with $m \geq 2$. Then*

$$|\mathbb{R}C| \leq \begin{cases} 2k^2 + g(C) + 3 & \text{if } m = 2k \\ 2k^2 + 4k + g(C) & \text{if } m = 2k+1 \end{cases}. \tag{8}$$

In particular we have

$$|\mathbb{R}C| \leq \begin{cases} 3k^2 - 2k + 2 & \text{if } m = 2k \\ 3k^2 + 2k & \text{if } m = 2k+1 \end{cases}. \tag{9}$$

Proof. In order to apply Theorem 2.5, we note that $T_{2,1}(\Sigma_0) = -h^{1,1}(\Sigma_0) = -2$ and that the real locus of (Σ_0, c_e) being a sphere, $\chi(\mathbb{R}\Sigma_0) = 2$.

The case when $m = 2k$ is then provided by Theorem 2.5 and Corollary 2.6. Indeed, in this case, $[C] = m(e_1 + e_2) = 2k(e_1 + e_2)$ and letting $e = k(e_1 + e_2)$, we get $e^2 = 2k^2$ and $e \cdot c_1(\Sigma_0) = 2k(e_1 + e_2)(e_1 + e_2) = 4k$.

So suppose that $m = 2k + 1$ and let $p \in \mathbb{R}C$. Let E_1 and E_2 be a pair of conjugate generatrices which meet C at p. Let $\widetilde{C} = C \cup E_1 \cup E_2$ and let $\overline{C}$ be the strict transform of $\widetilde{C}$ in the blow-up $\overline{\Sigma}_0$ of Σ_0 at p. The class of the auxiliary curve $\widetilde{C}$ in $H_2(\Sigma_0;\mathbb{Z})$ is then $[\widetilde{C}] = 2(k+1)(e_1 + e_2)$. Let $e = (k+1)(e_1+e_2)$, we get $e^2 = 2(k+1)^2$. Let $\overline{e}$ be half the class of $\overline{C}$ in $H_2(\overline{\Sigma}_0;\mathbb{Z})$, we get $\overline{e}^2 \leq e^2-4$, as the point p is of multiplicity at least 4 in $\widetilde{C}$. Furthermore, we have $g(\overline{C}) = g(\widetilde{C}) = g(C) - 2$ and $|\mathbb{R}C| = |\mathbb{R}\widetilde{C}| = |\mathbb{R}\overline{C}| + 1$. In order to apply Theorem 2.5 for the curve $\overline{C}$ on $\overline{\Sigma}_0$, it remains to note that $T_{2,1}(\overline{\Sigma}_0) = T_{2,1}(\Sigma_0) - 1$ and $\chi(\mathbb{R}\overline{\Sigma}_0) = \chi(\mathbb{R}\Sigma_0) - 1$. Hence we obtain (8) from Theorem 2.5 applied to the curve $\overline{C}$ on $\overline{\Sigma}_0$.

To get (9), it suffices to remark that, C being a curve of bidegree $(2k + 1, 2k + 1)$, we have $g(C) \leq 4k^2 - |\mathbb{R}C|$. □

Remark 6.2. Let us consider the following problem: given a smooth real projective surface (X, c) and a homology class $d \in H_2(X;\mathbb{Z})$, what is the maximal possible number of intersection points between C and $\mathbb{R}X$ for a non-real algebraic curve C in X realizing the class d?

Since any two distinct irreducible algebraic curves in X intersect positively, any non-real irreducible algebraic curve C in X intersects $c(C)$ in $-[C] \cdot c_*[C]$ points, and so intersects $\mathbb{R}X$ in at most $-[C] \cdot c_*[C]$ points. In $\mathbb{C}P^2$ this upper bound is sharp: it suffices to take a non-real member in the pencil generated by two generic real curves intersecting at real points only. Interestingly, Theorem 6.1 shows that this trivial upper bound is not sharp in the case of the quadric ellipsoid.

Any irreducible algebraic curve C in Σ_0 realizing the class $(m - 1, 1)$ with $m \geq 3$ is non real and rational. Since the union of C and $c_e(C)$ is a real algebraic curve of geometric genus -1 realizing the class (m, m), Theorem 6.1 implies that

$$|C \cap \mathbb{R}\Sigma_0| \leq \begin{cases} 2k^2 + 2 & \text{if } m = 2k \\ 2k^2 + 4k - 1 & \text{if } m = 2k+1 \end{cases},$$

whereas $(m - 1, 1) \cdot (1, m - 1) = m^2 - 2m + 2$ is at least twice as large.

