FAKE REAL PLANES: EXOTIC AFFINE ALGEBRAIC MODELS OF $\mathbb{R}^2$

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Abstract. We study real rational models of the euclidean plane $\mathbb{R}^2$ up to isomorphisms and up to birational diffeomorphisms. The analogous study in the compact case, that is the classification of real rational models of the real projective plane $\mathbb{RP}^2$ is well known: up to birational diffeomorphisms, there is only one model. A fake real plane is a nonsingular affine surface defined over the reals with homologically trivial complex locus and real locus diffeomorphic to $\mathbb{R}^2$ but which is not isomorphic to the real affine plane. We prove that fake planes exist by giving many examples and we tackle the question: does there exist fake planes whose real locus is not birationally diffeomorphic to the real affine plane?

Introduction

An algebraic complexification of a real smooth $\mathcal{C}^\infty$-manifold M is a smooth complex quasi-projective algebraic variety V endowed with an anti-regular involution σ such that M is diffeomorphic to the *real locus* V^σ of V. Equivalently V is the complex model $X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ of a smooth quasi-projective algebraic variety X defined over $\mathbb{R}$, whose set of $\mathbb{R}$-rational points is diffeomorphic to M when equipped with the euclidean topology. Some manifolds such as real projective spaces $\mathbb{RP}^n$ and real euclidean affine spaces $\mathbb{R}^n$ have natural algebraic complexifications, given by the complex projective and affine spaces $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{A}^n_{\mathbb{C}}$ respectively. But these also admit infinitely many other complexifications, and it is a natural problem to try to classify them up to appropriate notions of equivalence.

For example, the real projective plane $M = \mathbb{RP}^2$ is also the real locus of the quotient V_1 of the smooth quartic hypersurface $x^4 + y^4 + z^4 - w^4 = 0$ in $\mathbb{P}^3_{\mathbb{C}}$ by the fixed point-free involution $[x : y : z : w] \mapsto [-y : x : -w : z]$. The complex surface V_1 endowed with the induced real structure is not isomorphic to $V_0 = \mathbb{P}^2_{\mathbb{C}}$ endowed with the usual complex conjugation, and actually, since V_1 is an Enriques surface (see e.g. [1, Chapter VIII]), V_1 and V_0 are not even birational to each other. On the other hand, we obtain infinite families of pairwise non-isomorphic rational projective complexifications V of $\mathbb{RP}^2$ by blowing-up sequences of pairs of non-real complex conjugate points of $\mathbb{P}^2_{\mathbb{C}}$.

Thus whatever is the equivalence relation: biregular, birational or deformation, there are an infinite number of possibilities. It is natural to pay attention to complexifications with "minimal topology", say in the sense of Betti numbers. Recall that a *fake projective plane*, as defined by Mumford [29], is a nonsingular complex projective surface S, whose Betti numbers are those of $\mathbb{CP}^2$, and which is not biregularly isomorphic to $\mathbb{P}^2_{\mathbb{C}}$. One could define a *fake real projective plane* to be a complex fake projective plane with anti-regular involution, whose real locus is diffeomorphic to $\mathbb{RP}^2$ and which is not biregularly isomorphic to $\mathbb{P}^2_{\mathbb{R}}$, but despite of the existence of 100 fake projective planes up to biregular isomorphism [32, 33, 5], none of them admits a real structure as proved by Kulikov and Kharlamov [25, Thm. 5.1]. Thus there is no fake real projective plane at all.

In addition to $\mathbb{RP}^2$, the connected compact surfaces M diffeomorphic to the real locus of smooth rational projective surface minimal over $\mathbb{R}$ are the sphere S^2, the torus $S^1 \times S^1$ and the Klein bottle $K = \mathbb{RP}^2 \# \mathbb{RP}^2$. Their respective minimal complexifications are $\mathbb{P}^2_{\mathbb{C}}$, the quadric hypersurface $x^2 + y^2 + z^2 - w^2 = 0$ in $\mathbb{P}^3_{\mathbb{C}}$, the Hirzebruch surfaces of even index $\mathbb{F}_{2k}$, $k \geq 0$, and the Hirzebruch surfaces of odd index $\mathbb{F}_{2k+1}$, $k \geq 1$. The minimality of the complexification V endowed with the real structure is equivalent to the minimality of its topology as a compact complex manifold among all complexifications of M. The above description

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shows that, for each of the surfaces $\mathbb{RP}^2$, S^2, $S^1 \times S^1$ and K, complexifications with minimal topology are either unique, or all diffeomorphic to each others, belonging to a unique equivalence class of deformation.

Coming back to surfaces obtained by blowing-up sequences of pairs of non-real conjugated points of $\mathbb{P}^2_{\mathbb{C}}$, it follows from their construction that they are all $\mathbb{R}$-biregularly birationally equivalent: this means that for every pair V and V' of such complexifications, there exists birational maps $\varphi : V \dashrightarrow V'$ and $\varphi' \colon V' \dashrightarrow V$ inverse to each other and whose restrictions to the real loci of V and V' are diffeomorphisms inverse to each other. A striking result of Biswas and Huisman [4] (see also [19]) asserts that rational algebraic complexifications of a given smooth connected compact surface M are all $\mathbb{R}$-biregularly birationally equivalent. This classification result gave rise to many further discoveries, see the survey article [26] and the bibliography given there.

In this paper, we lay the importance on affine complexifications and we discover that, contrary to the projective case, the easiest example $M = \mathbb{R}^2$ possesses a lot of affine complexifications S with "minimal topology" which are not biregularly isomorphic to $\mathbb{A}^2$. We call them *fake real planes*. In contrast with the projective case, where the notion of rational complexification with minimal topology is unambiguous due to the fact that the topology of a smooth rational complexification V of a given compact surface M is fully determined by its Picard rank $\rho(V) = \mathrm{rk}(N_1(V))$, there are many possibilities to define a notion of minimality of the topology of an affine complexification, even in the rational case. For instance, there exists many rational affine complexifications of $\mathbb{R}^2$ with vanishing second homology group $H_2(S;\mathbb{Z})$ but nontrivial fundamental group. Inspired by the work of Totaro [37], who defined a *good affine complexification* of M to be a complexification S for which the inclusion of $M \hookrightarrow S$ of M as the real locus of S is a homotopy equivalence, a natural notion of minimality of affine complexifications S of $M = \mathbb{R}^2$ is to require that S is contractible as a smooth complex manifold. Here we mainly consider weaker variants in which we require only that the inclusion $M \hookrightarrow S$ induces an isomorphism between the respective homology groups of M and S, taken with integral or rational coefficients. So for $M = \mathbb{R}^2$, the corresponding smooth affine complexifications S are respectively $\mathbb{Z}$-acyclic and $\mathbb{Q}$-acyclic complex surfaces.

Our first goal is to show that fake real planes do exist by exhibiting many examples and to give elements of classification of these objects up to biregular isomorphism depending on natural algebro-geometric invariants such as their logarithmic Kodaira dimension. Any contractible affine complexification S of $\mathbb{R}^2$ of non positive logarithmic Kodaira dimension is isomorphic to $\mathbb{A}^2$ ([27, Theorem 4.7.1 (1), p. 244] and [28]), in particular, there is no good affine complexification of $\mathbb{R}^2$ of logarithmic Kodaira dimension 0. In contrast, we give several families of rational good affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 1 (see Examples 3.5) and 2 respectively, which are therefore not biregularly isomorphic to $\mathbb{A}^2$. Another striking family of examples is provided by some Ramanujam surfaces [34], a famous class of smooth complex contractible affine surfaces of logarithmic Kodaira dimension 2, which admit real structures with real locus diffeomorphic to $\mathbb{R}^2$ (see 3.8).

As a step towards a classification of fake planes, we establish a real counter-parts of a series of results due to thom Dieck and Petrie [10] describing the structure of $\mathbb{Z}$-acyclic and $\mathbb{Q}$-acyclic smooth complex affine surfaces in terms of blow-ups of arrangements of rational curves in $\mathbb{P}^2_{\mathbb{C}}$. As an application, we obtain a complete classification of $\mathbb{Z}$-acyclic smooth affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 1 (Theorem 3.2) and a precise description of the structure of $\mathbb{Q}$-acyclic smooth affine complexifications of $\mathbb{R}^2$ of logarithmic Kodaira dimension 2 (Theorem 3.9) formulated in terms of arrangements of $\mathbb{R}$-rational curves in $\mathbb{P}^2_{\mathbb{R}}$.

In a second step, we tackle the classification of fake real planes with $\mathbb{Q}$-acyclic smooth affine complexifications up to $\mathbb{R}$-biregular birational equivalence: we prove that a large class of such surfaces are $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2$. More precisely, we establish that every smooth $\mathbb{Q}$-acyclic complexification S of $\mathbb{R}^2$ with negative logarithmic Kodaira dimension admits a surjective morphism $\pi \colon S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, with general fiber isomorphic to the affine line (Theorem 4.1), and we show that every such fibered surface $\pi : S \to \mathbb{A}^1$ with at most one singular fiber is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2$ (Theorem 4.9).

We saw above that for $\mathbb{RP}^2$, S^2, $S^1 \times S^1$ and $K = \mathbb{RP}^2 \# \mathbb{RP}^2$, there exists either a unique minimal rational complexification or at most one family of pairwise non isomorphic but $\mathbb{R}$-biregularly birationally equivalent minimal rational complexifications. We conclude the paper with the construction of an infinite number of fake planes with moduli of arbitrary positive dimension of pairwise non isomorphic, deformation equivalent, $\mathbb{Q}$-acyclic euclidean planes all $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$ (see § 5.2).

In contrast with the projective case, the main difficulty to understand the notion of $\mathbb{R}$-biregular birational equivalence comes from the lack of natural numerical invariants to distinguish classes. In particular, neither the topology of the complexification of a given fake real plane S nor its logarithmic Kodaira dimension are invariants of its $\mathbb{R}$-biregular birational class. And even though Theorem 4.9 is a significant step towards a complete classification of fake real planes up to $\mathbb{R}$-biregular birational equivalence, the fact that its proof depends on the construction of explicit elementary $\mathbb{R}$-biregular birational links between appropriate projective models of the affine complexification of S does not give any clear insight on possible numerical invariants of $\mathbb{R}$-biregular birational equivalence classes. As a consequence, the question of existence of fake real planes not $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2$ remains open, a good candidate being the real Ramanujam surfaces mentioned above (see also § 5.1 for another candidate of logarithmic Kodaira dimension 0).

Contents

1. Preliminaries

In this article, the term **k**-variety will always refer to a geometrically integral scheme X of finite type over a base field **k** of characteristic zero. A morphism of **k**-varieties is a morphism of **k**-schemes. In the sequel, **k** will be most of the time equal to either $\mathbb{R}$ or $\mathbb{C}$, and we will say that X is a real, respectively complex, algebraic variety.

A complex algebraic variety X will be said to be defined over $\mathbb{R}$ if there exists a real algebraic variety X_0 and an isomorphism of complex algebraic varieties between X and the *complexification* $X_{0,\mathbb{C}} = X_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ of X_0, where $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is the morphism induced by the usual inclusion $\mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[x]/(x^2+1)$. We will often use the shorter notation $X_{0,\mathbb{C}} = X_0 \otimes_{\mathbb{R}} \mathbb{C}$. A complex variety of the form $X_{0,\mathbb{C}}$ is naturally endowed with an additional action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ given by the anti-regular involution $\sigma = \mathrm{id}_{X_0} \times (x \mapsto -x)$, which we call the *real structure* on $X_{0,\mathbb{C}}$.

1.1. **Points and curves on surfaces.**

Notation 1.1. Given a real algebraic surface S, we denote by $S(\mathbb{R})$ and $S(\mathbb{C})$ the sets of $\mathbb{R}$-rational and $\mathbb{C}$-rational points respectively. We always consider $S(\mathbb{R})$ as a subset of $S(\mathbb{C})$ via the map induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. In what follows, the elements of $S(\mathbb{R})$, $S(\mathbb{C})$ and $S(\mathbb{C}) \setminus S(\mathbb{R})$ will be frequently referred to as *real points*, *complex points* and *non-real points* of S respectively.

The subset $(S_{\mathbb{C}}(\mathbb{C}))^{\sigma}$ of $S_{\mathbb{C}}(\mathbb{C})$ consisting of $\mathbb{C}$-rational points of $S_{\mathbb{C}}$ that are fixed by the real structure σ is called the *real locus* of $S_{\mathbb{C}}$, and we identify it with $S(\mathbb{R})$ in the natural way.

By a *curve* on a surface S defined over $\mathbf{k}$, we mean a geometrically reduced closed sub-scheme $C \subset S$ of pure codimension 1 defined over $\mathbf{k}$. We denote by $\mathbb{Z}\langle C\rangle$ the free abelian group generated by the irreducible components of C.

Definition 1.2. 1) A *Smooth Normal Crossing* (*SNC*) *divisor* B on a smooth surface S defined over $\mathbf{k}$ is a curve B on S whose base extension $B_{\overline{\mathbf{k}}}$ to the algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$ has smooth irreducible components and ordinary double points only as singularities. Equivalently, for every closed point $p \in B_{\overline{\mathbf{k}}} \subset S_{\overline{\mathbf{k}}}$, the local equations of the irreducible components of $B_{\overline{\mathbf{k}}}$ passing through p form a part of a regular sequence in the maximal ideal $\mathfrak{m}_{S_{\overline{\mathbf{k}}},p}$ of the local ring $\mathcal{O}_{S_{\overline{\mathbf{k}}},p}$ of $S_{\overline{\mathbf{k}}}$ at p.

We say that B is a *strictly SNC divisor* if every two of the irreducible components of $B_{\overline{\mathbf{k}}}$ intersect in at most one point.

2) The *dual graph* $\Gamma B = (\Gamma_v B, \Gamma_e B)$ of an SNC divisor B on a smooth surface S defined over an algebraically closed field is the graph with vertex set $\Gamma_v B$ the set of irreducible components of B and with edges set $\Gamma_e B$ the set of double points of B. An edge in ΓB connects the two vertices which intersect in it. Note that B is a strictly SNC divisor if and only if any two vertices of its dual graph are connected by at most one edge.

Definition 1.3. Let V be a smooth surface defined over $\mathbf{k}$, and let $\overline{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$.

1) A *geometrically rational tree* on V is an SNC divisor B defined over $\mathbf{k}$ such that every irreducible component of $B_{\overline{\mathbf{k}}} \subset S_{\overline{\mathbf{k}}}$ is a $\overline{\mathbf{k}}$-rational complete curve and the dual graph $\Gamma(B_{\overline{\mathbf{k}}})$ is a tree.

2) A *geometrically rational chain* on V is a geometrically rational tree B defined over $\mathbf{k}$ such that $\Gamma(B_{\overline{\mathbf{k}}})$ is a chain. The irreducible components $B_0, \ldots, B_r$ of $B_{\overline{\mathbf{k}}}$ can be ordered in such a way that $B_i \cdot B_j = 1$ if $|i - j| = 1$ and 0 otherwise. A geometrically rational chain B with such an ordering on the set of irreducible components of $B_{\overline{\mathbf{k}}}$ is said to be *oriented*. The components B_0 and B_r are called respectively the left and right boundaries of B, and we say by extension that an irreducible component B_i of $B_{\overline{\mathbf{k}}}$ is on the left of another one B_j when $i < j$. The sequence of self-intersections $[B_0^2, \ldots, B_r^2]$ is called the *type* of the oriented geometrically rational chain B.

An *oriented* $\mathbf{k}$*-subchain* of B is a geometrically rational chain Z whose support is contained in that of B. We say that an oriented geometrically rational chain B is composed of $\mathbf{k}$-subchains $Z_1, \ldots, Z_s$ and we write $B = Z_1 \triangleright \cdots \triangleright Z_s$ if the Z_i are oriented $\mathbf{k}$-subchains of B whose union is B and the irreducible components of $Z_{i,\overline{\mathbf{k}}}$ precede those of $Z_{j,\overline{\mathbf{k}}}$ for $i < j$.

1.2. **Birational maps and log-resolutions.**

Recall that the domain of definition of a rational map $\varphi : X \dashrightarrow Y$ between two $\mathbf{k}$-schemes X and Y is the largest open subset dom_{φ} on which φ is represented by a morphism. We say that φ is regular at a closed point x if $x \in \mathrm{dom}_{\varphi}$. A rational map $\varphi : X \dashrightarrow Y$ is called *birational* if it admits a rational inverse $\psi : Y \dashrightarrow X$.

In the sequel, we will frequently make use of the following type of birational morphisms:

Example 1.4. (Subdivisional expansion of a surface at a point [10, § 2.4]). Let S be a smooth surface defined over $\mathbf{k}$ and let $p \in S$ be a closed point with residue field $\kappa(p)$. A *subdivisionial expansion with center at* p is a birational morphism $\tau : S' \to S$ restricting to an isomorphism over $S \setminus \{p\}$ and such that $\tau^{-1}(p)$ is a chain of smooth $\kappa(p)$-rational curves, containing a unique irreducible component $A_0(p) \simeq \mathbb{P}^1_{\kappa(p)}$ with normal bundle of degree $-\deg \kappa(p)/\mathbf{k}$. Given an ordered sequence of regular parameters $(x_-, x_+) \in \mathfrak{m}_{S,p}$ in the local ring $\mathcal{O}_{S,p}$ of S at p, there exists a unique pair of coprime integers $1 \leq \mu_- \leq \mu_+$ such that $\tau : S' \to S$ coincides with the minimal resolution of the indeterminacies at p of the rational map $x_+^{\mu_+}/x_-^{\mu_-} : S \dashrightarrow \mathbb{P}^1$ (see [39, Theorem 2.6 (d)]). For instance, the particular case $\mu_{\pm} = 1$ is nothing but the blow-up $\tau : S' = \mathrm{Proj}_S(\bigoplus_{n\geq 0} \mathcal{I}_p^n) \to S$ of S at p, where $\mathcal{I}_p \subset \mathcal{O}_S$ denotes the ideal sheaf of p.

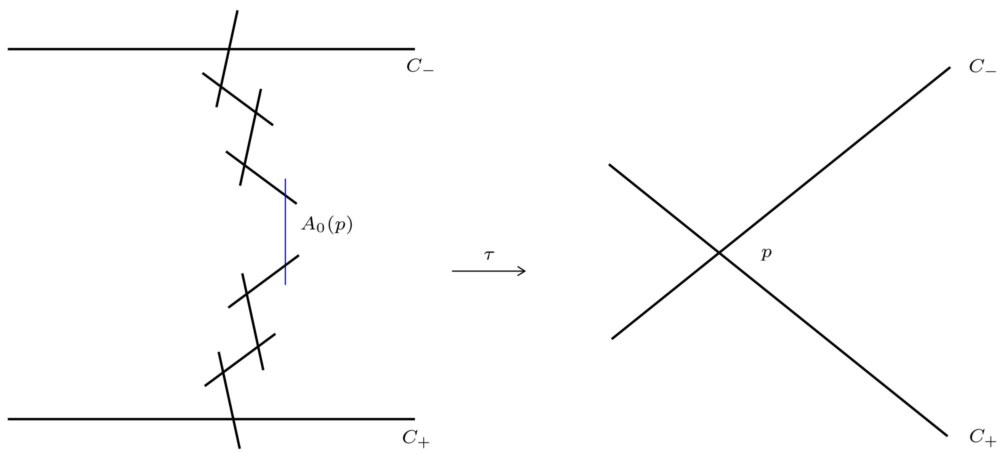

FIGURE 1.1. A subdivisional expansion at a point p. The non horizontal components represent the exceptional divisor of τ.

In the sequel, we will mostly use such birational morphisms in the particular case where x_- and x_+ are the respective local equations of integral curves C_- and C_+ on S intersecting transversally at p. The integer $\mu_\pm$ is then equal to the coefficient of $A_0(p)$ in the total transform of $C_\pm$ and will say that $\tau : S' \to S$ is the subdivisional expansion of S at the $\kappa(p)$-rational point $(C_- \cap C_+)_p$, with multiplicities (μ_-, μ_+).

Convention 1.5. In the rest of the article, we will often use the same notation for a divisor on a surface and its proper transform on another surface by a birational map.

Definition 1.6. A $\mathbf{k}$-variety X is called $\mathbf{k}$-*rational* if there exist a birational map $\varphi : \mathbb{P}^{\dim X}_{\mathbf{k}} \dashrightarrow X$. A geometrically reduced $\mathbf{k}$-scheme of finite type X is called *geometrically rational* if every irreducible component of $X_{\overline{\mathbf{k}}}$ is $\overline{\mathbf{k}}$-rational, where $\overline{\mathbf{k}}$ denotes an algebraic closure of $\mathbf{k}$.

Definition 1.7. An SNC (resp. strictly SNC) divisor B on a smooth complete surface V defined over $\mathbf{k}$ is said to be *SNC-minimal* (resp. *strictly SNC-minimal*) over $\mathbf{k}$ if there does not exist any projective strictly birational morphism $\tau : V \to V'$ onto a smooth surface defined over $\mathbf{k}$ with exceptional locus contained in B such that $\tau_*(B)$ is SNC (resp. stricly SNC).

Example 1.8. Let V be a smooth projective surface defined over $\mathbb{R}$ and let B_0 and $\overline{B}_0$ be a pair of smooth $\mathbb{C}$-rational curves in $V_{\mathbb{C}}$ exchanged by the real structure σ and intersecting transversally in a single point. Then $B_0 \cup \overline{B}_0$ is the complexification of a geometrical rational chain B on V which is SNC minimal over $\mathbb{R}$ even if $B_0^2 = \overline{B}_0^2 = -1$.

