BIRATIONAL INVOLUTIONS OF THE REAL PROJECTIVE PLANE FIXING AN IRRATIONAL CURVE

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1. Examples of birational involutions of the projective plane $\mathbb{P}^2$

Let $\Bbbk = \mathbb{R}, \mathbb{C}$.

The first way to describe an element of $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ is to explicitly express it in homogeneous coordinates.

Example 1 (Standard Cremona involution)**.** *The birational map from $\mathbb{P}^2$ to $\mathbb{P}^2$ given by*

$$\alpha_0 \colon [x:y:z] \dashrightarrow [yz:xz:xy]$$

is called the Standard Cremona involution*. It is well-defined except at the three points* $[1:0:0], [0:1:0], [0:0:1]$ (base points) *and is conjugate in* $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ *to the linear involution*

$$\tau_0 \colon [x:y:z] \dashrightarrow [x:y:-z] \, .$$

Another way to define a birational involution of $\mathbb{P}^2$ is to start with a biregular involution on a smooth rational surface which is called a *model*, and then pull it back to $\mathbb{P}^2$ by a birational map.

Example 2 (Geiser involution)**.** *Let S_2 be a complex del Pezzo surface of degree 2 with anti-canonical double cover*

$$\pi \colon S_2 \xrightarrow{2:1} \mathbb{P}^2$$

with branch locus a smooth quartic plane curve C. The surface S_2 is defined by an equation of the form

$$w^2 = f_4(x,y,,z)$$

in $\mathbb{P}(2,1,1,1)$ where f_4 is the homogeneous polynomial of degree 4 whose zero locus in $\mathbb{P}^2$ is the curve C.

The biregular involution $\tau \in \mathrm{Aut}_{\mathbb{C}}(S_2)$ given by

$$\tau \colon [w:x:y:z] \dashrightarrow [-w:x:y:z]$$

exchanges the two sheets of the double cover. The surface S_2 is rational over $\mathbb{C}$ which means there is a birational map

$$\varphi \colon S_2 \dashrightarrow \mathbb{P}^2 \, .$$

Hence the composed map $\alpha = \varphi\tau\varphi^{-1}$ is an order 2 element of the group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ classically called the Geiser *involution.*

Assume now that S_2 is defined over $\mathbb{R}$ (i.e. f_4 has real coefficients). If the real locus $S_2(\mathbb{R})$ is non empty and connected (for the euclidean topology), then S_2 is rational over $\mathbb{R}$ by Comessatti's Theorem (see e.g. [Man20]*), and we can assume that φ is defined over $\mathbb{R}$.*

The biregular automorphism $\tau \in \mathrm{Aut}_{\mathbb{R}}(S_2)$ leads to a birational involution $\alpha \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Observe that α is only defined up to conjugation in $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ as the map φ is not uniquely defined. when we say "the" Geiser involution, we speak in fact of the conjugacy class of α in $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$.

This review is an elaboration of a presentation given at the *Real algebraic geometry and singularities conference in honor of Wojciech Kucharz's 70th birthday* in Krakow in 2022.

2. Reduction to G-birational classification of minimal del Pezzo surfaces and conic bundles

In fact we can reverse and generalize the process used in the second example by *regularizing* any birational involution of $\mathbb{P}^2$.

From now on, we let $G = \mathbb{Z}/2$.

Proposition. *Let $\Bbbk = \mathbb{R}, \mathbb{C}$ and $\alpha \in \mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ be an element of order 2. There exists a smooth rational surface S and a birational map defined over $\Bbbk$*

$$\varphi \colon S \dashrightarrow \mathbb{P}^2 .$$

such that $\tau := \varphi^{-1}\alpha\varphi \in \mathrm{Aut}_{\Bbbk}(S)$ is a biregular involution of S.

Definition. *Let S be a smooth surface endowed with a nontrivial biregular involution τ. The subgroup $H = \langle \tau \rangle \subset \mathrm{Aut}_{\Bbbk}(S)$ is then isomorphic to G and the pair (S,H) is called a G-*surface.

