Approximation des courbes sur les surfaces rationnelles réelles

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Approximating by algebraic maps

Weierstrass (1885)
Every C^∞-map $f\colon \mathbb{R} \to \mathbb{R}$ is approximated by polynomials.

$S^1 := \{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$
Every C^∞-map $f\colon S^1 \to S^1$ is approximated by rational maps

$$\Phi\colon (x,y) \mapsto \left(\frac{p_1(x,y)}{q_1(x,y)}, \frac{p_2(x,y)}{q_2(x,y)}\right)$$

X real algebraic variety
Is a given C^∞-map $f\colon S^1 \to X$ approximated by rational curves?
[Recall: $\mathbb{P}^1(\mathbb{R}) \sim S^1$.]

Rational curves

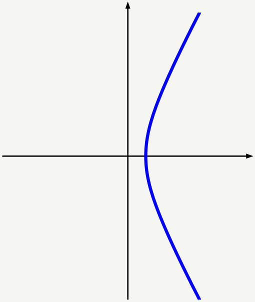
Example

$$f : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \big(t^2+1, t(t^2+1)\big)$$

Compactification

$\mathbb{R} \hookrightarrow \mathbb{P}^1(\mathbb{R}) \xrightarrow{\hat{f}} X \xleftarrow{bir} \mathbb{R}^2$

X rational surface



Approximating by rational curves

X nonsingular real algebraic variety
$\mathcal{C}^\infty(S^1, X)$:= space of maps endowed with the $\mathcal{C}^\infty$-topology
$\mathcal{A}_X \subset \mathcal{C}^\infty(S^1, X)$:= subset of rational curves $\mathbb{P}^1(\mathbb{R}) \to X$

Definition

Let $f \in \mathcal{C}^\infty(S^1, X)$ be a $\mathcal{C}^\infty$-map
f is approximated by rational curves
$\Leftrightarrow$
$f \in \overline{\mathcal{A}_X}$.

Theorem (Bochnak, Kucharz, 1999)

Let X be a nonsingular real rational variety, then any $\mathcal{C}^\infty$-map $\mathbb{P}^1(\mathbb{R}) \to X$ is approximated by rational curves.

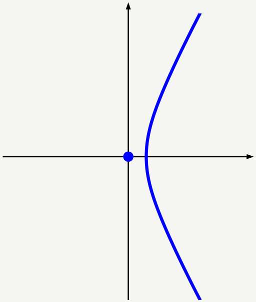
Smoothness?

Remark

$$\begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ t & \longmapsto & \big(t^2+1, t(t^2+1)\big) \end{array}$$

$$y^2 = x^2(x-1)$$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C}^2 \end{array}$$



Approximating by smooth rational curves

$\mathcal{B}_X \subset \mathcal{A}_X \subset \mathcal{C}^\infty(S^1, X)$:= subset of real-smooth rational curves $\mathbb{P}^1(\mathbb{R}) \to X$

Definition

$f \in \mathcal{C}^\infty(S^1, X)$ is approximated by real-smooth rational curves
$\Leftrightarrow$
$f \in \overline{\mathcal{B}_X}$.

Proposition

Let $C \subset \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ be a real-smooth rational curve. Then $[C] \in H_1(S^1 \times S^1, \mathbb{Z}/2)$ is nonzero.

Proof

$E_1, E_2 :=$ horizontal (resp. vertical) complex line on $\mathbb{P}^1 \times \mathbb{P}^1$.
$\forall D$ complex algebraic curve, $D = a_1 E_1 + a_2 E_2$ for $a_1, a_2 \geq 0$.

D is defined over $\mathbb{R} \Rightarrow a_i = (D \cdot E_{3-i}) \equiv \big(D(\mathbb{R}) \cdot E_{3-i}(\mathbb{R})\big) \mod 2$.
$[D(\mathbb{R})] = 0$ in $H_1(S^1 \times S^1, \mathbb{Z}/2) \Rightarrow a_1, a_2$ even.