Definition 1.9. Let (S, C) be a pair consisting of a smooth surface S and a curve $C \subset S$ defined over $\mathbf{k}$. A *log-resolution* of (S, C) is a projective birational morphism $\tau : S' \to S$ defined over $\mathbf{k}$ such that S' is smooth and the union C' of the reduced total transform $\tau^{-1}(C)$ of C with the exceptional locus $\mathrm{Ex}(\tau)$ of τ is an SNC divisor on S'. We say that $\tau : (S', C') \to (S, C)$ is a *strict* log-resolution if C' is strictly SNC.

1.3. Smooth projective completions and logarithmic Kodaira dimension.

By virtue of Nagata compactification Theorem [30] and classical desingularization theorems (see e.g. [38]), every smooth surface S defined over $\mathbf{k}$ admits an open immersion $S \hookrightarrow V$ into a smooth projective surface with SNC boundary divisor $B = V \setminus S$, both defined over $\mathbf{k}$. In what follows the term *smooth projective completion* of a surface S will be used to refer to any pair (V, B) consisting of a smooth projective surface V and a reduced SNC divisor B on it such that $V \setminus B$ is isomorphic to S. A smooth projective completion (V, B) of S will be called SNC-minimal if B is an SNC-minimal divisor on V.

The *(logarithmic) Kodaira dimension* $\kappa(S)$ of S is then defined as the Iitaka dimension [20] of the pair $(V; \omega_V(\log B))$ where $\omega_V(\log B) = (\det \Omega^1_{V/\mathbf{k}}) \otimes \mathcal{O}_V(B)$. The so-defined element $\kappa(S) \in \{-\infty, 0, 1, 2\}$ is independent of the choice of a smooth complete model (V, B) [21], and it coincides with the usual notion of Kodaira dimension in the case where S is already complete. Furthermore, it is invariant under arbitrary extensions of the base field $\mathbf{k}$, as a consequence of the flat base change theorem [17, Proposition III.9.3]. A surface of Kodaira dimension 2 is usually said to be of *general type*.

1.4. **Euclidean topology.**
Recall that when $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, the set of $\mathbf{k}$-rational point of a $\mathbf{k}$-variety X can be endowed with the euclidean topology. Namely, every $\mathbf{k}$-rational point p admits an affine open neighborhood U_p and the choice of a closed immersion $j : U_p \hookrightarrow \mathbb{A}^N_{\mathbf{k}}$ enables to equip $j(U_p(\mathbf{k}))$ with the subspace topology induced by the usual euclidean topology on $\mathbb{A}^N_{\mathbf{k}}(\mathbf{k}) \simeq \mathbf{k}^N$. The so-constructed topology on $X(\mathbf{k})$ is well-defined and independent of the choices made [35, Lemme 1 and Proposition 2]. When X is smooth, $X(\mathbf{k})$ is a $\mathcal{C}^\infty$-manifold when equipped with the structure locally inherited by the standard $\mathcal{C}^\infty$-structure on $\mathbf{k}^N$.

Convention 1.10. Given a real algebraic variety X, we always consider the sets $X(\mathbb{R})$ and $X_{\mathbb{C}}(\mathbb{C})$ as equipped with their respective euclidean topologies. The real structure σ on $X_{\mathbb{C}}$ is in particular a continuous involution of $X_{\mathbb{C}}(\mathbb{C})$, and we consider $X(\mathbb{R})$ as a subspace of $X_{\mathbb{C}}(\mathbb{C})$ via its identification with the set $X^\sigma_{\mathbb{C}}(\mathbb{C})$ of fixed points of σ.

Recall that given a coefficient ring A, a topological manifold M is called *A-acyclic* if all its homology groups $H_i(M; A)$, $i \geq 1$, are trivial. Recall the following classical topological characterization of $\mathbb{R}^2$ as a smooth manifold:

Proposition 1.11. *A smooth 2-dimensional real manifold M is diffeomorphic to $\mathbb{R}^2$ if and only if it is connected and $\mathbb{Z}_2$-acylic.*

2. Algebro-topological characterizations of $\mathbb{Q}$-homology euclidean planes

Definition 2.1. A *homology* (resp. $\mathbb{Q}$-homology) *euclidean plane* is a smooth real algebraic surface S such that $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ and whose complexification $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Z}$-acyclic (resp. $\mathbb{Q}$-acyclic).

Recall that by virtue of respective results of Fujita [13] and Gurjar-Pradeep-Shastri [15, 16], a $\mathbb{Q}$-acyclic complex surface is affine and rational.

Proposition 2.2. *A $\mathbb{Q}$-homology euclidean plane is affine and $\mathbb{R}$-rational.*

Proof. Let S be $\mathbb{Q}$-homology euclidean plane and let (V, B) be a smooth projective completion of S defined over $\mathbb{R}$. Then V is geometrically rational, with non empty connected real locus $V(\mathbb{R})$ as $S(\mathbb{R}) \approx \mathbb{R}^2$, hence is $\mathbb{R}$-rational by virtue of [6] (see also [36, Corollary VI.6.5]). The $\mathbb{R}$-rationality of S follows. ☐

2.1. **Criteria for $\mathbb{Q}$-acyclicity and structure of the real locus.** Every smooth affine surface S admits a smooth projective completion (V, B) with geometrically connected SNC boundary divisor B. The following well-known lemma (see e. g. [27, Lemma 4.2.1]) provides a characterization of the $\mathbb{Q}$-acyclicity of $S_{\mathbb{C}}(\mathbb{C})$ in terms of the geometry of B and the properties of the natural map $j_{\mathbb{C}} : \mathbb{Z}\langle B_{\mathbb{C}}\rangle \to \mathrm{Cl}(V_{\mathbb{C}})$ associating to an irreducible component of $B_{\mathbb{C}}$ its class in the divisor class group $\mathrm{Cl}(V_{\mathbb{C}})$ of $V_{\mathbb{C}}$.

Lemma 2.3. *Let (V, B) be a smooth projective completion of a smooth complex surface S. Then $S(\mathbb{C})$ is $\mathbb{Q}$-acyclic if and only if B is a rational tree and the map $j \otimes \mathrm{id} : \mathbb{Z}\langle B\rangle \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Cl}(V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. Furthermore, $S(\mathbb{C})$ is $\mathbb{Z}$-acyclic if and only if $j : \mathbb{Z}\langle B\rangle \to \mathrm{Cl}(V)$ is an isomorphism.*

Not all smooth real algebraic surfaces S with $\mathbb{Q}$-acyclic complexifications $S_{\mathbb{C}}$ have their real locus diffeomorphic to $\mathbb{R}^2$. For instance, the real locus of the complement S of a smooth conic C in $\mathbb{P}^2_{\mathbb{R}}$ is either diffeomorphic to $\mathbb{RP}^2$ if $C(\mathbb{R}) = \emptyset$ or to the disjoint union of $\mathbb{R}^2$ with a Möebius band otherwise. In this context, the algebro-topological criterion of Lemma 2.3 can be refined as follows: since the pair $(V_{\mathbb{C}}, B_{\mathbb{C}})$ is defined over $\mathbb{R}$, the free abelian groups $\mathbb{Z}\langle B_{\mathbb{C}}\rangle$ and $\mathrm{Cl}(V_{\mathbb{C}})$ both inherits a structure of G-module for the group $G = \{1, \sigma\} \simeq \mathbb{Z}_2$ generated by the real structure σ on $V_{\mathbb{C}}$[1]. Furthermore, the complexification of divisors gives rise to a homomorphism $\mathrm{Cl}(V) \to \mathrm{Cl}(V_{\mathbb{C}})$ whose image is contained in the subgroup $\mathrm{Cl}(V_{\mathbb{C}})^\sigma$ of σ-invariant classes. Recall that for every G-module M, the Galois cohomology

[1]On $\mathrm{Cl}(V_{\mathbb{C}})$, we consider the natural action induced by σ: if d is the class in $\mathrm{Cl}(V_{\mathbb{C}})$ of a real divisor D, then $\sigma(d) = d$. Recall that if $[D]$ is the fundamental class of D in $H_2(V_{\mathbb{C}}(\mathbb{C}); \mathbb{Z})$, then $\sigma_*([D]) = -[D]$, in fact the cycle map $\mathrm{cl} : \mathrm{Cl}(V_{\mathbb{C}}) \to H_2(V_{\mathbb{C}}(\mathbb{C}); \mathbb{Z})$ is anti-equivariant.

groups $H^1(G,M) = \mathrm{Ker}(\mathrm{id}_M + \sigma)/\mathrm{Im}(\mathrm{id}_M - \sigma)$ and $H^2(G,M) = \mathrm{Ker}(\mathrm{id}_M - \sigma)/\mathrm{Im}(\mathrm{id}_M + \sigma)$ are both $\mathbb{Z}_2$-vector spaces. We have the following criterion:

Proposition 2.4. *Let (V,B) be a smooth projective completion defined over $\mathbb{R}$ of an $\mathbb{R}$-rational real algebraic surface S. Suppose that B is a geometrically rational tree and let $j_{\mathbb{C}} : \mathbb{Z}\langle B_{\mathbb{C}}\rangle \to \mathrm{Cl}(V_{\mathbb{C}})$ be the natural homomorphism. Then the following hold:*

1) $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ if and only if $B(\mathbb{R})$ is non empty and the homomorphism $H^2(j_{\mathbb{C}}) : H^2(G,\mathbb{Z}\langle B_{\mathbb{C}}\rangle) \to H^2(G,\mathrm{Cl}(V_{\mathbb{C}}))$ induced by $j_{\mathbb{C}}$ is an isomorphism.

2) If in addition $\mathrm{Cl}(V) \to \mathrm{Cl}(V_{\mathbb{C}})$ is an isomorphism, then the second condition can be replaced by the requirement that $j_{\mathbb{C}} \otimes \mathrm{id} : \mathbb{Z}\langle B_{\mathbb{C}}\rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathrm{Cl}(V_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is an isomorphism.

Proof. Because V is $\mathbb{R}$-rational, the cycle map $\mathrm{cl} : \mathrm{Cl}(V_{\mathbb{C}}) \to H_2(V_{\mathbb{C}}(\mathbb{C});\mathbb{Z})$ is an isomorphism by the Lefschetz Theorem on $(1,1)$-cycles and we get an isomorphism between $\mathrm{Cl}(V)$ and the group $H_2(V_{\mathbb{C}}(\mathbb{C});\mathbb{Z})^{-\sigma}$ of anti-invariant classes. The Borel-Haefliger homomorphism, [3, §5], induces an isomorphism between $H^2(G,\mathbb{Z}\langle C_{\mathbb{C}}\rangle)$ and $H_1(C(\mathbb{R});\mathbb{Z}_2)$ for any geometrically rational curve C with non empty real locus, hence for any geometrically rational tree with non empty real locus. As a consequence, passing to a minimal model, we get an isomorphism between $H^2(G,\mathrm{Cl}(V_{\mathbb{C}}))$ and $H_1(V(\mathbb{R});\mathbb{Z}_2)$ if V is $\mathbb{R}$-rational (see e. g. [36, Proposition 3.2 and Theorem 3.4] for further details).

From Proposition 1.11, we have that $S(\mathbb{R}) \approx \mathbb{R}^2$ if and only if $H^*(S(\mathbb{R});\mathbb{Z}_2) = \mathbb{Z}_2$. The long exact sequence of homology for the pair $(V(\mathbb{R}), B(\mathbb{R}))$ together with Poincaré duality

$$H_i(V(\mathbb{R}), B(\mathbb{R});\mathbb{Z}_2) \simeq H^{2-i}(S(\mathbb{R});\mathbb{Z}_2)$$

yields the exact sequence

$$\begin{aligned} 0 \;\to\;& H_2(V(\mathbb{R});\mathbb{Z}_2) \to H^0(S(\mathbb{R});\mathbb{Z}_2) \to H_1(B(\mathbb{R});\mathbb{Z}_2) \xrightarrow{i_{\mathbb{R}}} H_1(V(\mathbb{R});\mathbb{Z}_2) \to H^1(S(\mathbb{R});\mathbb{Z}_2) \to \\ \to\;& H_0(B(\mathbb{R});\mathbb{Z}_2) \to H_0(V(\mathbb{R});\mathbb{Z}_2) \to H^2(S(\mathbb{R});\mathbb{Z}_2) \to 0, \end{aligned}$$

Again, the $\mathbb{R}$-rationality of V implies that $V(\mathbb{R})$ is non empty and connected, so that $H_0(V(\mathbb{R});\mathbb{Z}_2) \simeq H_2(V(\mathbb{R});\mathbb{Z}_2) = \mathbb{Z}_2$. Furthermore, B is a geometrically rational tree in a smooth real algebraic surface, hence it follows from classification of involutions on a tree and classification of real structures on $\mathbb{P}^1_{\mathbb{C}}$ that $B(\mathbb{R})$ is either empty, or a point or a connected union of curves homeomorphic to S^1. So either $H_0(B(\mathbb{R});\mathbb{Z}_2) = 0$ if $B(\mathbb{R})$ is empty, and then $H^2(S(\mathbb{R});\mathbb{Z}_2) = \mathbb{Z}_2$, or the map $H_0(B(\mathbb{R});\mathbb{Z}_2) \to H_0(V(\mathbb{R});\mathbb{Z}_2)$ is an isomorphism. We conclude that $S(\mathbb{R}) \approx \mathbb{R}^2$ if and only if $B(\mathbb{R})$ is not empty and $i_{\mathbb{R}}$ is an isomorphism. The first assertion is then a consequence of the following commutative diagram (vertical isomorphism on the left is the Borel-Haefliger homomorphism)

$$\begin{array}{ccc} H_1(B(\mathbb{R});\mathbb{Z}_2) & \xrightarrow{i_{\mathbb{R}}} & H_1(V(\mathbb{R});\mathbb{Z}_2) \\ \uparrow\wr & & \uparrow\wr \\ H^2(G,\mathbb{Z}\langle B_{\mathbb{C}}\rangle) & \xrightarrow{H^2(j_{\mathbb{C}})} & \mathrm{H^2(G,Cl(V_{\mathbb{C}}))}. \end{array}$$

For the second assertion, it is enough to observe that if $\mathrm{Cl}(V) \to \mathrm{Cl}(V_{\mathbb{C}})$ is an isomorphism then $\mathrm{Cl}(V_{\mathbb{C}}) = \mathrm{Cl}(V_{\mathbb{C}})^{\sigma}$, $(\mathrm{id}_{\mathrm{Cl}(V_{\mathbb{C}})} + \sigma)\mathrm{Cl}(V_{\mathbb{C}}) = 2\mathrm{Cl}(V_{\mathbb{C}})$ and so $H^2(G,\mathrm{Cl}(V_{\mathbb{C}})) = \mathrm{Cl}(V_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. For the same reason, $H^2(G,\mathbb{Z}\langle B_{\mathbb{C}}\rangle) = \mathbb{Z}\langle B_{\mathbb{C}}\rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and the conclusion follows. □

2.2. **$\mathbb{Q}$-homology euclidean planes obtained from rational plane curves arrangements.** Here we setup a real counterpart of a general blow-up construction already used by tom Dieck and Petrie [10] in the complex case which leads to a rough description of $\mathbb{Q}$-homology euclidean planes in terms of a datum consisting of a suitable arrangement D of rational curves in $\mathbb{P}^2_{\mathbb{R}}$ and a subtree B of the total transform of D in a log-resolution $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ of the pair $(\mathbb{P}^2_{\mathbb{R}}, D)$. This construction will be refined later on in subsection 3.2.1 to describe in a more precise fashion the structure of homology euclidean planes of general type.

2.2.1. Let $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$ and let (V_0,D_0) be a pair consisting of a smooth $\mathbf{k}$-rational projective surface V_0 and a reduced curve D_0 defined over $\mathbf{k}$, with geometrically rational irreducible components. Let $\tau:(V,D)\to(V_0,D_0)$, where $D=\tau^{-1}(D_0)$ be a strict log-resolution of (V_0,D_0) such that the image of the exceptional locus of τ is contained in D_0.

Now suppose that there exists a geometrically rational subtree $B\subset D$ defined over $\mathbf{k}$ satisfying the following properties:

a) The support of B contains the proper transform $\tau_*^{-1}D_0$ of D_0,

b) $\mathrm{rk}(\mathbb{Z}\langle D_{\mathbb{C}}\rangle)-\mathrm{rk}(\mathbb{Z}\langle B_{\mathbb{C}}\rangle)=\mathrm{rk}(\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle)-\mathrm{rk}(\mathrm{Cl}(V_{0,\mathbb{C}}))$.

By assumption, the set $\mathcal{E}_0$ of irreducible components of $D_{\mathbb{C}}$ not contained in the support of $B_{\mathbb{C}}$ is a subset of the set $\mathcal{E}$ of exceptional divisors of $\tau_{\mathbb{C}}:V_{\mathbb{C}}\to V_{0,\mathbb{C}}$. Letting $\mathcal{E}_1=\mathcal{E}\setminus\mathcal{E}_0$, we have natural identifications $\mathbb{Z}\langle D_{\mathbb{C}}\rangle=\mathbb{Z}\langle B_{\mathbb{C}}\rangle\oplus\mathbb{Z}\langle\mathcal{E}_0\rangle$,

$$\mathbb{Z}\langle B_{\mathbb{C}}\rangle=\mathbb{Z}\langle\tau_*^{-1}D_{0,\mathbb{C}}\rangle\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle=\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle$$

and $\mathrm{Cl}(V_{\mathbb{C}})=\mathrm{Cl}(V_{0,\mathbb{C}})\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle\oplus\mathbb{Z}\langle\mathcal{E}_0\rangle$. We let R be the kernel of

$$\pi:=\mathrm{pr}_{\mathrm{Cl}(V_{0,\mathbb{C}})\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle}\circ j_{\mathbb{C}}=d_{\mathbb{C}}\oplus\mathrm{id}_{\mathbb{Z}\langle\mathcal{E}_1\rangle}:\mathbb{Z}\langle B_{\mathbb{C}}\rangle=\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle\to\mathrm{Cl}(V_{0,\mathbb{C}})\oplus\mathbb{Z}\langle\mathcal{E}_1\rangle,$$

and we let $\varphi=\mathrm{pr}_{\mathbb{Z}\langle\mathcal{E}_0\rangle}\circ j_{\mathbb{C}}|_R:R\to\mathbb{Z}\langle\mathcal{E}_0\rangle$.

When $\mathbf{k}=\mathbb{R}$, the fact that D_0 and B are defined over $\mathbb{R}$ guarantees that R, $\mathbb{Z}\langle\mathcal{E}_0\rangle$ and $\mathbb{Z}\langle\mathcal{E}_1\rangle$ have the additional structures of G-modules for the Galois group $G=\{1,\sigma\}\simeq\mathbb{Z}_2$ generated by the real structure σ on $V_{\mathbb{C}}$ and that π and φ are homomorphisms of G-module.

Lemma 2.5. (*See also* [10, 3.6-3.9]) *Let (V_0,D_0) be a pair consisting of a smooth $\mathbf{k}$-rational projective surface V_0 and a reduced curve D_0 defined over $\mathbf{k}$, with geometrically rational irreducible components. Let $\tau:(V,D)\to(V_0,D_0)$ be a strict log-resolution of (V_0,D_0) such that the image of the exceptional locus of τ is contained in D_0, and let $B\subset D$ be a geometrically rational subtree defined over $\mathbf{k}$ satisfying conditions a) and b) above. Then the following hold for the smooth quasi-projective surface $S=V\setminus B$:*

a) $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic if and only if $d_{\mathbb{C}}:\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathrm{Cl}(V_{0,\mathbb{C}})\otimes_{\mathbb{Z}}\mathbb{Q}$ is surjective and $\varphi\otimes\mathrm{id}:R\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathbb{Z}\langle\mathcal{E}_0\rangle\otimes_{\mathbb{Z}}\mathbb{Q}$ is an isomorphism.

b) $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Z}$-acylic if and only if $d_{\mathbb{C}}:\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle\to\mathrm{Cl}(V_{0,\mathbb{C}})$ is surjective and $\varphi:R\to\mathbb{Z}\langle\mathcal{E}_0\rangle$ is an isomorphism.

c) If $\mathbf{k}=\mathbb{R}$ then $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$ if and only if $H^2(d_{\mathbb{C}}):H^2(G,\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle)\to H^2(G,\mathrm{Cl}(V_{0,\mathbb{C}}))$ is surjective and $H^2(\varphi):H^2(G,R)\to H^2(G,\mathbb{Z}\langle\mathcal{E}_0\rangle)$ is an isomorphism. Furthermore, when $d:\mathbb{Z}\langle D_0\rangle\to\mathrm{Cl}(V_0)$ is surjective and $\mathrm{Cl}(V)\to\mathrm{Cl}(V_{\mathbb{C}})$ is an isomorphism, the second condition is satisfied if and only if $\varphi\otimes\mathrm{id}:R\otimes_{\mathbb{Z}}\mathbb{Z}_2\to\mathbb{Z}\langle\mathcal{E}_0\rangle\otimes_{\mathbb{Z}}\mathbb{Z}_2$ is an isomorphism.

