ON G-BIRATIONAL RIGIDITY OF PROJECTIVE SPACES

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Abstract. In this paper, we study finite subgroups $G \subset \mathrm{Aut}(\mathbb{P}^n)$ such that $\mathbb{P}^n$ is G-birationally rigid. For each $n \geqslant 3$, we prove that $\mathrm{Aut}(\mathbb{P}^n)$ contains at most finitely many such subgroups up to conjugation. For $n = 4$, we prove that $\mathbb{P}^4$ is G-birationally superrigid if $G \simeq \mathrm{PSp}_4(\mathbf{F}_3)$.

Contents

1. Introduction

1.1. **Birational rigidity.** The notion of birational rigidity originated in the work of Iskovskikh and Manin on the irrationality of smooth quartic threefolds — implicitly they proved that every smooth complex hypersurface of degree 4 in $\mathbb{P}^4$ is birationally superrigid and, in particular, it is not rational. Recall that a Fano variety X is said to be birationally rigid if

(1) X is a Mori fiber space over a point (X has terminal singularities and $\mathrm{Cl}(X)$ is of rank 1),
(2) and X is the only Mori fiber space that is birational to X.

If X is birationally rigid and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, we say that X is birationally superrigid. Being birationally rigid is an obstruction to rationality: this notion is an extreme opposite of rationality.

At the moment, birational rigidity is proved for many complex Fano varieties. However, birationally rigid Fano varieties are rare among all Fanos. For instance, there are no birationally rigid two-dimensional Fano varieties (del Pezzo surfaces), and only 3 out of 105 deformation families of smooth three-dimensional Fano varieties (Fano threefolds) contain birationally rigid smooth members. The latter are smooth sextic hypersurfaces in $\mathbb{P}(1,1,1,1,3)$ (all of them are birationally superrigid), smooth complete intersections in $\mathbb{P}(1,1,1,1,1,2)$ of a quadric and a quartic hypersurfaces (all of them are birationally rigid), and complete intersections in $\mathbb{P}^5$ of a quadric and a cubic hypersurfaces (a general member of this deformation family is known to be birationally rigid). Smooth complex hypersurfaces in $\mathbb{P}^n$ of degree $d \leqslant n$ are birationally rigid only for $d = n \geqslant 4$.

All varieties are assumed to be projective, normal and defined over the field of complex numbers $\mathbb{C}$ unless stated otherwise.

Irrationality problems for Fano varieties defined over non-algebraically closed fields and study of finite subgroups of Cremona groups lead to an equivariant generalization of the notion of birational rigidity. To state it, we fix a Fano variety X such that X has terminal singularities, and we fix a finite subgroup $G \subset \mathrm{Aut}(X)$.

Definition 1.1 ([18, Definition 3.1.1])**.** The Fano variety X is G-birationally rigid if the following two conditions are satisfied:

(1) X is a G-Mori fiber space over a point (X has terminal singularities and $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$),
(2) if Y is a G-Mori fiber space that is G-birational to X, then Y is G-equivariantly isomorphic to X.

The Fano variety X is said to be G-birationally superrigid if X is G-birationally rigid and

$$\mathrm{Bir}^G(X) = \mathrm{Aut}^G(X),$$

where $\mathrm{Aut}^G(X)$ is the subgroup in $\mathrm{Aut}(X)$ consisting of all G-automorphisms (the normalizer of the group G in $\mathrm{Aut}(X)$), and $\mathrm{Bir}^G(X)$ is the subgroup in $\mathrm{Bir}(X)$ consisting of all G-birational selfmaps (the normalizer of the group G in $\mathrm{Bir}(X)$).

Remark 1.2. Definition 1.1 goes back to Manin–Segre theorem: if X is a smooth del Pezzo surface such that $(-K_X)^2 \leqslant 3$ and $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$, then X is G-birationally rigid. For the complete classification of all G-birationally rigid del Pezzo surfaces, see [22, 52].

The simplest example of a Fano variety of dimension n is the projective space $\mathbb{P}^n$. Moreover, finite subgroups of the group $\mathrm{Aut}(\mathbb{P}^n) \simeq \mathrm{PGL}_{n+1}(\mathbb{C})$ have been extensively studied for $n \leqslant 7$, see for instance [6, 7, 25, 36, 37, 38, 50, 51]. Thus, it is natural to pose the following question:

Question 1.3. *For which finite subgroups $G \subset \mathrm{Aut}(\mathbb{P}^n)$, $\mathbb{P}^n$ is G-birationally rigid?*

For $\mathbb{P}^2$ and $\mathbb{P}^3$, the complete answer to this question has been obtained in [16, 17, 20, 47], and we know very simple geometric criteria for being G-birationally rigid in these two cases:

Theorem 1.4 ([47])**.** *Let G be a finite subgroup in $\mathrm{PGL}_3(\mathbb{C})$. Then $\mathbb{P}^2$ is G-birationally rigid if and only if the following conditions are satisfied: G does not fix points in $\mathbb{P}^2$, $G \not\simeq \mathfrak{A}_4$, and $G \not\simeq \mathfrak{S}_4$.*

Theorem 1.5 ([16, 17, 20])**.** *Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3$ is G-birationally rigid if and only if the following conditions are satisfied:*

(1) *G does not fix points in $\mathbb{P}^3$,*
(2) *$\mathbb{P}^3$ does not contain G-invariant pair of skew lines,*
(3) *there exists no G-orbit in $\mathbb{P}^3$ of length 4,*
(4) *$G \not\simeq \mathfrak{A}_5$ and $G \not\simeq \mathfrak{S}_5$.*

In particular, we see that $\mathrm{PGL}_3(\mathbb{C})$ contains infinitely many finite subgroups G up to conjugation such that $\mathbb{P}^2$ is G-birationally rigid [25]. Indeed, if G is the subgroup in $\mathrm{PGL}_3(\mathbb{C})$ generated by

$$\begin{pmatrix} e^{\frac{2\pi\sqrt{-1}}{m}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi\sqrt{-1}}{m}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $G \simeq \boldsymbol{\mu}_m^2 \rtimes \mathfrak{S}_3$, where $\boldsymbol{\mu}_m$ denotes the cyclic group of order m. We see that $\mathbb{P}^2$ is G-birationally rigid for $m \geqslant 3$ by Theorem 1.4. On the other hand, it follows from Theorem 1.5 that $\mathrm{PGL}_4(\mathbb{C})$ contains finitely many subgroups G (up to conjugation) such that $\mathbb{P}^3$ is G-birationally rigid. Using the classification of finite subgroups of $\mathrm{PGL}_4(\mathbb{C})$ provided in [6] (cf. [20, Appendix A]), it is not difficult to list all of them.

The geometric conditions (1), (2), (3) in Theorem 1.5 mean that the subgroup G is *primitive*. Namely, we recall the following terminology.

Definition 1.6 (cf. Definition A.1)**.** A finite subgroup $\widehat{G} \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is said to be primitive if there exists no non-trivial decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^{r} V_i$$

such that for any $g \in \widehat{G}$ and any i there is some $j = j(g)$ such that $g(V_i) = V_j$. Similarly, a finite subgroup $G \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ is said to be primitive if the subgroup G is an image of some primitive finite subgroup $\widehat{G} \subset \mathrm{GL}_{n+1}(\mathbb{C})$ via the natural projection.

In Section 3, we will prove the following theorem.

Theorem A. *If G is a finite subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$ for $n \geqslant 3$ such that $\mathbb{P}^n$ is G-birationally rigid, then the subgroup G is primitive.*

The proof of Theorem A uses geometric and combinatorial results about toric symmetry of $\mathbb{P}^n$, which are presented in Section 3. Recall that the group $\mathrm{PGL}_{n+1}(\mathbb{C})$ contains only finitely many primitive finite subgroups up to conjugation, see for instance [6, §§61,73,74].

Corollary B. *Fix $n \geqslant 3$. Up to conjugation, the group $\mathrm{PGL}_{n+1}(\mathbb{C})$ contains finitely many finite subgroups G such that $\mathbb{P}^n$ is G-birationally rigid.*

Remark 1.7. For higher-dimensional Fano varieties different from $\mathbb{P}^n$, an analogue of Corollary B does not always hold. For instance, if $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then it follows from [10, Corollary 1.4] that $\mathrm{Aut}(X)$ contains infinitely many finite subgroups G up to isomorphism such that $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$ and X is G-birationally superrigid.

Using Theorem A, we can list all possibilities for $G \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ such that $\mathbb{P}^n$ has a chance to be G-birationally rigid in the case when n is small. For instance, in Proposition 5.2, we will see that up to conjugation there are at most 8 finite subgroups $G \subset \mathrm{PGL}_5(\mathbb{C})$ such that $\mathbb{P}^4$ can a priori be G-birationally rigid. One of them is the subgroup $G \simeq \mathrm{PSp}_4(\mathbf{F}_3)$ that preserves the famous Burkhardt quartic hypersurface. In Section 5, we will prove the following result.

Theorem C. *Let G be a finite primitive subgroup in $\mathrm{PGL}_5(\mathbb{C})$ such that $G \simeq \mathrm{PSp}_4(\mathbf{F}_3)$. Then $\mathbb{P}^4$ is G-birationally superrigid.*

It would be interesting to find out whether $\mathbb{P}^4$ is G-birationally rigid in each of the 8 cases listed in Proposition 5.2. Unfortunately, our proof of Theorem C cannot be adapted to show that $\mathbb{P}^4$ is G-birationally rigid in the remaining 7 cases. It should be pointed out that, except for Theorem A, there are other obstructions for $\mathbb{P}^n$ to be G-birationally rigid, see e.g. Lemmas 2.4 and 2.5.

In dimensions $n \geqslant 5$, we do not know examples of finite groups $G \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ such that $\mathbb{P}^n$ is G-birationally rigid. See Sections 6 and 7 for a more detailed discussion of possible candidates in dimensions $n = 5$ and $n = 6$. We find the next question particularly interesting.

Question 1.8 (cf. [15])**.** *Let $G \subset \mathrm{PGL}_6(\mathbb{C})$ be the simple Hall–Janko group. Is it true that $\mathbb{P}^5$ is G-birationally rigid?*

We can naturally generalize Definition 1.1 for Fano varieties that are defined over an algebraically non-closed field $\Bbbk$, and we can pose Question 1.3 for $\mathbb{P}^n_{\Bbbk}$. The next theorem gives an analogue of Theorem A over the field of real numbers (but, somewhat surprisingly, its proof is much simpler).

Theorem D. *Let $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$ be a finite group such that $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. Then G is a primitive subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$.*

In Lemma 8.10, we will show that $\mathrm{PGL}_3(\mathbb{R})$ contains a unique subgroup G up to conjugation such that $\mathbb{P}^2_{\mathbb{R}}$ is G-birationally rigid. This subgroup is isomorphic to $\mathfrak{A}_5$. Similarly, in Proposition 8.13, we will describe finite subgroups $G \subset \mathrm{PGL}_4(\mathbb{R})$ such that $\mathbb{P}^3_{\mathbb{R}}$ is G-birationally rigid. On the other hand, using the classification of primitive finite subgroups in $\mathrm{PGL}_{n+1}(\mathbb{C})$ for $n \in \{4, 5, 6\}$, we will prove the following surprising result.

Theorem E. *Let G be a finite subgroup in $\mathrm{PGL}_{n+1}(\mathbb{R})$. Suppose that $n \in \{4, 5, 6\}$. Then $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid.*

Keeping in mind the proof of Theorem E, we expect the following to hold.

Conjecture 1.9. *Suppose that $n \geqslant 4$ and $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$ is a finite subgroup. Then $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid.*

Plan of the paper. In Section 2, we recall some definitions, establish auxiliary facts concerning primitive and imprimitive finite groups acting on projective spaces, and present a couple of simple obstructions for $\mathbb{P}^n$ to be G-birationally rigid. In Section 3, we discuss equivariant birational geometry of the projective space $\mathbb{P}^n$ with respect to the action of an infinite group that contains a maximal torus in $\mathrm{PGL}_{n+1}(\mathbb{C})$ as a normal subgroup and, as an application, we prove Theorem A modulo one technical result (Theorem 3.6), which is proved later in Section 4. In Section 5, we discuss primitive subgroups of $\mathrm{PGL}_5(\mathbb{C})$ and prove Theorem C. In Sections 6 and 7, we discuss primitive subgroups of $\mathrm{PGL}_6(\mathbb{C})$ and $\mathrm{PGL}_7(\mathbb{C})$, respectively, and show that $\mathbb{P}^5$ and $\mathbb{P}^6$ are not G-birationally rigid for many such groups G. In Section 8, we discuss finite groups acting on projective spaces over the field of real numbers, and we prove Theorems D and E. In Appendix A, we present combinatorial results that are used in the proof of Theorem 3.6. Finally, we present Magma codes used in the paper in Appendix B.

Notation. Throughout the paper, we denote by $\boldsymbol{\mu}_n$ the cyclic group of order n, and by $\mathfrak{D}_n$ the dihedral group of order $2n \geqslant 6$. Similarly, we denote by $\mathfrak{S}_n$ the symmetric group of degree n, and we denote by $\mathfrak{A}_n$ its alternating normal subgroup. By $\mathrm{SU}_n(\mathbf{F}_q)$ we denote the group of all unitary matrices with determinant 1 in $\mathrm{GL}_n(\mathbf{F}_{q^2})$, and by $\mathrm{PSU}_n(\mathbf{F}_q)$ we denote its image in $\mathrm{PGL}_n(\mathbf{F}_{q^2})$. If $G_1, \ldots, G_r$ are subgroups of a group Γ, and $g_1, \ldots, g_s$ are elements of Γ, then we denote by $\langle G_1, \ldots, G_r, g_1, \ldots, g_s\rangle$ the subgroup of Γ generated by $G_1, \ldots, G_r$ together with $g_1, \ldots, g_s$.

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2. Preliminaries

In this section we recall some definitions and establish auxiliary facts concerning primitive and imprimitive finite groups acting on projective spaces.

Let G be a finite subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$. Then there exists a finite subgroup $\widehat{G} \subset \mathrm{GL}_{n+1}(\mathbb{C})$ such that $\widehat{G}$ is mapped surjectively to G via the natural epimorphism $\mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$. Recall the following terminology.

Definition 2.1 (cf. Definitions 1.6 and A.1)**.** The group G is said to be transitive if the embedding

$$\widehat{G} \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{C})$$

is an irreducible representation. Otherwise, G is intransitive. Similarly, G is said to be imprimitive if there exists a non-trivial decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^{r} V_i \tag{2.1}$$

such that for any $g \in \widehat{G}$ and any i we have $g(V_i) = V_j$ for some $j = j(g)$. If G is transitive and not imprimitive, G is said to be primitive.

It is easy to see that the properties of the group G described in Definition 2.1 do not depend on the choice of the lifting $\widehat{G}$ of G to $\mathrm{GL}_{n+1}(\mathbb{C})$. The following result is a baby version of Theorem A.

Lemma 2.2. *Suppose that $\mathbb{P}^n$ is G-birationally rigid. Then the group G is transitive.*

Proof. Suppose that G is not transitive. Then there exists a G-invariant linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension $k \leqslant n-2$. Let $\pi\colon Y \to \mathbb{P}^n$ be the blow up of Λ, and let $m = n-k-1$. Then there exists a G-Sarkisov link

$$\begin{array}{ccc} & Y & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^n & & \mathbb{P}^m \end{array}$$

where ϕ is a $\mathbb{P}^{k+1}$-bundle, which is a G-Mori fiber space with a positive-dimensional base. This contradicts our assumption that $\mathbb{P}^n$ is G-birationally rigid. ∎

Thus, if $\mathbb{P}^n$ is G-birationally rigid, then G must be transitive. Theorem A says that G must also be primitive. In order to prove this in Section 3, we need the following result that says that if $\mathbb{P}^n$ is G-birationally rigid and G is not primitive, then G must have an orbit of length $n+1$.

Lemma 2.3. *Suppose that G is transitive and imprimitive, so that there exists a non-trivial decomposition* (2.1) *such that for any $g \in \widehat{G}$ and any i we have $g(V_i) = V_j$ for some $j = j(g)$. If $\mathbb{P}^n$ is G-birationally rigid, then $r = n+1$.*

Proof. Note that r divides $n+1$. Set $m = \frac{n+1}{r} - 1$. Suppose that $r < n+1$. Then, by definition, there exists a collection of r linear subspaces $\Lambda_1, \ldots, \Lambda_r \subset \mathbb{P}^n$ of dimension m such that their union spans $\mathbb{P}^n$, each Λ_i is disjoint from the linear span of the union of Λ_j with $j \neq i$, and the group G permutes $\Lambda_1, \ldots, \Lambda_r$. For each $i \in \{1, \ldots, r\}$, consider the linear projection

$$\psi_i\colon \mathbb{P}^n \dashrightarrow \Lambda_i$$

from the span of the union of all Λ_j with $j \neq i$. Set $\Lambda = \Lambda_1 \times \ldots \times \Lambda_r$, and consider the rational map

$$\psi = \psi_1 \times \ldots \times \psi_r\colon \mathbb{P}^n \dashrightarrow \Lambda.$$

Then ψ is dominant and G-equivariant. Furthermore, ψ fits into the following G-equivariant commutative diagram:

$$\begin{array}{ccc} & Y & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^n & \overset{\psi}{\dashrightarrow} & \Lambda \end{array}$$

where π is a blow up of the disjoint union $\Lambda_1 \cup \ldots \cup \Lambda_r$, and ϕ is a $\mathbb{P}^{r-1}$-bundle, which is a G-Mori fiber space with a positive-dimensional base. Hence, in particular, $\mathbb{P}^n$ is not G-birationally rigid. ∎

In the remaining part of this section, we present several obstructions for $\mathbb{P}^n$ to be G-birationally rigid, which will be used later in Sections 5, 6, and 7.

Lemma 2.4. *Suppose that $|\mathcal{O}_{\mathbb{P}^n}(d)|$ contains a G-invariant pencil $\mathcal{P}$ for some $d \leqslant n$. Then $\mathbb{P}^n$ is not G-birationally rigid.*

Proof. Replacing $\mathcal{P}$ by its mobile part, we may assume that $\mathcal{P}$ is mobile. The pencil $\mathcal{P}$ gives a G-equivariant rational map $\psi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^1$. Equivariantly resolving the indeterminacy of this map and the singularities of the resulting variety, we obtain a commutative diagram

$$\begin{array}{ccc} & Y & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^n & \overset{\psi}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

where Y is smooth. Furthermore, passing to the Stein factorization of ϕ, we may assume that a general fiber of η is irreducible. Observe that the fibers of ϕ are strict transforms of the elements of the pencil $\mathcal{P}$, which are (possibly singular) hypersurfaces of degree d. Such hypersurfaces are uniruled by [43, Theorem 1]. Therefore, applying equivariant relative Minimal Model Program over $\mathbb{P}^1$ to Y, we obtain a G-birational map from $\mathbb{P}^n$ to a G-Mori fiber space with positive-dimensional base, which implies that $\mathbb{P}^n$ is not G-birationally rigid. $\square$

For the simplest three-dimensional application of Lemma 2.4, see [13, Example 1.2]. For higher-dimensional applications, see Sections 5, 6, and 7 below.

Lemma 2.5. *Suppose that $\mathbb{P}^n$ contains a G-irreducible complete intersection $X = F_{d_1} \cap F_{d_2}$ such that X has at most isolated ordinary double singularities and $d_1 < d_2 \leqslant n$, where F_{d_1} and F_{d_2} are hypersurfaces in $\mathbb{P}^n$ of degree d_1 and d_2, respectively. Then $\mathbb{P}^n$ is not G-birationally rigid.*

Proof. Let $\pi\colon V \to \mathbb{P}^n$ be the blow up of the complete intersection X, and let $\widetilde{F}_{d_1}$ be the strict transform on V of the hypersurface F_{d_1}. Then V has at most ordinary double points, and we have the following G-equivariant diagram:

$$\begin{array}{ccc} & V & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^n & & Y \end{array}$$

where Y is a (singular) Fano variety, and η is a birational morphism that contracts $\widetilde{F}_{d_1}$ to a singular point of the variety Y. Observe that

$$K_V \sim_{\mathbb{Q}} \eta^* K_Y + \frac{n+1-d_2}{d_2-d_1}\widetilde{F}_{d_1}.$$

This implies that Y has terminal singularities. Moreover, by construction we have $\operatorname{rk}\operatorname{Cl}^G(Y) = 1$ and $Y \not\simeq \mathbb{P}^n$, so that $\mathbb{P}^n$ is not G-birationally rigid. $\square$

Let us present the simplest application of Lemma 2.5 (for another application, see the proof of Corollary 6.12).

Example 2.6. Consider the action of the group $\mathfrak{S}_5$ on $\mathbb{P}^3$ given by its irreducible 4-dimensional representation. Then $\mathbb{P}^3$ contains a $\mathfrak{S}_5$-invariant smooth complete intersection $X = F_2 \cap F_3$ such that F_2 and F_3 are $\mathfrak{S}_5$-invariant quadric and cubic surfaces, respectively. Note that X is a smooth curve of genus 4 known as the Bring's curve [26]. By Lemma 2.5, the projective space $\mathbb{P}^3$ is not

$\mathfrak{S}_5$-birationally rigid. Moreover, arguing as in the proof of Lemma 2.5, we obtain the following $\mathfrak{S}_5$-equivariant diagram:

$$\begin{array}{ccc} & V & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^3 & & Y \end{array}$$

where π is the blow up of the curve X, the threefold Y is a cubic hypersurface in $\mathbb{P}^4$ with one isolated ordinary double point, and η is the blow up of this point. Note that $|\mathcal{O}_{\mathbb{P}^3}(d)|$ does not contain $\mathfrak{S}_5$-invariant pencils for $d \leqslant 3$, so Lemma 2.4 is not applicable here. On the other hand, there are other ways to see that $\mathbb{P}^3$ is not $\mathfrak{S}_5$-birationally rigid. For instance, one can show that $\mathbb{P}^3$ is $\mathfrak{S}_5$-birationally equivalent to a $\mathfrak{S}_5$-conic bundle [20, Example 2.7].