Adjunction: $2p_a(D) - 2 = \big(a_1 E_1 + a_2 E_2\big) \cdot \big((a_1 - 2)E_1 + (a_2 - 2)E_2\big)$
$= a_1(a_2 - 2) + a_2(a_1 - 2)$, hence $p_a(D) = (a_1 - 1)(a_2 - 1)$.

a_1, a_2 even $\Rightarrow p_a(D)$ odd.

If D is rational then it has an odd number of singular points and at least one of them has to be real. □

Approximating by smooth rational curves

Main Theorem

An embedded circle in a nonsingular real rational surface admits a $\mathcal{C}^\infty$-approximation by smooth rational curves if and only if is is not diffeomorphic to a null-homotopic circle on a torus.

Corollary

Let X be a nonsingular real rational variety, then an embedded circle is approximated by smooth rational curves if and only if is is not diffeomorphic to a null-homotopic circle on a 2-dimensional torus.

Real rational surfaces

Theorem (Comessatti, 1914)

X orientable nonsingular real rational surface
$\Rightarrow X$ diffeomorphic to the sphere S^2 or to the torus $S^1 \times S^1$

Conversely:
$S^2 \sim$ rational model $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$
$S^1 \times S^1 \sim$ rational model $\{x^2 + y^2 = z^2 + t^2 = 1\} \subset \mathbb{R}^4$
$\mathbb{RP}^2 \sim$ rational model $\mathbb{P}^2(\mathbb{R})$
$\#^h \mathbb{RP}^2 \sim$ rational model $B_{p_1,p_2,\dots,p_{h-1}}\mathbb{P}^2(\mathbb{R})$ (blow-up at $h-1$ points)

Classification of rational models

$S^1 := \{(x,y) \in \mathbb{R}^2,\ x^2+y^2=1\}$
Real algebraic manifold := compact connected submanifold of $\mathbb{R}^n$ defined by real polynomial equations, for some n.
X, Y real algebraic manifolds, $f\colon X \to Y$ map
f algebraic := (i) real rational (ii) defined $\forall x \in X$
f isomorphism := (i) algebraic, (ii) f^{-1} exists (iii) f^{-1} algebraic

$x \mapsto x + x^3$ algebraic self-diffeomorphism of $\mathbb{R}$ is not an isomorphism. (Implicit function Theorem does not hold in the algebraic setting.)

Theorem (Biswas, Huisman, 2007)

Two nonsingular real rational surfaces are isomorphic if and only if they are diffeomorphic.

Real (-1)-curves

Let $L \subset X$ be a real algebraic curve on a real algebraic surface

Definition

L is a (-1)-curve iff
$\exists$ birational morphism $\pi \colon X \to Y$ such that $\pi(L)$ is a smooth point on Y and π restricted to $X \setminus L \to Y \setminus \pi(L)$ is an isomorphism.

By Casteluovo's criterium, $\exists$ such a birational morphism $\pi \colon X \to Y$ iff there exists a real algebraic surface X' and a real algebraic isomorphism $\Phi \colon X \to X'$ such that $L' := \Phi(L)$ is rational, irreducible and nonsingular and $L' \cdot L' = -1$ (self-intersection over complex points).

Approximating by (-1)-curves

Theorem

X nonsingular real rational surface and $L \subset X$ a nonsingular curve, the following assertions are equivalent:

1. *X is nonorientable near L and one of the following is satisfied:*
   - *$X \setminus L$ is a punctured sphere, or*
   - *$X \setminus L$ is a punctured torus, or*
   - *$X \setminus L$ is nonorientable.*
2. *L is homotopic to a (-1)-curve*
3. *L admits $\mathcal{C}^\infty$-approximation by (-1)-curves*

Proof of the approximation by smooth rational curves

1. Classify all topological pairs (K, S) such that S closed surface either nonorientable or of genus ≤ 1 and K embedded circle in S.
2. Construct rational models for each topological pair $\neq (S^1 \times S^1, \partial\mathbb{D})$.
3. Get: $\forall$ pair (K, S), $\exists X$ nonsingular real rational surface $\exists \varphi \colon S \xrightarrow{\sim} X$ diffeomorphism such that $L := \varphi(K) \subset X$ nonsingular real rational curve.
4. The rest of the talk is devoted to deduce the approximation result!