Proof. Since a homomorphism of modules $f:M\to M'\oplus M''$ is an isomorphism if and only if the projection $\mathrm{pr}_{M'}\circ f$ is surjective and the kernel of this projection is mapped isomorphically onto M'' by the restriction of $\mathrm{pr}_{M''}\circ f$, the assertions are straightforward consequences of Lemma 2.3 and Proposition 2.4, applied to $j_{\mathbb{C}}\otimes\mathrm{id}$, $j_{\mathbb{C}}$ and $H^2(j_{\mathbb{C}})$. □

Proposition 2.6. (*See also* [10, Proposition 2.2]). *Let $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$ and let S be a smooth $\mathbf{k}$-rational surface such that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic. In the case where $\mathbf{k}=\mathbb{R}$, assume further that $S(\mathbb{R})$ is non compact. Then there exists an arrangement D_0 of reduced geometrically rational curves in $\mathbb{P}^2_{\mathbf{k}}$ and a rational subtree B of the reduced total transform of D_0 in a strict log-resolution $\tau:(V,D)\to(\mathbb{P}^2_{\mathbf{k}},D_0)$ of the pair $\left(\mathbb{P}^2_{\mathbf{k}},D_0\right)$ satisfying properties a) and b) in § 2.2.1 above such that $S\simeq V\setminus B$.*

Proof. Let (V',B') be a smooth projective completion of S defined over $\mathbf{k}$, with boundary geometrically rational tree B'. Since V' is $\mathbf{k}$-rational the output W of a MMP process $\alpha:V'\to W$ over $\mathbf{k}$ ran from V' is isomorphic over $\mathbf{k}$ either to $\mathbb{P}^2_{\mathbf{k}}$, or to a Hirzebruch surface $\pi_n:\mathbb{F}_n=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}(-n))\to\mathbb{P}^1_{\mathbf{k}}$, $n\in\mathbb{Z}_{\geq 0}\setminus\{1\}$, or to the smooth quadric $Q=\{x^2+y^2+z^2-t^2=0\}\subset\mathbb{P}^3_{\mathbf{k}}$, the latter being isomorphic to $\mathbb{F}_0$ when $\mathbf{k}=\mathbb{C}$, see [6] or [23, Theorem 33, p. 206].

Let us assume for the moment that $W\simeq\mathbb{P}^2_{\mathbf{k}}$. Since S is affine, the geometrically rational tree B' is the support of an ample divisor and hence it cannot be fully contained in the exceptional locus of α. Its image $D_0=\alpha_*B'$ is thus a reduced divisor defined over $\mathbf{k}$, with geometrically rational irreducible components, containing the image of the exceptional locus of α. Furthermore, since $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic, the map $j_{\mathbb{C}}\otimes\mathrm{id}:\mathbb{Z}\langle D_{0,\mathbb{C}}\rangle\otimes_{\mathbb{Z}}\mathbb{Q}\to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{C}})\otimes_{\mathbb{Z}}\mathbb{Q}$ is surjective, because $j_{\mathbb{C}}\otimes\mathrm{id}:\mathbb{Z}\langle B'_{\mathbb{C}}\rangle\otimes_{\mathbb{Z}}\mathbb{Q}\to \mathrm{Cl}(V'_{\mathbb{C}})\otimes_{\mathbb{Z}}\mathbb{Q}$ is an isomorphism. Let $\beta:V\to V'$ be a strict log-resolution of the pair $(V,\alpha^{-1}(D_0))$ defined over $\mathbf{k}$, with SNC-minimal exceptional locus, and restricting to an isomorphism over $V'\setminus\alpha^{-1}(D_0)$. By construction, $\tau=\alpha\circ\beta:V\to\mathbb{P}^2_{\mathbf{k}}$ is a strict log-resolution of $(\mathbb{P}^2_{\mathbf{k}},D_0)$ and S is isomorphic to the complement in V of the geometrically rational subtree tree $B=\beta^{-1}(B')$ of $\tilde{B}$. Furthermore, since the exceptional locus of α is a disjoint union of geometrically rational trees, the image in V' of the exceptional locus of β is contained in the support of B. The minimality of β then implies that $\mathrm{rk}(\mathbb{Z}\langle\tilde{B}_{\mathbb{C}}\rangle)$ and $\mathrm{rk}(\mathbb{Z}\langle B_{\mathbb{C}}\rangle)$ differ precisely by the number of irreducible components of the exceptional locus $\mathrm{Exc}(\alpha_{\mathbb{C}})$ of $\alpha_{\mathbb{C}}$ which are not contained in the support of $B'_{\mathbb{C}}$. Since $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic, we get $\mathrm{rk}(\mathbb{Z}\langle B'_{\mathbb{C}}\rangle)=\mathrm{rk}(\mathrm{Cl}(\mathrm{V}'_{\mathbb{C}}))=1+\mathrm{rk}(\mathbb{Z}\langle\mathrm{Exc}(\alpha_{\mathbb{C}})\rangle)$. Combined with $\mathrm{rk}(\mathbb{Z}\langle B'_{\mathbb{C}}\rangle)=\mathrm{rk}(\mathbb{Z}\langle D_{\mathbb{C}}\rangle)+\mathrm{rk}\mathbb{Z}\langle\mathrm{Exc}(\alpha_{\mathbb{C}})\cap B'_{\mathbb{C}}\rangle$, we conclude that $\mathrm{rk}(\mathbb{Z}\langle\tilde{B}_{\mathbb{C}}\rangle)-\mathrm{rk}(\mathbb{Z}\langle B_{\mathbb{C}}\rangle)=\mathrm{rk}(\mathbb{Z}\langle D_{\mathbb{C}}\rangle)-1$. So S is isomorphic to a surface of the form $V\setminus B$ constructed as in § 2.2.1 above.

Now it remains to show that the initial smooth projective completion (V',B') of S and the MMP process $\alpha:V'\to W$ can be chosen so that $W\simeq\mathbb{P}^2_{\mathbf{k}}$. Starting with an arbitrary smooth projective completion (V_0,B_0) of S with boundary geometrically rational tree B_0 and an arbitrary MMP process $\alpha_0:V_0\to W_0$, we proceed as follows.

Suppose first that W_0 is a Hirzebruch surface $\pi_n:\mathbb{F}_n\to\mathbb{P}^1_{\mathbf{k}}$, $n\geq 2$, with exceptional section $C\simeq\mathbb{P}^1_{\mathbf{k}}$ of self-intersection $-n$. First note that $D_0=(\alpha_0)_*B_0$ cannot be equal to C only, for otherwise $V_0\setminus\alpha_0^{-1}(D_0)\subset S=V_0\setminus B_0$ would contain a complete curve, for instance the inverse image of a section of π_n disjoint from C, in contradiction with the affineness of S. Furthermore every irreducible component of D_0 distinct from C intersects C in a finite number of closed points. So if $\mathbf{k}=\mathbb{R}$ (resp. $\mathbf{k}=\mathbb{C}$), it follows that there exists a non real $\mathbb{C}$-rational point p of $\mathbb{P}^1_{\mathbb{R}}$ (resp. a closed points $p\in\mathbb{P}^1_{\mathbb{C}}$) such that the intersection of $\pi_n^{-1}(p)$ with D_0 is nonempty and not fully contained in $D_0\cap C$. Let $\varphi:\mathbb{F}_n\dashrightarrow\mathbb{F}_{n-2}$ (resp. $\varphi:\mathbb{F}_n\dashrightarrow\mathbb{F}_{n-1}$) be the elementary birational map consisting of blowing-up a point $q\in\pi_n^{-1}(p)\cap(D_0\setminus C)$ and contracting the proper transform of $\pi_n^{-1}(p)$. We obtain a commutative diagram

$$\begin{array}{ccc} (V_0,B_0) & \stackrel{f}{\leftarrow} & (V_1,B_1) \\ \alpha_0\downarrow & & \downarrow\alpha_1 \\ W_0=\mathbb{F}_n & \stackrel{\varphi}{\dashrightarrow} & W_1=\begin{cases}\mathbb{F}_{n-2} & \text{if } \mathbf{k}=\mathbb{R}\\ \mathbb{F}_{n-1} & \text{if } \mathbf{k}=\mathbb{C}\end{cases} \end{array}$$

where f and B_1 are defined as follows: if q belongs to the image of the exceptional locus of α_0 then f is an isomorphism of pairs, otherwise f is the blow-up of the point $q\in B_0(\mathbb{C})$ and $B_1=f^{-1}(B_0)$. In each case, B_1 is a geometrically rational tree defined over $\mathbf{k}$, $V_1\setminus B_1\simeq S$ and the induced birational map $\alpha_1:V_1\to W_1$ is a process of MMP over $\mathbf{k}$. Arguing by induction, we reach a smooth projective completion (V_ℓ,B_ℓ) of S defined over $\mathbf{k}$ with geometrically rational tree boundary B_ℓ and a process of MMP over $\mathbf{k}$ $\alpha_\ell:V_\ell\to\mathbb{F}_\varepsilon$, where $\varepsilon=0,1$. In the case where $\varepsilon=1$, we eventually obtain the desired birational morphism $\tau:V_\ell\to\mathbb{P}^2_{\mathbf{k}}$ defined over $\mathbf{k}$ by contracting the negative section of π_1.

So it remains to treat the case where W_0 is isomorphic either to $\mathbb{F}_0=\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$ or to the smooth quadric $Q=\{x^2+y^2+z^2-t^2=0\}\subset\mathbb{P}^3_{\mathbf{k}}$. The hypothesis implies that $D_0=(\alpha_0)_*B_0$ has a $\mathbf{k}$-rational point p. Indeed, this is clear if $\mathbf{k}=\mathbb{C}$ and, in the case where $\mathbf{k}=\mathbb{R}$, the emptiness of $D_0(\mathbb{R})$ would imply that of $\alpha_0^{-1}(D_0)$, and we would have

$$S(\mathbb{R})=V_0(\mathbb{R})\setminus B_0(\mathbb{R})\supset V_0(\mathbb{R})\setminus\alpha_0^{-1}(D_0)(\mathbb{R})=W_0(\mathbb{R})\approx\begin{cases}\mathbb{T}^2 & \text{if } W_0=\mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}}\\ \mathbb{S}^2 & \text{if } W_0=Q,\end{cases}$$

in contradiction with the non compactness of $S(\mathbb{R})$. In the case where $W_0=\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$, we let $\varphi:\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}\dashrightarrow\mathbb{F}_1$ be the blow-up of p followed by the contraction of the fiber of pr_1 containing p. Similarly as in the previous case, we obtain a commutative diagram

$$\begin{array}{ccc} (V_0,B_0) & \stackrel{f}{\leftarrow} & (V_1,B_1) \\ \alpha_0 \downarrow & & \downarrow \alpha_1 \\ W_0=\mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}} & \stackrel{\varphi}{\dashrightarrow} & \mathbb{F}_1=W_1, \end{array}$$

where f is either an isomorphism of pairs if p belongs to the image of the exceptional locus of α_0, or the blow-up of the point $p\in B_0(\mathbf{k})$ in which case $B_1=f^{-1}(B_0)$. By construction, (V_1,B_1) is a smooth projective completion of S with geometrically rational tree boundary B_1 and $\alpha_1:V_1\to\mathbb{F}_1$ is a process of MMP over $\mathbf{k}$. The composition of α_1 with the contraction of the exceptional section of π_1 is the desired morphism $\tau:V_1\to\mathbb{P}^2_{\mathbf{k}}$.

Finally, in the remaining case where $\mathbf{k}=\mathbb{R}$ and $W_0=Q$, we let $\varphi:Q\dashrightarrow\mathbb{P}^2_{\mathbb{R}}$ be the blow-up of p followed by the contraction of the unique curve $\Delta\simeq\mathbb{P}^1_{\mathbb{C}}$ passing through p and whose complexification $\Delta_{\mathbb{C}}\simeq\mathbb{P}^1_{\mathbb{C}}\cup\mathbb{P}^1_{\mathbb{C}}$ is of type $(1,1)$ in $\mathrm{Cl}(Q_{\mathbb{C}})$. Again, we obtain a commutative diagram

$$\begin{array}{ccc} (V_0,B_0) & \stackrel{f}{\leftarrow} & (V_1,B_1) \\ \alpha_0 \downarrow & & \downarrow \alpha_1 \\ W_0=Q & \stackrel{\varphi}{\dashrightarrow} & W_1=\mathbb{P}^2_{\mathbb{R}} \end{array}$$

where f is the blow-up of $p\in B_0(\mathbb{R})$ and $B_1=f^{-1}(B_1)$ if p does not belong to the image of the exceptional locus of α_0 and an isomorphism of pairs otherwise. By construction, (V_1,B_1) is a smooth projective completion of S with geometrically rational tree boundary B_1 and $\tau=\alpha_1:V_1\to\mathbb{P}^2_{\mathbb{R}}$ is the desired morphism. □

3. Elements of classification of homology euclidean planes

In this section, we consider homology euclidean planes S up to biregular isomorphisms of schemes over $\mathbb{R}$ according to their Kodaira dimension. The cases where S have Kodaira dimension 0 or $-\infty$ are easily dispensed by the following observations: first there is no smooth complex $\mathbb{Z}$-acyclic surface of Kodaira dimension 0 at all (see e.g. [27, Theorem 4.7.1 (1), p. 244]) and second, a smooth complex $\mathbb{Z}$-acyclic surface of negative Kodaira dimension is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ by virtue of [28]. Combined with the fact that there is no nontrivial form of the affine 2-space over a field of characteristic zero [22], this implies that $\mathbb{A}^2_{\mathbb{R}}$ is the only smooth $\mathbb{Z}$-acyclic surface of non positive Kodaira dimension, up to isomorphisms of schemes over $\mathbb{R}$.

So we are left with the problem of classifying homology euclidean planes of Kodaira dimension 1 and 2. A complete classification in the first case is given in the next subsection, in the form of a real counterpart of the existing description for complex surfaces. In contrast, the classification of $\mathbb{Q}$-acyclic surfaces of general type remains much more elusive, already in the complex case. Therefore, we only establish a real counterpart of the "cutting-cycle construction" due to tom Dieck and Petrie [10] in the complex case, from which we derive real models of existing families of examples in the complex case.

3.1. **Homology euclidean planes of Kodaira dimension 1.** Here we establish the real counterpart of the classification of smooth complex $\mathbb{Z}$-acyclic surfaces of Kodaira dimension 1 following Gurjar and Miyanishi [14] (see also [8] for a complementary alternative construction of contractible such surfaces in the complex case).

Let us first briefly review the scheme of the classification in the complex case, following the presentation of Chapter 3, Section 4 in [27]. By [27, Theorem 4.7.1] a smooth $\mathbb{Z}$-acyclic complex surface of Kodaira dimension 1 has an untwisted $\mathbb{A}^1_*$-fibration $q:S\to\mathbb{P}^1_{\mathbb{C}}$, that is, a surjective morphism whose generic fiber is isomorphic to punctured affine line $\mathbb{A}^1_*$ over the field of rational functions on $\mathbb{P}^1_{\mathbb{C}}$. More precisely, it follows from [27, Chapter 2, Theorem 6.1.5] that given an arbitrary smooth projective completion (V,B) of S, q is the restriction to S of the rational map $V\dashrightarrow\mathbb{P}^1_{\mathbb{C}}$ defined by the complete linear system $|m(K_V+B)|$ for sufficiently big $m\geq 1$. In what follows we refer $q:S\to\mathbb{P}^1_{\mathbb{C}}$ to as the *log-canonical* $\mathbb{A}^1_*$-fibration of S. By [27, Lemma 4.5.1], every scheme theoretic fiber of q is isomorphic to $\mathbb{A}^1_{*,\mathbb{C}}$ when equipped with its reduced structure, except for one fiber which is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. The $\mathbb{Z}$-acyclicity of S then implies (see [27, §

4.5.2 and Theorem 4.6.1]) the existence of a smooth projective completion (V,B) into a projective surface equipped with a $\mathbb{P}^1$-fibration $\pi : V \to \mathbb{P}^1_{\mathbb{C}}$, that is a surjective morphism with generic fiber isomorphic to the projective line over the function field of $\mathbb{P}^1_{\mathbb{C}}$. The completion (V,B) is obtained by a very specific sequence of blow-ups $\tau : V \to \mathbb{P}^2_{\mathbb{C}}$ (see § 3.1.1 below), for which q coincides with the restriction of π. The classification in the complex case can then be summarized as follows (see also [39, Theorem 2.6]):

Theorem 3.1. *Let S be a smooth complex $\mathbb{Z}$-acyclic surface of Kodaira dimension 1 and let $q : S \to \mathbb{P}^1_{\mathbb{C}}$ be its log-canonical untwisted $\mathbb{A}^1_*$-fibration. Then there exists a pair (V,B), consisting of a smooth projective $\mathbb{P}^1$-fibered surface $\pi : V \to \mathbb{P}^1_{\mathbb{C}}$ and a rational tree $B \subset V$ constructed by the procedure described in § 3.1.1, and an isomorphism $S \simeq V \setminus B$ making the following diagram commutative*

$$\begin{array}{ccc} S & \xrightarrow{\simeq} & V \setminus B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi|_{V\setminus D}} \\ \mathbb{P}^1_{\mathbb{C}} & = & \mathbb{P}^1_{\mathbb{C}}. \end{array}$$

Our counterpart for surfaces defined over $\mathbb{R}$ reads as follows:

Theorem 3.2. *Let S be an integral homology euclidean plane S of Kodaira dimension 1. Then S admits an untwisted $\mathbb{A}^1_*$-fibration $\rho : S \to \mathbb{P}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, and there exists a pair (V,B), consisting of a smooth projective surface $\mathbb{P}^1$-fibered surface $\pi : V \to \mathbb{P}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ and a tree of $\mathbb{R}$-rational curves $B \subset V$ constructed by the procedure described in § 3.1.1, and an isomorphism $S \simeq V \setminus B$ defined over $\mathbb{R}$ making the following diagram commutative*

$$\begin{array}{ccc} S & \xrightarrow{\simeq} & V \setminus B \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \pi|_{V\setminus B}} \\ \mathbb{P}^1_{\mathbb{R}} & = & \mathbb{P}^1_{\mathbb{R}}. \end{array}$$

The rest of this subsection is devoted to the proof of Theorem 3.2. We first introduce the appropriate blow-up construction in § 3.1.1 and then proceed to the proof itself in § 3.1.2.

3.1.1. *A blow-up construction.* Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. We let $D_0 \subset \mathbb{P}^2_{\mathbf{k}}$ be the union of a collection $E_{0,0}, \ldots, E_{n,0} \simeq \mathbb{P}^1_{\mathbf{k}}$ of $n+1 \geq 3$ lines in $\mathbb{P}^2_{\mathbf{k}}$ intersecting in a same $\mathbf{k}$-rational point x and of a general line $C_1 \simeq \mathbb{P}^1_{\mathbf{k}}$. For every $i = 1, \ldots, n$, we choose a pair of coprime integers $1 \leq \mu_{i,-} < \mu_{i,+}$ in such a way that for $v_- = {}^t(\mu_{1,-}, \ldots, \mu_{n-}) \in \mathcal{M}_{n,1}(\mathbb{Z})$ and $\Delta_+ = \operatorname{diag}(\mu_{1,+}, \ldots, \mu_{n,+}) \in \mathcal{M}_{n,n}(\mathbb{Z})$, the following two conditions are satisfied[2]:

$$\text{(3.1)}\quad \text{a) } \eta = n - 1 - \sum_{i=1}^{n} \frac{1}{\mu_{i,+}} > 0 \quad \text{and} \quad b) \text{ The matrix } \mathcal{N} = \begin{pmatrix} -1 & -1 \cdots -1 \\ v_- & \Delta_+ \end{pmatrix} \text{ belongs to } \mathrm{GL}_{n+1}(\mathbb{Z}).$$

Then we let $\tau : V \to \mathbb{P}^2_{\mathbf{k}}$ be the smooth projective surface obtained by the following blow-up procedure:

1) We first blow-up x with exceptional divisor $C_0 \simeq \mathbb{P}^1_{\mathbf{k}}$. The resulting surface is isomorphic to the Hirzebruch surface $\pi_1 : \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}(-1)) \to \mathbb{P}^1_{\mathbf{k}}$ with C_0 as the negative section of π_1, the proper transforms of $E_{0,0}, \ldots, E_{n,0}$ are $\mathbf{k}$-rational fibers of π_1 while the strict transform of C_1 is a section of π_1 disjoint from C_0.

2) Then for every $i = 1, \ldots, n$, we perform the subdivisional expansion at the $\mathbf{k}$-rational point $p_i = C_1 \cap E_{i,0}$ with multiplicity $(\mu_{i,-}, \mu_{i,+})$ (see Example 1.4) and exceptional divisors $E_{i,1}, \ldots, E_{i,r_i-1}, E_{i,r_i} = A_0(p_i)$.

3) Finally, we perform a sequence of blow-ups starting with the blow-up of a $\mathbf{k}$-rational point $p_0 \in E_{0,0} \setminus (C_0 \cup C_1)$, with exceptional divisor $E_{0,1} \simeq \mathbb{P}^1_{\mathbf{k}}$ and continuing with a sequence of $r_0 - 1 \geq 0$ blow-ups of $\mathbb{R}$-rational points $p_{0,i} \in E_{0,i} \setminus E_{0,i-1}$, $i = 1, \ldots, r_0 - 1$, with successive exceptional divisors $E_{0,i+1}$. We let $A_0(p_0) = E_{0,r_0}$.