Proposition. *Let $\alpha, \alpha' \in \mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ be elements of order 2, $(S, \langle \tau \rangle)$ and $(S', \langle \tau' \rangle)$ associated rational G-surfaces. Then α, α' are conjugated in $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$ if and only if the associated G-surfaces are equivariantly birational. That is there exists a birational map $\varphi \colon S \dashrightarrow S'$ such that $\varphi\tau\varphi^{-1} = \tau'$ over a Zariski dense open subset of S.*

Hence to classify conjugacy classes of elements of order 2 in $\mathrm{Bir}_{\Bbbk}(\mathbb{P}^2)$, we classify equivariant birational classes of rational G-surfaces. For this purpose, take a G-surface $(S, \langle \tau \rangle)$ rational over $\Bbbk$ and run a G-MMP over $\Bbbk$ (see [Kol97]) which ends with a pair $(S^*, \langle \tau^* \rangle)$. There are two possibilities for S^*:

Proposition. *Let $\Bbbk = \mathbb{R}, \mathbb{C}$ and (S,H) be a G-surface rational over $\Bbbk$. Denote by (S^*, H^*) the output of a G-MMP over $\Bbbk$. Then S^* belong to one of the two following classes:*

(DP) S^* is a del Pezzo surface such that $\mathrm{Pic}^G(S^*) \simeq \mathbb{Z}$;

(CB) S^* admits a G-conic bundle structure over $\mathbb{P}^1$ and $\mathrm{Pic}^G(S^*) \simeq \mathbb{Z}^2$;

Here the action of G on $\mathrm{Pic}(S^*)$ is given by H^*.

When the hypothese on the invariant part of the Picard group is satisfied we say the G-surface, DP or CB, is *minimal*.

The initial problem of classification of conjugacy classes of birational involutions is now reduced to the G-equivariant birational classification of minimal G-surfaces belonging to the set $(DP) \cup (CB)$. In fact in [CMYZ24] we went further and gave explicit models of all such pairs.

The two former examples are in (DP): $(S,H) = (\mathbb{P}^2, \alpha_0)$ and $(S^*, H^*) = (\mathbb{P}^2, \tau_0)$ for the first example; $(S,H) = (\mathbb{P}^2, \alpha)$ and $(S^*, H^*) = (S_2, \tau)$ in the second example.

3. Main invariant: the fixed curve

Recall that a real variety X is *geometrically rational* if its complexification $X_{\mathbb{C}}$ is rational. For example, a smooth geometrically rational real curve C is rational if and only if $C(\mathbb{R}) \neq \varnothing$. A complex variety is rational if and only if it is geometrically rational.

Proposition. *Let (S,H) and (S',H') be G-surfaces and $\varphi \colon S \dashrightarrow S'$ a G-equivariant rational map. If C is a geometrically irrational curve on S, its proper transform $C' := \varphi(C)$ is a geometrically irrational curve on S'. If furthermore C is fixed by H then C' is fixed by H'. If φ is birational, the curves C and C' are birational. They are isomorphic if they are smooth.*

Proof. The proper transform C' is obtained from C in the following way: let C_0 be the image of φ of the open subset of C where φ is defined. The set C_0 is a curve because φ contracts only geometrically rational curves. Then let C' be the Zariski closure of C_0 in S'. □

Definition. *Let S be a rational surface over $\Bbbk$ and $\tau \in \mathrm{Aut}_{\Bbbk}(S)$ an element of ordre 2. Define $F(\tau)$ the normalization of the union of geometrically irrational curves fixed by τ. In particular, $F(\tau) = \varnothing$ if τ fixes no geometrically irrational curve.*

From the discussion above, $F(\tau)$ is a conjugacy invariant.

Remark. *In fact we can prove that in our context, we get exactly two cases, see* [CMYZ24, Lemma 2.7]*:*

(1) $F(\tau) = \varnothing$*, or*
(2) $F(\tau) = C$ *where* C *is a smooth geometrically irreducible curve of genus* $g \geq 1$.

Returning to the two former examples, we get $F(\tau) = \varnothing$ for the standard Cremona involution and $F(\tau)$ is the smooth non hyperelliptic curve of genus 3 given by the equation $f_4 = 0$ in $\mathbb{P}^2$ in the second example.