The next lemma will be used in Section 6.

Lemma 2.7. *Suppose that $n = 5$, the action of G on $\mathbb{P}^5$ is transitive, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(2)|$ contains a G-invariant two-dimensional linear subsystem $\mathcal{M}$. Suppose also that the action of G on $\mathcal{M} \simeq \mathbb{P}^2$ is faithful and transitive, and G cannot act faithfully on a rational curve. Then $\mathbb{P}^5$ is not G-birationally rigid.*

Proof. Since there are no G-invariant hyperplanes in $\mathbb{P}^5$, we see that the linear system $\mathcal{M}$ is mobile.

Let $\chi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ be the rational map given by the net $\mathcal{M}$. Then we obtain a G-equivariant commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^5 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where π is a G-equivariant resolution of indeterminacies of the rational map χ with smooth X (cf. the proof of Lemma 2.4). Recall that G acts faithfully on $\mathbb{P}^2$. Hence the morphism η is surjective, because otherwise the image of η would be either a point, or a rational curve. On the other hand, by our assumptions there are no G-invariant points in $\mathbb{P}^2$, and G cannot act faithfully on a rational curve.

Let F be a general fiber of the morphism η. Then π induces a birational morphism $F \to \pi(F)$, and

$$\pi(F) \subset M_1 \cap M_2$$

for two distinct quadric hypersurfaces M_1 and M_2 in the net $\mathcal{M}$. Hence, every irreducible component of $\pi(F)$ is an irreducible component of the intersection $M_1 \cap M_2$. This implies that all irreducible components of F are uniruled. Thus, applying G-equivariant Minimal Model Program to X over $\mathbb{P}^2$, we obtain a G-birational map from X to a G-Mori fibred space with a positive dimensional base. In particular, $\mathbb{P}^5$ is not G-birationally rigid. □

3. Toric symmetry of projective spaces

In this section, we discuss equivariant birational geometry of the projective space $\mathbb{P}^n$, $n \geqslant 2$, with respect to the action of a group that contains a maximal torus in $\mathrm{PGL}_{n+1}(\mathbb{C})$ as a normal subgroup. After some preparation, we formulate the technical result to be proved in Section 4, and use it to deduce Theorem A.

The groups we are interested in can be described as follows. Let $\mathbb{T}$ be the maximal torus in $\mathrm{PGL}_{n+1}(\mathbb{C})$ consisting of the transformations

$$(t_1, \ldots, t_n)\colon \big[x_1 : \ldots : x_n : x_{n+1}\big] \mapsto \big[t_1x_1 : \ldots : t_nx_n : x_{n+1}\big],$$

let $\mathbb{W}$ be its normalizer in $\mathrm{PGL}_{n+1}(\mathbb{C})$, and let $\mathbb{G}$ be a subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$ such that

$$\mathbb{T} \subseteq \mathbb{G} \subseteq \mathbb{W}.$$

Then both $\mathbb{G}$ and $\mathbb{W}$ permute the $\mathbb{T}$-invariant points:

$$P_1 = [1:0:\ldots:0], P_2 = [0:1:0:\ldots:0], \ldots, P_{n+1} = [0:\ldots:0:1].$$

The $\mathbb{W}$-action gives an epimorphism $\Upsilon\colon \mathbb{W} \to \mathfrak{S}_{n+1}$. Set $G = \Upsilon(\mathbb{G})$, and let $\upsilon\colon \mathbb{G} \to G$ be the induced epimorphism. Then we have the following exact sequences of groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{G} & \xrightarrow{\upsilon} & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{W} & \xrightarrow{\Upsilon} & \mathfrak{S}_{n+1} & \longrightarrow & 1 \end{array}$$

Since Υ and υ have natural sections, we have $\mathbb{W} = \mathbb{T} \rtimes \mathfrak{S}_{n+1}$ and $\mathbb{G} = \mathbb{T} \rtimes G$, so we will occasionally consider G and $\mathfrak{S}_{n+1}$ as subgroups in $\mathbb{G}$ and $\mathbb{W}$, respectively, consisting of projective transformations given by standard permutation matrices. On the other hand, $\mathfrak{S}_{n+1}$ acts on $\mathbb{T}$ by conjugations, which gives a monomorphism

$$\mathfrak{S}_{n+1} \hookrightarrow \mathrm{Aut}(\mathbb{T}) \simeq \mathrm{GL}_n(\mathbb{Z}).$$

Thus, occasionally, we may also consider G and $\mathfrak{S}_{n+1}$ as subgroups in $\mathrm{GL}_n(\mathbb{Z})$. We will say that the subgroup $G \subset \mathrm{GL}_n(\mathbb{Z})$ is irreducible, if $\mathbb{Z}^n$ is an irreducible G-module.

Proposition 3.1 ([10, Proposition 3.7])**.** *The following two conditions are equivalent:*

- *the image of the group G in $\mathrm{GL}_n(\mathbb{Z})$ is an irreducible subgroup;*
- *$\mathbb{P}^n$ is not $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base.*

Remark 3.2. In the notation of [10], Proposition 3.1 says that $\mathbb{P}^n$ is $\mathbb{G}$-birationally solid if and only if the image of the group G in $\mathrm{GL}_n(\mathbb{Z})$ is irreducible.

In particular, it follows from Proposition 3.1 that if $\mathbb{P}^n$ is not $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base, then G is a transitive subgroup of $\mathfrak{S}_{n+1}$ in the sense of Definition A.1. In fact, we can say more.

Lemma 3.3. *If the group G is not a primitive subgroup of $\mathfrak{S}_{n+1}$ in the sense of Definition A.1, then $\mathbb{P}^n$ is $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base.*

Proof. The group $\mathfrak{S}_{n+1}$ and its subgroup G act by permutations on the homogeneous coordinates $x_1, \ldots, x_{n+1}$. The $\mathfrak{S}_{n+1}$-module $\mathbb{Z}^n$ can be identified with the quotient of $\mathbb{Z}^{n+1}$ with the basis vectors $e_1, \ldots, e_{n+1}$ which are in bijection with these coordinates, by the submodule generated by the vector $e_1 + \ldots + e_{n+1}$.

First, suppose that G is not transitive in the sense of Definition A.1. Then for some integer $1 \leqslant r < n+1$ the group G permutes the vectors $e_1, \ldots, e_r$, so that their images in $\mathbb{Z}^n$ span a non-trivial submodule. Therefore, $\mathbb{P}^n$ is $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base by Proposition 3.1.

Now suppose that G is transitive but not primitive. Then there is a partition

$$\{e_1, \ldots, e_{n+1}\} = \Sigma^1 \sqcup \ldots \sqcup \Sigma^k$$

such that $1 < |\Sigma^i| < n+1$, and the sets Σ^i are permuted by G. We may assume that $n+1 = mk$, and

$$\Sigma^1 = \{e_1, \ldots, e_m\}, \ldots, \Sigma^k = \{e_{(m-1)k+1}, \ldots, e_{n+1}\}.$$

Thus, G preserves the submodule of $\mathbb{Z}^{n+1}$ spanned by the vectors

$$v_1 = e_1 + \ldots + e_m, \ldots, v_k = e_{(m-1)k+1} + \ldots + e_{n+1},$$

whose images span a non-trivial submodule of $\mathbb{Z}^n$. Therefore, $\mathbb{P}^n$ is again $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base by Proposition 3.1. □

Another useful result concerning $\mathbb{G}$-birational maps of toric varieties is as follows.

Proposition 3.4 ([10, Lemma 3.1]). *Let V_1 and V_2 be n-dimensional toric varieties with the action of the n-dimensional torus $\mathbb{T}$. Let G be a finite group acting on V_i, $i = 1, 2$, such that G normalizes the torus $\mathbb{T} \subset \mathrm{Aut}(V_i)$ and acts faithfully on $\mathbb{T}$. Denote by G_i the image of G in*

$$\mathrm{Aut}(\mathbb{T}) \simeq \mathrm{GL}_n(\mathbb{Z})$$

corresponding to these two actions on $\mathbb{T}$, and set $\mathbb{G}_i = \mathbb{T} \rtimes G_i$. The following two conditions are equivalent:

- *the groups G_1 and G_2 are conjugate in $\mathrm{GL}_n(\mathbb{Z})$;*
- *the groups $\mathbb{G}_i$ are isomorphic to each other, and there exists a birational map $V_1 \dashrightarrow V_2$ which is equivariant with respect to $\mathbb{G}_1 \simeq \mathbb{G}_2$.*

In Definition 1.1, we defined equivariant birational rigidity for a Fano variety with terminal singularities acted on by a finite group. The same definition applies for an algebraic group action, so we can define $\mathbb{G}$-birational rigidity for $\mathbb{P}^n$ exactly as in Definition 1.1. Our next goal is to prove the following purely toric theorem.

Theorem 3.5. *Suppose that $n \geqslant 3$. Then $\mathbb{P}^n$ is not $\mathbb{G}$-birationally rigid.*

To prove Theorem 3.5, we have to prove that

(1) either $\mathbb{P}^n$ is $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with a positive-dimensional base,
(2) or the projective space $\mathbb{P}^n$ is $\mathbb{G}$-birational to a Fano variety X with terminal singularities such that $\mathrm{rk}\,\mathrm{Cl}(X)^{\mathbb{G}} = 1$, but X is not $\mathbb{G}$-equivariantly isomorphic to $\mathbb{P}^n$.

This immediately follows from the following slightly stronger technical result.

Theorem 3.6. *Suppose that $n \geqslant 3$. Then either $\mathbb{P}^n$ is $\mathbb{G}$-birational to a $\mathbb{G}$-Mori fiber space with positive-dimensional base, or $\mathbb{P}^n$ is $\mathbb{G}$-birational to a Fano variety X with terminal singularities such that $\mathrm{rk}\,\mathrm{Cl}(X)^{\mathbb{G}} = 1$ and $X \not\simeq \mathbb{P}^n$.*

We will prove Theorem 3.6 later in Section 4.

Remark 3.7. In dimension 3, both Theorems 3.5 and 3.6 have been proved in [20, 10].

Let us repeat that Theorem 3.6 implies Theorem 3.5. In fact, using Theorem 3.6, we can also deduce Theorem A.

Proof of Theorem A. Suppose that G is not primitive. By Lemma 2.2, we may assume that G is transitive. Next, by Lemma 2.3, we may assume that G has an orbit of length $n+1$. Choosing appropriate coordinates on $\mathbb{P}^n$, we may assume that this orbit consists of the points

$$P_1 = [1 : 0 : \ldots : 0], P_2 = [0 : 1 : 0 : \ldots : 0], \ldots, P_{n+1} = [0 : \ldots : 0 : 1].$$

Let $\mathbb{T}$ be the maximal torus in $\mathrm{PGL}_{n+1}(\mathbb{C})$ consisting of automorphisms

$$\big[x_1 : \ldots : x_n : x_{n+1}\big] \mapsto \big[t_1x_1 : \ldots : t_nx_n : x_{n+1}\big],$$

and let $\mathbb{G}$ be the subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$ generated by G and $\mathbb{T}$. Since $n \geqslant 3$, it follows from Theorem 3.6 that there exists a $\mathbb{G}$-equivariant birational map $\chi\colon \mathbb{P}^n \dashrightarrow X$ such that $X \not\simeq \mathbb{P}^n$ is a toric $\mathbb{G}$-Mori fiber space over some base Z (which may be a point or have positive dimension); in particular, X is $\mathbb{G}\mathbb{Q}$-factorial, i.e.

$$\mathrm{rk}\,\mathrm{Cl}(X)^{\mathbb{G}} = \mathrm{rk}\,\mathrm{Pic}(X)^{\mathbb{G}},$$

and one has $\operatorname{rk}\operatorname{Pic}(X/Z)^{\mathbb{G}} = 1$. Note that χ is also G-equivariant, and the torus $\mathbb{T}$ acts trivially on the lattices $\operatorname{Cl}(X)$ and $\operatorname{Pic}(X)$. Therefore, one has

$$\operatorname{rk}\operatorname{Cl}(X)^G = \operatorname{rk}\operatorname{Cl}(X)^{\mathbb{G}} = \operatorname{rk}\operatorname{Pic}(X)^{\mathbb{G}} = \operatorname{rk}\operatorname{Pic}(X)^G,$$

so that X is $G\mathbb{Q}$-factorial. Similarly, we see that

$$\operatorname{rk}\operatorname{Pic}(X/Z)^G = \operatorname{rk}\operatorname{Pic}(X/Z)^{\mathbb{G}} = 1.$$

Thus, X is a G-Mori fiber space over Z. Since $X \not\cong \mathbb{P}^n$, this means that $\mathbb{P}^n$ is not G-birationally rigid. □

4. Proof of the main technical result

In this section we prove Theorem 3.6. Let us use the notation of Section 3. We may assume that the image of the group G in $\mathrm{GL}_n(\mathbb{Z})$ is an irreducible subgroup, since otherwise we are done by Proposition 3.1. Moreover, by Lemma 3.3 we may assume that G is a primitive subgroup of $\mathfrak{S}_{n+1}$ in the sense of Definition A.1. To prove Theorem 3.6, we are going to construct a $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow X$ such that the following conditions hold:

- X has terminal singularities,
- X is a toric Fano variety,
- $\operatorname{rk}\operatorname{Cl}(X)^{\mathbb{G}} = 1$,
- $X \not\cong \mathbb{P}^n$.

Remark 4.1. Observe that $\mathbb{T}$ acts trivially on $\operatorname{Cl}(X)$, so that

$$\operatorname{rk}\operatorname{Cl}(X)^{\mathbb{G}} = \operatorname{rk}\operatorname{Cl}(X)^G.$$

Thus, in fact we have to check whether $\operatorname{rk}\operatorname{Cl}(X)^G = 1$ or not.

The plan of our construction is as follows. First, we blow up $\mathbb{P}^n$ in a certain configuration of linear subspaces and run an $\mathbb{W}$-equivariant Minimal Model Program to construct a $\mathbb{W}$-equivariant birational map $\mathbb{P}^n \dashrightarrow X$ to a particular terminal Fano variety X which is not isomorphic to $\mathbb{P}^n$. It turns out that if the group G is large enough, then $\operatorname{rk}\operatorname{Cl}(X)^{\mathbb{G}} = 1$; this includes the cases when G is isomorphic to $\mathfrak{S}_{n+1}$ or $\mathfrak{A}_{n+1}$ and, more generally, when G acts transitively on the non-ordered pairs of indices from $\{1, \ldots, n+1\}$ (cf. Example 4.9). We expect that this step of the construction has an alternative more explicit description, but we cannot prove this yet (see Remark 4.2). In a more general case, when $\operatorname{rk}\operatorname{Cl}(X)^{\mathbb{G}} \neq 1$, we start with the latter variety X and proceed with a $\mathbb{G}\mathbb{Q}$-factorialization followed by a further $\mathbb{G}$-equivariant Minimal Model Program to obtain another $\mathbb{G}\mathbb{Q}$-Fano variety X' which is $\mathbb{G}$-birational to $\mathbb{P}^n$. If the group G is not too small, then it turns out that the variety X' is not isomorphic to $\mathbb{P}^n$ (see Lemma 4.10). This works unless the group G is either a cyclic group $\boldsymbol{\mu}_{n+1}$ or a dihedral group $\mathfrak{D}_{n+1}$, with certain particular action on $\mathbb{P}^n$. In these two cases we construct two special $\mathbb{G}\mathbb{Q}$-Fano varieties V and U, and check that there exists a $\mathbb{G}$-equivariant birational map from $\mathbb{P}^n$ to V or U, respectively (see Corollary 4.14 and Lemma 4.19).

To start with, let $\mathcal{M}$ be the linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(2n)|$ that consists of all hypersurfaces

$$x_1x_2\ldots x_{n+1}f(x_1,\ldots,x_{n+1}) + \sum_{i=1}^{n+1}\lambda_i\frac{\big(x_1x_2\ldots x_{n+1}\big)^2}{x_i^2} = 0,$$

where f is any polynomial of degree $n-1$, and $\lambda_1,\ldots,\lambda_{n+1}\in\mathbb{C}$. Set $N = \dim(\mathcal{M})$. Then

$$N = n + \binom{2n-1}{n},$$

the linear system $\mathcal{M}$ is mobile, its base locus is the union of all $(n-2)$-dimensional $\mathbb{T}$-invariant subvarieties, and a general member of $\mathcal{M}$ is singular along the base locus.

Remark 4.2. Since the linear system $\mathcal{M}$ is $\mathbb{W}$-invariant, it gives an $\mathbb{W}$-equivariant birational map

$$\psi\colon \mathbb{P}^n \dashrightarrow \mathscr{X}$$

such that $\mathscr{X}$ is a toric 3-fold in $\mathbb{P}^N$. We expect that $\mathscr{X}$ is a Fano variety with terminal singularities, one has $\operatorname{rk}\operatorname{Cl}(X)^{\mathbb{W}} = 1$, and $\mathscr{X} \not\simeq \mathbb{P}^n$. If this is the case, we can set $X = \mathscr{X}$, at least when $G \simeq \mathfrak{S}_{n+1}$. But we were unable to prove this for all $n \geqslant 3$.

The properties of the variety $\mathcal{X}$ introduced in Remark 4.2 have been described only for small values of n.

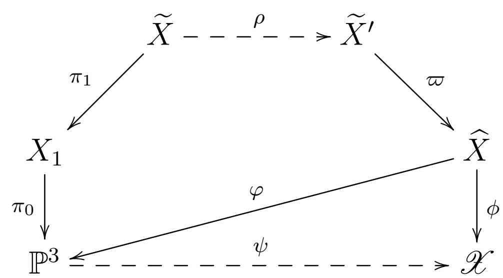
Example 4.3 (cf. [4, 10, 13, 20, 42, 48])**.** Suppose that $n = 3$. Then $\mathcal{M}$ consists of all sextic surfaces that are singular along six $\mathbb{T}$-invariant lines forming a toric tetrahedron in $\mathbb{P}^3$, and

$$\mathscr{X} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1/\boldsymbol{\mu}_2,$$

where $\boldsymbol{\mu}_2$ acts on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as

$$\big([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]\big) \mapsto \big([x_1 : -y_1], [x_2 : -y_2], [x_3 : -y_3]\big).$$

Therefore, the threefold $\mathscr{X}$ has 8 cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$. Moreover, we have the following $\mathbb{W}$-equivariant commutative diagram:



where

- π_0 blows up four $\mathbb{T}$-invariant points;
- π_1 blows up the strict transforms of the six $\mathbb{T}$-invariant lines;
- ρ flops strict transforms of $\mathbb{T}$-invariant curves contained in π_0-exceptional surfaces;
- ϖ contracts the strict transforms of π_0-exceptional surfaces;
- ϕ contracts the strict transforms of four $\mathbb{T}$-invariant planes in $\mathbb{P}^3$;
- φ symbolically blows up the ideal sheaf of the union of six $\mathbb{T}$-invariant lines [45, §6.1].

Moreover, if $G \simeq \mathfrak{S}_4$ or $G \simeq \mathfrak{A}_4$, then one has $\operatorname{rk}\operatorname{Cl}(\mathcal{X})^{\mathbb{G}} = \operatorname{rk}\operatorname{Cl}(\mathcal{X})^G = 1$.

Example 4.4. For $n = 4$, we used Magma code provided by Andrea Petracci (see Appendix B) to verify the properties of $\mathscr{X}$. Namely, it turns out that $\mathscr{X}$ is a toric Fano 4-fold with terminal (non-isolated) singularities such that $\operatorname{rk}\operatorname{Cl}(\mathscr{X}) = 6$, $\operatorname{rk}\operatorname{Pic}(\mathscr{X}) = 1$, and $(-K_{\mathscr{X}})^4 = 70$. Moreover, if $G \simeq \mathfrak{S}_5$ or $G \simeq \mathfrak{A}_5$, then one has $\operatorname{rk}\operatorname{Cl}(\mathcal{X})^{\mathbb{G}} = \operatorname{rk}\operatorname{Cl}(\mathcal{X})^G = 1$. The singular locus of $\mathscr{X}$ consists of 5 isolated quotient singularities of type $\frac{1}{3}(1, 1, 1, 1)$, and 10 curves such that $\mathscr{X}$ has a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ at their general points. In particular, $\mathscr{X}$ is not isomorphic to $\mathbb{P}^4$. In this case, the linear system $\mathcal{M}$ has been studied in [29, 49].

Now we construct a $\mathbb{W}$-birational map $\mathbb{P}^n \dashrightarrow X$ using equivariant toric Minimal Model Program. Keeping in mind Remark 3.7, we will further assume that $n \geqslant 4$. To start with, consider the following sequence of $n - 1 \geqslant 3$ toric $\mathbb{W}$-equivariant blowups

$$\widetilde{X} \xrightarrow{\pi_{n-2}} X_{n-2} \xrightarrow{\pi_{n-3}} \ldots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} \mathbb{P}^n,$$

where

- π_0 is the blow up of all $\mathbb{T}$-invariant points in $\mathbb{P}^n$,

- π_1 is the blow up of the strict transforms of all $\mathbb{T}$-invariant lines in $\mathbb{P}^n$,
- π_2 is the blow up of the strict transforms of all $\mathbb{T}$-invariant planes in $\mathbb{P}^n$, etc.

Let $\pi\colon \widetilde{X} \to \mathbb{P}^n$ be the composition $\pi_0 \circ \pi_1 \circ \ldots \circ \pi_{n-2}$. Then $\widetilde{X}$ is toric, and π is $\mathbb{W}$-equivariant.