[2]These conditions guarantee respectively that the open surface S resulting from the construction has Kodaira dimension 1 and $\mathbb{Z}$-acyclic complexification, see the proof of Proposition § 3.3.

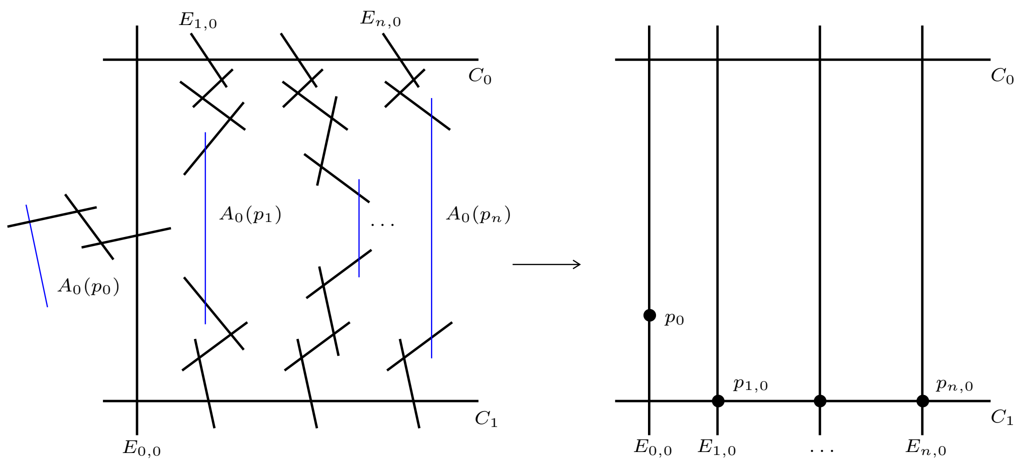

FIGURE 3.1. Construction of homology euclidean plane of logarithmic Kodaira dimension 1.

The union B of the proper transforms of C_0, C_1, and the divisors $E_{i,j}$, $i = 0, \dots, n$, $j = 0, \dots r_i - 1$, is a rational subtree of the total transform $\tilde{B}$ of D_0 by the so-constructed morphism $\tau : V \to \mathbb{P}^2_{\mathbf{k}}$. We let $\tau_1 : V \to \mathbb{F}_1$ be the induced birational morphism, we let $\pi = \pi_1 \circ \tau_1 : V \to \mathbb{P}^1_{\mathbf{k}}$, and we let $S = V \setminus B$. By construction, $\pi|_S : S \to \mathbb{P}^1_{\mathbf{k}}$ is an untwisted $\mathbb{A}^1_*$-fibration defined over $\mathbf{k}$ whose unique degenerate fibers are $A_0(p_0) \cap S \simeq \mathbb{A}^1_{\mathbf{k}}$ occuring with multiplicity one, and the curves $A_0(p_i) \cap S \simeq \mathbb{A}^1_{*,\mathbf{k}}$, $i = 1, \dots, n$, with respective multiplicities $\mu_{i,+}$.

Proposition 3.3. *With the notation above, the surface S is smooth, $\mathbf{k}$-rational, geometrically integral. Its Kodaira dimension is equal to 1 and $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Z}$-acyclic. If $\mathbf{k} = \mathbb{R}$ then $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$.*

Proof. The fact that S is defined over $\mathbf{k}$, smooth, $\mathbf{k}$-rational and geometrically integral is clear from the construction. Condition 3.1 a) in the blow-up construction implies by virtue of Theorem 4.6.1 (1) in [27] that $\kappa(S) = \kappa(S_{\mathbb{C}}) = 1$. The morphism $d_{\mathbb{C}} : \mathbb{Z}\langle D_{0,\mathbb{C}}\rangle \to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{C}})$ is clearly surjective and the elements $C_1 - E_{0,0}$ and $E_{i,0} - E_{0,0}$, $i = 1, \dots, n$ form a basis of its kernel R. With the notation of Lemma 2.5, the abelian group $\mathbb{Z}\langle \mathcal{E}_0\rangle$ is generated by the curves $A_0(p_i)$, $i = 0, \dots n$, and by construction, the matrix of the homomorphism $\varphi : R \to \mathbb{Z}\langle \mathcal{E}_0\rangle$ is equal to $\mathcal{N}$. Since $\mathcal{N}$ is invertible by hypothesis, the $\mathbb{Z}$-acylicity of $S_{\mathbb{C}}(\mathbb{C})$ follows from Lemma 2.5 a). In the case where $\mathbf{k} = \mathbb{R}$, the fact that $S(\mathbb{R}) \approx \mathbb{R}^2$ follows from c) in the same lemma and the surjectivity of $j : \mathbb{Z}\langle D\rangle \to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{R}})$. ☐

3.1.2. *Classification of homology euclidean planes of Kodaira dimension* 1. We now proceed to the proof of Theorem 3.2. So let S be a homology euclidean plane of Kodaira dimension 1 and let (V_0, B_0) be a smooth projective completion of S with SNC boundary defined over $\mathbb{R}$. Since $\kappa(S) = 1$, a multiple of the positive part of the Zariski decomposition of $K_{V_0} + B_0$ induces a rational map $\overline{\rho}_0 : V_0 \dashrightarrow Z$ over a smooth projective curve Z defined over $\mathbb{R}$ whose restriction to S is a morphism $\rho : S \to U$ over a Zariski open subset U of Z defined over $\mathbb{R}$. Furthermore, the generic fiber of ρ is a form of the punctured affine line over the function field of Z [27, Theorem 6.1.5]. Since S, whence V_0 is $\mathbb{R}$-rational and contains an $\mathbb{R}$-rational point, it follows that $Z \simeq \mathbb{P}^1_{\mathbb{R}}$. Letting $\beta : V_1 \to V_0$ be a minimal resolution of the indeterminacies of $\overline{\rho}_0$ and $B_1 = \beta^{-1}(B_0)$, the composition $\overline{\rho}_1 = \overline{\rho}_0 \circ \beta : V_1 \to Z = \mathbb{P}^1_{\mathbb{R}}$ is a surjective fibration with generic fiber isomorphic to a form of the projective line over the function field $\mathbb{R}(t)$ of Z, which restricts to ρ on the open subset $V_1 \setminus B_1 \simeq S$. This implies that B_1 contains a 2-section C' of $\overline{\rho}_1$. After the base change to $\mathrm{Spec}(\mathbb{C})$, we obtain a $\mathbb{P}^1$-fibration $\overline{\rho}_{1,\mathbb{C}} : V_{1,\mathbb{C}} \to Z_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$ whose restriction $\rho_{1,\mathbb{C}} : S_{\mathbb{C}} \simeq V_{1,\mathbb{C}} \setminus B_{1,\mathbb{C}} \to U_{\mathbb{C}}$ coincides with the log-canonical $\mathbb{A}^1_*$-fibration of $S_{\mathbb{C}}$. Since by [27, Theorem 4.7.1], $\rho_{1,\mathbb{C}}$ is an untwisted $\mathbb{A}^1_*$-fibration, it follows that $C'_{\mathbb{C}}$ consists of a pair of distinct irreducible sections of $\overline{\rho}_{1,\mathbb{C}}$. By replacing (V_1, B_1) by the surface obtained by blowing-up $\mathbb{R}$-rational points on C', including infinitely near ones, we may assume that the irreducible components of $C'_{\mathbb{C}}$ are disjoint. Then we let (V', B') be the smooth projective completion of S obtained by contracting if necessary all possible exceptional curves of the first kind supported simultaneously in B_1 and in the fibers of $\overline{\rho}_1$ while keeping the following properties: a) $C'_{\mathbb{C}}$

consists of two disjoint irreducible components $(C_0')_{\mathbb{C}}$ and $(C_1')_{\mathbb{C}}$ and b) the successive proper transforms of B_1 are simple normal crossing divisors.

After the base change to $\mathrm{Spec}(\mathbb{C})$, we obtain a $\mathbb{P}^1$-fibration $\overline{\rho}'_{\mathbb{C}} : V'_{\mathbb{C}} \to Z_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$ whose restriction $\rho'_{\mathbb{C}} : S_{\mathbb{C}} \simeq V'_{\mathbb{C}} \setminus B'_{\mathbb{C}} \to U_{\mathbb{C}}$ coincides with the log-canonical $\mathbb{A}^1_*$-fibration of $S_{\mathbb{C}}$. By virtue of Theorem 3.1, there exists a pair (V, B) obtained by a sequence of blow-ups $\tau : V \to \mathbb{P}^2_{\mathbb{C}}$ as in § 3.1.1 and a commutative diagram

$$\begin{array}{ccc} S_{\mathbb{C}} & \xrightarrow{\simeq} & V \setminus B \\ {\scriptstyle \rho'_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \pi|_{V\setminus B}} \\ \mathbb{P}^1_{\mathbb{C}} & = & \mathbb{P}^1_{\mathbb{C}}. \end{array}$$

Lemma 3.4. *The birational map $\psi : V \dashrightarrow V'_{\mathbb{C}}$ induced by the isomorphism $V \setminus B \simeq S_{\mathbb{C}} \simeq V'_{\mathbb{C}} \setminus B'_{\mathbb{C}}$ is an isomorphism of $\mathbb{P}^1$-fibered surfaces.*

Proof. Indeed, let $V \stackrel{\alpha}{\leftarrow} Y \stackrel{\alpha'}{\to} V'_{\mathbb{C}}$ be a minimal resolution of ψ, where α consists of a possibly empty sequence of blow-ups of points supported on the successive total transforms of B, and let $B_Y = \alpha^{-1}(B)$. Then we have a commutative diagram

$$\begin{array}{ccc} & Y & \\ \alpha \swarrow & & \searrow \alpha' \\ V & \stackrel{\psi}{\dashrightarrow} & V'_{\mathbb{C}} \\ \pi \downarrow & & \downarrow \overline{\rho}'_{\mathbb{C}} \\ \mathbb{P}^1_{\mathbb{C}} & \stackrel{\simeq}{\to} & \mathbb{P}^1_{\mathbb{C}}. \end{array}$$

Note that the proper transforms of C_0 and C_1 in Y are cross-sections of the $\mathbb{P}^1$-fibrations $\pi \circ \alpha$ and $\overline{\rho}'_{\mathbb{C}} \circ \alpha'$. So ψ must restrict to an isomorphism in a Zariski neighborhood of them and since $(C_0')_{\mathbb{C}}$ and $(C_1')_{\mathbb{C}}$ are the only cross-sections of $\overline{\rho}'_{\mathbb{C}} \circ \alpha'$ supported on $B'_{\mathbb{C}}$, we have $\psi_*(C_0 + C_1) = (C_0')_{\mathbb{C}} + (C_1')_{\mathbb{C}}$. If α' is not an isomorphism, then it factors through the contraction of a (-1)-curve supported on the proper transform of B in Y. By construction, the only possible such curves are the proper transforms of the curves C_1, C_0 and $E_{0,0}$ of $\mathbb{F}_1$. Since ψ does not contract any of the first two, the only possibility would be that the proper transform of $E_{0,0}$ in Y is a (-1)-curve, and this can occur if and only if no proper base point of α^{-1} is supported on the proper transform of $E_{0,0}$ in V. This implies in turn that the proper transform of $E_{0,0}$ in Y still intersects those of C_0 and C_1. But then, after the contraction of $E_{0,0}$, the images of C_0 and C_1 would intersect each other. This would imply in turn that $\psi_*(C_0)$ and $\psi_*(C_1)$ intersect each other, in contradiction with the construction of (V', B'). So α' is an isomorphism and we may assume from now on that $(Y, B_Y) = (V'_{\mathbb{C}}, B'_{\mathbb{C}})$. Now suppose that α is not an isomorphism. Then at least one of its exceptional divisor is a (-1)-curve E supported on the boundary $B'_{\mathbb{C}}$ and since B is SNC, E intersects at most two other irreducible components of $B'_{\mathbb{C}}$.

By construction of B, this implies that either E intersects $(C_0')_{\mathbb{C}}$ and $(C_1')_{\mathbb{C}}$ simultaneously or that E is an irreducible component of a fiber of $\overline{\rho}'_{\mathbb{C}}$ whose image $\overline{E}$ by the real structure on $V'_{\mathbb{C}}$ intersects E and is contained in $B'_{\mathbb{C}}$. The first case is impossible as $(C_0')_{\mathbb{C}}$ and $(C_1')_{\mathbb{C}}$ coincide with the proper transforms of C_0 and C_1. In the second case, the image of $\overline{E}$ by the contraction of E would be a curve with self-intersection 0 contained simultaneously in B and in a fiber of the $\mathbb{P}^1$-fibration π, hence would be equal to this fiber. But this would contradict the surjectivity of the restriction $q : S \to \mathbb{P}^1_{\mathbb{C}}$ of π to S. So ψ is an isomorphism of $\mathbb{P}^1$-fibered surfaces. □

To complete the proof of Theorem 3.2, it remains to observe the following. First since since $A_0(p_0) \cap S_{\mathbb{C}}$ is the unique fiber of $\pi|_{S_{\mathbb{C}}}$ isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, we conclude that ρ' has a unique degenerate fiber isomorphic to $\mathbb{A}^1_{\mathbb{R}}$. Its image by ρ' is thus necessarily an $\mathbb{R}$-rational point of $Z \simeq \mathbb{P}^1_{\mathbb{R}}$ and the structure of $\pi^{-1}(\pi(A_0(p_0)))$ then implies further that the irreducible components of the fiber of $\overline{\rho}'$ over this point are all $\mathbb{R}$-rational.

The other degenerate fibers F_ℓ of ρ' are isomorphic to forms of $\mathbb{A}^1_*$ over the corresponding residue fields when equipped with their reduced structure. If ρ' has such a degenerate fiber over a $\mathbb{C}$-rational point of Z then there exists a pair of distinct points $p_i, p_j \in \mathbb{P}^1_{\mathbb{C}}$, $i, j \in \{1, \ldots, n\}$ such that the scheme-theoretic fibers of π over p_i and p_j are isomorphic. With the notation of § 3.1.1, this implies in particular that $\mu_{i,\pm} = \mu_{j,\pm}$. But since $\mu_{i,+} \geq 2$, the matrix $\mathcal{N}$ would not belong to $\mathrm{GL}_{n+1}(\mathbb{Z})$, a contradiction. So every other degenerate fiber F_ℓ is isomorphic to a form of $\mathbb{A}^1_{*,\mathbb{R}}$. In addition, the fiber of $\overline{\rho}'_{\mathbb{C}}$ over $\rho'_{\mathbb{C}}(F_{\ell,\mathbb{C}})$ is a chain of rational curves invariant under the real structure on $V'_{\mathbb{C}}$ and containing the closure of $F_{\ell,\mathbb{C}}$ as a unique invariant (-1)-curve. So we are left with two possibilities: either $(C_0)'_{\mathbb{C}}$ and $(C_1)'_{\mathbb{C}}$ are exchanged by the real structure on $V'_{\mathbb{C}}$ and then the fiber of $\overline{\rho}'_{\mathbb{C}}$ over $\rho'_{\mathbb{C}}(F_{\ell,\mathbb{C}})$ consists of a chain of type $[-2,-1,-2]$, or $(C_0)'_{\mathbb{C}}$ and $(C_1)'_{\mathbb{C}}$ are both invariant under the real structure and then the fiber of $\overline{\rho}'$ over $\rho'(F_\ell)$ is a chain of $\mathbb{R}$-rational curves. In the first case, the fibers of π over all the points p_i, $i \in \{1,\ldots,n\}$, would all be chains of the same type $[-2,-1,-2]$, and since $n \geq 2$ the matrix $\mathcal{N}$ would not belong to $\mathrm{GL}_{n+1}(\mathbb{Z})$, a contradiction.

So all degenerate fibers of $\overline{\rho}'$ are chains of $\mathbb{R}$-rational curves. By contracting all successive possible (-1)-curves in these fibers in the reverse order of their creation by $\tau : V \to \mathbb{P}^2_{\mathbb{C}}$ and then contracting the image of C'_0, we obtain a morphism $\tau' : V' \to \mathbb{P}^2_{\mathbb{R}}$ defined over $\mathbb{R}$ which presents V' as a surface obtained by a sequence of blow-ups of $\mathbb{R}$-rational points as described in § 3.1.1.

3.1.3. *Examples.*

Example 3.5. (Homotopy euclidean planes of Kodaira dimension 1). Let $1 < a < b$ be a pair of coprime integers and let $\psi : \mathbb{P}^2_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[x,y,z]) \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ be the pencil generated by the curves $\{x^a z^{b-a} = 0\}$ and $\{y^b = 0\}$. So ψ has two proper base points $b_0 = [0:0:1]$ and $b_1 = [1:0:0]$, a general geometrically irreducible member of ψ is an $\mathbb{R}$-rational cuspidal curve and ψ has precisely two degenerate members: $\psi^{-1}([1:0])$ which is the union of the lines $L_x = \{x = 0\}$ and $L_z = \{z = 0\}$ counted with multiplicities a and $b-a$ respectively, and $\psi^{-1}([0:1])$ which is equal to the line $L_y = \{y = 0\}$ counted with multiplicity b. Up to exchanging the roles of x and z, we assume from now on that $a > b-a$.

Let $E_{0,0}$ be a general member q, for instance $E_{0,0} = q^{-1}([1:1]) = \{x^a z^{b-a} + y^b = 0\}$ and let $D = E_{0,0} \cup L_z$. Let $p_0 \in E_{0,0}(\mathbb{R}) \setminus \{b_1\}$ be a smooth $\mathbb{R}$-rational point, for instance $p_0 = [1:-1:1]$, let $\beta_{r_0} : X(a,b;r_0) \to \mathbb{P}^2_{\mathbb{R}}$ be the birational morphism obtained by first blowing-up p_0 with exceptional divisor $E_{0,1}$, and then performing a sequence of $r_0 - 1 \geq 0$ blow-ups of $\mathbb{R}$-rational points $p_{0,i} \in E_{0,i} \setminus E_{0,i-1}$, $i = 1,\ldots,r_0-1$ with exceptional divisor $E_{0,i+1}$. We let $S(a,b,;r_0) = X \setminus \{E_{0,0} \cup \cdots \cup E_{0,r_0-1} \cup L_z\}$ where we identified a curve in $\mathbb{P}^2_{\mathbb{R}}$ with its proper transform in X. A minimal resolution $\alpha : V \to X(a,b;r_0)$ of the induced rational pencil $\beta_{r_0} \circ q : X(a,b;r_0) \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ is isomorphic to a surface $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ obtained by the construction of § 3.1.1 with $\mathbf{k} = \mathbb{R}$, $n = 2$ and multiplicities $(\mu_{1,-}, a)$ and $(\mu_{2,-}, b)$, where $1 \leq \mu_{1,-} < a$ and $1 \leq \mu_{2,-} < b$ are uniquely determined in terms of a and b (see [8, (2.7)] and [10, (5.3)] for the computation).

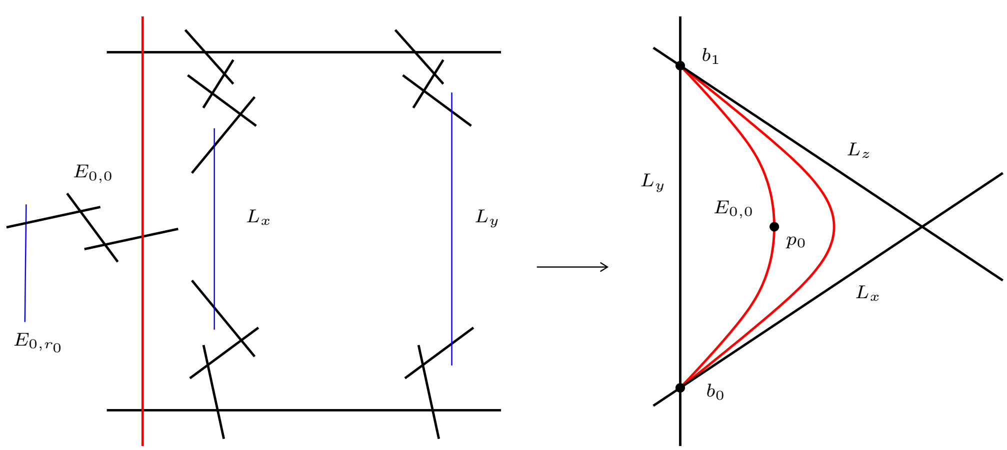

FIGURE 3.2

Via this isomorphism, the boundary B coincides with the total transform of $E_{0,0} \cup \cdots \cup E_{0,r_0-1} \cup L_z$, the sections C_0 and C_1 coincide respectively with the last exceptional divisors of α of the points q_0 and q_1 and the curves A_1 and A_2 are the proper transforms of L_x and L_y respectively. The surface $S(a,b;r_0)_{\mathbb{C}}$

is thus $\mathbb{Z}$-acylic with $S(a,b;r_0)(\mathbb{R}) \approx \mathbb{R}^2$. In fact, $S(a,b;r_0)_{\mathbb{C}}(\mathbb{C})$ is even contractible [8], and, using the same method as in the proof of Theorem 3.2 above, one can deduce from the classification of smooth complex contractible surfaces of Kodaira dimension 1 given in *loc. cit.* that every homology euclidean plane S of Kodaira dimension 1 such that $S_{\mathbb{C}}(\mathbb{C})$ is contractible is isomorphic over $\mathbb{R}$ to $S(a,b;r_0)$ for some parameters a,b,r_0 as above.