Density of $\mathrm{Aut}(X)$

Recall: $f: X \to X$ automorphism $\Leftrightarrow$
(i) f birational map, (ii) f is a self-diffeomorphism on the real locus
$\mathrm{Aut}(X) :=$ group of real algebraic automorphisms $X \to X$
Remark: let $V|_{\mathbb{R}}$ such that $V(\mathbb{R}) = X$, then
$\mathrm{Aut}_{\mathbb{R}}(V) \subset \mathrm{Aut}(X) \subset \mathrm{Bir}_{\mathbb{R}}(V)$

Theorem (Kollár, M. 2009)

- *$S = S^2$, $S^1 \times S^1$, or any non-orientable surface,*
  $\Rightarrow \exists$ *real algebraic model $X \sim S$ such that* $\overline{\mathrm{Aut}(X)} = \mathrm{Diff}(X)$
  for the C^∞-topology.
- *S any orientable surface of genus ≥ 2,*
  $\Rightarrow \forall$ *model $X \sim S$, $\mathrm{Aut}(X)$ is not dense in $\mathrm{Diff}(X)$,*
  even for the C^0-topology.

Cremona transformation (around 1860)

On $\mathbb{P}^3$ take $(x:y:z:t) \mapsto \left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{t}\right) = (yzt : ztx : txy : xyz)$

Base locus = 6 edges of a tetraedron T.
Move vertices to $(1, \pm i, 0, 0), (0, 0, 1, \pm i)$, get:
$\sigma\colon (x:y:z:t) \mapsto \big((x^2+y^2)z : (x^2+y^2)t : (z^2+t^2)x : (z^2+t^2)y\big)$

σ diffeomorphism of $\mathbb{P}^3(\mathbb{C}) \setminus T$

Each quadric
$Q_{abcdef} := a(x^2+y^2) + b(z^2+t^2) + cxz + dyt + ext + fyz$
(i) passes through the vertices of T,
(ii) has no real points on T.

$\sigma\colon Q_{abcdef}(\mathbb{R}) \xrightarrow{\cong} Q_{abcdfe}(\mathbb{R})$

Action on spheres

$S^2 := \{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1\}$
$Q_0 := \{(x, y, z, t) \in \mathbb{P}^3,\ x^2 + y^2 + z^2 - t^2 = 0\}$

Take Q_{abcdef} with $Q_{abcdef}(\mathbb{R}) \sim S^2$, $\Rightarrow Q_{abcdfe}(\mathbb{R}) \sim S^2$,
then both are equivalent to Q_0 up to linear change of coordinates.
Get: $\sigma_{abcdef} : S^2 \xrightarrow{\cong} S^2$, well defined up to $O(3, 1)$.

Theorem

The Cremona transformations with imaginary base points and $O(3, 1)$ generate $\mathrm{Aut}(S^2)$ which is dense in $\mathrm{Diff}(S^2)$.

Theorem (Lukackiĭ 1977)

$SO(m + 1, 1)$ is a maximal closed subgroup of $\mathrm{Diff}_0(S^m)$.

Rational models of non-orientable surfaces: $(\chi(R_g) = 2 - g)$
$R_g \sim B_{p_1,\dots,p_g} S^2$, the sphere blown-up at g points
Let $q_1, \dots, q_n \in R_g$ n distinct points (n can be zero.)

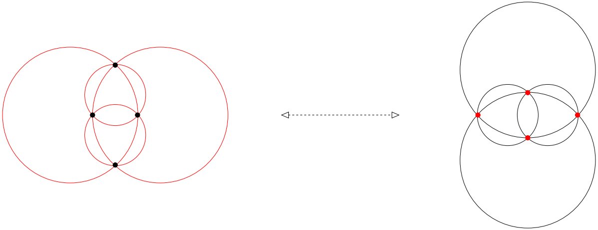
Theorem

$\mathrm{Aut}(R_g, q_1, \dots, q_n)$ *is dense in* $\mathrm{Diff}(R_g, q_1, \dots, q_n)$ *in the* C^∞*-topology on* R_g*.*

Steps of the proof:

1. Marked points
[Huisman, M. 2007: $\mathrm{Aut}(S^m)$ acts ∞-transitively on S^m, $\forall m > 1$]
$\Rightarrow \mathrm{Aut}(S^2, p_1, \dots, p_{g+n})$ is dense in $\mathrm{Diff}(S^2, p_1, \dots, p_{g+n})$ for any finite set of distinct points $p_1, \dots, p_{g+n} \in S^2$.
2. Identity components
[Fragmentation Lemma]
$\Rightarrow \mathrm{Aut}_0(R_g, q_1, \dots, q_n)$ is dense in $\mathrm{Diff}_0(R_g, q_1, \dots, q_n)$.
3. Mapping class group
$\mathrm{Aut}(R_g, q_1, \dots, q_n)$ surjects to $\mathcal{M}(R_g, q_1, \dots, q_n)$.

Cremona transformation with real base points



Factored as:
$S^2 \longleftarrow B_{p_1,\dots,p_4}S^2 \cong B_{q_1,\dots,q_4}S^2 \longrightarrow S^2$

Proposition

Cremona transformations act transitively on isotopy classes of g disjoint Möbius bands in R_g.

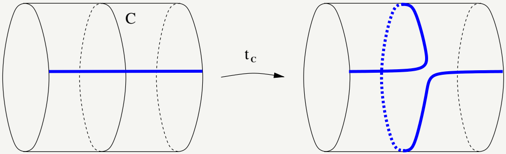
Cremona $\sigma\colon B_{p_1,\dots,p_4}S^2 \cong B_{q_1,\dots,q_4}S^2$, $\exists \Phi \in \mathrm{Aut}(S^2)$ such that $\Phi(p_i) = q_i$,
get $\Phi \circ \sigma$:
$B_{p_1,\dots,p_4}S^2 \xrightarrow{\sigma} B_{q_1,\dots,q_4}S^2 \xrightarrow{\Phi} B_{p_1,\dots,p_4}S^2$

The mapping class group

R smooth compact surface
$\mathcal{M}(R, q_1, \dots, q_n) := \pi_0(\mathrm{Diff}(R, q_1, \dots, q_n))$

Theorem (Dehn 1938)

When R orientable, $\mathcal{M}$ is generated by Dehn twists around simple closed curves:



Theorem

When R non-orientable, Dehn twists generate an index 2 subgroup of $\mathcal{M}$, need to add cross-cap slides.

Reduction of the set of generators

Chillingworth (1969), and Korkmaz (2002) with base points
Recall $R_g = B_{p_1,\dots,p_g} S^2$

Theorem

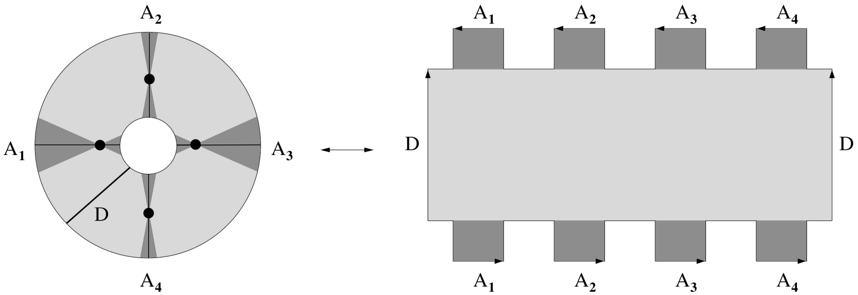
Dehn twists around lifts of simple closed curves of S^2 passing through an even number of the p_i (no self-intersection at the p_i) suffice.

With lantern relation $\Rightarrow$

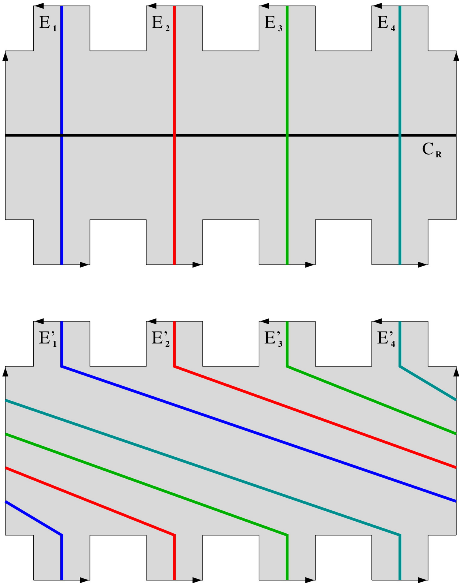
Corollary

Dehn twists around lifts of simple closed curves of S^2 passing through 0, 2 or 4 of the p_i suffice.