Remark 4.5. The variety $\widetilde{X}$ is known as permutohedron [31, 34].

For every $k \in \{0, \ldots, n-2\}$, let $\widetilde{E}_k$ be the $\mathbb{W}$-irreducible π-exceptional divisor whose irreducible components are mapped to $\mathbb{T}$-invariant linear subspaces of $\mathbb{P}^n$ that have dimension k. Similarly, we let $\widetilde{E}_{n-1}$ be the $\mathbb{W}$-irreducible divisor in $\widetilde{X}$ whose irreducible components are strict transforms of $\mathbb{T}$-invariant hyperplanes in $\mathbb{P}^n$. Denote by $\Xi \subset \mathbb{P}^n$ the union of the latter hyperplanes, i.e. the divisor in $\mathbb{P}^n$ given by equation $x_1x_2\ldots x_{n+1} = 0$. Then π induces an isomorphism

$$\widetilde{X} \setminus \mathrm{Supp}\left(\sum_{i=0}^{n-1} \widetilde{E}_i\right) \simeq \mathbb{P}^n \setminus \Xi \simeq \big(\mathbb{C}^*\big)^n,$$

so the union $\widetilde{E}_0 \cup \widetilde{E}_1 \cup \ldots \cup \widetilde{E}_{n-1}$ forms the toric boundary in $\widetilde{X}$. Note also that

$$\widetilde{E}_{n-1} \sim \pi^*\big(-K_{\mathbb{P}^n}\big) - \sum_{i=0}^{n-2}(n-i)\widetilde{E}_i. \tag{4.1}$$

Remark 4.6. Let $\tau_{\mathbb{P}^n}$ be the standard Cremona birational involution in $\mathrm{Bir}^{\mathbb{W}}(\mathbb{P}^n)$. Then we have the following $\mathbb{W}$-equivariant commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \tau_{\widetilde{X}}\ } & \widetilde{X} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^n & \overset{\tau_{\mathbb{P}^n}}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

where $\tau_{\widetilde{X}}$ is a biregular involution that swaps $\widetilde{E}_k$ and $\widetilde{E}_{n-1-k}$ for every $k \in \{0, \ldots, n-1\}$.

Let $\widetilde{\mathcal{M}}$ be the strict transform on $\widetilde{X}$ of the linear system $\mathcal{M}$. Then

$$\widetilde{\mathcal{M}} \sim \pi^*(\mathcal{M}) - \sum_{i=0}^{n-2}(n-i)\widetilde{E}_i. \tag{4.2}$$

This gives

$$K_{\widetilde{X}} + \frac{n+1}{2n}\widetilde{\mathcal{M}} \sim_{\mathbb{Q}} \pi^*\Big(K_{\mathbb{P}^n} + \frac{n+1}{2n}\mathcal{M}\Big) + \sum_{i=0}^{n-2} a_i\widetilde{E}_i \sim_{\mathbb{Q}} \sum_{i=0}^{n-2} a_i\widetilde{E}_i,$$

where

$$
(4.3) \qquad \begin{cases} a_0 = \dfrac{n-3}{2}, \\ a_1 = \dfrac{n^2-4n+1}{2n}, \\ \vdots \\ a_i = \dfrac{n^2-(i+3)n+i}{2n}, \\ \vdots \\ a_{n-3} = \dfrac{n-3}{2n}, \\ a_{n-2} = -\dfrac{1}{n}. \end{cases}
$$

Observe that $a_i > 0$ for all $0 \leqslant i \leqslant n-3$, because we assume that $n \geqslant 4$. On the other hand, we have $a_{n-2} = -\frac{1}{n} < 0$. In particular, we see that the singularities of the log pair $(\mathbb{P}^n, \frac{n+1}{2n}\mathcal{M})$ are not canonical.

Lemma 4.7. *The singularities of the log pair $(\widetilde{X}, \frac{n+1}{2n}\widetilde{\mathcal{M}})$ are terminal.*

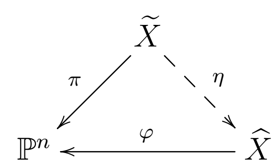
Proof. Since $\frac{n+1}{2n} < 1$, it is enough to show that $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$ for every point P contained in the base locus of the linear system $\widetilde{\mathcal{M}}$. On the other hand, since $\widetilde{\mathcal{M}}$ is $\mathbb{W}$-invariant and mobile, its base locus is contained in $\widetilde{E}_0 \cup \ldots \cup \widetilde{E}_{n-2}$. Moreover, it follows from the local computations that the base locus of the linear system $\widetilde{\mathcal{M}}$ is contained in $\widetilde{E}_0 \cup \ldots \cup \widetilde{E}_{n-3}$. So, let us fix a point $P \in \widetilde{E}_0 \cup \ldots \cup \widetilde{E}_{n-3}$ and show that $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$.

Suppose that $P \in \widetilde{E}_0$. Let $\widetilde{E}$ be the irreducible component of the divisor $\widetilde{E}_0$ that contains P, and let E be its strict transform on X_1. Then $E \simeq \mathbb{P}^{n-1}$, the variety $\widetilde{E}$ is a permutohedron, and the induced morphism $\widetilde{E} \to E$ is an analogue of the constructed birational morphism $\pi\colon \widetilde{X} \to \mathbb{P}^n$. Moreover, it follows from (4.2) and (4.1) that the restriction $\widetilde{\mathcal{M}}|_{\widetilde{E}}$ is the disjoint union of the strict transforms on $\widetilde{E}$ of torus invariant hyperplanes in E. This gives $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$.

Now, we suppose that $P \in \widetilde{E}_1 \setminus \widetilde{E}_0$. Let $\widetilde{E}$ be the irreducible component of $\widetilde{E}_1$ containing P, let E be its strict transform on X_2, let Z be its image on X_1, let $\widetilde{F}$ be the fiber of the induced morphism $\widetilde{E} \to Z$ such that $P \in \widetilde{F}$, and let F be its strict transform on X_2. Then $F \simeq \mathbb{P}^{n-2}$, and $\widetilde{F}$ is a permutohedron that is obtained from F via the induced birational morphism $\widetilde{F} \to F$. Now, using (4.2) and (4.1), we conclude that the restriction $\widetilde{\mathcal{M}}|_{\widetilde{F}}$ is the disjoint union of the strict transforms on $\widetilde{F}$ of all torus invariant hyperplanes in F. As above, we see that $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$.

Hence, if $P \in \widetilde{E}_0 \cup \widetilde{E}_1$, we have $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$. Thus, we may assume that $n \geqslant 5$. Now, arguing exactly as in the previous case, we see that $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$ if $P \in \widetilde{E}_2 \setminus (\widetilde{E}_0 \cup \widetilde{E}_1)$. Continuing in this way, we see that $\mathrm{mult}_P(\widetilde{\mathcal{M}}) \leqslant 1$ for every $P \in \widetilde{E}_0 \cup \ldots \cup \widetilde{E}_{n-3}$. ☐

Applying the relative $\mathbb{W}$-equivariant Minimal Model Program to the log pair $(\widetilde{X}, \frac{n+1}{2n}\widetilde{\mathcal{M}})$ over $\mathbb{P}^n$, we obtain a toric $\mathbb{W}$-equivariant commutative diagram



such that η is a composition of divisorial contractions and log flips, φ is a birational morphism, and one has

$$\operatorname{rk}\operatorname{Cl}\big(\widehat{X}\big)^{\mathbb{W}} = \operatorname{rk}\operatorname{Pic}\big(\widehat{X}\big)^{\mathbb{W}}.$$

Furthermore, by Lemma 4.7 the singularities of the log pair $(\widetilde{X}, \frac{n+1}{2n}\widetilde{\mathcal{M}})$ are terminal, which implies that the pair $(\widehat{X}, \frac{n+1}{2n}\widehat{\mathcal{M}})$ also has terminal singularities, and $-(K_{\widehat{X}} + \frac{n+1}{2n}\widehat{\mathcal{M}})$ is φ-nef, where $\widehat{\mathcal{M}}$ is the strict transform on $\widehat{X}$ of the linear system $\mathcal{M}$. Moreover, since $-(K_{\widehat{X}} + \frac{n+1}{2n}\widehat{\mathcal{M}})$ is φ-nef, the map η contracts every $\mathbb{W}$-irreducible π-exceptional divisor $\widetilde{E}_k$ with $a_k > 0$. Therefore, using (4.3), we conclude that the constructed $\mathbb{W}$-birational map $\eta\colon \widetilde{X} \dashrightarrow \widehat{X}$ contracts all $\mathbb{W}$-irreducible π-exceptional divisors except for $\widetilde{E}_{n-2}$. Hence, we see that $\operatorname{rk}\operatorname{Cl}(\widehat{X})^{\mathbb{W}} = 2$ and

$$K_{\widehat{X}} + \frac{n+1}{2n}\widehat{\mathcal{M}} \sim_{\mathbb{Q}} \pi^*\Big(K_{\mathbb{P}^n} + \frac{n+1}{2n}\mathcal{M}\Big) - \frac{1}{n}\widehat{E}_{n-2} \sim_{\mathbb{Q}} -\frac{1}{n}\widehat{E}_{n-2},$$

where $\widehat{E}_{n-2}$ is the strict transform on $\widehat{X}$ of the divisor $\widetilde{E}_{n-2}$.

Now, applying the absolute $\mathbb{W}$-equivariant Minimal Model Program to the log pair $(\widehat{X}, \frac{n+1}{2n}\widehat{\mathcal{M}})$, and using Proposition 3.1, we obtain the following $\mathbb{W}$-equivariant Sarkisov link:

$$\begin{array}{ccc} \widehat{X} & \overset{\rho}{\dashrightarrow} & \overline{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{P}^n & \overset{\chi}{\dashrightarrow} & X \end{array}$$

Here ρ is a small $\mathbb{W}$-birational map, ϕ is an $\mathbb{W}$-equivariant divisorial contraction that contracts the strict transform of the divisor $\widetilde{E}_{n-1}$, and X is a toric Fano variety such that

$$\operatorname{rkCl}(X)^{\mathbb{W}} = \operatorname{rkCl}(X)^{\mathfrak{S}_{n+1}} = 1.$$

By construction, the toric boundary of the toric variety X is the strict transform of the divisor $\widetilde{E}_{n-2}$, and its irreducible components generate the class group $\operatorname{Cl}(X)$. Thus, since ϕ contracts the strict transform of the divisor $\widetilde{E}_{n-1}$, we see that

$$\operatorname{rk}\operatorname{Cl}(X) = \operatorname{rk}\operatorname{Cl}(\widehat{X}) - n - 1 = \binom{n+1}{2} - n.$$

In particular, one has $X \not\simeq \mathbb{P}^n$.

Remark 4.8. We expect that X is isomorphic to the variety $\mathscr{X}$ introduced in Remark 4.2, but we are not able to prove this at the moment.

The above construction proves Theorem 3.6 in the case when

$$\operatorname{rkCl}(X)^{\mathbb{G}} = \operatorname{rkCl}(X)^{G} = 1. \tag{4.4}$$

This often happens when G is large enough, although there are cases when the group does not satisfy these properties.

Example 4.9. If the group G acts transitively on the set of all non-ordered pairs $\{P_i, P_j\}$ consisting of distinct points in

$$\big\{P_1, \ldots, P_{n+1}\big\} \subset \mathbb{P}^n,$$

then it acts transitively on irreducible components of $\widetilde{E}_{n-2}$, and so condition (4.4) holds. For instance, this is always the case when $G \simeq \mathfrak{S}_{n+1}$ or $G \simeq \mathfrak{A}_{n+1}$. Furthermore, if $n = 4$, then G is isomorphic to one of the following groups:

$$\boldsymbol{\mu}_5,\ \mathfrak{D}_5,\ \boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4,\ \mathfrak{A}_5,\ \mathfrak{S}_5,$$

because G is a primitive subgroup in $\mathfrak{S}_5$ in the sense of Definition A.1. If G is isomorphic to $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\mathfrak{A}_5$, or $\mathfrak{S}_5$, then condition (4.4) holds. However, if $G \simeq \boldsymbol{\mu}_5$ or $G \simeq \mathfrak{D}_5$, then explicit computations show that $\mathrm{rkCl}(X)^{\mathbb{G}} = \mathrm{rk}\,\mathrm{Cl}(X)^G = 2$.

Thus, to complete the proof of Theorem 3.6, we may assume that condition (4.4) fails, i.e. one has

$$\mathrm{rkCl}(X)^{\mathbb{G}} = \mathrm{rkCl}(X)^G \neq 1.$$

Let $\gamma\colon X^\sharp \to X$ be a $\mathbb{G}\mathbb{Q}$-factorialization, i.e. γ is a small $\mathbb{G}$-equivariant birational morphism (possibly an isomorphism) such that U has terminal $\mathbb{G}\mathbb{Q}$-factorial singularities. One has

$$\mathrm{rkCl}(X^\sharp)^G = \mathrm{rkCl}(X^\sharp)^{\mathbb{G}} = \mathrm{rkPic}(X^\sharp)^{\mathbb{G}} = \mathrm{rkPic}(X^\sharp)^G.$$

Then, applying $\mathbb{G}$-equivariant Minimal Model Program, and using Proposition 3.1 again, we obtain a $\mathbb{G}$-equivariant birational map $\theta\colon X^\sharp \dashrightarrow X'$ such that θ is a composition of divisorial contractions and flips, and X' is a toric Fano variety with terminal singularities such that

$$\mathrm{rk}\,\mathrm{Cl}\big(X'\big)^{\mathbb{G}} = \mathrm{rk}\,\mathrm{Cl}\big(X'\big)^G = 1.$$

If $X' \not\simeq \mathbb{P}^n$, we are done. However, we do not know whether X' is isomorphic to $\mathbb{P}^n$ or not. On the other hand, there exists a very simple combinatorial condition on G which guarantees that $X' \not\simeq \mathbb{P}^n$.

Lemma 4.10. *Let Σ be the set of non-ordered pairs $\{P_i, P_j\}$ consisting of distinct points in*

$$\big\{P_1, \ldots, P_{n+1}\big\} \subset \mathbb{P}^n.$$

Then G naturally acts on Σ. Suppose that Σ contains no G-orbits of length at most $n+1$. Then $X' \not\simeq \mathbb{P}^n$.

Proof. The constructed $\mathbb{G}$-equivariant birational map θ contracts some (but not all) irreducible components of the strict transform of the $\mathbb{T}$-invariant divisor $\widetilde{E}_{n-2}$. The remaining irreducible components are mapped to $\mathbb{T}$-invariant divisors in X', and form its toric boundary. By assumption, the number of these irreducible components is at least $n+2$, so X' cannot be isomorphic to $\mathbb{P}^n$, since $\mathbb{P}^n$ has exactly $n+1$ torus invariant irreducible divisors. □

If the subgroup $G \subset \mathfrak{S}_{n+1}$ satisfies the combinatorial condition of Lemma 4.10, we are done. Hence, to complete the proof of Theorem 3.6, we may assume that it does not satisfy this condition. Recall that G is a primitive subgroup in $\mathfrak{S}_{n+1}$ in the sense of Definition A.1. Thus, it follows from Corollary A.4 that either $G \simeq \boldsymbol{\mu}_{n+1}$ or $G \simeq \mathfrak{D}_{n+1}$, and G is conjugate in $\mathfrak{S}_{n+1}$ to one of the following two subgroups:

(1) the cyclic group generated by the cycle $(1\ 2\ \ldots\ n+1)$;
(2) the dihedral group generated by the cycle $(1\ 2\ \ldots\ n+1)$ and the permutation

$$\big(1\ n+1\big)\big(2\ n\big)\ldots\left(\left\lfloor\frac{n+1}{2}\right\rfloor\ \left\lceil\frac{n+1}{2}\right\rceil\right).$$

Hence, either $\mathbb{G}$ is conjugate in $\mathbb{W}$ to the subgroup generated by $\mathbb{T}$ and σ such that

$$\sigma\big([x_1 : x_2 : x_3 : \ldots : x_{n-1} : x_n : x_{n+1}]\big) = [x_{n+1} : x_1 : x_2 : \ldots : x_{n-2} : x_{n-1} : x_n],$$

or $\mathbb{G}$ is conjugate in $\mathbb{W}$ to the subgroup generated by $\mathbb{T}$, σ, and ι such that

$$\iota\big([x_1 : x_2 : x_3 : \ldots : x_{n-1} : x_n : x_{n+1}]\big) = [x_{n+1} : x_n : x_{n-1} : \ldots : x_3 : x_2 : x_1].$$

Hence, to complete the proof of Theorem 3.6, we may assume that

(1) either $\mathbb{G} \simeq \mathbb{T} \rtimes \boldsymbol{\mu}_{n+1}$ and $\mathbb{G} = \langle \mathbb{T}, \sigma\rangle$,
(2) or $\mathbb{G} \simeq \mathbb{T} \rtimes \mathfrak{D}_{n+1}$ and $\mathbb{G} = \langle \mathbb{T}, \sigma, \iota\rangle$.

Lemma 4.11. *The number $n+1$ is prime.*

Proof. If $n+1$ is not prime, then it follows from Remark A.3 that G is not a primitive subgroup in $\mathfrak{S}_{n+1}$ in the sense of Definition A.1, which contradicts our assumptions. $\square$

To deal with the case $\mathbb{G} = \langle \mathbb{T}, \sigma \rangle$, let $V = \mathbb{P}^n/\boldsymbol{\mu}_{n+1}$ such that $\boldsymbol{\mu}_{n+1}$ acts on $\mathbb{P}^n$ as

$$[x_1 : \ldots : x_n : x_{n+1}] \mapsto [\zeta x_1 : \ldots : \zeta^n x_n : x_{n+1}], \tag{4.5}$$

where ζ is a primitive $(n+1)$-th root of unity. Then V is a toric Fano variety.

Lemma 4.12. *The variety V has terminal singularities.*

Proof. By construction, V has $n+1$ cyclic quotient singularities of type

$$\frac{1}{n+1}(1, \ldots, n).$$

Hence, according to Reid–Tai criterion [46, Theorem 4.11], the variety V is terminal if and only if for each $r \in \{1, 2, \ldots, n\}$, one has

$$\sum_{i=1}^{n} \overline{ri} > n+1, \tag{4.6}$$

where $\overline{a}$ denotes the integer such that $0 \leqslant \overline{a} \leqslant n$ and $a \equiv \overline{a} \mod n+1$. Since $n+1$ is a prime number by Lemma 4.11, we have

$$\{\overline{r}, \overline{2r}, \ldots, \overline{nr}\} = \{1, 2, \ldots, n\}.$$

Thus, we compute

$$\sum_{i=1}^{n} \overline{ri} = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} > n+1.$$

Therefore, inequality (4.6) holds, so that V has terminal singularities. $\square$

Note that the quotient map $\mathbb{P}^n \to V$ is $\mathbb{G}$-equivariant, and the action of the group $\mathbb{G}$ on the toric variety V gives a group homomorphism $\mathbb{G} \to \mathrm{Aut}(V)$, whose kernel is isomorphic to $\boldsymbol{\mu}_{n+1}$. Let $\mathbb{G}_V$ be its image. Then

$$\mathbb{G}_V = \mathbb{T}_V \rtimes G_V,$$

where $\mathbb{T}_V$ is the maximal torus in $\mathrm{Aut}(V)$, and G_V is the image of the group G. We have $G_V \simeq G$. Recall that we have a natural monomorphism $G \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$. Similarly, the $\mathbb{G}_V$-action on V gives a monomorphism

$$G_V \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{T}_V).$$

By Proposition 3.4, the following are equivalent:

- there exists a $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow V$, where the $\mathbb{G}$-action on V is given by $\mathbb{G}_V$;
- the images of the groups G and G_V in $\mathrm{GL}_n(\mathbb{Z})$ are conjugate.

Lemma 4.13. *If $\mathbb{G} = \langle \mathbb{T}, \sigma \rangle$, then the images of the groups G and G_V in $\mathrm{GL}_n(\mathbb{Z})$ are conjugate.*

Proof. Recall that $\mathbb{P}^n$ is a toric variety whose fan in the standard lattice $\mathbb{Z}^n$ has $n+1$ one-dimensional rays with primitive vectors

$$\begin{aligned} v_1 &= (1, 0, \ldots, 0, 0), \\ &\ldots \\ v_n &= (0, 0, \ldots, 0, 1), \\ v_{n+1} &= (-1, -1, \ldots, -1, -1). \end{aligned}$$

Then the action of σ on $\mathbb{P}^n$ corresponds to the action of the linear operator

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

on the lattice $\mathbb{Z}^n$. In other words, A acts as

$$v_1 \mapsto v_2 \mapsto \ldots \mapsto v_n \mapsto v_{n+1} \mapsto v_1.$$

On the other hand, V is a toric variety whose fan has $n+1$ one-dimensional rays with primitive vectors $v_1, \ldots, v_{n+1}$ in the lattice Λ generated by the standard lattice $\mathbb{Z}^n$ and the vector

$$v = \frac{1}{n+1}(1, 2, \ldots, n) = \frac{1}{n+1}(v_1 + 2v_2 + \ldots + nv_n).$$

The vectors $v, v_2, \ldots, v_n$ form a basis in the lattice $\Lambda \simeq \mathbb{Z}^n$; one has

$$v_1 = (n+1)v - 2v_2 - 3v_3 - \ldots - nv_n$$

so that

$$A(v) = -nv + v_2 + 2v_3 + \ldots + (n-1)v_n,$$

and

$$A(v_n) = v_{n+1} = -(n+1)v + v_2 + 2v_3 + \ldots + (n-1)v_n.$$

This means that the action of σ on V corresponds to the action of the linear operator

$$B = \begin{pmatrix} -n & 0 & 0 & \dots & 0 & -n-1 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 2 & 1 & 0 & \dots & 0 & 2 \\ 3 & 0 & 1 & \dots & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & 0 & 0 & \dots & 1 & n-1 \end{pmatrix}$$

on the lattice Λ. Consider the matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}).$$

It is straightforward to check that

$$CA = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & -n \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 2 \\ -1 & -1 & 0 & \dots & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 0 & n-1 \end{pmatrix} = BC.$$

Thus, if $\mathbb{G} = \langle \mathbb{T}, \sigma \rangle$, then the images of the subgroups G and G_V in $\mathrm{GL}_n(\mathbb{Z})$ are conjugate. $\square$

Corollary 4.14. *Suppose that $\mathbb{G} = \langle \mathbb{T}, \sigma \rangle$. Then there exists a $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow V$, where the $\mathbb{G}$-action on V is given by $\mathbb{G}_V$.*

Proof. Follows from Lemma 4.13 and Proposition 3.4. $\square$

Recall that V has terminal singularities, $\operatorname{rk}\operatorname{Cl}(V)=1$, and $V\not\simeq\mathbb{P}^n$, because V is singular. Hence in the case when $\mathbb{G}=\langle\mathbb{T},\sigma\rangle$ we see that Theorem 3.6 follows from Corollary 4.14. Therefore, to complete the proof of Theorem 3.6, we may assume that $\mathbb{G}=\langle\mathbb{T},\sigma,\iota\rangle$, and thus $G\simeq\mathfrak{D}_{n+1}$.