Example 3.6. Specializing the values (a,b,r_0) to $(2,3,1)$ in the previous example, $E_{0,0}$ is the cuspidal cubic $\{x^2z+y^3=0\} \subset \mathbb{P}^2_{\mathbb{R}}$ and the fact that the real locus of the corresponding surface $S = S(2,3,1)$ is homeomorphic to $\mathbb{R}^2$ can be seen directly as follows. Since $\beta_1 : X \to \mathbb{P}^2_{\mathbb{R}}$ consists only of the blow-up of the point $p_0 = [1:-1:1]$, $X(\mathbb{R})$ is a Klein bottle which we view as a circle bundle $\theta\colon X(\mathbb{R}) \to S_1$ with fibers equal to the set of $\mathbb{R}$-rational point of the lines through p_0 in $\mathbb{P}^2_{\mathbb{R}}$. The sets $E_{0,1}(\mathbb{R})$ and $L_z(\mathbb{R})$ are two sections of θ which do no intersect each other. On the other hand $E_{0,0}(\mathbb{R})$ is a connected closed curve which intersects $E_{0,1}(\mathbb{R})$ and $L_z(\mathbb{R})$ transversally in one point and in one point with multiplicity 3 respectively.

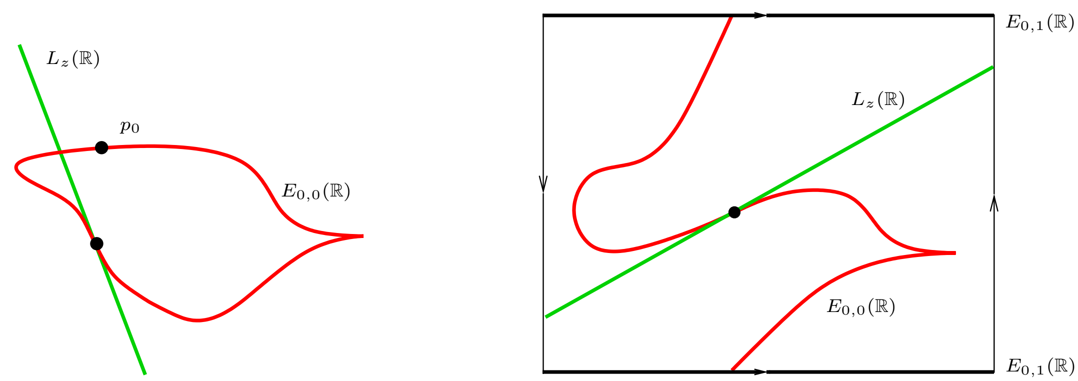

FIGURE 3.3. The initial arrangement in $\mathbb{P}^2_{\mathbb{R}}$ and the corresponding curves in the Klein bottle $X(\mathbb{R})$

The pair $(X(\mathbb{R}), E_{0,0}(\mathbb{R}) \cup L_z(\mathbb{R}))$ is thus homotopically equivalent to $(X(\mathbb{R}), \ell \cup E_{0,1}(\mathbb{R}))$ where ℓ is a fiber of θ. So $S(\mathbb{R})$ is homotopically equivalent to $X(\mathbb{R})$ minus a fiber and a section of θ whence to a disc, implying that $S(\mathbb{R})$ is homeomorphic to $\mathbb{R}^2$.

3.2. **Homology euclidean planes of general type.** In this subsection, we establish the real counterpart of a refined procedure to construct homology euclidean homology planes of general type due to tom Dieck and Petrie [10] in the complex case. We then study the possible real forms of certain known complex families.

3.2.1. *Cycle-cutting construction.* Let again $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, let $D \subset \mathbb{P}^2_{\mathbf{k}}$ be a reduced curve defined over $\mathbf{k}$, with geometrically rational irreducible components, and let $\beta : V_0 \to \mathbb{P}^2_{\mathbf{k}}$ be a minimal log-resolution of the pair (V, D). Given a partition $\mathcal{E}(\beta) = \mathcal{E}_0 \sqcup \mathcal{E}_1$ of the set $\mathcal{E}(\beta)$ of irreducible exceptional divisors of β, with associated indicator function $\chi : \mathcal{E}(\beta) \to \{0,1\}$, we let $R_0 = \sum_{E\in\mathcal{E}_0} E$, $R_1 = \sum_{E\in\mathcal{E}_1} E$ and we let $D(\chi)$ be the SNC divisor on V_0 defined by

$$D(\chi) = \beta_*^{-1}(D) + R_1 \subset \beta^{-1}(D) = \beta_*^{-1}(D) + R_1 + R_0.$$

Definition 3.7. A *cutting datum* for a pair $(\mathbb{P}^2_{\mathbf{k}}, D)$ as above consists of

a) A partition of $\mathcal{E}(\beta)$ with indicator function $\chi : \mathcal{E}(\beta) \to \{0,1\}$ such that $D(\chi)_{\mathbb{C}}$ is connected and

$$\mathrm{rk}\mathbb{Z}\langle (R_0)_{\mathbb{C}}\rangle + s(D(\chi)_{\mathbb{C}}) = \mathrm{rk}\mathbb{Z}\langle D_{\mathbb{C}}\rangle - 1,$$

where $s(D(\chi)_{\mathbb{C}})$ denote the number of independent cycles of the dual graph $\Gamma(D(\chi)_{\mathbb{C}}) = (\Gamma_0(D(\chi)_{\mathbb{C}}), \Gamma_1(D(\chi)_{\mathbb{C}}))$ of $D(\chi)_{\mathbb{C}}$.

b) A subset Φ of the set of double points of $\mathrm{Supp}(D(\chi))$ such that the subgraph $(\Gamma_0(D(\chi)_{\mathbb{C}}), \Gamma_1(D(\chi)_{\mathbb{C}}) \setminus \Phi_{\mathbb{C}})$ of $\Gamma(D(\chi)_{\mathbb{C}})$ is a tree.

3.2.1.1. Given a cutting datum (χ,Φ) for a pair $(\mathbb{P}^2_{\mathbf{k}},D)$, we denote by $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}},D,\chi,\Phi)$ the set of isomorphy classes of birational morphisms $\alpha: V=V(\alpha)\to V_0$ restricting to isomorphisms over $V_0\setminus\Phi$ and such that for every $p\in\Phi$, there exists an open neighborhood $V_{0,p}$ of p over which α restricts to a subdivisional expansion of $V_{0,p}$ with center at p (see § 1.4). For every $(\alpha: V(\alpha)\to V_0)\in\mathcal{B}(\mathbb{P}^2_{\mathbf{k}},D,\chi,\Phi)$, we let $B(\alpha)=\alpha^{-1}(D(\chi))-\sum_{p\in\Phi}A_0(p)$ and $S(\alpha)=V(\alpha)\setminus B(\alpha)$.

Example 3.8. (Ramanujam Surfaces [34], [10, Example 3.15]). Let $D\subset\mathbb{P}^2_{\mathbb{R}}=\mathrm{Proj}(\mathbb{R}[x,y,z])$ be the union of the cuspidal cubic $C=\{x^2z+y^3=0\}$ with its osculating conic Q at a general $\mathbb{R}$-rational point $q\in C(\mathbb{R})$. So Q is a smooth $\mathbb{R}$-rational conic intersecting C at q with multiplicity 5 and transversally at a second $\mathbb{R}$-rational point p.

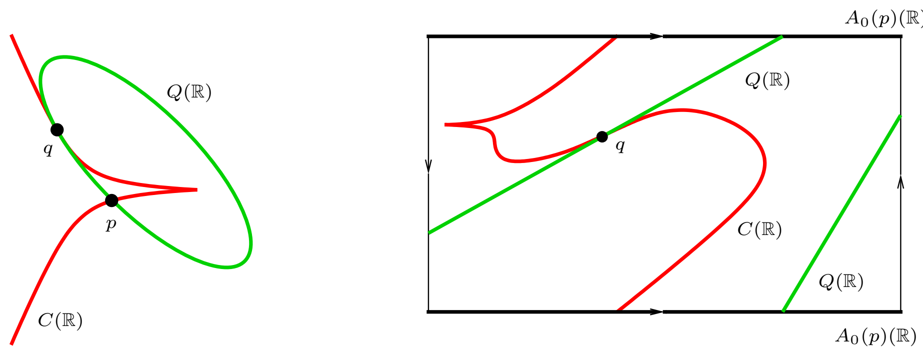

FIGURE 3.4. The Ramanujam surface for the choice $(\mu_-,\mu_+)=(1,1)$

Let $1\le\mu_-\le\mu_+$ be a pair of integers such that $2\mu_--3\mu_+=\pm1$, let $\gamma: V'\to\mathbb{P}^2_{\mathbb{R}}$ be the subdivisional expansion with center at the $\mathbb{R}$-rational point $(C\cap Q)_p$ with multiplicities (μ_-,μ_+) and last exceptional divisor $A_0(p)$, let $B'=\gamma^{-1}(D)-A_0(p)$ and let $S=V'\setminus B'$. Choosing $r=2C-3Q$ as the generator of the kernel R of the surjective homomorphism $d:\mathbb{Z}\langle D\rangle\to\mathrm{Cl}(\mathbb{P}^2_{\mathbb{R}})$, the choice of (μ_-,μ_+) guarantees that the coefficient of $A_0(p)$ in $\gamma^*(r)$ is equal to $2\mu_--3\mu_+=\pm1$, whence that the induced homomorphism $\varphi: R\to\mathbb{Z}\langle A_0(p)\rangle$ (see § 2.2.1) is an isomorphism. Since $D(\mathbb{R})$ is not empty, we deduce from Lemma 2.5 that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Z}$-acyclic and that $S(\mathbb{R})\approx\mathbb{R}^2$. In fact it is known that $S_{\mathbb{C}}(\mathbb{C})$ is even contractible. The reduced total transforms of D and B' in the minimal log-resolutions $\beta: V_0\to\mathbb{P}^2_{\mathbb{R}}$ and $\beta': V_0'\to V'$ of the pairs $(\mathbb{P}^2_{\mathbb{R}},D)$ and (V',B') respectively have the following structures:

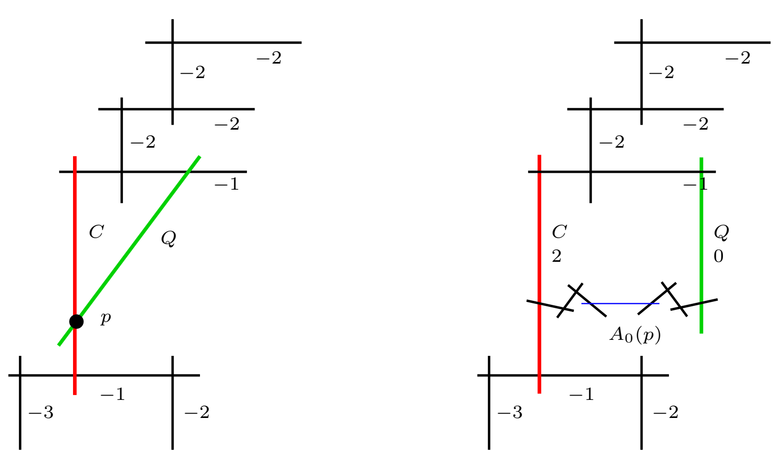

FIGURE 3.5. Resolved boundaries

So S belongs to $\mathcal{B}(\mathbb{P}^2_{\mathbb{R}},D,\chi,\Phi)$ where $(\chi,\Phi)=(\mathbf{1}_{\mathcal{E}(\beta)},\{q\})$. Clearly, the pair $(V_0',(\beta')^{-1}(B')$ cannot be birationally equivalent to either $(\mathbb{P}^2_{\mathbb{R}},\mathrm{Line})$ or a pair (V,B) described in §3.1.1 via a birational map restricting to an isomorphism on S. So Theorem 3.2 and the fact that $\mathbb{A}^2_{\mathbb{R}}$ is the only homology euclidean plane of Kodaira dimension ≤ 0 imply that S is a homology euclidean plane of general type.

Theorem A in [10] admits the following real counterpart:

Theorem 3.9. *Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ and let S be a smooth $\mathbf{k}$-rational surface of general type such that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic. In the case where $\mathbf{k} = \mathbb{R}$, assume further that $S(\mathbb{R})$ is non compact. Then there exists an arrangement D of reduced geometrically rational curves in $\mathbb{P}^2_{\mathbf{k}}$, such that S is isomorphic over $\mathbf{k}$ to the surface $S(\alpha)$ associated with a birational morphism $(\alpha : V(\alpha) \to W) \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}}, D, \chi, \Phi)$ for a suitable cutting datum (χ, Φ).*

Proof. By virtue of Proposition 2.6, there exists an arrangement D of geometrically reduced and geometrically rational curves in $\mathbb{P}^2_{\mathbf{k}}$ and a birational morphism $\alpha : V \to \mathbb{P}^2_{\mathbf{k}}$ with the property that $\tau^{-1}(D)$ is SNC and that B is a subtree of $\tau^{-1}(D)$ containing the proper transform $\tau_*^{-1}D$ of D. Furthermore, the image of the exceptional locus of τ is supported on D and, because $S_{\mathbb{Q}}$ is $\mathbb{Z}$-acyclic, we have $\mathrm{rk}(\mathbb{Z}\langle\tau^{-1}(D)_{\mathbb{C}}\rangle) - \mathrm{rk}(\mathbb{Z}\langle B_{\mathbb{C}}\rangle) = \mathrm{rk}(\mathbb{Z}\langle D_{\mathbb{C}}\rangle) - 1$. Since $\tau : V \to \mathbb{P}^2_{\mathbf{k}}$ is a log-resolution of the pair $(\mathbb{P}^2_{\mathbf{k}}, D)$, there exists a unique birational morphism $\alpha : V \to V_0$ such that $\alpha_*\tau^{-1}(D) = \beta^{-1}(D)$. The function $\chi : \mathcal{E}(\beta) \to \{0, 1\}$ defined by $\chi(E) = 1$ if and only if $E \in \alpha_* B$ defines a partition of $\mathcal{E}(\beta)$ and letting Φ be the image of the exceptional locus of α, it is enough to check that (χ, Φ) is a cutting datum for the pair $(\mathbb{P}^2_{\mathbb{R}}, D)$ for which $(\alpha : V \to V_0)$ belongs to $\mathcal{B}(\mathbb{P}^2_{\mathbb{R}}, D, \chi, \Phi)$. The proof is a verbatim of that of Proposition 2.3 in [10]. □

3.2.2. *Real forms of homology euclidean planes of general type.* It follows from the proof of Theorem 3.2 that a homology euclidean plane S of Kodaira dimension 1 does not admit non trivial real forms. Furthermore, it always admits a smooth projective completion (V, B) obtained from $\mathbb{P}^2_{\mathbb{R}}$ by blowing-up $\mathbb{R}$-rational points only and whose rational boundary tree consists of $\mathbb{R}$-rational curves only. Here we construct examples of homology euclidean plane of general type for which both properties fail.

3.2.2.1. Consider the nodal cubic curves $C_1, C_2 \subset \mathbb{P}^2_{\mathbb{R}}$ with respective equations $(x-y)(x^2+y^2) - xyz = 0$ and $(x-y)(x^2-4y^2) - xyz = 0$. Both have the $\mathbb{R}$-rational point $[1:1:0]$ as a flex but C_1 has a second $\mathbb{C}$-rational flex $C_1 \cap \{x^2+y^2=0\}$ while C_2 possesses two other $\mathbb{R}$-rational flexes $[2:1:0]$ and $[-2:1:0]$. So C_1 and C_2 are not $\mathbb{R}$-isomorphic but their complexifications are both projectively equivalent over $\mathbb{C}$ to the curve with equation $x^3+y^2-xyz = 0$. The projective duals Γ_1 and Γ_2 of C_1 and C_2 respectively are cuspidal quartics, which are nontrivial $\mathbb{R}$-forms of each other, with projectively equivalent complexifications. They both have an ordinary $\mathbb{R}$-rational cusp p_0 corresponding to the common $\mathbb{R}$-rational flex of C_1 and C_2, and either a second $\mathbb{C}$-rational ordinary cusp q for Γ_1, or a pair of additional $\mathbb{R}$-rational ordinary cusps q_1 and q_2 for Γ_2. The tangent L to Γ_1 (resp. Γ_2) at p_0 intersects Γ_1 (resp. Γ_2) transversally in a second $\mathbb{R}$-rational point p.

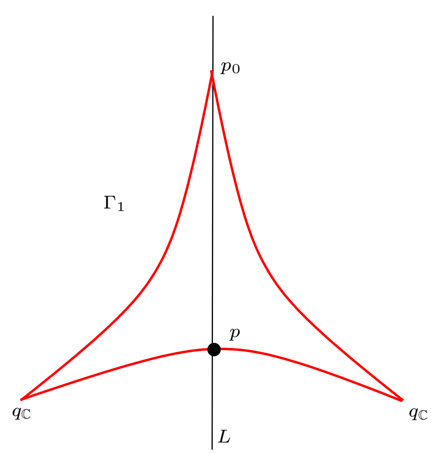

FIGURE 3.6. Real forms of a tricuspidal quartic

Let $D_i = \Gamma_i \cup L$, $i = 1, 2$ and let $1 \leq \mu_- \leq \mu_+$ be a pair of integers such that $\mu_- - 4\mu_+ = \pm 1$. Then let $\gamma_i : V_i' \to \mathbb{P}^2_{\mathbb{R}}$, $i = 1, 2$, be the projective surface obtained from $\mathbb{P}^2_{\mathbb{R}}$ by the subdivisional expansion with center at the $\mathbb{R}$-rational point $(\Gamma_i \cap L)_p$ with multiplicities (μ_-, μ_+) and last exceptional divisor $A_0(p)$, let

$B_i' = \gamma_i^{-1}(D_i) - A_0(p)$ and let $S_i = V_i' \setminus B_1'$. Letting $W_i \to V_i'$ be a minimal resolution of the pair (V_i', B_i'), the reduced total transform B_2 of B_2' consists of $\mathbb{R}$-rational curves only, while the reduced total transform B_1 of B_1' contains a chain of $\mathbb{C}$-rational curves arising from the resolution of the $\mathbb{C}$-rational cuspidal point q.

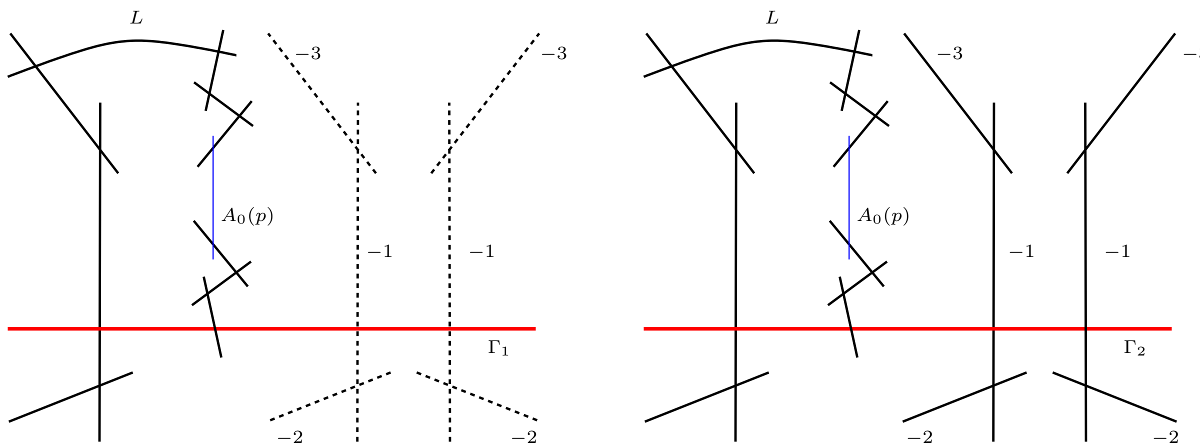

FIGURE 3.7. Total transforms of the boundaries in the miminal resolution

The complexifications $B_{i,\mathbb{C}}$ of B_i have the same structure: every irreducible component of $B_{2,\mathbb{C}}$ is invariant under the real structure σ, while σ acts on $B_{1,\mathbb{C}}$ by permuting the two "cuspidal branches" $[-3, -1, -2]$ and leaving all other irreducible component invariant.

Proposition 3.10. *With the notation above, the following hold:*

1) The surfaces S_1 and S_2 are non isomorphic homology euclidean planes of general type, with isomorphic complexifications $(S_1)_{\mathbb{C}}$ and $(S_2)_{\mathbb{C}}$.

2) The surface S_1 does not admit any smooth SNC-minimal completion (V, B) defined over $\mathbb{R}$ for which B consists of $\mathbb{R}$-rational curves only.