4. Main result

4.1. **Classification over** $\mathbb{C}$. Using an equivariant version of Mori theory in dimension two as discussed above, L. Bayle and A. Beauville obtained a very precise classification (see [BB00]):

Theorem (Bayle-Beauville 2000)**.** *Let* α *be an element of order* 2 *in the group* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$*. Then* α *is conjugate in* $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ *to one and only one of the following* 4 *classes of involution:*

(1) The linear involution on $\mathbb{P}^2$ *given by* $\tau_0\colon [x:y:z] \dashrightarrow [x:y:-z]$.
   $F(\alpha) = \varnothing$.
(2) A Bertini involution (analogously to Example 2, a biregular model is the deck involution of a del Pezzo surface of degree 1 *given by a double cover of the quadric cone with branch locus the fixed curve of the involution).*
   $F(\alpha)$ *is a smooth non hyperelliptic curve of genus* $g = 4$ *canonically embedded in a quadric cone.*
(3) A Geiser involution (see Example 2).
   $F(\alpha)$ *is a smooth non hyperelliptic curve of genus* $g = 3$.
(4) A de Jonquières involution (see Section 5.1).
   $F(\alpha)$ *is a smooth hyperelliptic curve of genus* $g \geq 1$.

Except for the case (1), all these involutions have moduli [BB00]. Namely, conjugacy classes of de Jonquières involutions of genus $g \geqslant 1$ are parametrized by hyperelliptic curves of genus $g \geqslant 1$. Conjugacy classes of Geiser involutions are parametrized by non-hyperelliptic curves of genus 3, and conjugacy classes of Bertini involutions are parametrized by non-hyperelliptic curves of genus 4 canonically embedded in a quadric cone.

4.2. **Classification over** $\mathbb{R}$**.** The first new involution in this context is the antipodal map on the quadric sphere. In [CMYZ24], we discovered 7 additional classes of involutions in $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and called them d-twisted Trepalin involutions, $d = 0,1,2$, Kowalevskaya involution and d-twisted Iskovskikh involutions $d = 0,1,2$, see Section 5.2 for Iskovskikh involutions and [CMYZ24] for the definitions of the others.

Main Theorem (Cheltsov-Mangolte-Yasinsky-Zimmermann 2024)**.** *Let* α *be an element of order* 2 *in the group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$*. Then* α *is conjugate in* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *to one of the following* 12 *classes of involution:*

(1) The linear involution τ_0 *or the antipodal involution on the quadric sphere or a* t*-twisted Trepalin involution,* $t = 0,1,2$.
   $F(\alpha) = \varnothing$*:*
(2) A Bertini involution.
   $F(\alpha)$ *is a smooth non hyperelliptic curve of genus* $g = 4$ *canonically embedded in a quadric cone.*
(3) A Geiser involution.
   $F(\alpha)$ *is a smooth non hyperelliptic curve of genus* $g = 3$.
(4) A Kowalevskaya involution.
   $F(\alpha)$ *is a smooth elliptic curve.*
(5) A de Jonquières involution or a t*-twisted Iskovskikh involutions* $t = 0,1,2$ *(see Section 5.2).*
   $F(\alpha)$ *is a smooth hyperelliptic curve of genus* $g \geq 1$.

Furthermore, involutions in different classes are not conjugate except some exceptions when the fixed curve is elliptic, see [CMYZ24, Main Theorem] for details.

In contrast with the complex case, fixed curves does not parametrize conjugacy classes. See [CMYZ24, Main Corollary]

Main Corollary. *Let* C *be a real smooth projective hyperelliptic curve of genus* $g \geqslant 2$ *such that the locus* $C(\mathbb{R})$ *consists of at least* 2 *connected components. Then* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *contains uncountably many non-conjugate* *involutions that all fix a curve isomorphic to the curve C. Besides, the real plane Cremona group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *contains uncountably many non-conjugate involutions that fix no geometrically irrational curves.*

5. BIRATIONAL MODELS OF G-CONIC BUNDLES OVER $\mathbb{R}$

To illustrate the methods used in the proof of the main theorem above, we will focus now on the case where S^* is in the case (CB), see Section 2.

Let $\Bbbk = \mathbb{R}, \mathbb{C}$ and S be a smooth surface defined over $\Bbbk$ endowed with a biregular involution τ. Assume that S admits a G-equivariant morphism $\pi\colon S \to \mathbb{P}^1$ whose fibers are conics. Assume furthermore that $\mathrm{Pic}^G(S^*) \simeq \mathbb{Z}^2$. We have the following, [CMYZ24, Lemma 2.7] for a proof.

Lemma. *If $C = F(\tau)$ is a geometrically irrational curve, then τ acts trivially on the base $\mathbb{P}^1$ and C is a double section of π.*

Assume from now on that τ acts trivially on the base of the G-conic bundle $\pi\colon S \to \mathbb{P}^1$. In this case, a general complex fiber of π is a smooth conic on which τ restricts to an involution and there is a finite number of singular fibers which are unions of two smooth complex rational curves F_1, F_2 intersecting transversally in one point. Each F_i, $i = 1,2$ is a (-1)-curve on the complexification $S_{\mathbb{C}}$ of S.