Remark 4.15. Note that $\mathbb{P}^n$ and the constructed toric Fano variety V are not always $\mathbb{G}$-birational in the case when $\mathbb{G}=\langle\mathbb{T},\sigma,\iota\rangle$. Indeed, in the notation of the proof of Lemma 4.13, the action of the involution ι on the projective space $\mathbb{P}^n$ corresponds to the action of the linear operator

$$S=\begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0\\ -1 & 0 & 0 & \ldots & 0 & 1\\ -1 & 0 & 0 & \ldots & 1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & 1 & \ldots & 0 & 0\\ -1 & 1 & 0 & \ldots & 0 & 0\end{pmatrix}$$

on the lattice $\mathbb{Z}^n$, and the action of ι on V corresponds to the action of the linear operator

$$T=\begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0\\ 1 & 0 & 0 & \ldots & 0 & 1\\ 1 & 0 & 0 & \ldots & 1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 0 & 1 & \ldots & 0 & 0\\ 1 & 1 & 0 & \ldots & 0 & 0\end{pmatrix}$$

on the lattice Λ. Using this, one can show that the images of the groups G and G_V in $\mathrm{GL}_n(\mathbb{Z})$ are not always conjugate (it is highly likely that they are never conjugate). For instance, if $n=4$, this can be verified by the following Magma code:

```
Z := IntegerRing();
G := GL(4,Z);
A := G ! Matrix([[0,0,0,-1],[1,0,0,-1],[0,1,0,-1],[0,0,1,-1]]);
B := G ! Matrix([[-4,0,0,-5],[1,0,0,1],[2,1,0,2],[3,0,1,3]]);
S := G ! Matrix([[-1,0,0,0],[-1,0,0,1],[-1,0,1,0],[-1,1,0,0]]);
T := G ! Matrix([[-1,0,0,0],[1,0,0,1],[1,0,1,0],[1,1,0,0]]);
G1 := sub<G | {A,S}>;
G2 := sub<G | {B,T}>;
IsGLZConjugate(G1,G2);
```

Now, using Proposition 3.4, we see that there exists no $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow V$.

By Remark 4.6, the action of the standard Cremona involution $\tau_{\mathbb{P}^n}$ can be regularized on the permutohedron $\widetilde{X}$, i.e. the involution

$$\tau_{\widetilde{X}}=\pi^{-1}\circ\tau_{\mathbb{P}^n}\circ\pi$$

is biregular. Using this and the fact that π is $\mathbb{W}$-equivariant, one can show that

$$\operatorname{Aut}(\widetilde{X})\simeq\mathbb{T}\rtimes\big(\mathfrak{S}_{n+1}\times\boldsymbol{\mu}_2\big).$$

Now, applying equivariant Minimal Model Program to $\widetilde{X}$, we get an $\operatorname{Aut}(\widetilde{X})$-equivariant birational map $\nu\colon\widetilde{X}\dashrightarrow Y$ such that ν contracts toric divisors $\widetilde{E}_1,\ldots,\widetilde{E}_{n-2}$ (but ν does not contract $\widetilde{E}_0$), and Y is a smooth toric Fano variety, which is known as centrally symmetric n-dimensional toric del Pezzo variety [8, 27, 41], cf. also [2, §4.4]. The toric Fano variety Y can be obtained from X_1 by antiflipping the strict transforms of the torus invariant subspaces in $\mathbb{P}^n$ of dimension $1,\ldots,\frac{n}{2}-1$

in this order (recall that $n+1 \geqslant 5$ is prime by Lemma 4.11, so that in particular n is even). To be precise, we have the following $\mathbb{W}$-equivariant commutative diagram

$$\begin{array}{ccccccc} X_1 & \dashrightarrow & Y & \xrightarrow{\tau_Y} & Y & \dashleftarrow & X_1 \\ \downarrow{\scriptstyle \pi_0} & & & & & & \downarrow{\scriptstyle \pi_0} \\ \mathbb{P}^n & & & \overset{\tau_{\mathbb{P}^n}}{\dashrightarrow} & & & \mathbb{P}^n \end{array}$$

where $X_1 \dashrightarrow Y$ is the small birational map described above, and τ_Y is a biregular involution. Thus, we may identify $\mathbb{W}$ with a subgroup in $\mathrm{Aut}(Y)$. Then

$$\mathrm{Aut}(Y) = \langle \mathbb{W}, \tau_Y \rangle \simeq \mathbb{T} \rtimes \big(\mathfrak{S}_{n+1} \times \boldsymbol{\mu}_2\big).$$

By construction, the torus invariant divisors in Y are strict transforms on Y of the irreducible components of the divisors $\widetilde{E}_0$ and $\widetilde{E}_{n-1}$ described earlier, which form one $\mathrm{Aut}(Y)$-irreducible divisor. In particular, Y is a toric $\mathrm{Aut}(Y)\mathbb{Q}$-Fano variety. Note that

$$\langle \mathbb{G}, \tau_Y \rangle = \mathbb{T} \rtimes \langle G, \tau_Y \rangle \simeq \mathbb{T} \rtimes \big(\mathfrak{D}_{n+1} \times \boldsymbol{\mu}_2\big) \simeq \mathbb{T} \rtimes \mathfrak{D}_{2n+2},$$

and $\langle G, \tau_Y \rangle \simeq \mathfrak{D}_{n+1} \times \boldsymbol{\mu}_2 \simeq \mathfrak{D}_{2n+2}$.

Remark 4.16. In the standard lattice $\mathbb{Z}^n$, the fan of the toric variety Y has $2n+2$ one-dimensional rays with primitive vectors

$$\begin{aligned} v_1 &= (1, 0, \ldots, 0, 0), \\ u_1 &= (-1, 0, \ldots, 0, 0), \\ &\ldots \\ v_n &= (0, 0, \ldots, 0, 1), \\ u_n &= (0, 0, \ldots, 0, -1), \\ v_{n+1} &= (-1, -1, \ldots, -1, -1), \\ u_{n+1} &= (1, 1, \ldots, 1, 1). \end{aligned}$$

If $n = 4$, then Y is the smooth toric Fano 4-fold №118 in [3].

The constructed birational map $\mathbb{P}^n \dashrightarrow Y$ is equivariant with respect to the action of the group $\boldsymbol{\mu}_{n+1}$ on $\mathbb{P}^n$ given by (4.5), which we used earlier to construct the toric variety $V = \mathbb{P}^n / \boldsymbol{\mu}_{n+1}$. Now, we let $U = Y / \boldsymbol{\mu}_{n+1}$.

Lemma 4.17. *The variety U has terminal singularities.*

Proof. Let $v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}$ be the vectors listed in Remark 4.16, and let

$$w = \frac{1}{n+1}(1, \ldots, n).$$

Let $\mathcal{N}$ be the lattice in $\mathbb{R}^n$ spanned by the vectors $v_1, \ldots, v_n$ and w. Thus, U is a $\mathbb{Q}$-factorial toric variety whose fan sits in the lattice $\mathcal{N}$, and the one-dimensional cones of this fan are generated by the vectors $v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}$. We have to show that U has terminal singularities. In fact, we are going to show a stronger assertion: if $\breve{U}$ is a toric variety defined by a simplicial fan $\mathcal{F}$ in the lattice $\mathcal{N}$, such that $\mathcal{F}$ has $2n+2$ one-dimensional cones generated by the vectors $v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}$, then $\breve{U}$ has terminal singularities.

Consider an n-dimensional cone of the fan $\mathcal{F}$, and let $w_1, \ldots, w_n$ be the generators of its edges; thus, $w_1, \ldots, w_n$ are n vectors among $v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}$. Note that the vectors v_i and $u_i = -v_i$ cannot appear simultaneously among $w_1, \ldots, w_n$. Next, observe that the lattice $\mathcal{N}$

and the collection of the vectors $v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}$ are invariant with respect to the action (4.5) of the group $\boldsymbol{\mu}_{n+1}$ (this group acts by cyclic permutation of the vectors $v_1, \dots, v_{n+1}$ and the vectors $u_1, \dots, u_{n+1}$). Thus, replacing $w_1, \dots, w_n$ by their images under a suitable element of $\boldsymbol{\mu}_{n+1}$, we may assume that none of the vectors v_{n+1} and u_{n+1} appears among $w_1, \dots, w_n$. Furthermore, replacing $w_1, \dots, w_n$ by $-w_1, \dots, -w_n$ if necessary, we may assume that at least one of the vectors $v_1, \dots, v_n$ is in $\{w_1, \dots, w_n\}$. We note also that one cannot have

$$\{w_1, \dots, w_n\} = \{v_1, \dots, v_n\},$$

because the cone generated by $v_1, \dots, v_n$ cannot be contained in any fan which has a one-dimensional cone generated by v_{n+1}. Therefore, it remains to consider the case when $w_1, \dots, w_n$ is a collection of n vectors among $v_1, \dots, v_n, u_1, \dots, u_n$ where v_i and u_i do not appear simultaneously. In other words, for some splitting of the set of indices

$$\{1, \dots, n\} = I_v \sqcup I_u$$

one has $w_i = v_i$ when $i \in I_v$ and $w_i = u_i$ when $i \in I_u$.

Observe that the interior of the cone generated by $w_1, \dots, w_n$ contains the vector

$$w' = w + \sum_{i \in I_u} u_i = \frac{1}{n+1}\left(\sum_{i \in I_v} i v_i + \sum_{i \in I_u} (n+1-i) u_i\right) \in \mathcal{N}.$$

Moreover, the vectors $w_1, \dots, w_n$ and w' generate the lattice $\mathcal{N}$. It follows that our cone describes an affine toric variety U° with a cyclic quotient singularity of type

$$\frac{1}{n+1}\left(\overline{\varsigma_1 1}, \dots, \overline{\varsigma_n n}\right),$$

where $\varsigma_i = 1$ for $i \in I_v$ and $\varsigma_i = -1$ for $i \in I_u$. As in the proof of Lemma 4.12, by $\overline{a}$ we denote the integer such that $0 \leqslant \overline{a} \leqslant n$ and $a \equiv \overline{a} \mod n+1$.

Now, according to Reid–Tai criterion [46, Theorem 4.11], we see that the variety U° is terminal if and only if for each $r \in \{1, 2, \dots, n\}$, one has

$$\sum_{i=1}^{n} \overline{\varsigma_i r i} > n+1. \tag{4.7}$$

Since $n+1 \geqslant 5$ is a prime number by Lemma 4.11, we have

$$\{\overline{r}, \overline{2r}, \dots, \overline{nr}\} = \{1, 2, \dots, n\}.$$

Furthermore, $n = 2k$ is an even number, and one has $k \geqslant 2$. Thus, we compute

$$\sum_{i=1}^{n} \overline{\varsigma_i r i} \leqslant \sum_{i=1}^{k} i + \sum_{i=k+1}^{2k} \overline{-i} = 2\sum_{i=1}^{k} i = 2 \cdot \frac{k(k+1)}{2} =$$
$$= (k^2 - 2k + 1) + (k-2) + (2k+1) = (k-1)^2 + (k-2) + (2k+1) > 2k+1 = n+1.$$

Therefore, inequality (4.7) holds, so U° (and hence also $\breve{U}$, and in particular our initial variety U) has terminal singularities. □

The quotient map $Y \to U$ is $\langle \mathbb{G}, \tau_Y \rangle$-equivariant, and the action of the group $\langle \mathbb{G}, \tau_Y \rangle$ on U gives a group homomorphism $\langle \mathbb{G}, \tau_Y \rangle \to \mathrm{Aut}(U)$, whose kernel is $\boldsymbol{\mu}_{n+1}$. Let $\mathbb{G}_U$ be its image. Then

$$\mathbb{G}_U = \mathbb{T}_U \rtimes \langle G_U, \tau_U \rangle,$$

where $\mathbb{T}_U$ is the maximal torus in $\mathrm{Aut}(U)$, the group G_U is the image of the group G, and τ_U is the image of the involution τ_Y. We have

$$G_U \simeq \mathfrak{D}_{n+1},$$
$$\langle G_U, \tau_U\rangle \simeq \mathfrak{D}_{n+1} \times \boldsymbol{\mu}_2 \simeq \mathfrak{D}_{2n+2}.$$

By construction, U is a toric $\langle \mathbb{G}_U, \tau_U\rangle\mathbb{Q}$-Fano variety, i.e., we have

$$\mathrm{rk}\,\mathrm{Cl}(U)^{\langle \mathbb{G}_U, \tau_U\rangle} = \mathrm{rk}\,\mathrm{Cl}(U)^{\langle G_U, \tau_U\rangle} = 1.$$

However, U is not a $\mathbb{G}_U\mathbb{Q}$-Fano variety, since $\mathrm{rk}\,\mathrm{Cl}(U)^{\mathbb{G}_U} = \mathrm{rk}\,\mathrm{Cl}(U)^{G_U} = 2$.

Remark 4.18. Recall that $V = \mathbb{P}^n/\boldsymbol{\mu}_{n+1}$. Thus, there exists a $\mathbb{G}_U$-birational map $U \dashrightarrow V$, where the $\mathbb{G}_U$-action on V is given by $\mathbb{G}_V \simeq \mathbb{G}_U$.

Note that the group $\langle G_U, \tau_U\rangle \simeq \mathfrak{D}_{2n+2}$ contains two (normal) subgroups isomorphic to $\mathfrak{D}_{n+1}$. One of them is our group G_U. Let G'_U be the other one. Set $\mathbb{G}'_U = \langle \mathbb{T}_U, G'_U\rangle$. Then

$$\mathrm{rk}\,\mathrm{Cl}(U)^{\mathbb{G}'_U} = \mathrm{rk}\,\mathrm{Cl}(U)^{G'_U} = 1,$$

because the group G'_U acts transitively on irreducible torus invariant divisors in U.

Lemma 4.19. *There exists a $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow U$, where the $\mathbb{G}$-action on U is given by the action of the group $\mathbb{G}'_U \simeq \mathbb{G}$.*

Proof. The action of the group $\mathbb{G}$ on $\mathbb{P}^n$, gives a natural monomorphism $G \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{T})$. Similarly, the action of the group $\langle \mathbb{G}_U, \tau_U\rangle$ on U gives a monomorphism $\langle G_U, \tau_U\rangle \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ such that the image of the involution τ_U is the scalar matrix $-I_n$. It follows from Proposition 3.4 that the following two conditions are equivalent:

- there exists a $\mathbb{G}$-birational map $\mathbb{P}^n \dashrightarrow U$, where $\mathbb{G}$-action on U is given by $\mathbb{G}'_U$;
- the images of the groups G and G'_U in $\mathrm{GL}_n(\mathbb{Z})$ are conjugate.

On the other hand, arguing as in the proof of Lemma 4.13 and using Remark 4.15, we see that the image of the group G in $\mathrm{GL}_n(\mathbb{Z})$ is generated by the following two $n \times n$ matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 1 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, the image of the group G'_U in $\mathrm{GL}_n(\mathbb{Z})$ is generated by the following two $n\times n$ matrices:

$$B = \begin{pmatrix} -n & 0 & 0 & \dots & 0 & 0 & -n-1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 2 & 1 & 0 & \dots & 0 & 0 & 2 \\ 3 & 0 & 1 & \dots & 0 & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n-2 & 0 & 0 & \dots & 1 & 0 & n-2 \\ n-1 & 0 & 0 & \dots & 0 & 1 & n-1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 0 \\ -1 & 0 & 0 & \dots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & -1 & \dots & 0 & 0 & 0 \\ -1 & -1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Consider the following $n \times n$ matrix:

$$C = \begin{pmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 & \dots & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & \dots & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{pmatrix}.$$

Then $C \in \mathrm{GL}_n(\mathbb{Z})$. Moreover, we have $C^{-1}AC = B$ and $C^{-1}SC = T$. □

Since $U \not\simeq \mathbb{P}^n$, an application of Lemma 4.19 completes the proof of Theorem 3.6.

5. Four-dimensional projective space

The goal of this section is to discuss primitive subgroups of $\mathrm{PGL}_5(\mathbb{C})$ and prove Theorem C.

Two finite primitive subgroups of the group $\mathrm{PGL}_5(\mathbb{C})$ are conjugate if and only if they are isomorphic. Moreover, up to conjugation, the group $\mathrm{PGL}_5(\mathbb{C})$ contains 11 primitive finite subgroups, see [7, 28]. These subgroups can be described as follows.

Up to conjugation, $\mathrm{PGL}_5(\mathbb{C})$ contains a unique simple subgroup isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$, which can be described as the automorphism group of the quartic 3-fold in $\mathbb{P}^4$ given by

$$x_1\big(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3\big) + 3x_2x_3x_4x_5 = 0, \tag{5.1}$$

and $\mathrm{PGL}_5(\mathbb{C})$ contains a unique simple subgroup isomorphic to $\mathrm{PSL}_2(\mathbf{F}_{11})$, which can be described as the automorphism group of the cubic 3-fold in $\mathbb{P}^4$ given by

$$x_1x_2^2 + x_2x_3^2 + x_3x_4^4 + x_4x_5^2 + x_5x_1^2 = 0. \tag{5.2}$$

These two simple subgroups of the group $\mathrm{PGL}_5(\mathbb{C})$ are primitive. Note that the quartic (5.1) is known as the Burkhardt quartic 3-fold, and the cubic (5.2) is known as the Klein cubic 3-fold.

Recall that $\mathfrak{S}_6$ faithfully acts on $\mathbb{P}^4$ via its standard five-dimensional irreducible representation. This gives a primitive subgroup in $\mathrm{PGL}_5(\mathbb{C})$, which is isomorphic to $\mathfrak{S}_6$. This subgroup contains a unique subgroup isomorphic to $\mathfrak{A}_6$, which is also primitive. Up to conjugation, it also contains two subgroups isomorphic to $\mathfrak{S}_5$, which are called standard and non-standard — the non-standard subgroup is primitive, and the standard subgroup is intransitive.

Lemma 5.1 (cf. [19, Corollary 2.5(ii)] and [11])**.** *Suppose that $G \subset \mathrm{PGL}_5(\mathbb{C})$ is a primitive subgroup isomorphic to $\mathfrak{S}_6$, $\mathfrak{A}_6$, or $\mathfrak{S}_5$. Then the linear system $|\mathcal{O}_{\mathbb{P}^4}(4)|$ contains a unique G-invariant pencil.*

Proof. This follows from standard results on symmetric functions (or from a direct computation of characters of the fourth symmetric power of the corresponding representations). □

To describe the remaining 6 primitive finite subgroups of the group $\mathrm{PGL}_5(\mathbb{C})$, let $\mathbb{H}_5$ be the subgroup in $\mathrm{PGL}_5(\mathbb{C})$ generated by the following projective transformations:

$$[x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_2 : x_3 : x_4 : x_5, x_1],$$
$$[x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [e^{\frac{2\pi\sqrt{-1}}{5}}x_1 : e^{\frac{4\pi\sqrt{-1}}{5}}x_2 : e^{\frac{6\pi\sqrt{-1}}{5}}x_3 : e^{\frac{8\pi\sqrt{-1}}{5}}x_4 : x_5],$$

and let $\mathbb{N}_5$ be the normalizer of the subgroup $\mathbb{H}_5$ in $\mathrm{PGL}_5(\mathbb{C})$. Then $\mathbb{H}_5$ is transitive and imprimitive, one has $\mathbb{H}_5 \simeq \boldsymbol{\mu}_5^2$, and it follows from [30] that

$$\mathbb{N}_5/\mathbb{H}_5 \simeq \mathrm{SL}_2(\mathbf{F}_5).$$

Moreover, we have $\mathbb{N}_5 \simeq \mathbb{H}_5 \rtimes \mathrm{SL}_2(\mathbf{F}_5)$, the subgroup $\mathbb{N}_5$ is primitive [7], and it contains 5 more primitive subgroups G_1, G_2, G_3, G_4, G_5 such that all of them contain $\mathbb{H}_5$, and

$$
\begin{aligned}
G_1/\mathbb{H}_5 &\simeq \boldsymbol{\mu}_3, \\
G_2/\mathbb{H}_5 &\simeq \boldsymbol{\mu}_6, \\
G_3/\mathbb{H}_5 &\simeq \mathfrak{Q}_8, \\
G_4/\mathbb{H}_5 &\simeq \boldsymbol{\mu}_3 \rtimes \boldsymbol{\mu}_4, \\
G_5/\mathbb{H}_5 &\simeq \mathrm{SL}_2(\mathbf{F}_3),
\end{aligned}
$$

where $\mathfrak{Q}_8$ denotes the quaternion group of order 8. Explicit generators of $\mathbb{N}_5$ and its five primitive subgroups G_1, G_2, G_3, G_4, G_5 can be found in [33].