Proof. Choosing $r_i = \Gamma_i - 4L$ as the generator of the kernel R_i of the surjective homomorphism $\mathbb{Z}\langle(D_i)\rangle \to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{R}})$, the choice of (μ_-, μ_+) guarantees that the coefficient of $A_0(p)$ in $\gamma_i^*(r_i)$ is equal to $\mu_- - 4\mu_+ = \pm 1$, whence that the induced homomorphism $\varphi_i : R_i \to \mathbb{Z}\langle A_0(p)\rangle$(see § 2.2.1) is an isomorphism. Since $D_i(\mathbb{R})$ is not empty, we deduce from Lemma 2.5 that $S_{i,\mathbb{C}}(\mathbb{C})$ is $\mathbb{Z}$-acyclic and that $S_i(\mathbb{R}) \approx \mathbb{R}^2$. The fact that S_i is of general type follows from the same argument as in Example 3.8, by comparing the structure of the minimal rational boundary tree B_i in Figure 3.7 above with those described in § 3.1.1. Since $\Gamma_{1,\mathbb{C}}$ and $\Gamma_{2,\mathbb{C}}$ are projectively equivalent, $S_{1,\mathbb{C}}$ and $S_{2,\mathbb{C}}$ are isomorphic by construction. Now suppose that S_1 admits smooth projective completion (V, B) defined over $\mathbb{R}$ for which B is SNC-minimal and consists of $\mathbb{R}$-rational curves only. Then there would exists a birational map of pairs $\varphi : (V_{1,\mathbb{C}}, B_{1,\mathbb{C}}) \dashrightarrow (V_{\mathbb{C}}, B_{\mathbb{C}})$ defined over $\mathbb{R}$ and restricting to an isomorphism $V_{1,\mathbb{C}} \setminus B_{1,\mathbb{C}} \stackrel{\cong}{\to} V_{\mathbb{C}} \setminus B_{\mathbb{C}}$. Since every irreducible component of B is $\mathbb{R}$-rational, the real structure σ on $V_{\mathbb{C}}$ acts trivially on the set of irreducible components $B_{\mathbb{C}}$. So φ cannot be an isomorphism of pairs because, as observed before, the real structure σ on $V_{1,\mathbb{C}}$ acts non trivially on the set of irreducible components of $B_{1,\mathbb{C}}$. So φ must be strictly birational and, letting $V_{1,\mathbb{C}} \stackrel{\alpha}{\leftarrow} X \stackrel{\alpha'}{\to} V_{\mathbb{C}}$ be a minimal resolution of φ defined over $\mathbb{R}$, the morphism $\alpha' : X \to V_{\mathbb{C}}$ would consists of a sequence of blow-downs of either σ-invariant (-1)-curves or pairs of disjoint (-1)-curves exchanged by σ supported on the strict transform of $B_{1,\mathbb{C}}$ by α. The structure of $B_{1,\mathbb{C}}$ depicted in Figure 3.7 above implies that the only possible such curves are the proper transforms of the last exceptional divisors of the minimal log-resolution of the pair $(V'_{1,\mathbb{C}}, B'_{1,\mathbb{C}})$ over the three singular points of $\Gamma_{1,\mathbb{C}}$. But the image of $\alpha^{-1}(B_{1,\mathbb{C}})$ after their contraction would no longer be SNC, which is excluded since $B_{\mathbb{C}}$ is an SNC divisor by hypothesis. So the boundary B of every smooth projective SNC-minimal completion (V, B) of S_1 must contain at least one non $\mathbb{R}$-rational component, which shows 2). Since the pair (V_2, B_2) constructed in § 3.2.2.1 is a smooth projective SNC-minimal completion of S_2 for which B_2 consists of $\mathbb{R}$-rational curves only, we deduce in turn that S_1 and S_2 are not isomorphic as schemes over $\mathbb{R}$. ☐

Remark 3.11. The surfaces S_1 and S_2 above can also be obtained by a cutting-cycle construction from arrangements Δ_1 and Δ_2 in $\mathbb{P}^2_{\mathbb{R}}$ consisting of lines and conics and whose complexifications are both projectively equivalent to the following arrangement Δ of 7 lines with 3 double points and 6 triple points (see [9]).

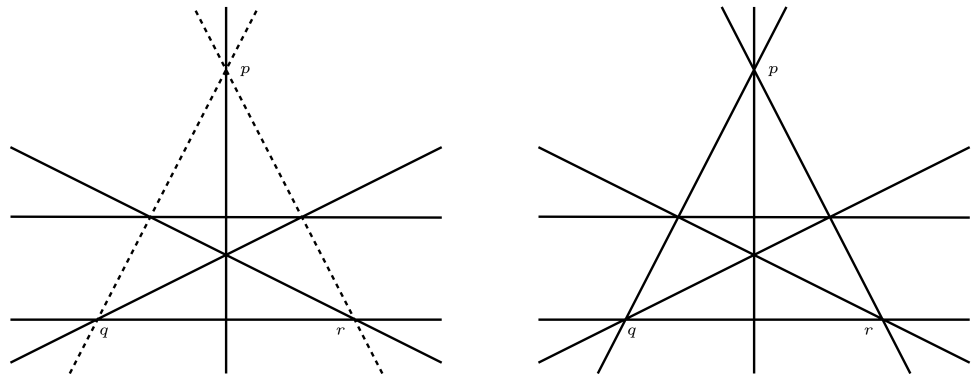

FIGURE 3.8. Arrangement of lines associated to tricuspidal quartics

In the case of S_1 the real structure σ on $\mathbb{P}^2_{\mathbb{C}}$ acts on Δ by exchanging the lines pq and pr and fixing the others while in the case of S_2, the real structure acts trivially on Δ.

4. $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension

This section is devoted to the study of $\mathbb{Q}$-acyclic euclidean planes S of negative Kodaira dimension. We first give a complete classification of these up to isomorphisms of schemes over $\mathbb{R}$. More precisely, we show that they all admit an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, that is, a surjective morphism with generic fiber isomorphic to the affine line of the function field of $\mathbb{A}^1_{\mathbb{R}}$ and we characterize the $\mathbb{Q}$-acyclicity of $S_{\mathbb{C}}(\mathbb{C})$ and the property that $S(\mathbb{R}) \approx \mathbb{R}^2$ in terms of the degenerate fibers of π.

We then consider the classification of such euclidean planes up to $\mathbb{R}$-biregular birational equivalence. Recall that a rational map $\varphi : X \dashrightarrow X'$ between real algebraic varieties is called $\mathbb{R}$-*regular* if it is regular at every $\mathbb{R}$-rational point of X, equivalently, the real locus of X is contained in the domain of definition of φ. We say that φ is $\mathbb{R}$-*biregular* if it admits an $\mathbb{R}$-regular birational inverse ψ. If this holds the induced morphisms $\varphi(\mathbb{R}) : X(\mathbb{R}) \to X'(\mathbb{R})$ and $\psi(\mathbb{R}) : X'(\mathbb{R}) \to X(\mathbb{R})$ are diffeomorphisms for the euclidean topology, and are inverse to each other.

We establish that a large class of $\mathbb{Q}$-acyclic euclidean plane of negative Kodaira dimension are $\mathbb{R}$-regularly birationally equivalent to the affine plane $\mathbb{A}^2_{\mathbb{R}}$.

4.1. **Structure of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension.**

A deep result of Miyanishi-Sugie and Fujita (see e.g. [27, Chapter 2, Theorem 2.1.1]) asserts that every smooth complex affine surface of negative Kodaira dimension admits an $\mathbb{A}^1$-fibration over a smooth curve. But it is wrong in general that a smooth real affine surface of negative Kodaira dimension admits such an $\mathbb{A}^1$-fibration defined over $\mathbb{R}$. For instance, the complement S of a smooth conic $Q \subset \mathbb{P}^2_{\mathbb{R}}$ without $\mathbb{R}$-rational point has negative Kodaira dimension but no $\mathbb{A}^1$-fibration defined over $\mathbb{R}$. Indeed if S admitted such an $\mathbb{A}^1$-fibration $\pi : S \to C$ over a smooth curve defined over $\mathbb{R}$ then C would be geometrically rational, hence rational since S has $\mathbb{R}$-rational points. But then the closure in $\mathbb{P}^2_{\mathbb{R}}$ of a fiber of π over a general $\mathbb{R}$-rational point of C would intersect Q in a unique point, necessarily $\mathbb{R}$-rational, which is impossible.

This subsection is devoted to the proof of the following characterization which shows in particular that $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension do admit $\mathbb{A}^1$-fibrations defined over $\mathbb{R}$.

Theorem 4.1. *For a smooth geometrically integral surface S defined over $\mathbb{R}$ the following are equivalent:*

1) S is a $\mathbb{Q}$-acyclic euclidean plane of negative Kodaira dimension,

2) S admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, whose closed degenerate fibers are all isomorphic to the affine line over the corresponding residue fields when equipped with their reduced structure and whose degenerate fibers over $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ all have odd multiplicities.

The proof requires several steps which occupy the next subsections. After recalling basic properties of $\mathbb{A}^1$-fibrations and their completions into $\mathbb{P}^1$-fibrations on smooth projective surfaces in § 4.1.1, we establish in Proposition 4.3 of § 4.1.2 a characterization of $\mathbb{Q}$-acyclic euclidean plane among real surfaces S admitting an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$. The proof of the existence of an $\mathbb{A}^1$-fibration defined over $\mathbb{R}$ on every $\mathbb{Q}$-acyclic euclidean plane (Proposition 4.7 in § 4.1.4 below) then follows from a careful analysis of the structure of smooth real surfaces admitting a smooth projective completion (V, B) defined over $\mathbb{R}$ for which B is a rational chain which is done in § 4.1.3.

4.1.1. *Basic properties of $\mathbb{A}^1$-fibrations and their completions.* The geometry of smooth complex affine surfaces admitting an $\mathbb{A}^1$-fibration is well-understood through the study of their completions into $\mathbb{P}^1$-fibered surfaces (see e.g. [27, Chapter 3, § 1]). Let us recollect basic properties of these fibrations and their completions in a form which also covers the case of real affine surface.

Lemma 4.2. *Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, let S be a smooth geometrically integral surface defined over $\mathbf{k}$ and let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration over a smooth curve, either affine or projective, defined over $\mathbf{k}$. Then there exists a smooth projective completion (V, B) of S defined over $\mathbf{k}$ with the following properties:*

a) V admits a $\mathbb{P}^1$-fibration $\overline{\pi} : V \to \overline{C}$ defined over $\mathbf{k}$ over a smooth projective completion $\overline{C}$ of C and there is a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & V \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ C & \longrightarrow & \overline{C}. \end{array}$$

.

b) If S is affine, then the divisor $B = V \setminus S$ is a tree which can be written in the form

$$B = \bigcup_{c \in \overline{C} \setminus C} F_c \cup \overline{C}_0 \cup \bigcup_{p \in C} H_p,$$

where $\overline{C}_0$ is a section of $\overline{\pi}$, $F_c = \overline{\pi}^{-1}(c) \simeq \mathbb{P}^1_{\kappa(c)}$ and H_p is either empty or a proper strictly SNC-minimal rational subtree of $\overline{\pi}^{-1}(p)$ containing a $\kappa(p)$-rational component intersecting $\overline{C}_0$, the full fiber $\overline{\pi}^{-1}(p)$ being equal to the union of H_p and of the closure in V of $\pi^{-1}(p)$.

Furthermore, the closure in V of every irreducible component of $\pi^{-1}(p)$ is isomorphic to the projective line over a finite extension κ' of $\kappa(p)$ and it intersects H_p transversally in a unique κ'-rational point.

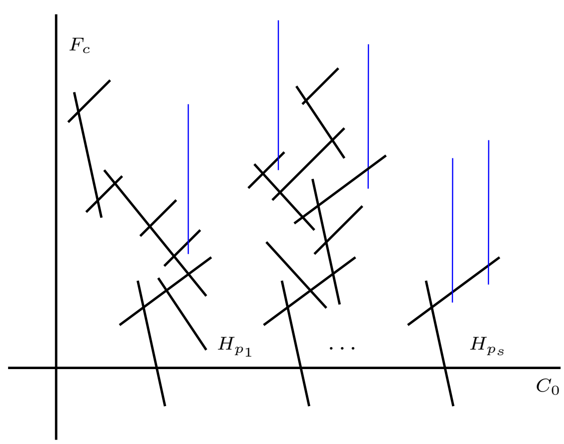

FIGURE 4.1. Structure of degenerate fibers in a minimal completion

Proof. Letting V be any smooth projective completion of S defined over $\mathbf{k}$, the fibration $\pi : S \to C$ extends to a rational map $\overline{\pi} : V \dashrightarrow \overline{C}$ over the smooth projective model $\overline{C}$ or C over $\mathbf{k}$, and, after taking a log-resolution of the indeterminacies of $\overline{\pi}$ and of the boundary divisor $V \setminus S$, we may assume that V is a smooth projective completion of S with SNC boundary B on which π extends to a morphism

$\overline{\pi} : V \to \overline{C}$. Up to performing a sequence of blow-downs defined over $\mathbf{k}$, we may further assume that B does not contain any irreducible component $B_i \simeq \mathbb{P}^1_\kappa$ where $\kappa = \mathbb{R}$ or $\mathbb{C}$, with self-intersection $-\deg(\kappa/\mathbf{k})$ contained in a fiber of $\overline{\pi}$ and intersecting at most two other irreducible components of B. Since V is smooth and the generic fiber of π is isomorphic to the affine line over the function field of C, the generic fiber of $\overline{\pi}$ is isomorphic to the projective line over the function field of $\overline{C}$, and the boundary divisor B contains precisely one irreducible component, say $\overline{C}_0$, which is a section of $\overline{\pi}$. So there exists a birational morphism $\tau : V \to \mathbb{P}(E)$ defined over $\mathbf{k}$ to a ruled surface $\mathbb{P}(E) \to \overline{C}$ for a certain rank 2 vector bundle E over $\overline{C}$. Arguing for instance as in the proof of [2, Lemma 1.0.7], we may, and do, further assume up to changing τ for a different birational morphism, that τ restricts to an isomorphism in a open neighborhood of $\overline{C}_0$. As a consequence, every fiber of $\overline{\pi}$ over a closed point $c \in \overline{C}$ is a geometrically rational tree defined over of the residue field $\kappa(c)$ of c, containing at least one irreducible component isomorphic to $\mathbb{P}^1_{\kappa(c)}$. In the case where S is affine, assertion b) then immediately follows from our minimality assumptions. □

4.1.2. *Topology of $\mathbb{A}^1$-fibered affine surfaces.* The following proposition provides in particular a characterization of $\mathbb{Q}$-acyclic euclidean plane among real surfaces S admitting an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_\mathbb{R}$ defined over $\mathbb{R}$.

Proposition 4.3. *Let S be a smooth geometrically integral surface defined over $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ and let $\pi : S \to C$ be an $\mathbb{A}^1$-fibration defined over $\mathbf{k}$.*

1) The surface $S_\mathbb{C}(\mathbb{C})$ is $\mathbb{Q}$-acyclic if and only if $C \simeq \mathbb{A}^1_\mathbf{k}$ and every closed fiber of π is isomorphic to the affine line when equipped with its reduced structure.

2) If $\mathbf{k} = \mathbb{R}$, S is affine and $C \simeq \mathbb{A}^1_\mathbb{R}$, then $S(\mathbb{R}) \approx \mathbb{R}^2$ if and only if the scheme theoretic fiber of π over every $\mathbb{R}$-rational point is of the form $mR + R'$, where $R \simeq \mathbb{A}^1_\mathbb{R}$, $m \geq 1$ is odd, and R' is an effective divisor whose support is disjoint from R and consists of a disjoint union of affine lines defined over $\mathbb{C}$.

Proof. The first assertion follows from [27, Theorem 4.3.1 p. 231] and its proof, together with the fact that there is no nontrivial real form of $\mathbb{A}^1_\mathbb{C}$.

For the second assertion, we consider a smooth projective completion (V, B) of S as in Lemma 4.2 above. Since S is affine and rational, V is obtained from a Hirzebruch surface $\rho_n : \mathbb{F}_n \to \mathbb{P}^1_\mathbb{R}$ by a sequence of blow-ups $\tau : V \to \mathbb{F}_n$ defined over $\mathbb{R}$ mapping the irreducible component $\overline{C}_0$ of B isomorphically onto a section of ρ_n. Up to changing V for another smooth projective completion obtained by making elementary transformation with center on the fiber $F_\infty \simeq \mathbb{P}^1_\mathbb{R}$ of $\overline{\pi}$ over $\infty = \mathbb{P}^1_\mathbb{R} \setminus \mathbb{A}^1_\mathbb{R}$, we can assume that $n = 1$ and that $\tau_*(\overline{C}_0)$ is the exceptional section of ρ_1 with self-intersection -1. Let $p_1, \ldots, p_s$ and $q_1, \ldots, q_r$ be respectively the $\mathbb{R}$-rational and $\mathbb{C}$-rational points of $\mathbb{A}^1_\mathbb{R}$ over which the fiber of $\pi : S \to \mathbb{A}^1_\mathbb{R}$ is degenerate. Let $F_i \simeq \mathbb{P}^1_\mathbb{R}$ and $G_j \simeq \mathbb{P}^1_\mathbb{C}$ be the fibers of ρ_1 over these points. The images of F_∞, F_i and G_j in $\mathbb{P}^2_\mathbb{R}$ by the contraction of $\tau_*(\overline{C}_0)$ form an arrangement D of lines and geometrically reducible conics meeting each other in a unique $\mathbb{R}$-rational point. This yields a presentation of $(V_\mathbb{C}, B_\mathbb{C})$ as the blow-up of an arrangement $D_\mathbb{C}$ of $s + 2r + 1$ lines meeting each other in a unique point. The image of $F_\infty \simeq \mathbb{P}^1_\mathbb{R}$ in $\mathbb{P}^2_\mathbb{R}$ is a generator of $\mathrm{Cl}(\mathbb{P}^2_\mathbb{R})$, and a basis of the kernel R of the homomorphism $j_\mathbb{C} : \mathbb{Z}\langle D_\mathbb{C}\rangle \to \mathrm{Cl}(\mathbb{P}^2_\mathbb{C})$ consists for instance of the classe of the images in $\mathbb{P}^2_\mathbb{C}$ of the divisors $F_{i,\mathbb{C}} - F_{\infty,\mathbb{C}}$, $T_j - F_{\infty,\mathbb{C}}$ and $\overline{T}_j - F_{\infty,\mathbb{C}}$, where T_j and $\overline{T}_j$ denotes the two connected components of $G_{j,\mathbb{C}}$, exchanged by the real structure σ. Let $\mathcal{E}_0$ be the set consisting of the complexifications of the closures $A_{i,\ell}(p_i)$, $\ell = 1, \ldots, n_i$ and $A_{j,\ell}(q_j)$, $\ell = 1, \ldots, m_j$ in V of the irreducible components of the fibers of π over the points p_i and q_j. We let $\mu_{i,\ell} \geq 1$ and $\nu_{j,\ell} \geq 1$ be the multiplicities $A_{i,\ell}(p_i)$ and $A_{j,\ell}(q_j)$ in the fibers $\overline{\pi}^*(p_i)$ and $\overline{\pi}^*(q_j)$ respectively.

Since $B(\mathbb{R})$ is not empty, it follows from c) in Lemma 2.5 that $S(\mathbb{R}) \approx \mathbb{R}^2$ if and only if the map $H^2(\varphi) : H^2(\mathbb{Z}_2, R) \to H^2(\mathbb{Z}_2, \mathbb{Z}\langle\mathcal{E}_0\rangle)$ is an isomorphism. The Galois cohomology groups $H^2(\mathbb{Z}_2, R)$ and $H^2(\mathbb{Z}_2, \mathbb{Z}\langle\mathcal{E}_0\rangle)$ have bases consisting respectively of the classes of the σ-invariant curves $F_{i,\mathbb{C}} - F_{\infty,\mathbb{C}}$ and of the classes of the complexifications of the $\mathbb{R}$-rational curves among the $A_{i,\ell}(p_i)$. By construction, $H^2(\varphi)([F_{i,\mathbb{C}} - F_{\infty,\mathbb{C}}]) = \sum_{\ell=1}^{n_i} \mu_{i,\ell}[A_{i,\ell}(p_i)_\mathbb{C}]$ where the sum is taken over these $\mathbb{R}$-rational irreducible components. So $H^2(\varphi)$ is an isomorphism if and only if for every $i = 1, \ldots, s$, there exists exactly one $\mathbb{R}$-rational curve among the $A_{i,\ell}(p_i)$, say $A_{i,1}(p_i)$, and the residue class of $\mu_{i,1}$ modulo 2 is nonzero. This proves 2). □

4.1.3. *Affine surfaces completable by a rational chain.* Here we establish auxiliary results concerning the existence of "normal forms" for boundary chains of smooth affine surfaces S defined over $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$ admitting a smooth completion (V,B) whose boundary B is a rational chain.

Lemma 4.4. *Let $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, let (V,B) be a pair defined over $\mathbf{k}$ consisting of a smooth projective surface V and a geometrically rational chain B supporting a effective ample divisor on V and let $S=V\setminus B$. Then the following hold:*

1) If every irreducible component of B is $\mathbf{k}$-rational, then there exists a smooth projective completion (V_1,B_1) of S defined over $\mathbf{k}$ whose boundary B_1 is a chain of $\mathbf{k}$-rational curves of the form $B_1=F\triangleright C\triangleright E$ where $F^2=0$, $C^2=-1$ and E is either empty, or an irreducible curve with self-intersection 0, or a chain of rational curves with self-intersections ≤ -2.