Over $\mathbb{C}$, for each singular fiber, we must have $\tau(F_i) = F_{3-i}$ because $\mathrm{Pic}^G(S^*) \simeq \mathbb{Z}^2$.

Over $\mathbb{R}$, denoting by σ the real structure on the complexification $S_{\mathbb{C}}$ (σ is an anti-holomorphic involution on $S_{\mathbb{C}}$) , at least one of the two involutions τ or σ must exchanges F_1 and F_2 for the same reason.

5.1. De Jonquières involutions.

Firstly assume that π admits a section Z defined over $\Bbbk$. Then $Z + \tau(Z)$ is G-invariant and there exists a G-equivariant birational map $\chi\colon S \dashrightarrow X$ that fits into the following commutative G-equivariant diagram:

$$\begin{tikzcd}
S \arrow[r, dashed, "\chi"] \arrow[d, "\pi"'] & X \arrow[r, "\rho"] \arrow[d, "\eta"] & Y \arrow[dl, dashed] \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 &
\end{tikzcd}$$

where X is a smooth surface, η is a conic bundle such that $\mathrm{Pic}(X)^G \simeq \mathbb{Z}^2$, Y is a hypersurface in $\mathbb{P}(d,d,1,1)$ of degree $2d = 8 - K_S^2$ that is given by

$$xy = f(z,t)$$

for some homogeneous polynomial $f(z,t)$ of degree $2d$ that has no multiple roots. The map $Y \dashrightarrow \mathbb{P}^1$ is given by

$$\big[x : y : z : t\big] \mapsto \big[z : t\big],$$

where x, y, z and t are coordinates on $\mathbb{P}(d,d,1,1)$ of weights d, d, 1 and 1, respectively.

The curves Z and $\tau(Z)$ are $\rho \circ \chi$-exceptional, the involution τ acts on the surface Y as

$$\big[x : y : z : t\big] \mapsto \big[y : x : z : t\big],$$

and the morphism ρ is a minimal resolution of singularities.

The fixed locus of τ is the curve $C \simeq \rho(C)$, where $\rho(C)$ is given by

$$\begin{cases} x = y, \\ x^2 = f(z,t). \end{cases}$$

If $d \geqslant 3$, then $\rho(C)$ is a real hyperelliptic curve of genus $g = d - 1$ with hyperelliptic covering

$$\nu\colon C \to \mathbb{P}^1, \quad [x : y : z : t] \mapsto [z : t].$$

Similarly, if $d = 2$, then $\rho(C)$ is an elliptic curve. The number of real roots r of f is even and the number of connected components of $C(\mathbb{R})$ is $\frac{1}{2}r$.

If $\Bbbk = \mathbb{C}$, forgetting the action of G, we can always contract one of the (-1)-curves in any singular fiber of π and obtain a locally trivial $\mathbb{P}^1$-fibration $S' \to \mathbb{P}^1$ (S' is an Hirzebruch surface). Any such fibration has a complex section whose pullback Z is a section of $\pi\colon S \to \mathbb{P}^1$. Hence any G-surface for which S^* is in the case (CB) admits such a model $Y \dashrightarrow \mathbb{P}^1$. In this case, τ is called a *de Jonquières* involution.

If $\Bbbk = \mathbb{R}$, we cannot contract a (-1)-curve if it's not defined over $\mathbb{R}$ (case $\sigma(F_i) = F_{3-i}$). So we need to consider another model when $\pi\colon S \to \mathbb{P}^1$ has no real section.

As a step in the classification, we get the following characterization of de Jonquières involutions over $\mathbb{R}$, see [CMYZ24, Proposition 6.5].

Proposition. *Let S be a real rational surface and $\pi\colon S \to \mathbb{P}^1$ a G-conic bundle with biregular involution τ acting trivially on the base and such that $\mathrm{Pic}^G(S) \simeq \mathbb{Z}^2$. Then τ is a de Jonquières involution if and only if*

$$\pi(S(\mathbb{R})) = \mathbb{P}^1(\mathbb{R}) \approx S^1 .$$

5.2. **d-twisted Iskovskikh involutions.** In the case $\pi(S(\mathbb{R})) \subsetneq \mathbb{P}^1(\mathbb{R})$, we prove first the existence of a good model in Theorem 1, then its unicity in Theorem 2, see [CMYZ24, Theorems 7.1 and 7.6].