The described 11 primitive subgroups in $\mathrm{PGL}_5(\mathbb{C})$ are all primitive finite subgroups in $\mathrm{PGL}_5(\mathbb{C})$. Note that the original classification in [7] also listed a primitive subgroup isomorphic to $\mathfrak{A}_5$, but later it has been determined that this subgroup is transitive and imprimitive.

Proposition 5.2. *If $\mathbb{P}^4$ is G-birationally rigid for some finite subgroup $G \subset \mathrm{PGL}_5(\mathbb{C})$, then G is conjugate to one of the following subgroups:* $\mathrm{PSp}_4(\mathbf{F}_3)$, $\mathrm{PSL}_2(\mathbf{F}_{11})$, $\mathbb{N}_5$, G_1, G_2, G_3, G_4, G_5.

Proof. We know from Theorem A that G is primitive. Thus, it is conjugate to one of the 11 subgroups described above. On the other hand, if G is a primitive subgroup of $\mathrm{PGL}_5(\mathbb{C})$ isomorphic to $\mathfrak{S}_6$, $\mathfrak{A}_6$, or $\mathfrak{S}_5$, then $\mathbb{P}^4$ is not G-birationally rigid by Lemma 2.4, because $\mathbb{P}^4$ contains a G-invariant pencil of quartic hypersurfaces, see Lemma 5.1. ◻

Remark 5.3. If $G \subset \mathrm{PGL}_5(\mathbb{C})$ is one of the groups $\mathrm{PSp}_4(\mathbf{F}_3)$, $\mathrm{PSL}_2(\mathbf{F}_{11})$, $\mathbb{N}_5$, G_1, G_2, G_3, G_4, G_5, then there are no G-invariant quadrics in $\mathbb{P}^4$. Indeed, let $\widehat{\mathbb{H}}_5$ denote the preimage of the group $\mathbb{H}_5$ in $\mathrm{SL}_5(\mathbb{C})$. Then $\widehat{\mathbb{H}}_5$ is a transitive group of order 125 with center $Z(\widehat{\mathbb{H}}_5) \simeq \boldsymbol{\mu}_5$. Moreover, every irreducible representation of $\widehat{\mathbb{H}}_5$ where $Z(\widehat{\mathbb{H}}_5)$ acts non-trivially has dimension 5. Hence there are no one-dimensional subrepresentations in the space of quadratic polynomials on $\mathbb{C}^5$. This implies that there are no quadric hypersurfaces in $\mathbb{P}^4$ which are invariant with respect to $\mathbb{H}_5$ or with respect to any group containing $\mathbb{H}_5$. As for the groups $\mathrm{PSp}_4(\mathbf{F}_3)$ and $\mathrm{PSL}_2(\mathbf{F}_{11})$, the absence of invariant quadrics can be deduced either from a direct computation of the character of the second symmetric power of the corresponding representation, or from the fact that these groups cannot be embedded into the birational automorphism group of $\mathbb{P}^3$, see [44].

In the remaining part of this section, we prove Theorem C. Let X_4 be the quartic 3-fold in $\mathbb{P}^4$ that is given by equation (5.1). Then

$$\mathrm{Aut}(X_4) \simeq \mathrm{PSp}_4\big(\mathbf{F}_3\big).$$

Set $G = \mathrm{Aut}(X_4)$. We have to show that $\mathbb{P}^4$ is G-birationally superrigid. Before doing this, let us present some facts about the action of the group G on X_4 and $\mathbb{P}^4$.

First, we recall that the singular locus $\mathrm{Sing}(X_4)$ consists of 45 isolated ordinary double points (nodes), which form a G-orbit. Recall also that

- a line in $\mathbb{P}^4$ can contain 1, 2, or 3 nodes,
- a plane in $\mathbb{P}^4$ can contain 1, 2, 3, 4, 6, or 9 nodes,
- a hyperplane in $\mathbb{P}^4$ can contain 1, 2, 3, 4, 7, 10, 12, or 18 nodes.

The planes in $\mathbb{P}^4$ containing 9 nodes of X_4 are called Jacobi planes. These planes are contained in X_4; conversely, every plane contained in X_4 is a Jacobi plane. In total, the quartic X_4 contains 40 Jacobi planes, and there are exactly 8 Jacobi planes in X_4 that pass through a given node. The union of all Jacobi planes in X_4 is a divisor in $|-10K_{X_4}|$, which we denote by $\mathbf{J}$. Two distinct Jacobi planes either intersect by a point or by a line. If they intersect by a line, then this line contains three nodes of X_4, and there are no more Jacobi planes passing through this line. Furthermore, any line containing three nodes of X_4 is an intersection of two Jacobi planes. The number of such lines is 240, and they form one G-irreducible curve. We refer the reader to [1] and [32] for more details on configurations of nodes, lines, and planes on X_4.

The following facts are well known to experts and are easy to check.

Lemma 5.4 (see e.g. [23, p. 26])**.** *The indices of proper maximal subgroups of G equal 27, 36, 40, and 45. The subgroups of indices 27 and 36 are isomorphic to $\boldsymbol{\mu}_2^4 \rtimes \mathfrak{A}_5$ and $\mathfrak{S}_6$, respectively.*

Lemma 5.5. *There are no G-orbits of length less than 40 in $\mathbb{P}^4$.*

Proof. From Lemma 5.4, we see that a stabilizer of a point in an orbit of length less than 40 must be isomorphic to $\boldsymbol{\mu}_2^4 \rtimes \mathfrak{A}_5$ and $\mathfrak{S}_6$. However, the restrictions of the corresponding representation to these subgroups are irreducible. ☐

Lemma 5.6. *Then there are no G-invariant hypersurfaces of degree of degree 1, 2, 3, and 5 in $\mathbb{P}^4$. There exists a unique G-invariant hypersurface of degree 4, and a unique G-invariant hypersurface of degree 6 in $\mathbb{P}^4$.*

Proof. A direct computation of characters of the symmetric powers of the corresponding representation. ☐

The unique G-invariant hypersurface of degree 4 is our Burkhardt quartic X_4. Let us denote by X_6 the unique G-invariant hypersurface of degree 6. The next fact is probably well-known, but we could not find an appropriate reference.

Lemma 5.7. *Let S be a G-invariant surface in $\mathbb{P}^4$ such that $\deg(S) \leqslant 24$. Then $\deg(S) = 24$, the surface S is irreducible, and $S = X_4 \cap X_6$.*

Proof. One has $\mathrm{Cl}^G(X_4) = \mathbb{Z}[-K_{X_4}]$, see e.g. [12, §2] or [21, Theorem 2(1)]. Thus, if $S \subset X_4$, then $d = 4k$ and S is cut out on X_4 by a G-invariant hypersurface of degree k. On the other hand, by Lemma 5.6 there are no G-invariant hypersurfaces of degree 1, 2, 3, and 5 in $\mathbb{P}^4$, and the only G-invariant hypersurface of degree 4 in $\mathbb{P}^4$ is the quartic X_4. Hence, if $S \subset X_4$ and $\deg(S) \leqslant 24$, then $\deg(S) = 24$ and $S = X_4 \cap X_6$. In particular, we see that S is G-irreducible, because otherwise there would exist a G-invariant surface of smaller degree in X_4. Furthermore, by Lemma 5.4 there are no subgroups of index at most 24 in G, and thus there are no non-trivial homomorphisms from G to $\mathfrak{S}_m$ with $m \leqslant 24$. Therefore, we see that S is actually irreducible.

To complete the proof, we may assume that $S \not\subset X_4$. Moreover, we can assume that the surface S is G-irreducible. Let us seek for a contradiction. Set $Z = S \cdot X_4$. Then Z is a possibly G-reducible and possibly non-reduced G-invariant curve of degree at most 96. Let Z_1 be one of its irreducible components, and let C be a G-irreducible curve such that Z_1 is an irreducible component of C. Note that C is not contained in a hyperplane in $\mathbb{P}^4$, since otherwise the linear span of C would be a G-invariant linear subspace of $\mathbb{P}^4$. Hence, if C is irreducible, then G acts faithfully on C.

We claim that $C \not\subset \mathbf{J}$. Indeed, recall that two distinct Jacobi planes either intersect by a point or by a line, and the lines contained in at least two Jacobi planes form one G-irreducible curve of degree 240. Thus, if $C \subset \mathbf{J}$, then each irreducible component of C is contained in exactly one Jacobi plane. Now, we let Π be a Jacobi plane, and let Γ_{648} be its stabilizer in G. Then

$$\Gamma_{648} \simeq \mathrm{SU}_3(\mathbf{F}_2).\boldsymbol{\mu}_3,$$

its GAP ID is [648,533], and its action on $\Pi \simeq \mathbb{P}^2$ gives a homomorphism

$$\Gamma_{648} \to \mathrm{Aut}(\Pi) \simeq \mathrm{PGL}_3(\mathbb{C}).$$

The image of this homomorphism is the so-called the Hessian group of order 216 isomorphic to $\mathrm{PSU}_3(\mathbf{F}_2).\boldsymbol{\mu}_3$, its GAP ID is [216,153], and the kernel is isomorphic to $\boldsymbol{\mu}_3$. It is straightforward to see that this group does not act on rational and elliptic curve, and does not act on a union of three lines in $\mathbb{P}^2$. Thus, the plane Π does not contain Γ_{648}-invariant curves of degree at most 3. This shows that $C \not\subset \mathbf{J}$, because the degree of C is at most 96.

We claim that $\mathrm{Sing}(X_4) \not\subset C$. Indeed, suppose that $\mathrm{Sing}(X_4) \subset C$. Let $f\colon \widetilde{X}_4 \to X_4$ be the blow up of all singular points of X_4, let $\widetilde{\mathbf{J}}$ be the strict transform on $\widetilde{X}_4$ of the divisor $\mathbf{J}$, let E be the union of all f-exceptional prime divisors, and let $\widetilde{C}$ be the strict transform on $\widetilde{X}_4$ of the curve C. Then f is G-equivariant and

$$0 \leqslant \widetilde{\mathbf{J}} \cdot \widetilde{C} = \Big(f^*\big(-10K_{X_4}\big) - 4E\Big) \cdot \widetilde{C} = 10\mathrm{deg}(C) - 4E \cdot \widetilde{C} \leqslant 960 - 4E \cdot \widetilde{C} \leqslant 960 - 4|E \cap \widetilde{C}|,$$

which gives $|E \cap \widetilde{C}| \leqslant 240$. Let E_1 be an irreducible component of the divisor E, and let Γ_{576} be the stabilizer in G of the point $f(E_1) \in \mathrm{Sing}(X_4)$. Then

$$\Gamma_{576} \simeq \boldsymbol{\mu}_2.\mathfrak{A}_4 \wr \boldsymbol{\mu}_2,$$

the GAP ID of Γ_{576} is [576,8277], and the kernel of the Γ_{576}-action on E_1 is the center of the group Γ_{576}, which is isomorphic to $\boldsymbol{\mu}_2$. Hence, the image in $\mathrm{Aut}(E_1)$ of the group Γ_{576} is isomorphic to $\mathfrak{A}_4 \wr \boldsymbol{\mu}_2$, and its GAP ID is [288,1025]. The latter group contains a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$, which acts on $E_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ via the product action. This implies that E_1 does not contain Γ_{576}-orbits of length at most 15, because the minimal length of a $\mathfrak{A}_4$-orbit in $\mathbb{P}^1$ is 4. On the other hand, it follows from the inequality $|E \cap \widetilde{C}| \leqslant 240$ that $|E_1 \cap \widetilde{C}| \leqslant 5$, which is a contradiction. Hence, we see that $\mathrm{Sing}(X_4) \not\subset C$. Since the action of G on $\mathrm{Sing}(X_4)$ is transitive, this means that every point of the intersection $\mathbf{J} \cap C$ is a smooth point of the quartic X_4.

Now, let us revisit our Jacobi plane Π, and its stabilizer Γ_{648}. Note that $\mathrm{Sing}(X_4) \cap \Pi$ is a Γ_{648}-orbit of length 9, this is the only Γ_{648}-orbit in Π of length less than 24, the plane Π contains a Γ_{648}-orbit of length 24, and the stabilizer of a point in this orbit is not cyclic. Thus, since $\mathrm{Sing}(X_4) \cap C = \varnothing$, we have

$$\Pi \cdot C \geqslant |C \cap \Pi| \geqslant 24,$$

which gives

$$960 \geqslant 10\mathrm{deg}(Z) \geqslant 10\mathrm{deg}(C) = -10K_{X_4} \cdot C = \mathbf{J} \cdot C \geqslant 40 \cdot 24 = 960.$$

This implies that $C = Z$, the equality $\mathrm{deg}(C) = 96$ holds, the intersection $C \cap \Pi$ form one Γ_{648}-orbit of length 24, and the curve C is smooth at every point of the intersection $C \cap \Pi$. If C is irreducible, this immediately leads to a contradiction, because the stabilizer in $\mathrm{Aut}(C)$ of a smooth point is cyclic. Hence, we conclude that C is a reducible G-irreducible curve of degree 96. On the other hand, G does not contain proper subgroups whose index divides 96 by Lemma 5.4. The obtained contradiction completes the proof of the lemma. ☐

Now, we are ready to prove that $\mathbb{P}^4$ is G-birationally superrigid. Suppose it is not. Then it follows from [18, Corollary 3.3.3] that there exists a non-empty G-invariant mobile linear subsystem $\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^4}(n)|$, for some positive integer n, such that the singularities of the log pair $(\mathbb{P}^4, \frac{5}{n}\mathcal{M})$ are not canonical. If this log pair is not canonical at a general point of a G-irreducible surface S, then

$$\mathrm{mult}_S\big(\mathcal{M}\big) > \frac{n}{5},$$

which implies that $\mathrm{deg}(S) < 25$. So, we have $S = X_4 \cap X_6$ by Lemma 5.7. Therefore, the restriction $\mathcal{M}|_{X_4}$ is a linear subsystem of $|nH|$, where H is a hyperplane section of X_4, whose fixed part contains a divisor

$$\mu S \sim_{\mathbb{Q}} 6\mu H$$

with $\mu > \frac{n}{5}$, which gives a contradiction.

Hence, the singularities of the log pair $(\mathbb{P}^4, \frac{5}{n}\mathcal{M})$ are canonical away from a codimension 3 subset in $\mathbb{P}^4$. Then it follows from [14, Remark 3.6] that the singularities of the log pair $(\mathbb{P}^4, \frac{15}{2n}\mathcal{M})$ are not log canonical. If this log pair is not log canonical at a general point of a G-irreducible surface S, then it follows from [24, Theorem 3.1] that

$$\mathrm{mult}_S\big(M_1 \cdot M_2\big) > 4 \cdot \left(\frac{2n}{15}\right)^2$$

for two general members M_1 and M_2 of the linear system $\mathcal{M}$. This implies that

$$4 \cdot \left(\frac{2n}{15}\right)^2 \cdot \deg(S) < n^2,$$

and so

$$\deg(S) < \frac{225}{16} < 15,$$

which is impossible by Lemma 5.7. Thus, the singularities of the log pair $(\mathbb{P}^4, \frac{15}{2n}\mathcal{M})$ are log canonical away from a codimension 3 subset in $\mathbb{P}^4$.

Now, let $\lambda < \frac{15}{2n}$ be the log canonical threshold of the log pair $(\mathbb{P}^4, \mathcal{M})$, let Z_1 be a minimal center of log canonical singularities of the log pair $(\mathbb{P}^4, \lambda\mathcal{M})$, let Z be the G-irreducible subvariety in $\mathbb{P}^4$ such that Z_1 is its irreducible component, and let ϵ be a very small positive rational number such that $(1+\epsilon)\lambda < \frac{15}{2n}$. Then it follows from [18, Lemma 2.4.10] that there exists a mobile G-invariant linear system $\mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^4}(k)|$ for some $k \gg 0$ and two very small positive rational numbers ϵ_1 and ϵ_2 such that

$$(1-\epsilon_1)\lambda\mathcal{M} + \epsilon_2\mathcal{D} \sim_{\mathbb{Q}} (1+\epsilon)\lambda\mathcal{M},$$

the log pair $(\mathbb{P}^4, (1-\epsilon_1)\lambda\mathcal{M}+\epsilon_2\mathcal{D})$ is log canonical, and the irreducible components of Z are the only centers of log canonical singularities of this log pair. By [39, Proposition 1.5], the irreducible components of Z are disjoint, since Z is a union of minimal centers of log canonical singularities of the log pair $(\mathbb{P}^4, (1-\epsilon_1)\lambda\mathcal{M} + \epsilon_2\mathcal{D})$.

Observe that either Z_1 is a point and Z is a G-orbit, or Z_1 is a curve and Z is a G-irreducible curve. In the latter case, the curve Z_1 is smooth by [40, Theorem 1]. Moreover, in both cases, it follow from the Nadel's vanishing theorem [35, Theorem 9.4.8] that

$$h^1\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}_Z\big) = 0,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z on $\mathbb{P}^4$. Thus, we have the following exact sequence of G-representations:

$$0 \longrightarrow H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}_Z\big) \longrightarrow H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)\big) \longrightarrow H^0\big(\mathcal{O}_Z(\mathcal{O}_{\mathbb{P}^4}(3)\big|_Z)\big) \longrightarrow 0. \tag{5.3}$$

Thus, if Z is a G-orbit, then

$$|Z| \leqslant h^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)\big) = 35.$$

This is a contradiction, because $\mathbb{P}^4$ does not contain G-orbits of length less than 40 by Lemma 5.5.

Thus, we see that Z is a smooth curve. Let r be the number of irreducible components of the curve Z, let d be the degree of the curve Z_1, let g be the genus of the curve Z_1, let H be a general hyperplane in $\mathbb{P}^4$, and let ε be sufficiently small positive rational number. Then it follows from Kawamata's subadjunction theorem [40, Theorem 1] that

$$\big(K_{\mathbb{P}^4} + (1-\epsilon_1)\lambda\mathcal{M} + \epsilon_2\mathcal{D} + \varepsilon H\big)|_{Z_1} \sim_{\mathbb{Q}} K_{Z_1} + \Delta_{Z_1}$$

for some effective $\mathbb{Q}$-divisor Δ_{Z_1} on the curve Z_1. Now, comparing the degrees of these divisors on Z_1, we get

$$2g - 2 < \frac{5}{2}d. \tag{5.4}$$

In particular, the divisor $3H|_{Z_1}$ of degree $3d$ is not special, so we have

$$h^0\big(\mathcal{O}_Z(\mathcal{O}_{\mathbb{P}^4}(3)\big|_Z)\big) = rh^0\big(\mathcal{O}_{Z_1}(3H\big|_{Z_1})\big) = r(3d - g + 1),$$

so (5.3) gives $35 \geqslant r(3d - g + 1)$. Hence, using inequality (5.4), we get

$$35 > \frac{7rd}{4},$$

so that $rd < 20$. This gives $r = 1$, because G does not have proper subgroups of index less than 20 by Lemma 5.4. Therefore, again using (5.4), we obtain

$$35 \geqslant 3d - g + 1 > \frac{12(g-1)}{5} - g + 1 = \frac{7g-7}{5},$$

which implies that $g < 32$. On the other hand, since $Z = Z_1$ is irreducible, the group G acts faithfully on it, because Z is not contained in a hyperplane. Thus, using the Hurwitz bound

$$\mathrm{Aut}(Z) \leqslant 84(g-1),$$

we immediately get a contradiction in the case when $g \geqslant 2$. Finally, we observe that G cannot act faithfully on a rational or elliptic curve. The obtained contradiction completes the proof of Theorem C.

6. Five-dimensional projective space

The goal of this section is to discuss primitive subgroups $G \subset \mathrm{PGL}_6(\mathbb{C})$ and show that $\mathbb{P}^5$ is not G-birationally rigid for many of them. First, we recall the classification of these subgroups [38, §3]. Namely, let G be a finite primitive subgroup in $\mathrm{PGL}_6(\mathbb{C})$. Then either

(I) G leaves invariant a Segre cubic scroll $\mathbb{P}^1 \times \mathbb{P}^2 \simeq Y \subset \mathbb{P}^5$,

or there exists a finite primitive subgroup $\widehat{G} \subset \mathrm{SL}_6(\mathbb{C})$ such that $\widehat{G}$ is mapped to G via the natural projection $\mathrm{SL}_6(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$, and $\widehat{G}$ is isomorphic to one of the following groups:

(II) $\mathrm{SL}_2(\mathbf{F}_5)$,
(III) $2.\mathfrak{S}_5$,
(IV) (i) $3.\mathfrak{A}_6$,
(ii) $3.\mathfrak{A}_6 \rtimes \boldsymbol{\mu}_2$,
(V) $6.\mathfrak{A}_6$,
(VI) $\mathfrak{A}_7$ or $\mathfrak{S}_7$,
(VII) $3.\mathfrak{A}_7$,
(VIII) $6.\mathfrak{A}_7$,
(IX) (i) $\mathrm{PSL}_2(\mathbf{F}_7)$,
(ii) $\mathrm{PGL}_2(\mathbf{F}_7)$,
(X) (i) $\mathrm{SL}_2(\mathbf{F}_7)$,
(ii) $\mathrm{SL}_2(\mathbf{F}_7) \rtimes \boldsymbol{\mu}_2$,
(XI) $\mathrm{SL}_2(\mathbf{F}_{11})$,
(XII) $\mathrm{SL}_2(\mathbf{F}_{13})$,
(XIII) (i) $\mathrm{PSp}_4(\mathbf{F}_3)$,
(ii) $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
(XIV) (i) $\mathrm{SU}_3(\mathbf{F}_3)$,
(ii) $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
(XV) (i) $6.\mathrm{PSU}_4(\mathbf{F}_3)$,
(ii) $6.\mathrm{PSU}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
(XVI) 2.HaJ, where HaJ is the Hall–Janko simple group,
(XVII) (i) $6.\mathrm{PSL}_3(\mathbf{F}_4)$,

(ii) $6.\mathrm{PSL}_3(\mathbf{F}_4) \rtimes \boldsymbol{\mu}_2$.