2) If B has a non $\mathbf{k}$-rational irreducible component then there exists a smooth projective completion (V_1,B_1) of S defined over $\mathbf{k}$ whose boundary B_1 is a geometrically rational chain such that $B_{1,\mathbb{C}}$ has one of the following forms:

a) $B_{1,\mathbb{C}}=H\triangleright\overline{H}$ where H is irreducible with self-intersection 1 and $\overline{H}$ is its image by the real structure σ on $V_{1,\mathbb{C}}$.

b) $B_{1,\mathbb{C}}=H\triangleright\overline{H}$ where $H=E\triangleright G$ is a chain consisting of an irreducible curve G with self-intersection 0 and a possible empty chain E of curves with self-intersections ≤ -2 and $\overline{H}$ is the image of H by the real structure σ on $V_{1,\mathbb{C}}$.

b') $B_{1,\mathbb{C}}=H\triangleright\overline{H}$ where $H=E\triangleright G$ is a chain consisting of an irreducible curve G with self-intersection -1 and a nonempty chain of curves with self-intersections ≤ -2 and $\overline{H}$ is the image of H by the real structure σ on $V_{1,\mathbb{C}}$.

c) $B_{1,\mathbb{C}}=H\triangleright C\triangleright\overline{H}$ where C is an irreducible curve of self-intersection 0 invariant by the real structure σ on $V_{1,\mathbb{C}}$, H is either an irreducible curve with self-intersection 0 or a chain of curves with self-intersections ≤ -2, except maybe its right boundary which is a (-1)-curve, and $\overline{H}$ is the image of H by the real structure σ on $V_{1,\mathbb{C}}$.

Proof. We may assume from the very beginning that B is minimal over $\mathbf{k}$, i.e. that it does not contain any irreducible component $B_i\simeq\mathbb{P}^1_\kappa$, where $\kappa=\mathbb{R}$ or $\mathbb{C}$ with self-intersection $-\deg(\kappa/\mathbf{k})$. Being the support of an effective ample divisor on $V_{\mathbb{C}}$, $B_{\mathbb{C}}$ must then contain an irreducible component with self-intersection ≥ -1.

The first assertion is well-known, so we only sketch the argument, referring the reader for instance to [7] for the detail. If every irreducible component of B is $\mathbf{k}$-rational, then the previous observation together implies that B contains at least one irreducible component of nonnegative self-intersection. Fixing an orientation on B, we let D be the leftmost irreducible component of B with this property and we let $D_{\leq}\subset B$ be the subchain of B consisting of D and the components on the left of it. Then by a sequence of birational transformations, consisting of blow-ups of double points of the support of the successive total transforms of $D_{\leq}$ and contractions of irreducible components of them, we can transform (V,B) into a smooth projective completion (V',B') of S defined over $\mathbf{k}$, whose boundary B' consists of $\mathbf{k}$-rational curves, in such a way that the left boundary F of B' has self-intersection 0. (e.g. see [11, Lemma 2.7] for a description of such kind of birational transformations). The surface $V'\setminus B'\simeq V\setminus B$ being affine, B' has at least a second irreducible component and up to making additional elementary transformations with centers at $\mathbf{k}$-rational points of F, we may assume that the irreducible component of B' intersecting F, say C, has self-intersection -1. It follows that the complete linear system $|F|$ generates a $\mathbb{P}^1$-fibration $\overline{\pi}:V'\to\mathbb{P}^1_{\mathbf{k}}$ having F as a full fiber, C as a section, and the remaining irreducible components of B' form a possibly empty chain T contained in a single other fiber of $\overline{\pi}$. If T is not empty then after contracting all successive (-1)-curves contained in its support and performing additional elementary transformations with centers at $\mathbf{k}$-rational points of F if necessary, we reach a smooth projective completion (V_1,B_1) defined over $\mathbf{k}$, whose boundary is a chain $B_1=F\triangleright C\triangleright T$ of $\mathbf{k}$-rational curves with the desired properties. Note for further use that the initial boundary B contained two disjoint irreducible components with non negative self-intersections if and only it consisted of a chain of three irreducible components $D\triangleright C\triangleright E$ with $D^2=(E)^2=0$ and C^2 arbitrary. Indeed, since the birational transformations involved in the construction the pair (V',B') restricts to isomorphisms outside $D_{\leq}$ and its successive total transforms, the proper transform E' in V' of an irreducible component $E \subset B$ disjoint from D is disjoint from F and has self-intersection $(E')^2 = E^2$. The only possibility is thus that E' is a fiber of $\overline{\pi} : V' \to \mathbb{P}^1_{\mathbf{k}}$. Thus E has self-intersection 0 and is necessarily the right boundary of B, and the same argument implies that $B = D \triangleright C \triangleright E$ as desired.

Now suppose that at least one of the irreducible component of B is not $\mathbf{k}$-rational. So $\mathbf{k} = \mathbb{R}$ and we have the following alternative: either $B(\mathbb{R})$ consists of a unique point p and then $B_{\mathbb{C}}$ is a chain of the form $H \triangleright \overline{H}$ where H and $\overline{H}$ are chains intersecting each others in p and exchanged by the real structure σ on $V_{\mathbb{C}}$, or B contains a unique geometrically irreducible component with empty real locus, or a unique $\mathbb{R}$-rational irreducible component, say C_0, and $B_{\mathbb{C}}$ is a chain of the form $H \triangleright C_{0,\mathbb{C}} \triangleright \overline{H}$ where $C_{0,\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$ and H and $\overline{H}$ are possibly empty chains exchanged by the real structure σ on $V_{\mathbb{C}}$. We consider two sub-cases:

1) If $B_{\mathbb{C}}$ is SNC-minimal, then by the observation at the beginning of the proof, there exists an irreducible component D_0 of $B_{\mathbb{C}}$ with non negative self-intersection. If $B(\mathbb{R}) = \{p\}$ then $B_{\mathbb{C}} = H \cup \overline{H}$ and since B defined over $\mathbb{R}$, it follows that $B_{\mathbb{C}}$ contains at least two irreducible components with non negative self-intersection, D_0 and its image $\overline{D}_0$ by the real structure σ on $V_{\mathbb{C}}$. If D_0 and $\overline{D}_0$ are disjoint then $B_{\mathbb{C}} = D_0 \cup C \cup \overline{D}_0$ where $C \simeq \mathbb{P}^1_{\mathbb{C}}$. But then it would follow that $B(\mathbb{R})$ is either empty or homeomorphic to S^1, a contradiction. So up to the choice of an ordering of $B_{\mathbb{C}}$ and the exchange of D_0 and $\overline{D}_0$, we may assume that $B_{\mathbb{C}} = G \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{G}$ where D_0 and $\overline{D}_0$ intersect in $\{p\}$ and G and $\overline{G}$ are possibly empty chains of rational curves with self-intersection ≤ -2 exchanged by the real structure σ. Furthermore $D_0^2 \leq 1$ for otherwise, by blowing-up $\{p\}$ with exceptional E, we would obtain a new smooth projective completion (V', B') of S defined over $\mathbb{R}$ whose boundary chain B' would have the property that $B'_{\mathbb{C}} = G \triangleright D_0 \triangleright E \triangleright \overline{D}_0 \triangleright \overline{G}$ contains two disjoint irreducible components with positive self-intersection. For the same reason we conclude that either $D_0^2 = 1$ and then $B = D_0 \triangleright \overline{D}_0$ or $D_0^2 = \overline{D}_0^2 = 0$ and then $B_{\mathbb{C}} = G \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{G}$ where G and $\overline{G}$ are possibly empty chains of rational curves with self-intersection ≤ -2 exchanged by the real structure σ. This corresponds to cases a) and b) respectively.

Otherwise, if $B(\mathbb{R}) = \emptyset$ or S^1 then $B_{\mathbb{C}} = H \triangleright C_{0,\mathbb{C}} \triangleright \overline{H}$. If H contains an irreducible component with non negative self-intersection then by the same argument as above, we conclude that $B = D_0 \triangleright C_{0,\mathbb{C}} \triangleright \overline{D}_0$ with $D_0^2 = \overline{D}_0^2 = 0$ and by elementary transformations defined over $\mathbb{R}$ with centers on $D_0 \cup \overline{D}_0$, we may assume that $C_{0,\mathbb{C}}^2 = 0$ or -1, the second case being then reduced further to the one $B = D_0 \cup \overline{D}_0$ with $D_0^2 = \overline{D}_0^2 = 1$ by contracting $C_{0,\mathbb{C}}$. The only other possibility is that $C_{0,\mathbb{C}}^2 \geq 0$ and that H and $\overline{H}$ consists of chains of curves with self-intersections ≤ -2 exchanged by the real structure σ. By blowing-up pairs of double points on $C_{0,\mathbb{C}}$ exchanged by the real structure σ, we may reduce to either case c), that is, $B_{\mathbb{C}} = H \triangleright C_{0,\mathbb{C}} \triangleright \overline{H}$ where $C_{0,\mathbb{C}}^2 = 0$, H consists of a chain of curves with self-intersection ≤ -2 except maybe its right boundary which is a (-1)-curve, and $\overline{H}$ is the image of H by the real structure σ, or to the situation that $C_{0,\mathbb{C}}^2 = -1$ from which we reach case b) by contracting C_0.

2) If $B_{\mathbb{C}}$ is not minimal over $\mathbb{C}$, then the hypothesis that B is minimal over $\mathbb{R}$ implies that $B_{\mathbb{C}} = E \triangleright D_0 \triangleright \overline{D}_0 \triangleright \overline{E}$ where D_0 and $\overline{D}_0$ are irreducible with self-intersection -1 and E is a chain of curves with self-intersection different from -1. Since $B_{\mathbb{C}}$ is the support of an ample divisor on $V_{\mathbb{C}}$, E cannot be empty. Furthermore, it cannot contain any irreducible component with nonnegative self-intersection for otherwise $B_{\mathbb{C}}$ would contain two disjoint such components. This yields case b'). □

Corollary 4.5. *Let (V, B) be a pair defined over $\mathbb{R}$ consisting of a smooth projective surface V and a geometrically rational chain B supporting a effective ample divisor on V. If the irreducible component of B are not all $\mathbb{R}$-rational then there exists a smooth projective completion (V_2, B_2) of $S_{\mathbb{C}} = V_{\mathbb{C}} \setminus B_{\mathbb{C}}$ defined over $\mathbb{C}$ whose boundary B_2 is a chain of $\mathbb{C}$-rational curves of the form $B_2 = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and where E is either empty, or an irreducible curve with self-intersection 0, or a chain of the one of the following types*

i) $[-e_1, -e_2, \cdots, -e_n, -e_n, \ldots, -e_1]$, *where $n \geq 1$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.*

ii) $[-e_1, \ldots, -e_{n-1}, -2e_n, -e_{n-1}, \ldots, -e_1]$ *where $n \geq 2$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.*

iii) $[-e_1, \ldots, -e_{n-1}, -2e_n + 1, -e_{n-1}, \ldots, -e_1]$ *where $n \geq 2$ and $e_i \geq 2$ for every $i = 1, \ldots, n$.*

Proof. Let (V_1, B_1) be the smooth projective completion (V_1, B_1) of S defined over $\mathbb{R}$ with geometrically rational chain boundary B_1 constructed in the previous lemma. In case a) we reach a chain of three irreducible components with self-intersection $(0,-1,0)$ by blowing-up the intersection point of $H \triangleright \overline{H}$. In case b) and E is empty, then $B_{1,\mathbb{C}}$ consists of a pair of irreducible curves G and $\overline{G}$ with self-intersection 0 which can be transformed into a pair of curves with self-intersection 0 and -1 by performing an elementary transformation at their intersection point. Otherwise, if $E = E_1 \triangleright \cdots \triangleright E_n$ is not empty, we let $e_i = -E_i^2 = -(\overline{E}_i)^2 \geq 2$ for every $i = 1, \ldots, n$. We desired smooth projective completion of $S_{\mathbb{C}}$ is obtained from $(V_{1,\mathbb{C}}, B_{1,\mathbb{C}})$ by performing the following sequence of birational transformations with centers and exceptional curves all supported on the successive total transforms of $B_{1,\mathbb{C}}$. We first blow-up the point $G \cap \overline{G}$ and contract the proper transform of G. The self-intersection of E_n increased by 1, the self-intersection of $\overline{G}$ decreased by one and the proper transform of the exceptional divisor of the blow-up has self-intersection 0. We repeat the same operation again $e_n - 2$ times until the proper transform of E_n has self-intersection -1. Then we blow-up again the intersection point of the proper transform of the last exceptional divisor E with $\overline{G}$ to get a chain of the form $E_1 \triangleright E_2 \triangleright \cdots \triangleright E_n \triangleright E \triangleright E' \triangleright \overline{G} \triangleright \overline{E}_n \triangleright \cdots \triangleright \overline{E}_1$ with $E_n^2 = E^2 = -1$ and $\overline{G}^2 = -e_n$. Then we contract E_n to get a chain of the form $E_1 \triangleright E_2 \triangleright \cdots \triangleright E_{n-1} \triangleright E \triangleright E' \triangleright \overline{G} \triangleright \overline{E}_n \cup \cdots \triangleright \overline{E}_1$ where $E_{n-1}^2 = -e_{n-1}+1$, $E^2 = 0$ and $(E')^2 = -1$. We continue by induction until we reach a chain of the form $E_1 \triangleright E \triangleright E' \triangleright \tilde{T}$ where $E_1^2 = E^2 = -1$, $(E')^2 = -e_1$ and $\tilde{T}$ is a chain of irreducible curves of type $[-e_2, \cdots, -e_n, -e_n, \ldots, -e_1]$. By contracting E_1, we eventually reach the desired smooth completion with boundary chain of type i).

The remaining two cases, corresponding respectively to smooth projective completions (V_1, B_1) of the form c) and b') in Lemma 4.4, follow from similar arguments. We leave the detail to the reader. □

Example 4.6. ($\mathbb{Q}$-acyclic surfaces completable by a chain of rational curves). Let S be smooth complex $\mathbb{Q}$-acyclic surface non isomorphic to $\mathbb{A}^2_{\mathbb{C}}$, with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{C}}$ and admitting a smooth projective completion (V, B) as in Lemma 4.2 for which $B = F_\infty \triangleright \overline{C}_0 \triangleright E$ is a chain, where $\infty = \mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{A}^1_{\mathbb{C}}$. Since $S \not\simeq \mathbb{A}^2_{\mathbb{C}}$, E is not empty. Letting $q = \overline{\pi}(E)$, it follows from Lemma 4.2 and Proposition 4.3 that $\pi^{-1}(q)$ is the unique degenerate fiber of π. Its closure in V is irreducible, of multiplicity $\mu \geq 2$ as a component of $\overline{\pi}^*(q)$ and is the unique (-1)-curve contained in $\overline{\pi}^{-1}(q)$. It follows that V is obtained from the Hirzebruch surface $\rho_n : \mathbb{F}_n \to \mathbb{P}^1_{\mathbb{C}}$, where $n = \overline{C}_0^2$, by a sequence of blow-ups $\tau : V \to \mathbb{F}_n$ of the following type: the first step is the blow-up of a point of $F_q = \rho_n^{-1}(q)$ distinct from its intersection with the exceptional section of ρ_n, say with exceptional divisor E_0. The second step consists of a subdivisional expansion with center at the point $p = (F_q \cap E_0)$ with last exceptional divisor $A_0(p)$ and multiplicities (μ, ν) for a certain integer $\nu \geq 1$. The final step consists of the blow-up of a simple point r of the total transform of F_q supported on $A_0(p)$, with exceptional divisor D. We then have $E = \tau^{-1}(F_q) \setminus D$.

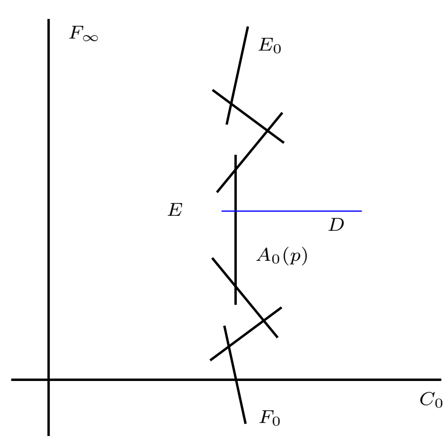

FIGURE 4.2. Boundary chain of a $\mathbb{Q}$-acyclic surface with negative Kodaira dimension

Note in particular that the left and right boundary curves of the chain E have distinct self-intersections, except if E consists of three irreducible components with self-intersection -2. In this case, the contraction of the chain $\overline{C}_0 \triangleright E$ defines a birational morphism $\beta : V \to \mathbb{P}^2_{\mathbb{C}}$ which maps S isomorphically onto the complement of the image $\beta_*(F_\infty)$ of F_∞, which is a smooth conic.

In the case where q is a real point of $\mathbb{P}^1_{\mathbb{C}}$ equipped with the standard real structure and r is a real point of $A_0(p)$, the pair (V, B) is defined over $\mathbb{R}$ and so is the $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{C}}$. By virtue of Proposition 4.3, S is a then a $\mathbb{Q}$-acyclic euclidean plane if and only if μ is odd.

4.1.4. *Existence of real $\mathbb{A}^1$-fibrations.* The next proposition completes the proof of Theorem 4.1.

Proposition 4.7. *Let S be a smooth geometrically integral surface defined over $\mathbb{R}$ of negative Kodaira dimension such that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic and $S(\mathbb{R})$ is non compact. Then S admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$.*

Proof. By [27, Chapter 4, Theorem 4.3.1] $S_{\mathbb{C}}$ admits an $\mathbb{A}^1$-fibration $q : S_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. If q is the unique such $\mathbb{A}^1$-fibration on $S_{\mathbb{C}}$ up to composition by automorphisms of $\mathbb{A}^1_{\mathbb{C}}$, it must be the complexification of a morphism $\pi : S \to C$ defined over $\mathbb{R}$. Since the affine line does not have nontrivial forms over fields of characteristic zero, we conclude that $C = \mathbb{A}^1_{\mathbb{R}}$ and that the generic fiber of π is isomorphic to the affine line over the function field of C. So $\pi : S \to C \simeq \mathbb{A}^1_{\mathbb{R}}$ is the desired $\mathbb{A}^1$-fibration defined over $\mathbb{R}$.

So it remains to consider the case where $S_{\mathbb{C}}$ admits at least two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{\mathbb{C}}$ with distinct general fibers. By virtue of [11, Theorem 2.4] this holds if and only if $S_{\mathbb{C}}$ is not isomorphic to $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ and admits a smooth projective completion whose boundary divisor consists of a chain of smooth proper rational curves. Furthermore, the boundary in any other smooth projective completion of $S_{\mathbb{C}}$ with SNC-minimal boundary is then again a chain of rational curves. Now let (V, B) be a smooth projective completion of S defined over $\mathbb{R}$ and such that B is a geometrically rational tree minimal over $\mathbb{R}$. The tree $B_{\mathbb{C}}$ is then minimal over $\mathbb{C}$ unless it contains pairs of non-disjoint (-1)-curves exchanged by the real structure σ, in which case a minimal smooth projective completion (V_0, B_0) over $\mathbb{C}$ is obtained from $(V_{\mathbb{C}}, B_{\mathbb{C}})$ by blowing-down all possible successive (-1)-curves in $B_{\mathbb{C}}$. Since B_0 must be a chain, it follows that $B_{\mathbb{C}}$ is a chain too, and so, B is a geometrically rational chain. Indeed, if $B_{\mathbb{C}}$ contains an irreducible component R intersecting at least three other components, then because B_0 is a chain, at least one the connected component of $B_{\mathbb{C}} \setminus R$, say D, is contracted to a smooth point by the above sequence of blow-downs. So it must contain at least a (-1)-curve E intersecting at most two other irreducible components of $B_{\mathbb{C}}$, one of which being, by the minimality assumption, its image $\overline{E}$ by the real structure σ. But then D contains a pair of non-disjoint (-1)-curves, contradicting the negative definiteness of its intersection matrix.

a) If B consists of $\mathbb{R}$-rational components only then by virtue of 1) in Lemma 4.4, there exists a smooth projective completion (V', B') of S defined over $\mathbb{R}$ whose boundary B' is a chain of $\mathbb{R}$-rational curves of the form $B' = F \triangleright C \triangleright T$ where $F^2 = 0$, $C^2 = -1$ and T is either empty, or an $\mathbb{R}$-rational curve with self-intersection 0, or a chain of $\mathbb{R}$-rational curves with self-intersections ≤ -2. Since V' is $\mathbb{R}$-rational, the complete linear system $|F|$ defines a $\mathbb{P}^1$-fibration $V' \to \mathbb{P}^1_{\mathbb{R}}$ having F as a fiber and C as a section and whose restriction to S is an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ if T does not consists of a unique curve, or an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$ otherwise, the second case being excluded by the fact that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic by hypothesis.

b) If B contains at least one non $\mathbb{R}$-rational component then by virtue of Corollary 4.5, there exists a smooth projective completion (V', B') of $S_{\mathbb{C}}$ whose boundary B' is a chain of $\mathbb{C}$-rational curves $B' = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and where E is either empty, or an irreducible curve with self-intersection 0 or a chain of curves with negative self-intersections listed in the corollary. In the first two cases, it follows that $S_{\mathbb{C}}$ is either isomorphic to $\mathbb{A}^2_{\mathbb{C}}$, in which case $S \simeq \mathbb{A}^2_{\mathbb{R}}$ and we are done, or it admits an $\mathbb{A}^1$-fibration over $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. But the second possibility is again excluded by the hypothesis that $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic. In the remaining cases, it follows from the description given in Corollary 4.5 that the sequences of self-intersections of the irreducible components of E are all symmetric. By virtue of [7, Corollary 2] (see also [2, Corollary 3.2.3]), such sequences up to reversion of the ordering are invariants of the isomorphism type of $S_{\mathbb{C}}$. Comparing with the sequences obtained in Example 4.6 above for $\mathbb{Q}$-acyclic complex surfaces admitting a smooth projective completion whose boundary is a chain, we conclude that the only possibility is that E consists of a chain of three curves with self-intersection -2, whence that $S_{\mathbb{C}}$ is isomorphic to the complement of a smooth conic in $\mathbb{P}^2_{\mathbb{C}}$. This implies in turn that S is isomorphic to the complement of a smooth conic D in $\mathbb{P}^2_{\mathbb{R}}$, and since $S(\mathbb{R})$ is not compact, D has an $\mathbb{R}$-rational point p. It follows that S admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, induced by the restriction of pencil of conics osculating D at p, i.e the pencil generated by D and twice its tangent line at p. □

Remark 4.8. The proof of Proposition 4.7 shows more generally that a smooth affine geometrically integral surface S defined over $\mathbb{R}$ and of negative Kodaira dimension admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ provided that $S_{\mathbb{C}}$ does not admit any smooth projective completion (V, B) not defined over $\mathbb{R}$ and whose boundary is a chain of rational curves of the form $B = F \triangleright C \triangleright E$ where $F^2 = 0$, $C^2 = -1$ and E is a nonempty ordered chain of rational curves whose type $[-a_1, \ldots, -a_n]$, $a_i \geq 2$, is symmetric, in the sense that the sequences $(-a_1, \ldots, -a_n)$ and $(-a_n, \ldots, -a_1)$ are equal. Chains with this symmetry property are called palindromes in [2].