Theorem 1. *Let $\pi\colon S \to \mathbb{P}^1$ be a minimal real rational G-conic bundle with biregular involution τ acting trivially on the base. Assume that $\pi(S(\mathbb{R})) \subsetneq \mathbb{P}^1(\mathbb{R})$. Then there exists G-equivariant commutative diagram*

$$\begin{tikzcd} S \arrow[r, dashed, "\chi"] \arrow[d, "\pi"'] & X \arrow[d, "\eta"] \\ \mathbb{P}^1 \arrow[r, "\phi"'] & \mathbb{P}^1 \end{tikzcd}$$

where χ is a birational map, $\phi \in \mathrm{PGL}_2(\mathbb{R})$, X is a smooth surface, η is a G-minimal conic bundle, the fiber $\eta^{-1}([1:0])$ is smooth and does not have real points, the quasi-projective surface $Y = X \setminus \eta^{-1}([1:0])$ is given in $\mathbb{P}^2 \times \mathbb{A}^1$ with coordinates $([x:y:z],t)$

$$A(t)x^2 + B(t)xy + C(t)y^2 = H(t)z^2$$

for some polynomials $A,B,C,H \in \mathbb{R}[t]$ such that $\Delta = (B^2 - 4AC)H$ does not have multiple roots and $\deg(\Delta)$ *is even, the involution τ acts on the surface Y by*

$$([x:y:z],t) \mapsto ([x:y:-z],t),$$

and the restriction map $\eta|_Y\colon Y \to \mathbb{P}^1 \setminus [1:0] = \mathbb{A}^1$ is the map given by $([x:y:z],t) \mapsto t$. Moreover, the following holds:

- *the polynomial $H(t)$ has only real roots and its leading coefficient is negative,*
- *fibers of η over roots of the polynomial $H(t)$ are singular irreducible conics ($\sigma(F_i) = F_{3-i}$).*

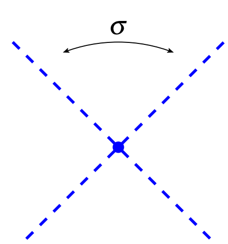

FIGURE 1. Singular irreducible real fiber

Definition. *In the assumptions of Theorem 1, let $d = \deg H$. We call τ a d-*twisted Iskovskikh involution*.*

The fixed curve $C = F(\tau)$ is given by

$$w^2 = B^2 - 4AC . \tag{1}$$

It is elliptic if $\deg(B^2 - 4AC) = 4$ and hyperelliptic if $\deg(B^2 - 4AC) \geq 6$.

Indeed, the fixed curve is given by $z = 0$ which gives $Ax^2 + Bxy + Cy^2 = 0$. Letting $w := 2(\frac{x}{y}A + \frac{1}{2}B)$ we get (1).

Theorem 2. *In the assumptions of Theorem 1, two G-conic bundles $\eta : X \to \mathbb{P}^1, \eta : X' \to \mathbb{P}^1$ are G-equivariantly birational if and only if*

(1) They have same discriminant loci $\{\Delta = 0\} = \{\Delta' = 0\}$ *;*
(2) They have the same real interval $\eta(X(\mathbb{R})) = \eta'(X'(\mathbb{R}))$.
(3) Sign conditions: $B^2 - 4AC = \lambda(B'^2 - 4A'C')$ *and* $H = \mu H'$ *for some positive real numbers* λ, μ.

Note that the third condition implies the first one. I put them like this for didactical purposes.

We conclude this note by giving explicit proof of the main corollary in the case C irrational (see [CMYZ24, Section 8.C] for details).

Let C be an hyperelliptic curve given by

$$w^2 = -4f(t)$$

where $f \in \mathbb{R}[t]$ has even degree ≥ 6, only simple roots and at least 4 real roots. For given real numbers a, b, let $S_{a,b}$ be the surface with equation

$$x^2 + f(t)y^2 = -(t-a)(t-b)z^2 \ .$$

Let $\tau_{a,b}$ be the corresponding involution. Then for general a, b, a', b' in a given interval (we need to preserve the connectedness of the real locus of the surface), $\tau_{a,b}$ and $\tau_{a',b'}$ are not conjugate by Theorem 2.

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