In each of the cases (II)–(XVII), the subgroup $\widehat{G}$ is uniquely determined by its isomorphism class up to conjugation in $\mathrm{SL}_6(\mathbb{C})$; note however that in some cases $\widehat{G}$ has a more than one six-dimensional representation such that $\widehat{G}$ is a primitive subgroup of $\mathrm{GL}_6(\mathbb{C})$.

Remark 6.1. Suppose that $\mathbb{P}^5$ contains a G-invariant Segre cubic scroll $Y \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Then the linear system which consists of all quadrics in $\mathbb{P}^5$ passing through Y provides a rational map $\chi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$. This map fits into the following G-equivariant commutative diagram:

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^5 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Here π is the blow up of the scroll Y, and η is a $\mathbb{P}^3$-bundle. In particular, $\mathbb{P}^5$ is not G-birationally rigid.

Lemma 6.2. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{SL}_2(\mathbf{F}_5)$. Then there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a G-invariant pencil.*

Proof. The GAP ID of $\widehat{G}$ is [120,5], and $\widehat{G}$ has a unique irreducible 6-dimensional representation. A direct computation of characters of the symmetric powers shows that $|\mathcal{O}_{\mathbb{P}^5}(2)|$ does not contain G-invariant hypersurfaces, and $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a G-invariant pencil. For instance, we can use the following GAP script to do this:

```
G:=SmallGroup(120,5);
T:=CharacterTable(G);
Ir:=Irr(T);
U:=Ir[9];
S:=SymmetricParts(T,[U],2);
Print(MatScalarProducts(Ir,S));
S:=SymmetricParts(T,[U],3);
Print(MatScalarProducts(Ir,S));
```

□

Lemma 6.3. *Suppose that $\widehat{G}$ is isomorphic to $2.\mathfrak{S}_5$. Then there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(4)|$ contains a unique G-invariant pencil such that the hypersurfaces in this pencil are given by $\widehat{G}$-invariant quartic polynomials.*

Proof. The GAP ID of $\widehat{G}$ is [240,90], and the action of the group G on $\mathbb{P}^5$ is given by one of two (complex conjugate) faithful irreducible 6-dimensional representations of the group $\widehat{G}$. A direct computation of characters of the symmetric powers shows that $|\mathcal{O}_{\mathbb{P}^5}(2)|$ does not contain G-invariant hypersurfaces, and $|\mathcal{O}_{\mathbb{P}^5}(4)|$ contains a unique G-invariant pencil such that the hypersurfaces in this pencil are given by $\widehat{G}$-invariant quartic polynomials. □

Lemma 6.4. *Suppose that $\widehat{G}$ is isomorphic to $3.\mathfrak{A}_6$ or to $3.\mathfrak{A}_6 \rtimes \boldsymbol{\mu}_2$. Then there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a unique G-invariant pencil.*

Proof. The GAP ID of $3.\mathfrak{A}_6$ is [1080,260], and this group has exactly two irreducible 6-dimensional representations, which are complex conjugate. A direct computation of characters of the symmetric powers shows that $|\mathcal{O}_{\mathbb{P}^5}(2)|$ does not contain $\mathfrak{A}_6$-invariant hypersurfaces, and $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a unique $\mathfrak{A}_6$-invariant pencil. This proves the lemma in the case when $\widehat{G} \simeq 3.\mathfrak{A}_6$. If $\widehat{G} \simeq 3.\mathfrak{A}_6 \rtimes \boldsymbol{\mu}_2$, then the required assertions follow from the case $\widehat{G} \simeq 3.\mathfrak{A}_6$. □

Lemma 6.5. *Suppose that $\widehat{G}$ is isomorphic to $\mathfrak{A}_7$ or $\mathfrak{S}_7$. Then the linear system $|\mathcal{O}_{\mathbb{P}^5}(4)|$ contains a unique G-invariant pencil.*

Proof. Each of these groups has a unique irreducible 6-dimensional representation, which is a summand of the 7-dimensional permutation representation. Now the well-known results about symmetric functions imply that $|\mathcal{O}_{\mathbb{P}^5}(4)|$ contains a unique G-invariant pencil. $\square$

Lemma 6.6. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{PSL}_2(\mathbf{F}_7)$ or $\mathrm{PGL}_2(\mathbf{F}_7)$. Then the linear system $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a unique G-invariant pencil.*

Proof. The GAP ID of $\mathrm{PSL}_2(\mathbf{F}_7)$ is $[168, 42]$, and this group has a unique irreducible 6-dimensional representation. A direct computation of the character of the third symmetric power shows that $|\mathcal{O}_{\mathbb{P}^5}(3)|$ contains a unique $\mathrm{PSL}_2(\mathbf{F}_7)$-invariant pencil. If $\widehat{G} \simeq \mathrm{PGL}_2(\mathbf{F}_7)$, then the required assertion follows from the case $\widehat{G} \simeq \mathrm{PSL}_2(\mathbf{F}_7)$. $\square$

Corollary 6.7. *Suppose that $\widehat{G}$ is isomorphic to one of the groups $\mathrm{SL}_2(\mathbf{F}_5)$, $2.\mathfrak{S}_5$, $3.\mathfrak{A}_6$, $3.\mathfrak{A}_6 \rtimes \boldsymbol{\mu}_2$, $\mathfrak{A}_7$, $\mathfrak{S}_7$, $\mathrm{PSL}_2(\mathbf{F}_7)$, or $\mathrm{PGL}_2(\mathbf{F}_7)$. Then $\mathbb{P}^5$ is not G-birationally rigid.*

Proof. According to Lemmas 6.2, 6.3, 6.4, 6.5, and 6.6, the linear system $|\mathcal{O}_{\mathbb{P}^5}(n)|$ contains a G-invariant pencil for $n = 3$ or $n = 4$. Therefore, $\mathbb{P}^5$ is not G-birationally rigid by Lemma 2.4. $\square$

Lemma 6.8. *Suppose that $\widehat{G} \simeq 6.\mathfrak{A}_6$. Then there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(2)|$ contains a two-dimensional linear subsystem.*

Proof. The group $6.\mathfrak{A}_6$ can be described as the Schur cover of $\mathfrak{A}_6$, and it has four faithful irreducible 6-dimensional representations. The required assertions follow from a direct computation of the character of the second symmetric power. $\square$

Lemma 6.9. *Suppose that $\widehat{G} \simeq \mathrm{SL}_2(\mathbf{F}_7)$. Then there are no G-invariant hypersurfaces of degree 1, 2, 3, and 5 in $\mathbb{P}^5$, there exist a unique G-invariant hypersurface of degree 4, and the linear system $|\mathcal{O}_{\mathbb{P}^5}(2)|$ contains a two-dimensional linear subsystem.*

Proof. The GAP ID of $\mathrm{SL}_2(\mathbf{F}_7)$ is $[336, 114]$, and this group has two faithful irreducible 6-dimensional representations. The required assertions follow from a direct computation of the characters of the symmetric powers. $\square$

Corollary 6.10. *Suppose that $\widehat{G} \simeq 6.\mathfrak{A}_6$ or $\widehat{G} \simeq \mathrm{SL}_2(\mathbf{F}_7)$. Then $\mathbb{P}^5$ is not G-birationally rigid.*

Proof. By Lemmas 6.8 and 6.9, the linear system $|\mathcal{O}_{\mathbb{P}^5}(2)|$ contains a two-dimensional linear subsystem $\mathcal{M}$. Moreover, there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$, which means that the action of G on $\mathcal{M} \simeq \mathbb{P}^2$ is transitive. In particular, G acts on $\mathcal{M}$ faithfully, because G is a simple group. Furthermore, by the classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ the group G does not admit a faithful action on a rational curve. Therefore, $\mathbb{P}^5$ is not G-birationally rigid by Lemma 2.7. $\square$

Lemma 6.11. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$ or $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$. Then $\mathbb{P}^5$ contains a unique G-invariant quadric hypersurface Q and a unique G-invariant quintic hypersurface X. Moreover, the complete intersection $Q \cap X$ is reduced, irreducible, and its singularities consists of 330 isolated ordinary double points.*

Proof. The group $\mathrm{PSp}_4(\mathbf{F}_3)$ has a unique irreducible 6-dimensional representation. A direct computation of characters of the symmetric powers shows that $\mathbb{P}^5$ contains a unique $\mathrm{PSp}_4(\mathbf{F}_3)$-invariant quadric hypersurface Q and a unique G-invariant quintic hypersurface X. Using the the Magma code from Appendix B (kindly provided to us by Zhijia Zhang), we see that the complete intersection $Q \cap X$ is reduced, irreducible, and its singularities consists of 330 isolated ordinary double points. Since $\mathrm{PSp}_4(\mathbf{F}_3)$ is a normal subgroup in $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$, we conclude that Q and X are also G-invariant in the case when $\widehat{G} \simeq \mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$. □

Corollary 6.12. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$ or $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$. Then $\mathbb{P}^5$ is not G-birationally rigid.*

Proof. Apply Lemma 6.11 together with Lemma 2.5. □

Remark 6.13. If $\widehat{G}$ is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$ or $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$, then the linear system $|\mathcal{O}_{\mathbb{P}^5}(n)|$ does not contain G-invariant pencils for $n \leqslant 5$. Thus, one cannot deduce Corollary 6.12 using Lemma 2.4 instead of Lemma 2.5.

Summarizing, we obtain the following corollary.

Corollary 6.14. *Let G be a finite subgroup in $\mathrm{PGL}_6(\mathbb{C})$ such that $\mathbb{P}^5$ is G-birationally rigid. Then there exists a finite primitive subgroup $\widehat{G} \subset \mathrm{SL}_6(\mathbb{C})$ such that $\widehat{G}$ is mapped to G via the natural projection $\mathrm{SL}_6(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$, and $\widehat{G}$ is isomorphic to one of the following groups:*

- (VII) $3.\mathfrak{A}_7$,
- (VIII) $6.\mathfrak{A}_7$,
- (X) (ii) $\mathrm{SL}_2(\mathbf{F}_7) \rtimes \boldsymbol{\mu}_2$
- (XI) $\mathrm{SL}_2(\mathbf{F}_{11})$,
- (XII) $\mathrm{SL}_2(\mathbf{F}_{13})$,
- (XIV) (i) $\mathrm{SU}_3(\mathbf{F}_3)$,
  (ii) $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
- (XV) (i) $6.\mathrm{PSU}_4(\mathbf{F}_3)$,
  (ii) $6.\mathrm{PSU}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
- (XVI) $2.\mathrm{HaJ}$, *where* HaJ *is the Hall–Janko simple group,*
- (XVII) (i) $6.\mathrm{PSL}_3(\mathbf{F}_4)$,
  (ii) $6.\mathrm{PSL}_3(\mathbf{F}_4) \rtimes \boldsymbol{\mu}_2$.

Proof. By Theorem A, the subgroup G is primitive. Therefore, it follows from Remark 6.1 and Corollaries 6.7, 6.10, and 6.12 that we can choose the lift $\widehat{G}$ to be isomorphic to one of the groups in the required list. □

Unfortunately, we do not know whether $\mathbb{P}^5$ is G-birationally rigid or not in any of the 12 cases listed in Corollary 6.14 (cf. Question 1.8).

Remark 6.15 (cf. [15, Theorem 3.3]). If $\widehat{G}$ is a primitive subgroup in $\mathrm{SL}_6(\mathbb{C})$ that is isomorphic to one of the groups listed in Corollary 6.14, then direct computations show that there are no G-invariant quadric hypersurfaces in $\mathbb{P}^5$. Moreover, if $\widehat{G} \simeq 3.\mathfrak{A}_7$, then $|\mathcal{O}_{\mathbb{P}^5}(n)|$ does not contain G-invariant hypersurfaces for $n \in \{1, 2, 4, 5\}$ (but there exists a unique G-invariant cubic hypersurface in $\mathbb{P}^5$). If $\widehat{G} \simeq \mathrm{SL}_2(\mathbf{F}_{11})$ or $\widehat{G} \simeq \mathrm{SL}_2(\mathbf{F}_{13})$, then $|\mathcal{O}_{\mathbb{P}^5}(n)|$ does not contain G-invariant divisors for $n \in \{1, 2, 3, 5\}$ (but there exists a unique G-invariant quartic hypersurface in $\mathbb{P}^5$). If $G \simeq \mathrm{SL}_2(\mathbf{F}_7) \rtimes \boldsymbol{\mu}_2$, the same assertion follows from Lemma 6.9. If $\widehat{G}$ is isomorphic to one of the groups $6.\mathfrak{A}_7$, $\mathrm{SU}_3(\mathbf{F}_3)$, $6.\mathrm{PSU}_4(\mathbf{F}_3)$, $2.\mathrm{HaJ}$, or $6.\mathrm{PSL}_3(\mathbf{F}_4)$, then $|\mathcal{O}_{\mathbb{P}^5}(n)|$ does not contain G-invariant divisors for $n \leqslant 5$. Hence, if G is isomorphic to $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$, $6.\mathrm{PSU}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$, or $6.\mathrm{PSL}_3(\mathbf{F}_4) \rtimes \boldsymbol{\mu}_2$, then the linear system $|\mathcal{O}_{\mathbb{P}^5}(n)|$ also does not contain G-invariant divisors for $n \leqslant 5$.

7. Six-dimensional projective space

The goal of this section is to discuss primitive subgroups $G \subset \mathrm{PGL}_7(\mathbb{C})$ and show that $\mathbb{P}^6$ is not G-birationally rigid for many of them. First, we recall the classification of primitive finite subgroups of $\mathrm{PGL}_7(\mathbb{C})$ from [50, Theorem 4.1] and [51, Theorem I]. Let $\mathbb{H}_7$ be the subgroup in $\mathrm{PGL}_7(\mathbb{C})$ generated by the following projective transformations:

$$[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7] \mapsto [x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_1],$$

$$[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7] \mapsto$$

$$\mapsto [e^{\frac{2\pi\sqrt{-1}}{7}} x_1 : e^{\frac{4\pi\sqrt{-1}}{7}} x_2 : e^{\frac{6\pi\sqrt{-1}}{7}} x_3 : e^{\frac{8\pi\sqrt{-1}}{7}} x_4 : e^{\frac{10\pi\sqrt{-1}}{7}} x_5 : e^{\frac{12\pi\sqrt{-1}}{7}} x_6 : x_7],$$

and let $\mathbb{N}_7$ be the normalizer of $\mathbb{H}_7$ in $\mathrm{PGL}_7(\mathbb{C})$. Then $\mathbb{H}_7 \simeq \boldsymbol{\mu}_7^2$, the subgroup $\mathbb{H}_7$ is transitive and imprimitive, and

$$\mathbb{N}_7/\mathbb{H}_7 \simeq \mathrm{SL}_2(\mathbf{F}_7).$$

Moreover, we have $\mathbb{N}_7 \simeq \mathbb{H}_7 \rtimes \mathrm{SL}_2(\mathbf{F}_7)$, and the subgroup $\mathbb{N}_7$ is primitive.

Now, let G be a finite primitive subgroup in $\mathrm{PGL}_7(\mathbb{C})$. Then either

(I) G is conjugate to a subgroup in $\mathbb{N}_7$ that contains $\mathbb{H}_7$,

or there exists a finite primitive subgroup $\widehat{G} \subset \mathrm{SL}_7(\mathbb{C})$ such that $\widehat{G}$ is mapped to G via the natural projection $\mathrm{SL}_7(\mathbb{C}) \to \mathrm{PGL}_7(\mathbb{C})$, and $\widehat{G}$ is isomorphic to one of the following groups:

(II) $\mathrm{PSL}_2(\mathbf{F}_{13})$,

(III) (i) $\mathrm{SL}_2(\mathbf{F}_8)$,
(ii) $\mathrm{SL}_2(\mathbf{F}_8) \rtimes \boldsymbol{\mu}_3$,

(IV) $\mathfrak{A}_8$ or $\mathfrak{S}_8$,

(V) (i) $\mathrm{PSL}_2(\mathbf{F}_7)$,
(ii) $\mathrm{PGL}_2(\mathbf{F}_7)$,

(VI) (i) $\mathrm{SU}_3(\mathbf{F}_3)$,
(ii) $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,

(VII) $\mathrm{Sp}_6(\mathbf{F}_2)$.

Let us show that $\mathbb{P}^6$ can be G-birationally rigid only in case (I).

Lemma 7.1. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{PSL}_2(\mathbf{F}_{13})$. Then the linear system $|\mathcal{O}_{\mathbb{P}^6}(4)|$ contains a unique G-invariant pencil.*

Proof. The GAP ID of $\mathrm{PSL}_2(\mathbf{F}_{13})$ is [1092,25], and this group has two irreducible 7-dimensional representation. A direct computation shows that $|\mathcal{O}_{\mathbb{P}^6}(4)|$ contains a unique G-invariant pencil. □

Lemma 7.2. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{SL}_2(\mathbf{F}_8)$ or $\mathrm{SL}_2(\mathbf{F}_8) \rtimes \boldsymbol{\mu}_3$. Then either the linear system $|\mathcal{O}_{\mathbb{P}^6}(4)|$, or the linear system $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains at least two G-invariant hypersurfaces. In particular, either the linear system $|\mathcal{O}_{\mathbb{P}^6}(4)|$, or the linear system $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a G-invariant pencil.*

Proof. The GAP ID of $\mathrm{SL}_2(\mathbf{F}_8)$ is [504,156], and this group has four irreducible 7-dimensional representations. (Note that three of them give rise to the same embedding of $\mathrm{SL}_2(\mathbf{F}_8)$ to $\mathrm{SL}_7(\mathbb{C})$, and both of the resulting subgroups $\mathrm{SL}_2(\mathbf{F}_8) \subset \mathrm{SL}_7(\mathbb{C})$ are primitive, because $\mathrm{SL}_2(\mathbf{F}_8)$ does not have subgroups of index 7.) A direct computation shows that for one of these representations the vector space of polynomials of degree 6 contains a five-dimensional vector subspace which consists of invariant polynomials, while for each of the remaining three representations the vector space of polynomials of degree 4 contains a two-dimensional vector subspace which consists of invariant polynomials (i.e., there is a unique G-invariant pencil in $|\mathcal{O}_{\mathbb{P}^6}(4)|$ in this case). The action of the group $\mathrm{SL}_2(\mathbf{F}_8) \rtimes \boldsymbol{\mu}_3$ on this subspace factors through the action of the abelian group $\boldsymbol{\mu}_3$, and thus splits into a sum of one-dimensional representations, which correspond to G-invariant hypersurfaces. □

Lemma 7.3. *Suppose that $\widehat{G}$ is isomorphic to $\mathfrak{A}_8$ or $\mathfrak{S}_8$. Then the linear system $|\mathcal{O}_{\mathbb{P}^6}(4)|$ contains a unique G-invariant pencil.*

Proof. Each of these groups has a unique irreducible 7-dimensional representation, which is a summand of the 8-dimensional permutation representation. The well-known results about symmetric functions imply that $|\mathcal{O}_{\mathbb{P}^6}(4)|$ contains a unique G-invariant pencil. □

Lemma 7.4. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{PSL}_2(\mathbf{F}_7)$ or $\mathrm{PGL}_2(\mathbf{F}_7)$. Then the linear system $|\mathcal{O}_{\mathbb{P}^6}(5)|$ contains a unique G-invariant pencil.*

Proof. The GAP ID of $\mathrm{PSL}_2(\mathbf{F}_7)$ is $[168, 42]$, and this group has a unique irreducible 7-dimensional representation. A direct computation shows that $|\mathcal{O}_{\mathbb{P}^6}(5)|$ contains a unique $\mathrm{PSL}_2(\mathbf{F}_7)$-invariant pencil. If $\widehat{G} \simeq \mathrm{PGL}_2(\mathbf{F}_7)$, then the required assertion follows from the case $\widehat{G} \simeq \mathrm{PSL}_2(\mathbf{F}_7)$. □

Lemma 7.5. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{SU}_3(\mathbf{F}_3)$ or $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$. Then the linear system $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a unique G-invariant pencil.*

Proof. The group $\mathrm{SU}_3(\mathbf{F}_3)$ has three irreducible 7-dimensional representations. A direct computation shows that $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a unique $\mathrm{SU}_3(\mathbf{F}_3)$-invariant pencil. This also implies that $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a unique G-invariant pencil in the case when $\widehat{G} \simeq \mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$. □

Lemma 7.6. *Suppose that $\widehat{G}$ is isomorphic to $\mathrm{Sp}_6(\mathbf{F}_2)$. Then the linear system $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a unique G-invariant pencil.*

Proof. The group $\mathrm{Sp}_6(\mathbf{F}_2)$ has a unique irreducible 7-dimensional representation. A direct computation shows that $|\mathcal{O}_{\mathbb{P}^6}(6)|$ contains a unique G-invariant pencil. □

Summarizing, we obtain the following corollary.

Corollary 7.7. *Let G be a finite subgroup in $\mathrm{PGL}_7(\mathbb{C})$ such that $\mathbb{P}^6$ is G-birationally rigid. Then G is conjugate to a subgroup in $\mathbb{N}_7$ that contains $\mathbb{H}_7$.*

Proof. By Theorem A, the subgroup G is primitive. Suppose that G is not conjugate to a subgroup in $\mathbb{N}_7$ that contains $\mathbb{H}_7$. Then it follows from Lemmas 7.1, 7.2, 7.3, 7.4, 7.5, and 7.6 that the linear system $|\mathcal{O}_{\mathbb{P}^6}(n)|$ contains a G-invariant pencil for some $n \in \{4, 5, 6\}$. Therefore, $\mathbb{P}^6$ is not G-birationally rigid by Lemma 2.4. □

Unfortunately, we do not know whether $\mathbb{P}^6$ is G-birationally rigid in the case when G is conjugate to a subgroup in $\mathbb{N}_7$ that contains $\mathbb{H}_7$.