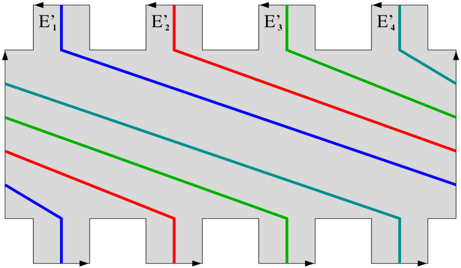
Two models of the annulus blown up in 4 points



The 4 exceptional curves and Dehn twist around C_R



Deformation



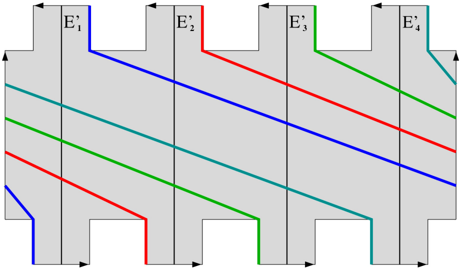
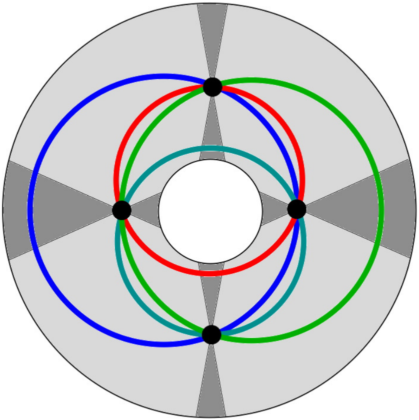


Image of the four exceptional curves



Cremona with 4 real base points represents the Dehn twist around C_R passing through the 4 base points.

Generalizations: geometrically rational surfaces

$S :=$ degree 2 Del Pezzo surface with $\rho(S) = 1$
$C \subset S$ a curve $\Rightarrow C \sim -aK_S$ for some $a \in \mathbb{N}$
So $p_a(C) = (C(C + K_S) - 2)/2 = a(a-1) - 1$ is odd
C real rational $\Rightarrow$ odd number of singular points on $S(\mathbb{C})$.
can not all be complex conjugate $\Rightarrow$ no smooth rational curves on S at all.

Conjecture

Let S be a geometrically rational surface whose real locus $S(\mathbb{R}) \neq S^1 \times S^1$ and that is not isomorphic to a degree 2 Del Pezzo surface with Picard number 1, then every embedded circle can be approximated by smooth rational curves.

Generalizations: rationally connected varieties

We believe that usually not every homotopy class of $X(\mathbb{R})$ can be represented by rational curves.

Let q_1, q_2, q_3 be quadrics such that $C := (q_1 = q_2 = q_3 = 0) \subset \mathbb{P}^4$ is a smooth curve with $C(\mathbb{R}) \neq \emptyset$. Consider the family of 3–folds

$$X_t := \left(q_1^2 + q_2^2 + q_3^2 - t\left(x_0^4 + \cdots + x_4^4\right) = 0\right) \subset \mathbb{P}^4$$

For $0 < t \ll 1$, the real points $X_t(\mathbb{R})$ form an S^2-bundle over $C(\mathbb{R})$.

Conjecture

For $0 < t \ll 1$, every rational curve $g\colon \mathbb{P}^1 \to X_t$ gives a contractible map $g\colon \mathbb{RP}^1 \to X_t(\mathbb{R})$.

Conjecture

Let X be a smooth, rationally connected variety defined over $\mathbb{R}$. Then a $\mathcal{C}^\infty$ map $S^1 \to X(\mathbb{R})$ can be approximated by rational curves iff it is homotopic to a rational curve $\mathbb{RP}^1 \to X(\mathbb{R})$.