Next theorem is an immediate consequence of Theorem 4.2 and Proposition 3.7

Theorem 6.3. *There exists a sequence of finite real algebraic curves C_m of bidegree (m, m) in the quadric ellipsoid such that*

$$\lim_{m\to\infty} \frac{1}{2m^2}|\mathbb{R}C_m| = \frac{4}{3}.$$

6.2. **Curves of low bidegree.** Next statement shows in particular that Theorem 6.1 is not sharp for $m = 2$ and $m = 5$.

Proposition 6.4. *For $m \leq 5$, the maximal possible value $\delta_e(m)$ of $|\mathbb{R}C|$* (cf. *Remark 1.2*) *for a finite real algebraic curve of bidegree (m, m) in the quadric ellipsoid is*

m	1	2	3	4	5
$\delta_e(m)$	1	2	5	10	15

Proof. We start by constructing real algebraic curves with a number of real points as stated in the proposition. For $m \leq 4$, such a curve is constructed by taking the union of two complex conjugated curves of bidegree $(m-1, 1)$ and $(1, m-1)$ intersecting $\mathbb{R}\Sigma_0$ in $(m-1)^2 + 1$ points. For $m \leq 3$, such a curve exists since $2m - 1$ points determine a pencil of curves of bidegree $(m-1, 1)$. For the case $m = 4$, consider 8 points in $\mathbb{R}P^2$ such that there exists a non-real rational cubic $C_0 \subset \mathbb{C}P^2$ passing through these 8 points (such configuration of 8 points exist). Since C_0 has a unique nodal point, it has to be non-real. Furthermore, since C_0 intersects $\mathbb{R}P^2$ in an odd number of points, it has to intersect $\mathbb{R}P^2$ in a ninth point. Hence the union of C_0 with its complex conjugate is a real algebraic curve of degree 6 with 9 solitary points and two complex conjugate nodal points. Denote by O the line passing through the two latter. Blowing up the two nodes and blowing down the strict transform of O, we obtain a real algebraic curve of bidegree $(4, 4)$ in the quadric ellipsoid whose real part has exactly 10 points.

The case $m = 5$ is treated by applying the deformation to the normal cone construction to a non-singular real hyperplane section B, with $\mathbb{R}B \neq \varnothing$, in the quadric ellipsoid X. Here we use notations from Section 3.3. According to Proposition 5.3, there exists a real algebraic curve C_B of bidegree $(4, 2)$ in $E_B = \Sigma_2$ whose real part consists of 14 solitary nodes. Let C_X be a reducible curve of bidegree $(1, 1)$ in X passing through $X \cap E_B \cap C_B$, and let us define $C_0 = C_X \cup C_B$. The curve C_0 is a finite real algebraic curve with $|\mathbb{R}C_0| = 15$, hence Proposition 3.7 ensures the existence of a finite real algebraic curve C of bidegree $(5, 5)$ in X with $|\mathbb{R}C| = 15$.

We now prove that there does not exist finite real algebraic curves of bidegree $m \leq 5$ with a number of real points greater than the one stated in the proposition. By Bézout Theorem, a finite real algebraic curve of bidegree (m, m) with $m = 1$ or $m = 2$ has at most 1 or 2 real points respectively. According to Theorem 6.1, a finite real algebraic curve of bidegree $(3, 3)$, $(4, 4)$ or $(5, 5)$ in the quadric ellipsoid cannot have more that $5, 10$, or 16 real points respectively. Suppose that there exists a real algebraic curve of bidegree $(5, 5)$ in the quadric ellipsoid with 16 real points. By the genus formula, this curve is rational and its 16 real points are all ordinary nodes. By a small perturbation creating an oval for each node, we obtain a non-singular real algebraic curve of bidegree $(5, 5)$ in the quadric ellipsoid whose real part consists of exactly 16 connected components, each of them bounding a disc in the sphere. This contradicts the congruence [Mik91, Theorem 1b)]. □

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