Remark 7.8 (cf. Remark 5.3). If $G \subset \mathrm{PGL}_7(\mathbb{C})$ is conjugate to a subgroup in $\mathbb{N}_7$ that contains $\mathbb{H}_7$, then there are no G-invariant quadrics in $\mathbb{P}^6$. Indeed, let $\widehat{\mathbb{H}}_7$ denote the preimage of the group $\mathbb{H}_7$ in $\mathrm{SL}_7(\mathbb{C})$. Then $\widehat{\mathbb{H}}_7$ is a transitive group of order 343 with center $Z(\widehat{\mathbb{H}}_7) \simeq \boldsymbol{\mu}_7$. Every irreducible representation of $\widehat{\mathbb{H}}_7$ where $Z(\widehat{\mathbb{H}}_7)$ acts non-trivially has dimension 7. Hence there are no one-dimensional subrepresentations in the space of quadratic polynomials on $\mathbb{C}^7$.

8. Real projective spaces

In this section we discuss finite groups acting on projective spaces over the field of real numbers, and prove Theorems D and E. We fix a finite subgroup $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$, and a finite subgroup $\widehat{G} \subset \mathrm{GL}_{n+1}(\mathbb{R})$ that is mapped surjectively onto G.

Remark 8.1. If n is even, then the projection $\theta\colon \mathrm{SL}_{n+1}(\mathbb{R}) \to \mathrm{PGL}_{n+1}(\mathbb{R})$ is an isomorphism, so we can take $\widehat{G} \simeq G$. On the other hand, if n is odd, then it is possible that one cannot choose $\widehat{G}$ to be a subgroup of $\mathrm{SL}_{n+1}(\mathbb{R})$. For instance, if $n = 1$ and

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{R}),$$

then g is not contained in the image of θ. In this case one can consider the group

$$\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) \subset \mathrm{GL}_{n+1}(\mathbb{R})$$

of all matrices whose determinant equals either 1 or -1, note that it maps surjectively onto $\mathrm{PGL}_{n+1}(\mathbb{R})$, and construct $\widehat{G}$ as the preimage of G in $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$.

First, we present a very simple and well-known observation, which is nevertheless very useful.

Lemma 8.2. *The projective space $\mathbb{P}^n_{\mathbb{R}}$ contains a smooth pointless G-invariant quadric.*

Proof. Starting from a positive definite quadratic form and averaging it over $\widehat{G}$, we produce a positive definite $\widehat{G}$-invariant quadratic form q. Let Q be the hypersurface defined by q, so that Q is a G-invariant quadric. Then Q is smooth, because q is non-degenerate, and it has no real points, because q is positive definite. ∎

The next observation will be used in the proof of Theorem E.

Lemma 8.3. *Suppose that for some positive integer m the linear system $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(m)|$ contains at least two G-invariant hypersurfaces. Then the linear system $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(m)|$ contains a G-invariant pencil.*

Proof. If a G-invariant hypersurface $F \in |\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(m)|$ is not invariant under the action of the non-trivial element σ of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \boldsymbol{\mu}_2$, then F and $\sigma(F)$ generate a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant G-invariant pencil in $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(m)|$, which gives a G-invariant pencil in $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(m)|$. On the other hand, if each of the G-invariant hypersurfaces in $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(m)|$ is also $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant, then they are all defined over $\mathbb{R}$, and so we can take any two of them to generate a G-invariant pencil in $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(m)|$. ∎

Now, we are going to prove Theorem D. Namely, we aim to show that G is a primitive subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$ if $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. To do this, we need to prove several auxiliary lemmas, which are real counterparts of Lemmas 2.2, 2.3, 2.4, 2.5 proved in Section 2. We start with

Lemma 8.4 (cf. Lemmas 2.2 and 2.3)**.** *Suppose that there exists a decomposition*

$$\mathbb{C}^{n+1} = V' \oplus V''$$

such that the vector subspaces V' and V'' are interchanged by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \boldsymbol{\mu}_2$, *and at least one of the following two conditions holds:*

- *either both V' and V'' are invariant with respect to the action of the group $\widehat{G}$,*
- *or V' and V'' are interchanged by $\widehat{G}$.*

Then $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid.

Proof. By assumption, there exist two disjoint linear subspaces $\Lambda', \Lambda'' \subset \mathbb{P}^n_{\mathbb{C}}$ such that the union $\Lambda' \cup \Lambda''$ spans $\mathbb{P}^n_{\mathbb{C}}$, the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ interchanges Λ' and Λ'', and for any $g \in \widehat{G}$ one has either $g(\Lambda') = \Lambda'$, or $g(\Lambda') = \Lambda''$. Denote $k = \dim \Lambda' = \dim \Lambda''$, so that $n = 2k + 1$. Let

$$\psi'\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \Lambda'',$$
$$\psi''\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \Lambda'$$

be the linear projections from Λ' and Λ'', respectively. Set

$$\psi = \psi' \times \psi'' \colon \mathbb{P}^n \dashrightarrow \Lambda'' \times \Lambda'.$$

Then ψ fits into a G-equivariant $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant commutative diagram

(8.1)
$$\begin{array}{ccc} & Y & \\ {\scriptstyle \pi_{\mathbb{C}}} \swarrow & & \searrow {\scriptstyle \phi_{\mathbb{C}}} \\ \mathbb{P}^n_{\mathbb{C}} & \overset{\psi}{\dashrightarrow} & \Lambda'' \times \Lambda' \end{array}$$

where $\pi_{\mathbb{C}}$ is a blow up of Λ' and Λ'', and $\phi_{\mathbb{C}}$ is a $\mathbb{P}^1_{\mathbb{C}}$-bundle. Since Λ' and Λ'' are interchanged by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, we see that $\Lambda'' \times \Lambda'$ is obtained by extension of scalars from the Weil restriction of scalars $R_{\mathbb{C}/\mathbb{R}}\Lambda'$. Furthermore, the diagram (8.1) is obtained by extension of scalars from the G-equivariant commutative diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \phi} \\ \mathbb{P}^n_{\mathbb{R}} & \dashrightarrow & R_{\mathbb{C}/\mathbb{R}}\Lambda' \end{array}$$

where π is a blow up of the subscheme whose geometrically irreducible components are Λ' and Λ'', and ϕ is a real G-Mori fiber space whose fibers are $\mathbb{P}^1_{\mathbb{R}}$. Hence, $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid. $\square$

Lemma 8.5 (cf. Lemma 2.2)**.** *Suppose that $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. Then G is a transitive subgroup of* $\mathrm{PGL}_{n+1}(\mathbb{C})$*.*

Proof. Suppose that G is not a transitive subgroup of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Then there exists a G-invariant linear subspace $\Lambda \subset \mathbb{P}^n_{\mathbb{C}}$ of dimension $k \leqslant \lfloor \frac{n-1}{2} \rfloor$. Denote by σ the non-trivial element of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \boldsymbol{\mu}_2$, and set $\Lambda^\sigma = \sigma(\Lambda)$. If the intersection $\Xi = \Lambda \cap \Lambda^\sigma$ is not empty, then Ξ is a G-invariant $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant linear subspace of $\mathbb{P}^n_{\mathbb{C}}$ of dimension at most $\dim \Lambda$. Thus, Ξ is the complexification of a real G-invariant linear subspace $\Xi_{\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{R}}$. The linear projection from $\Xi_{\mathbb{R}}$ provides a G-equivariant commutative diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \phi} \\ \mathbb{P}^n_{\mathbb{R}} & \overset{\psi}{\dashrightarrow} & \Xi^{\perp}_{\mathbb{R}} \end{array}$$

where ϕ is a G-Mori fiber space over the projective space $\Xi^{\perp}_{\mathbb{R}}$ of dimension

$$n - 1 - \dim \Xi \geqslant n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor \geqslant 1.$$

This contradicts our assumption that $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. Hence, one has $\Lambda \cap \Lambda^\sigma = \varnothing$. Moreover, if $2k+1 = n$, then $\Lambda \cup \Lambda^\sigma$ spans $\mathbb{P}^n_{\mathbb{C}}$, which contradicts Lemma 8.4. Hence, we have $2k+1 < n$.

Let $\Theta \subset \mathbb{P}^n_{\mathbb{C}}$ be the linear span of the union $\Lambda \cup \Lambda^\sigma$. Then Θ is a G-invariant $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant linear subspace of $\mathbb{P}^n_{\mathbb{C}}$ of dimension

$$\dim \Lambda + \dim \Lambda^\sigma + 1 = 2k + 1.$$

Thus, Θ is the complexification of a real G-invariant subspace $\Theta_{\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{R}}$ of the same dimension. Furthermore, there exists a G-invariant linear subspace $\Theta^{\perp}_{\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{R}}$ of dimension

$$n - \dim \Theta - 1 = n - 2k - 2 \geqslant 0$$

such that the union $\Theta_{\mathbb{R}} \cup \Theta_{\mathbb{R}}^{\perp}$ spans $\mathbb{P}^n_{\mathbb{R}}$. The linear projection from $\Theta_{\mathbb{R}}^{\perp}$ provides a G-equivariant commutative diagram

$$\begin{array}{ccc} & Y & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^n_{\mathbb{R}} & \dashrightarrow & \Theta_{\mathbb{R}} \end{array}$$

where ϕ is a G-Mori fiber space over the projective space $\Theta_{\mathbb{R}}$ of dimension $2k+1 \geqslant 1$. This is impossible, since $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. □

Lemma 8.6 (cf. Lemma 2.3)**.** *Suppose that G is transitive and imprimitive as a subgroup of* $\mathrm{PGL}_{n+1}(\mathbb{C})$*, so that there exists a non-trivial decomposition*

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^{s} V_i \tag{8.2}$$

such that for any $g \in \widehat{G}$ and any i we have $g(V_i) = V_j$ for some $j = j(g)$. Suppose that $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. Then $s = n+1$, and the points of $\mathbb{P}^n_{\mathbb{C}}$ corresponding to the one-dimensional subspaces in the decomposition (8.2) *are real.*

Proof. The action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^{n+1}$ commutes with the action of $\widehat{G}$. Hence the vector subspaces in the decomposition (8.2) are permuted by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Therefore, the decomposition (8.2) provides a decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^{r} W_i \tag{8.3}$$

for some $r \leqslant s$, where the vector subspaces W_i are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant. Moreover, one has $r = s$ if and only if all the vector subspaces V_i are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant. Next, since $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \boldsymbol{\mu}_2$, we see that the case $r = 1$ is possible if and only if $s = 2$, and the vector subspaces V_1 and V_2 in the decomposition (8.2) are permuted by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Applying Lemma 8.4 to the projectivizations of V_1 and V_2, we see that $\mathbb{P}^n_{\mathbb{R}}$ is not birationally rigid in this case. Thus, we may assume that $r \geqslant 2$, so that the decomposition (8.3) is non-trivial.

Note that r divides $n+1$. Set $m = \frac{n+1}{r} - 1$. The decomposition (8.3) gives a collection of r linear subspaces $\Lambda_1, \ldots, \Lambda_r \subset \mathbb{P}^n_{\mathbb{C}}$ of dimension m such that their union spans $\mathbb{P}^n_{\mathbb{C}}$, each Λ_i is disjoint from the linear span of the union of Λ_j with $j \neq i$, and the group G permutes $\Lambda_1, \ldots, \Lambda_r$. Furthermore, the linear subspaces Λ_i are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant. This means that each Λ_i is the complexification of some linear subspace $\Lambda_{i,\mathbb{R}} \subset \mathbb{P}^n_{\mathbb{R}}$.

For each $i \in \{1, \ldots, r\}$, consider the linear projection

$$\psi_i \colon \mathbb{P}^n_{\mathbb{R}} \dashrightarrow \Lambda_{i,\mathbb{R}}$$

from the span of the union of all $\Lambda_{j,\mathbb{R}}$ with $j \neq i$. Set $\Lambda = \Lambda_{1,\mathbb{R}} \times \ldots \times \Lambda_{r,\mathbb{R}}$, and consider the rational map

$$\psi = \psi_1 \times \ldots \times \psi_r \colon \mathbb{P}^n_{\mathbb{R}} \dashrightarrow \Lambda.$$

Then ψ fits into the G-equivariant commutative diagram

$$\begin{array}{ccc} & Y & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^n_{\mathbb{C}} & \overset{\psi}{\dashrightarrow} & \Lambda \end{array}$$

where π is a blow up of the union $\Lambda_1 \cup \ldots \cup \Lambda_r$, and ϕ is a $\mathbb{P}^{r-1}_{\mathbb{R}}$-bundle. Since $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid, ϕ cannot be a G-Mori fiber space over a positive-dimensional base. Therefore, for all $1 \leqslant i \leqslant r$ we have $\dim \Lambda_i = 0$. The latter means that $r = n+1$. Since

$$n+1 \geqslant s \geqslant r \geqslant n+1,$$

this gives $s = r = n+1$, so that all the vector subspaces V_i in the decomposition (8.2) are one-dimensional and real. ☐

Lemma 8.7 (cf. Lemma 2.4). *Suppose that $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(d)|$ contains a G-invariant pencil $\mathcal{P}$ for some $d \leqslant n$. Then $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid.*

Proof. Identical to the proof of Lemma 2.4. ☐

Lemma 8.8 (cf. Lemma 2.5). *Suppose that $\mathbb{P}^n_{\mathbb{R}}$ contains a G-irreducible (over $\mathbb{R}$) complete intersection $X = F_{d_1} \cap F_{d_2}$ such that $X_{\mathbb{C}}$ has at most isolated ordinary double singularities and $d_1 < d_2 \leqslant n$, where F_{d_1} and F_{d_2} are hypersurfaces in $\mathbb{P}^n_{\mathbb{R}}$ of degree d_1 and d_2, respectively. Then $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid.*

Proof. Identical to the proof of Lemma 2.5. ☐

Now we are ready to prove Theorem D in all dimensions $n \geqslant 4$.

Proposition 8.9. *Suppose that $n \geqslant 4$ and $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid. Then G is a primitive subgroup in $\mathrm{PGL}_{n+1}(\mathbb{C})$.*

Proof. Suppose that G is not a primitive subgroup of the group $\mathrm{PGL}_{n+1}(\mathbb{C})$. Then it follows from Lemmas 8.5 and 8.6 that $\mathbb{P}^n_{\mathbb{R}}$ contains a G-orbit of length $n+1$ whose points span $\mathbb{P}^n_{\mathbb{R}}$. Choosing appropriate coordinates on $\mathbb{P}^n_{\mathbb{R}}$, we may assume that this G-orbit consists of the points

$$P_1 = [1:0:\ldots:0], P_2 = [0:1:0:\ldots:0], \ldots, P_{n+1} = [0:\ldots:0:1].$$

Then the G-action on these points gives a homomorphism $\upsilon\colon G \to \mathfrak{S}_{n+1}$, whose kernel consists of linear transformations

$$[x_1 : x_2 : \ldots : x_{n+1}] \mapsto [\pm x_1 : \pm x_2 : \ldots : \pm x_{n+1}].$$

Moreover, the group G is generated by $\ker(\upsilon)$ and linear transformations given by permutations matrices whose entries are scaled by some real numbers.

By Lemma 8.2 there exists a G-invariant pointless smooth quadric hypersurface $Q \subset \mathbb{P}^n_{\mathbb{R}}$. Set

$$\mathbb{V} = H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(2)),$$

and let $f \in \mathbb{V}$ be such that the quadric Q is given by equation $f = 0$. Let $\widehat{G}$ be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{R})$ that is surjectively mapped to G via the natural epimomorphism $\mathrm{GL}_{n+1}(\mathbb{R}) \to \mathrm{PGL}_{n+1}(\mathbb{R})$. Then $\mathbb{V}$ is a real representation of the group $\widehat{G}$, and the polynomial f spans its one-dimensional subrepresentation. On the other hand, $\mathbb{V}$ splits as a sum

$$\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-,$$

where $\mathbb{V}^+$ is the subrepresentation spanned by $x_1^2, x_2^2, \ldots, x_{n+1}^2$, and $\mathbb{V}^-$ is the subrepresentation spanned by $x_1x_2, x_1x_3, \ldots, x_nx_{n+1}$. Since $Q(\mathbb{R}) = \varnothing$, we have $f \notin \mathbb{V}^-$, because every polynomial from $\mathbb{V}^-$ vanishes at the points $P_1, \ldots, P_{n+1}$. Moreover, if $f \notin \mathbb{V}^+$, then both $\mathbb{V}^+$ and $\mathbb{V}^-$ contain one-dimensional subrepresentations of the group $\widehat{G}$, so, in particular, there exists a G-invariant pencil of quadrics in $\mathbb{P}^n_{\mathbb{R}}$. In this case $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid by Lemma 8.7. Hence, we conclude that $f \in \mathbb{V}^+$. Then

$$f = \sum_{i=1}^{n+1} \lambda_i x_i^2,$$

where each λ_i is a positive real number. Now, scaling our coordinates $x_1, \ldots, x_{n+1}$, we may further assume that $\lambda_1 = \ldots = \lambda_{n+1} = 1$, so

$$f = x_1^2 + x_2^2 + \ldots + x_{n+1}^2.$$

This implies that G is generated by $\ker(\upsilon)$ and linear transformations given by permutation matrices whose entries are scaled by ± 1. Set

$$g = x_1^4 + x_2^4 + \ldots + x_{n+1}^4.$$

Then the quartic hypersurface given by equation $g = 0$ is G-invariant, so $|\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(4)|$ contains a G-invariant pencil generated by $2Q$ and the latter quartic. Therefore, keeping in mind that $n \geqslant 4$ and applying Lemma 8.7, we get a contradiction. □

To finish the proof of Theorem D, let us describe all finite subgroups $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$ such that $\mathbb{P}^n_{\mathbb{R}}$ is G-birationally rigid in the cases when $n = 2$ and $n = 3$.

Lemma 8.10. *Suppose that $n = 2$. Then $\mathbb{P}^2_{\mathbb{R}}$ is G-birationally rigid if and only if $G \simeq \mathfrak{A}_5$.*

Proof. We know from Lemma 8.2 that $\mathbb{P}^2_{\mathbb{R}}$ contains a G-invariant smooth pointless conic C. This gives a monomorphism

$$G \hookrightarrow \mathrm{Aut}(C) \simeq \mathrm{SO}_3(\mathbb{R}).$$

On the other hand, we know that every finite subgroup in $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to one of the following groups:

- the cyclic group $\boldsymbol{\mu}_m$,
- the Klein four group $\boldsymbol{\mu}_2^2$,
- the dihedral group $\mathfrak{D}_m$ of order $2m \geqslant 6$,
- the alternating group $\mathfrak{A}_4$,
- the symmetric group $\mathfrak{S}_4$,
- the alternating group $\mathfrak{A}_5$.

Moreover, finite subgroups of the group $\mathrm{SO}_3(\mathbb{R})$ are conjugate if and only if they are isomorphic. Hence, arguing as in the proof of Theorem 1.4 given in [47], we see that $\mathbb{P}^2_{\mathbb{R}}$ is G-birationally rigid if and only if $G \simeq \mathfrak{A}_5$. □

Up to conjugation, $\mathrm{PGL}_3(\mathbb{R})$ contains is unique subgroup isomorphic to $\mathfrak{A}_5$, and this subgroup is primitive. Hence, Lemma 8.10 implies Theorem D in dimension 2.

Remark 8.11. Note also that $\mathbb{P}^2_{\mathbb{R}}$ is not $\mathfrak{A}_5$-birationally superrigid, see e.g. [9, Lemma B.15]. Indeed, $\mathbb{P}^2_{\mathbb{R}}$ contains a unique $\mathfrak{A}_5$-orbit Σ_6 of length 6. Blowing up this orbit, we obtain a real form of the famous Clebsch cubic. Then blowing down the strict transforms of six conics passing through quintuples of points of Σ_6, we obtain a $\mathfrak{A}_5$-equivariant birational map $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$.

In dimension 3, Theorem D follows from the next lemma.

Lemma 8.12. *Suppose that $n = 3$, and G is not a primitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3_{\mathbb{R}}$ is not G-birationally rigid.*

Proof. We know from Lemma 8.5 that G is a transitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$. Thus, $\mathbb{P}^3_{\mathbb{R}}$ contains a G-orbit of length 4 by Lemma 8.6. Let $\overline{G}$ be the image of the homomorphism $G \to \mathfrak{S}_4$ defined by the action of G on this orbit. Then $\overline{G}$ is one of the groups $\mathfrak{S}_4$, $\mathfrak{A}_4$, $\mathfrak{D}_4$, $\boldsymbol{\mu}_4$, or $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \subset \mathfrak{A}_4$. If $\overline{G} \simeq \mathfrak{S}_4$ or $\overline{G} \simeq \mathfrak{A}_4$, then it follows from Example 4.3 that $\mathbb{P}^3_{\mathbb{R}}$ is G-birational to a singular terminal Fano threefold $\mathcal{X}$ with $\mathrm{rk}\,\mathrm{Cl}(\mathcal{X})^G = 1$. Hence, $\mathbb{P}^3_{\mathbb{R}}$ is not G-birationally rigid in this case. On the other hand, if $\overline{G}$ is one of the groups $\mathfrak{D}_4$, $\boldsymbol{\mu}_4$, or $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, then $\mathbb{P}^3_{\mathbb{R}}$ contains a G-invariant pair of skew lines. This gives a G-birational map from $\mathbb{P}^3_{\mathbb{R}}$ to a $\mathbb{P}^1_{\mathbb{R}}$-bundle over $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, so that $\mathbb{P}^3_{\mathbb{R}}$ is again not G-birationally rigid. □

Actually, the assertion of Lemma 8.12 can be made more precise.

Proposition 8.13. *Suppose that $n = 3$. Then the following conditions are equivalent:*

(1) $\mathbb{P}^3_{\mathbb{R}}$ *is G-birationally rigid;*
(2) $\mathbb{P}^3_{\mathbb{R}}$ *is G-birationally superrigid;*
(3) *G contains a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$.*

Proof. We know from Lemma 8.2 that $\mathbb{P}^3_{\mathbb{R}}$ contains a G-invariant smooth pointless quadric surface Q. Observe that $Q \simeq C \times C$ for a pointless conic C, because all other real quadrics have points. Thus, we have a monomorphism

$$G \hookrightarrow \mathrm{Aut}(C \times C) \simeq \big(\mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R})\big) \rtimes \boldsymbol{\mu}_2.$$

Suppose that $\mathbb{P}^3_{\mathbb{R}}$ is G-birationally rigid. Then G is a primitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$ by Lemma 8.12. Now, arguing as in the proof of [20, Theorem 1.3] (cf. Theorem 1.5), we see that G contains a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$, and $\mathbb{P}^3_{\mathbb{R}}$ is G-birationally superrigid. This gives the implication (1) $\Rightarrow$ (3).

The implication (2) $\Rightarrow$ (1) is obvious. Hence, to complete the proof, we must establish the implication (3) $\Rightarrow$ (2). First, using the classification of finite subgroups of $\mathrm{SO}_3(\mathbb{R})$, we see that the group $(\mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R})) \rtimes \boldsymbol{\mu}_2$ contains a unique subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$ up to conjugation. Hence, if G contains a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$, then it follows from the proof of [20, Theorem 1.3] that $\mathbb{P}^3_{\mathbb{R}}$ is G-birationally superrigid. This gives the implication (3) $\Rightarrow$ (2). $\square$

As we mentioned in the proof of Proposition 8.13, the group $\mathrm{PGL}_4(\mathbb{R})$ contains a unique subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$ up to conjugation, and it is a primitive subgroup of $\mathrm{PGL}_4(\mathbb{C})$. Anyway, by means of Proposition 8.13 or Lemma 8.12, we obtain Theorem D in dimension 3. Therefore, Theorem D is completely proved.

Remark 8.14. It is not difficult to list all finite subgroups in $\mathrm{PGL}_4(\mathbb{R})$ that contain a subgroup isomorphic to $\mathfrak{A}_4 \times \mathfrak{A}_4$. Implicitly, this is done in [20].

We conclude this section by proving Theorem E.

Proof of Theorem E. We suppose that $n \in \{4, 5, 6\}$. We have to prove that $\mathbb{P}^n_{\mathbb{R}}$ is not G-birationally rigid. By Theorem D, we may assume that G is a primitive subgroup of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Recall from Lemma 8.2 that G leaves invariant a smooth pointless quadric $Q \subset \mathbb{P}^n_{\mathbb{R}}$. Thus, if $n = 4$, then, keeping in mind Remark 5.3 and using the description of primitive subgroups in $\mathrm{PGL}_5(\mathbb{C})$ presented in Section 5, we see that G is isomorphic to one of the following 3 groups:

$$\mathfrak{S}_6,\ \mathfrak{A}_6,\ \mathfrak{S}_5.$$

By Lemma 5.1 the linear system $|\mathcal{O}_{\mathbb{P}^4_{\mathbb{C}}}(4)|$ contains a unique G-invariant pencil. Since this pencil is unique, it is preserved by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and hence is defined over $\mathbb{R}$. Thus, $\mathbb{P}^4_{\mathbb{R}}$ is not G-birationally rigid by Lemma 8.7.

Now, we consider the case $n = 5$. Then it follows from the classification of primitive subgroups of $\mathrm{PGL}_6(\mathbb{C})$ presented in Section 6 together with Lemmas 6.2, 6.3, 6.4, 6.8, and 6.9, and Remark 6.15 that either

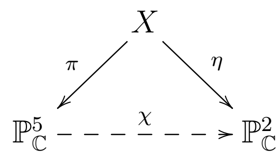
(I) G leaves invariant a (possibly complex) Segre cubic scroll in $\mathbb{P}^5_{\mathbb{C}}$,

or G is isomorphic to one of the following groups:

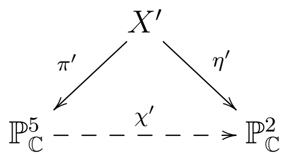
(VI) $\mathfrak{A}_7$ or $\mathfrak{S}_7$,
(IX) (i) $\mathrm{PSL}_2(\mathbf{F}_7)$,
(ii) $\mathrm{PGL}_2(\mathbf{F}_7)$,
(XIII) (i) $\mathrm{PSp}_4(\mathbf{F}_3)$,

(ii) $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$.

If G leaves invariant a real Segre cubic scroll in $\mathbb{P}^5_{\mathbb{R}}$, then it follows from Remark 6.1 that $\mathbb{P}^5_{\mathbb{R}}$ is not G-birationally rigid. If G leaves invariant a non-real Segre cubic scroll $Y \subset \mathbb{P}^5_{\mathbb{C}}$, then the complex conjugate Segre scroll Y' is also G-invariant. Thus, it follows from Remark 6.1 that we have two G-equivariant (complex) commutative diagrams:



and



where π and π' are blow ups of the scrolls Y and Y', respectively, both η and η' are $\mathbb{P}^3_{\mathbb{C}}$-bundles, and the maps χ and χ' are given by the linear systems of all quadrics passing through Y and Y', respectively. Note that the fibers of χ and χ' are three-dimensional linear subspaces in $\mathbb{P}^5_{\mathbb{C}}$. Thus, taking the product of the maps χ and χ', we obtain a real rational map $\rho\colon \mathbb{P}^5_{\mathbb{R}} \dashrightarrow W$, where W is a real form of $\mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}}$ obtained as the Weil restriction of scalars $R_{\mathbb{C}/\mathbb{R}}\mathbb{P}^2_{\mathbb{R}}$. By construction, a general fiber of ρ is an intersection of two (distinct) three-dimensional linear subspaces, so it is either a line or a plane. Thus, equivariantly resolving the indeterminacy of the map ρ and applying relative real G-equivariant Minimal Model Program, we obtain a G-birational map from $\mathbb{P}^5_{\mathbb{R}}$ to a real G-Mori fibre space with a positive dimensional base, which implies that $\mathbb{P}^5_{\mathbb{R}}$ is not G-birationally rigid in this case.

If G is isomorphic to $\mathfrak{A}_7$ or $\mathfrak{S}_7$, then by Lemma 6.5 the linear system $|\mathcal{O}_{\mathbb{P}^5_{\mathbb{C}}}(4)|$ contains a unique G-invariant pencil. Since this pencil is unique, it is preserved by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and hence is defined over $\mathbb{R}$. Thus, $\mathbb{P}^5_{\mathbb{R}}$ is not G-birationally rigid by Lemma 8.7. Similarly, if G is isomorphic to $\mathrm{PSL}_2(\mathbf{F}_7)$ or to $\mathrm{PGL}_2(\mathbf{F}_7)$, then it follows from Lemma 6.6 that $|\mathcal{O}_{\mathbb{P}^5_{\mathbb{R}}}(3)|$ contains a G-invariant pencil, so $\mathbb{P}^5_{\mathbb{R}}$ is again not G-birationally rigid by Lemma 8.7. Finally, if G is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$ or $\mathrm{PSp}_4(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$, then it follows from Lemma 6.11 that $\mathbb{P}^5_{\mathbb{R}}$ contains a G-invariant complete intersection of a quadric and a quintic which has only ordinary double points as singularities, so $\mathbb{P}^5_{\mathbb{R}}$ is not G-birationally rigid by Lemma 8.8.

Thus, to complete the proof of Theorem E, we may assume that $n = 6$. Then it follows from the classification of primitive subgroups of $\mathrm{PGL}_7(\mathbb{C})$ presented in Section 7 together with Remark 7.8 that G is isomorphic to one of the following groups:

(II) $\mathrm{PSL}_2(\mathbf{F}_{13})$,
(III) (i) $\mathrm{SL}_2(\mathbf{F}_8)$,
(ii) $\mathrm{SL}_2(\mathbf{F}_8) \rtimes \boldsymbol{\mu}_3$,
(IV) $\mathfrak{A}_8$ or $\mathfrak{S}_8$,
(V) (i) $\mathrm{PSL}_2(\mathbf{F}_7)$,
(ii) $\mathrm{PGL}_2(\mathbf{F}_7)$,
(VI) (i) $\mathrm{SU}_3(\mathbf{F}_3)$,
(ii) $\mathrm{SU}_3(\mathbf{F}_3) \rtimes \boldsymbol{\mu}_2$,
(VII) $\mathrm{Sp}_6(\mathbf{F}_2)$.

In each of these cases the linear system $|\mathcal{O}_{\mathbb{P}^6_{\mathbb{R}}}(n)|$ contains a G-invariant pencil for some $n \in \{4, 5, 6\}$. This follows in a straightforward way from Lemmas 7.1, 7.3, 7.4, 7.5, and 7.6 in all the cases except for $G \simeq \mathrm{SL}_2(\mathbf{F}_8)$ and $G \simeq \mathrm{SL}_2(\mathbf{F}_8) \rtimes \boldsymbol{\mu}_3$, while for the latter two groups the same assertion is implied by Lemmas 7.2 and 8.3. Now we can apply Lemma 8.7 to show that $\mathbb{P}^6_{\mathbb{R}}$ is not G-birationally rigid. □

Remark 8.15. In all the cases one comes across in the proof of Theorem E, the conjugacy class of the subgroup $G \subset \mathrm{PGL}_{n+1}(\mathbb{R})$ is uniquely determined by the conjugacy class of G in $\mathrm{PGL}_{n+1}(\mathbb{C})$. This can be deduced from [5, Proposition 2.2]. However, we do not use this fact in the proof.

Appendix A. Combinatorics

In the sequel, we will always interpret the symmetric group $\mathfrak{S}_r$ as the group of permutations of the set $\Sigma_r = \{1, \dots, r\}$. Recall the following widely-used definition.

Definition A.1 (cf. Definitions 1.6 and 2.1)**.** Let $G \subset \mathfrak{S}_r$ be a subgroup. One says that G is *transitive* if it acts transitively on the set Σ_r, i.e. Σ_r is a single G-orbit. One says that G is primitive if it is transitive, and for any partition

$$\Sigma_r = \Sigma_r^1 \sqcup \ldots \sqcup \Sigma_r^k$$

into non-empty subsets such that Σ_r^i are permuted by G, one has either $k = 1$ and $\Sigma_r^1 = \Sigma_r$, or $k = r$ and $|\Sigma_r^i| = 1$.

A cyclic subgroup $\boldsymbol{\mu}_r \subset \mathfrak{S}_r$ is said to be *regular*, if it is conjugate to the group whose generator acts by the cyclic permutation $(12 \dots n)$. It is easy to see that $\boldsymbol{\mu}_r$ is regular if and only if it is transitive. Thus, for instance, the subgroup of $\mathfrak{S}_6$ generated by the permutation $(12)(345)$ is a non-regular subgroup isomorphic to $\boldsymbol{\mu}_6$. Similarly, if $r \geqslant 3$, we say that a dihedral subgroup $\mathfrak{D}_r \subset \mathfrak{S}_r$ of order $2r$ is regular, if it is conjugate to the group generated by the cyclic permutation $(12 \dots r)$ and the involution which swaps 1 with r, 2 with $r-1$, etc. We will denote by $\Sigma_r^{(k)}$ the set of non-ordered k-tuples of distinct elements of Σ_r.

Lemma A.2. *Let G be a primitive subgroup of $\mathfrak{S}_r$. The following assertions hold.*

- *(i) Suppose that G has an orbit of length s in $\Sigma_r^{(2)}$. Then $s \geqslant r$.*
- *(ii) Suppose that G has an orbit of length r in $\Sigma_r^{(2)}$. Then G is a regular subgroup of $\mathfrak{S}_r$ isomorphic either to $\boldsymbol{\mu}_r$ or $\mathfrak{D}_r$.*

Proof. Let $\Xi = \{(a_1, b_1), \dots, (a_s, b_s)\}$ be a G-orbit of length $s \leqslant r$ in $\Sigma_r^{(2)}$. Since a primitive group is also transitive, the G-orbit of a_1 coincides with the whole set Σ_r. On the other hand, this orbit is contained in the set $\Theta = \{a_1, \dots, a_s, b_1, \dots, b_s\}$, so that $\Theta = \Sigma_r$. Furthermore, it follows from the transitivity of G that each element of Σ_r appears among $a_1, \dots, a_s, b_1, \dots, b_s$ the same number t of times. This gives

$$tr = 2s. \tag{A.1}$$

Suppose that $s < r$. Then (A.1) gives $t = 1$. In other words, the pairs $(a_1, b_1), \dots, (a_s, b_s)$ provide a partition of Σ_r into a union of subsets of cardinality 2 permuted by G. This is impossible for a primitive group. The obtained contradiction proves assertion (i).

Suppose that $s = r$. If $r = 2$, there is nothing to prove, and so we assume that $r \geqslant 3$. From (A.1) one obtains $t = 2$. In other words, each $c \in \Sigma_r$ appears in exactly two pairs (a_i, b_i) and (a_j, b_j). Consider the minimal equivalence relation under which a_i is equivalent to b_i for all $1 \leqslant i \leqslant r$. The equivalence classes under this relation form a partition of Σ_r into subsets permuted by G. Since G is primitive, we conclude that the whole Σ_r is the unique equivalence class. This means that after relabelling the elements of Σ_r if necessary, one has

$$\Xi = \{(1,2), (2,3), \dots, (r,1)\}.$$

Let $\Delta \subset \mathfrak{S}_r$ be the subgroup which consists of all permutations preserving the G-orbit Ξ. Then Δ is a regular dihedral group $\mathfrak{D}_r$. Indeed, such a group is obviously contained in Δ. On the other hand, one has

$$|\Delta| = r \cdot |\Delta_{(1,2)}|,$$

where $\Delta_{(1,2)}$ is the stabilizer of the non-ordered pair $(1,2)$ in Δ. On the other hand, the stabilizer $\Delta'_{(1,2)}$ of the *ordered* pair consisting of 1 and 2 in Δ fixes 1 and 2, hence fixes 3 as the unique element different from 1 which is contained together with 2 in a pair from Ξ, etc; in other words, the stabilizer $\Delta'_{(1,2)}$ is trivial. This gives

$$|\Delta_{(1,2)}| \leqslant 2|\Delta'_{(1,2)}| = 2,$$

and so $\Delta \simeq \mathfrak{D}_r$.

Thus, G is a subgroup of a regular dihedral group $\Delta \simeq \mathfrak{D}_r$. Since G has an orbit of length r, its order cannot be smaller than r. In other words, G either coincides with Δ, or is a subgroup of index 2 therein. However, any subgroup of index 2 in $\mathfrak{D}_r$ is either the cyclic group of order r (which is regular provided that $\mathfrak{D}_r$ is regular), or a dihedral group of order r if r is even. The latter subgroup of $\mathfrak{S}_r$ is not transitive (and in particular not primitive), which means that G is always a regular subgroup of $\mathfrak{S}_r$. This proves assertion (ii). ☐

Remark A.3. One can strengthen the assertion of Lemma A.2(ii) by observing that a regular dihedral subgroup of $\mathfrak{S}_r$ is primitive if and only if r is a prime number. In other words, under the assumptions of Lemma A.2, the group G is a regular subgroup of $\mathfrak{S}_r$ isomorphic either to $\boldsymbol{\mu}_r$ or $\mathfrak{D}_r$, and r is prime.

Lemma A.2 easily implies the following.

Corollary A.4. *Let G be a primitive subgroup of $\mathfrak{S}_r$. Suppose that G is neither a regular subgroup $\boldsymbol{\mu}_r$ nor a regular subgroup $\mathfrak{D}_r$ of $\mathfrak{S}_r$. Then the minimal length of the G-orbit in $\Sigma_r^{(r-2)}$ is greater than r.*

Proof. The lengths of the G-orbits in $\Sigma_r^{(r-2)}$ are the same as the lengths of the G-orbits in $\Sigma_r^{(2)}$. Now the assertion follows from Lemma A.2. ☐

Appendix B. Magma codes

In this appendix, we present Magma codes used in the paper.

B.1. **Code used in Example 4.4.** The following Magma code has been provided to us by Andrea Petracci.

```
x1 := Matrix([[1,0,0,0,0]]);
x2 := Matrix([[0,1,0,0,0]]);
x3 := Matrix([[0,0,1,0,0]]);
x4 := Matrix([[0,0,0,1,0]]);
x5 := Matrix([[0,0,0,0,1]]);
O1 := {x1,x2,x3,x4,x5};
O3 := {a+b+c : a in O1, b in O1, c in O1 };
Mat := Matrix([[1,1,0,0,0],[0,1,1,0,0],[0,0,1,1,0],[0,0,0,1,1],[1,0,0,0,1]]);
A := {x1+x2+x3+x4+x5+f : f in O1} join {f * Mat : f in O3};
M := ToricLattice(4);
L := [];
for a in A do
    v := M!([a[1][1], a[1][2], a[1][3], a[1][4]]);
```

```
    Append(~L,v);
end for;
P := Polytope(L);
Dimension(P);
#A;
#A eq #Points(P);
/* Now we check if the polytope P is normal */
S := ScalarLattice();
Mtilde,iM,iS,pM,pS := DirectSum(M,S);
/* Mtilde is the direct sum between M and \mathbb{Z} */
CP := Cone([iM(u) + iS(S![1]) : u in Vertices(P)]);
/* CP is the cone over P placed at height 1 */
#HilbertBasis(CP) eq #Points(P);
/* We test if the number of the minimal set of generators
of the monoid CP \cap Mtilde is equal to the number
of lattice points of P */
Sigma := NormalFan(P);
X := ToricVariety(Rationals(),Sigma);
X;
printf "Is X Fano? %o \n", IsFano(X);
printf "Is X smooth? %o \n", IsNonsingular(X);
printf "Is X terminal? %o \n", IsTerminal(X);
printf "Is X Gorenstein? %o \n", IsGorenstein(X);
printf "Does X have quotient singularities? %o \n", IsQFactorial(X);
printf "Pic(X) is a free abelian group of rank %o \n", Dimension(PicardLattice(X));
```

B.2. **Code used in the proof of Lemma 6.11.** The following Magma code has been provided to us by Zhijia Zhang.

```
function FindEigenspace(M)
    egspace:=<>;
    for lam in SetToSequence(Eigenvalues(M)) do
        ev:=lam[1];
        Append(~egspace,<Eigenspace(M,ev),<ev>>);
    end for;
    return egspace;
end function;

function EigenspaceIntersection(G,k)
    if #G eq 1 then
        return <<VectorSpace(BaseRing(G),Degree(G)),<>>>;
    end if;
    if k eq 1 then
        return FindEigenspace(G.1);
    else
        tempegs:=EigenspaceIntersection(G,k-1);
        newegs:=FindEigenspace(G.k);
        resegs:=<>;
        for i in [1..#newegs] do
            for j in [1..#tempegs] do
```

```
                V:=tempegs[j][1] meet newegs[i][1];
                if Dimension(V) gt 0 then
                    Append(~resegs,<V,tempegs[j][2] cat newegs[i][2] >);
                end if;
            end for;
        end for;
        return resegs;
    end if;
end function;

function Findfixlocus(G)
    KK:=BaseRing(G);
    n:=Ngens(G);
    egs:=EigenspaceIntersection(G,n);
    char:=<>;
    for y in egs do
        x:=[[i]:i in y[2]];
        S:=GModule(G, MatrixAlgebra<KK,1|x>);
        Append(~char, y cat <Representation(S)>);
    end for;

        return char;
end function;

function invd(P,d,G)
    PP:=CoordinateRing(P);
    mon:=MonomialsOfDegree(PP,d);
    n:=Dimension(P)+1;
    v:=Matrix(PP,1,n,[PP.i:i in [1..n]]);
    K:=BaseRing(G);
    gen:=[];
    for i in [1..Ngens(G)] do
        g:=Transpose(G.i);
        m:=[];
        for j in [1..#mon] do
            nf:=mon[j]^g;
            nfcoe:=Coefficients(nf);
            nfmon:=Monomials(nf);
            vv:=[K!0:i in [1..#mon]];

            for nm in [1..#nfmon] do

                vv[Position(mon,nfmon[nm])]:=nfcoe[nm];
            end for;
            m:=m cat vv;
        end for;
        ma:=(Matrix(K,#mon,#mon,m));
        Append(~gen,ma);
```

```
    end for;
    GM:=GModule(G,gen);
    cm:=CohomologyModule(G,GM);
    AG:=ActionGroup(GM);
    fx:=Findfixlocus(AG);
    poly:=[];
    vect:=<>;cahr:=<>;
    for x in fx do
        V:=x[1];Append(~cahr,x[2]);
        bas:=Basis(V);
        po:=[&+[mon[i]*xx[i]: i in [1..#mon]]: xx in bas];
        Append(~vect,V);
        Append(~poly,po);
    end for;

return poly,vect,mon,cahr;
end function;

G:=PSp(4,3);
K:=RationalField();
L:=IrreducibleModules(G,K);
GM:=L[2];
G:=ActionGroup(Transpose(GM));
P5<x1,x2,x3,x4,x5,x6>:=ProjectiveSpace(K,5);
a2:=invd(P5,2,G);
a5:=invd(P5,5,G);
print(a2[1][1]);
print(a5[1][1]);
X:=Scheme(P5,[a2[1][1],a5[1][1]]);
IsNonsingular(X);
Y:=SingularSubscheme(X);
Dimension(Y);
Degree(Y);
IsReduced(Y);
```

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