The $\mathbb{Q}$-acyclicity of $S_{\mathbb{C}}(\mathbb{C})$ and the non compactness of $S(\mathbb{R})$ play a crucial in the characterization obtained in this proposition. Indeed, as already observed in the proof of Proposition 4.7, the complexification of the complement S of a smooth conic $D \subset \mathbb{P}^2_{\mathbb{R}}$ without $\mathbb{R}$-rational point admits a smooth projective completion (V, B) not defined over $\mathbb{R}$ whose boundary B has the form $B = F \triangleright C \triangleright E$ where E is a palindrome of type $[-2, -2, -2]$, and this surface S does not admit any $\mathbb{A}^1$-fibration defined over $\mathbb{R}$.

One can show along the same lines that there even exists smooth surfaces S of negative Kodaira dimension with $S(\mathbb{R}) \approx \mathbb{R}^2$ but $S_{\mathbb{C}}(\mathbb{C})$ not $\mathbb{Q}$-acyclic which do not admit any $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$. We leave to the reader to check that this holds for instance for the nontrivial real form $S = \{x^2 + y^2 = z^3 - 1\} \subset \mathbb{A}^3_{\mathbb{R}}$ of the surface $S' = \{uv = z^3 - 1\} \subset \mathbb{A}^3_{\mathbb{R}}$, whose complexification has $H_2(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}^2$. The real loci of S and S' are both homeomorphic to $\mathbb{R}^2$ via the maps $\mathbb{R}^2 \to S(\mathbb{R})$, $(x, y) \mapsto (x, y, \sqrt[3]{x^2 + y^2 + 1})$ and $\mathbb{R}^2 \to S'(\mathbb{R})$, $(u, v) \mapsto (u, v, \sqrt[3]{uv + 1})$ respectively. The surface S' admits an $\mathbb{A}^1$-fibration $\pi' = \mathrm{pr}_u |_{S'} \colon S' \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ whose unique degenerate fiber $(\pi')^{-1}(0)$ consists of the disjoint union of the curves $\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[v])$ and $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{R}[z]/((z^2 + z + 1)[v]))$. So $\kappa(S) = \kappa(S_{\mathbb{C}}) = \kappa(S') = -\infty$. Furthermore, S' admits a smooth projective completion (V', B') defined over $\mathbb{R}$ whose boundary B is a chain of three $\mathbb{R}$-rational curves $F \triangleright C \triangleright E$ as above where E is palindrome consisting of a unique curve with self-intersection -3 (see e.g. [2, § 5.4]).

4.2. **$\mathbb{R}$-biregular birational rectification of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension.** In this subsection, we consider the question of classification of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension up to $\mathbb{R}$-biregular birational equivalence. We say for short that such an euclidean plane S is $\mathbb{R}$-*biregularly birationally rectifiable* if there exists an $\mathbb{R}$-biregular birational map $\varphi : S \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$, i.e. a birational map defined over $\mathbb{R}$, containing the real locus of S in its domain of definition and inducing a diffeomorphism $\varphi(\mathbb{R}) : S(\mathbb{R}) \xrightarrow{\simeq} \mathbb{R}^2 = \mathbb{A}^2_{\mathbb{R}}(\mathbb{R})$. The following theorem implies in particular that a large class of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira are indeed $\mathbb{R}$-biregularly birationally rectifiable.

Theorem 4.9. *Let S be a smooth affine geometrically integral surface defined over $\mathbb{R}$ with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$. Suppose that $S(\mathbb{R}) \approx \mathbb{R}^2$ and that all but at most one fibers of π over $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ contain a reduced $\mathbb{R}$-rational irreducible component. Then S is $\mathbb{R}$-biregularly birationally rectifiable.*

The assumptions of Theorem 4.9 being satisfied by any $\mathbb{Q}$-acyclic euclidean plane S admitting an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ with at most one degenereate, we obtain:

Corollary 4.10. *Every $\mathbb{Q}$-acyclic euclidean plane S with $S(\mathbb{R}) \approx \mathbb{R}^2$ admitting an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ with at most one degenerate fiber is $\mathbb{R}$-biregularly birationally rectifiable.*

The proof of Theorem 4.9 consists of two steps, given in § 4.2.1-4.2.3 below. We first reduce via suitable $\mathbb{R}$-biregular birational maps to the case of surfaces S equipped with an $\mathbb{A}^1_{\mathbb{R}}$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ with irreducible fibers and at most one degenerate $\mathbb{R}$-rational fiber. Then we show by induction on the number of irreducible components in the boundary B of a smooth projective SNC-minimal completion (V, B) of S defined over $\mathbb{R}$ that every such surface is $\mathbb{R}$-biregularly birationally rectifiable.

4.2.1. *Standard r-models.* The simplest surfaces $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ satisfying the hypotheses of Theorem 4.9 are those for which $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ restricts to a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$ and $\pi^*(\{0\})$ is geometrically irreducible, of odd multiplicity $m \geq 1$. Indeed, the fact that $S(\mathbb{R}) \approx \mathbb{R}^2$ is then guaranteed by 2) in Proposition 4.3.

4.2.1.1. When specialized to such surfaces, the general description given in § 4.1.1 provides a smooth projective completion (V, B) of S defined over $\mathbb{R}$ into a surface obtained from $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ with exceptional section $C_0 \simeq \mathbb{P}^1_{\mathbb{R}}$ and a pair of fixed $\mathbb{R}$-rational fibers $E_{-1} = \rho_1^{-1}(\{0\})$ and $F_\infty = \rho_1^{-1}(\mathbb{P}^1_{\mathbb{R}} \setminus \mathbb{A}^1_{\mathbb{R}})$ via a birational morphism $\tau : V \to \mathbb{P}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ of the following form:

- If $m = 1$ then $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ is isomorphic to the trivial $\mathbb{A}^1$-bundle $\mathrm{pr}_1 : S \simeq \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$, and we have an isomorphism $(V, B) = (\mathbb{F}_1, F_\infty \cup C_0)$.

- Otherwise, if $m \geq 2$ then $\tau = \tau_0 \circ \cdots \circ \tau_n$ is a sequence of blow-ups of $\mathbb{R}$-rational points, starting with the blow-up $\tau_0 : V_1 \to V_0 = \mathbb{F}_1$ of a point $p_0 \in E_{-1} \setminus C_0$, say with exceptional divisor E_0, followed by the blow-up $\tau_1 : V_2 \to V_1$ of the intersection point p_1 of E_0 with the proper transform of E_{-1}, with exceptional divisor E_1, and continuing with a sequence of blow-ups $\tau_i : V_{i+1} \to V_i$ of $\mathbb{R}$-rational points $p_i \in E_{i-1}$ with exceptional divisor E_i. The last step $\tau_n : V = V_{n+1} \to V_n$ is the blow-up of an $\mathbb{R}$-rational point $p_n \in E_{n-1}$ with exceptional divisor A_0. The surface S is then isomorphic to the complement in V of the SNC divisor $B = F_\infty \cup C_0 \cup E$, where $E = \bigcup_{i=0}^{n} E_{i-1}$ is a tree of $\mathbb{R}$-rational curves, the $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ coincides with the restriction to S of the $\mathbb{P}^1$-fibration $\overline{\pi} : \rho_1 \circ \tau : V \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi^{-1}(\{0\}) = A_0 \cap S$. Note that by construction $E_i^2 \leq -2$ for every $i = -1, \ldots, n-1$.

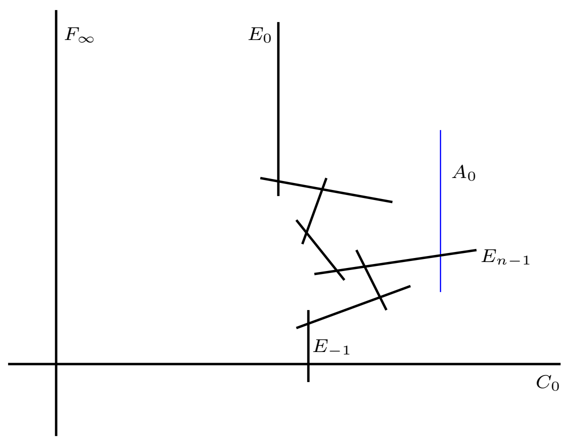

FIGURE 4.3. Structure of the divisor $B \cup A_0$

Since $m \geq 2$ is odd, the sequence τ_i is not completely arbitrary: for instance, the branching components of the tree E, i.e. the irreducible components of E intersecting at least two other irreducible components, must have odd multiplicities as irreducible components of the degenerate fiber $\overline{\pi}^*(\{0\})$ of the $\mathbb{P}^1$-fibration $\overline{\pi} : V \to \mathbb{P}^1_{\mathbb{R}}$. More precisely, we have the following description

Lemma 4.11. *For a pair $(V, B = F_\infty \cup C_0 \cup E)$ as in § 4.2.1.1, the following holds:*

1) Every branching component of E has odd multiplicity as an irreducible component of the degenerate fiber $\overline{\pi}^*(\{0\})$.

2) Let $L \subset C_0 \cup E$ be the unique minimal subchain containing C_0 and E_0 and let E_i be the unique branching component of E contained in L. If $E_i \cap E_0 \neq \emptyset$ then $i = 2p$ for some $p \geq 1$ and $L = C_0 \triangleright E_{-1} \triangleright \cdots \triangleright E_{2p} \triangleright E_0$ is a chain of type $(-1, -2, \ldots, -2, E_{2p}^2, -(2p+1))$.

Proof. If E has a branching component with even multiplicity, say E_i for some $i \geq 2$, then since $\tau_{i+1} : V_{i+2} \to V_{i+1}$ necessarily consists of the blow-up of a simple point of $E_{-1} \cup \cdots \cup E_i$ supported on E_i, the proper transform in V of its exceptional divisor E_{i+1} has the same multiplicity as E_i in $\overline{\pi}^*(\{0\})$. Since the center of the next blow-up τ_{i+2} is either the point $E_i \cap E_{i+1}$ or a simple point supported on E_{i+1}, it follows by induction that the proper transform in V of every divisor E_j, $j \geq i$ arises with even multiplicity in $\overline{\pi}^*(\{0\})$. As a consequence, A_0 would have even multiplicity as an irreducible component of $\overline{\pi}^*(\{0\})$, in contradiction with the fact that $A_0 \cap S$ has odd multiplicity m as a scheme theoretic fiber of π. The second assertion follows immediately from the observation that if $E_i \cap E_0 \neq \emptyset$ then the multiplicity of E_i as a component of $\overline{\pi}^*(\{0\})$ is equal to $-E_0^2 \geq 2$. $\square$

Definition 4.12. An *r-standard* $\mathbb{A}^1$*-fibered surface* is a smooth geometrically integral affine surface S with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, restricting to a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$ and such $\pi^{-1}(\{0\})$ is geometrically irreducible, of odd multiplicity $m \geq 1$. An *r-standard pair* is a pair (V, B) consisting of a smooth geometrically integral projective surface V and a geometrically rational tree B both defined over $\mathbb{R}$ isomorphic to the completion of an *r-standard* $\mathbb{A}^1$*-fibered surface* constructed in § 4.2.1.1.

The next proposition reduces the study of the $\mathbb{R}$-biregular birational rectifiability of surfaces considered in Theorem 4.9 to the case of r-standard surfaces:

Proposition 4.13. *Let S be a smooth affine geometrically integral surface defined over $\mathbb{R}$ with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$. Suppose that $S(\mathbb{R}) \approx \mathbb{R}^2$ and that all but at most one fiber of π over $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ contain a reduced $\mathbb{R}$-rational irreducible component. Then there exists an r-standard affine $\mathbb{A}^1$-fibered surface $\pi_0 : S_0 \to \mathbb{A}^1_{\mathbb{R}}$ and an $\mathbb{R}$-biregular birational map $\varphi : S \dashrightarrow S_0$ such that the following diagram commutes*

$$\begin{array}{ccc} S & \stackrel{\varphi}{\dashrightarrow} & S_0 \\ \pi \downarrow & & \downarrow \pi_0 \\ \mathbb{A}^1_{\mathbb{R}} & \stackrel{\simeq}{\to} & \mathbb{A}^1_{\mathbb{R}}. \end{array}$$

Proof. The strategy is of course to eliminate on the one hand all degenerate fibers of π over $\mathbb{C}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ and on the other hand all non $\mathbb{R}$-rational irreducible components of the degenerate fibers of π over $\mathbb{R}$-rational points. Since $S(\mathbb{R}) \approx \mathbb{R}^2$ it follows from 2) in Proposition 4.3 that for every $\mathbb{R}$-rational point $p \in \mathbb{A}^1_{\mathbb{R}}$, the fiber $\pi^*(p)$ has the form $mR + R'$, where $R \simeq \mathbb{A}^1_{\mathbb{R}}$, $m \geq 1$ is odd, and R' is an effective divisor whose support is disjoint from R and consists of a disjoint union of affine lines defined over $\mathbb{C}$. The complement S' in S of all non $\mathbb{R}$-rational irreducible components of $\pi^{-1}(p)$ where p runs through the finitely many $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ over which the fiber of π is degenerate is an affine open subset of S on which π restricts to an $\mathbb{A}^1$-fibration $\pi' : S' \to \mathbb{A}^1_{\mathbb{R}}$ whose fibers over $\mathbb{R}$-rational point of $\mathbb{A}^1_{\mathbb{R}}$ are all isomorphic to $\mathbb{A}^1_{\mathbb{R}}$ when equipped with their reduced structure. Furthermore, the hypotheses imply that there exists at most one $\mathbb{R}$-rational point of $\mathbb{A}^1_{\mathbb{R}}$ over which the scheme theoretic fiber of π', say $(\pi')^*(\{0\})$ is degenerate. By construction, the inclusion $S' \hookrightarrow S$ is an $\mathbb{R}$-biregular birational map. Now let (V', B') be a smooth projective completion of S' defined over $\mathbb{R}$ with geometrically rational boundary tree

$$B' = F'_\infty \cup \overline{C}'_0 \cup \bigcup_{p \in \mathbb{A}^1_{\mathbb{R}}} H'_p$$

as in §4.1.1, where $F_\infty \simeq \mathbb{P}^1_{\mathbb{R}}$ and $\overline{C}_0 \simeq \mathbb{P}^1_{\mathbb{R}}$ are respectively the fiber over the $\mathbb{R}$-rational closed point $\infty = \mathbb{P}^1_{\mathbb{R}} \setminus \mathbb{A}^1_{\mathbb{R}}$ and a section of the $\mathbb{P}^1$-fibration $\overline{\pi}' : V' \to \mathbb{P}^1_{\mathbb{R}}$ extending π'. Since $(\pi')^{-1}(\{0\})$ is the unique possibly degenerate fiber of π' over an $\mathbb{R}$-rational point of $\mathbb{A}^1_{\mathbb{R}}$, the divisor $\bigcup_{p \in \mathbb{A}^1_{\mathbb{R}}} H_p$ can be decomposed into the disjoint $\bigcup H_0 \coprod \bigcup_{q \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})} H_q$ where H_0 is a possibly empty tree of $\mathbb{R}$-rational curve supported on $(\overline{\pi}')^{-1}(\{0\})$. For every $\mathbb{C}$-rational point q of $\mathbb{A}^1_{\mathbb{R}}$ for which H_q is not empty, equivalently, for every $\mathbb{C}$-rational point q of $\mathbb{A}^1_{\mathbb{R}}$ over which the fiber $(\pi')^*(q)$ is degenerate, there exists a sequence of contractions $\beta_q : V' \to V'_q$ of curves defined over $\mathbb{R}$ supported on $(\overline{\pi}')^{-1}(q)$ such that $\beta_q((\overline{\pi}')^*(q)) \simeq \mathbb{P}^1_{\mathbb{C}}$ is a smooth fiber of the $\mathbb{P}^1$-fibration $\overline{\pi}'_q = \overline{\pi}' \circ \beta_q^{-1} : V'_q \to \mathbb{P}^1_{\mathbb{R}}$. Let V_0 be the $\mathbb{P}^1$-fibered surface obtained from V' by performing such sequences of contractions for every $\mathbb{C}$-rational point $q \in \mathbb{A}^1_{\mathbb{R}}$ such that $(\pi')^*(q)$ is degenerate, let $\beta : V' \to V_0$ be the corresponding birational morphism and let B_0 be the image of $F_\infty \cup \overline{C}_0 \cup H_0$ by β. By construction, $\overline{\pi}_0^*(\{0\})$ is the unique degenerate fiber of $\overline{\pi}_0 : V_0 \to \mathbb{P}^1_{\mathbb{R}}$ and the restriction of β to S induces an $\mathbb{R}$-biregular birational map $S \dashrightarrow S_0 = V_0 \setminus B_0$ which commutes with the $\mathbb{A}^1$-fibrations π and π_0 induced by $\overline{\pi}$ and $\overline{\pi}_0$ respectively. So $\pi_0 : S_0 \to \mathbb{A}^1$ is a standard $\mathbb{A}^1$-fibered surface, provided that S_0 is affine. First note that S_0 does not contain any complete algebraic curve. Indeed, otherwise such an irreducible curve D would not intersect F_∞ in V_0, whence would be contained in a fiber of $\overline{\pi}_0$. Since every fiber of $\overline{\pi}_0$ but $\overline{\pi}_0^{-1}(0)$ is smooth and $\overline{C}_0$ is a section of $\overline{\pi}_0$, it would follow that D is contained in $\overline{\pi}_0^{-1}(0)$. But since β restricts to an isomorphism in a neighborhood of $(\overline{\pi}')^{-1}(0)$, it would follow that $\beta^{-1}(D)$ is a complete curve in S', which is absurd since S' is affine. It remains to observe that B_0 is the support of an effective $\mathbb{Q}$-divisor Δ on V_0 whose intersection with every irreducible component of B_0 is strictly positive: since $F_0^2 = 0$ and the dual graph of B_0 is a tree, such a Δ is obtained by assigning a positive coefficient $a_0 \in \mathbb{Q}$ to F_0 and assigning to the other irreducible components of B_0 a sequence of positive rational coefficients decreasing rapidly with respect to the distance to F_0 in the dual graph of B_0. The so constructed Δ is ample by virtue of the Nakai-Moishezon criterion and so, S_0 is affine as desired. □

4.2.2. *Elementary birational links between r-standard pairs.* Let $(V, B = F_\infty \cup C_0 \cup E)$ be an r-standard pair with non empty tree E and let $\tau = \tau_0 \circ \cdots \circ \tau_m : V \to \mathbb{F}_1$ be the birational morphism constructed in § 4.2.1.1. Since the proper base point p_0 of τ^{-1} belongs to $E_{-1} \setminus C_0$, the pencil $\mathbb{F}_1 \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ lifting the projection $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ from p_0 via the contraction $\mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ of C_0 lifts to a $\mathbb{P}^1$-fibration $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ with a unique degenerate fiber supported by the closure in V of $(C_0 \cup E \setminus E_0) \cup A_0$ and having the proper transforms of F_∞ and E_0 as cross-sections. The restriction of ξ_{p_0} to $S = V \setminus B$ is thus a surjective fibration $S \to \mathbb{P}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ whose generic fiber is isomorphic to the 1-punctured affine line over the function field of $\mathbb{P}^1_{\mathbb{R}}$.

Definition 4.14. With the notation above, we call *elementary links* the birational transformations of pairs $\eta : (V, B) \dashrightarrow (V^{(1)}, B^{(1)})$ defined as follows: :

a) If $E_0^2 = -2s$ is even, we choose s distinct smooth fibers $\ell_i \simeq \mathbb{P}^1_{\mathbb{C}}$ of $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ over $\mathbb{C}$-rational points of $\mathbb{P}^1_{\mathbb{R}}$ and we let $\eta' : V \dashrightarrow V'$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of the $\mathbb{C}$-rational points $\ell_i \cap F_\infty$, with respective exceptional divisors $\tilde{\ell}_i \simeq \mathbb{P}^1_{\mathbb{C}}$, followed by the contraction of the proper transforms of the ℓ_i, $i = 1, \ldots, s$. The proper transforms of E_0 and F_∞ in V' have self-intersections 0 and $-2s$ respectively, while the self-intersections of the remaining irreducible components of B are left unchanged.

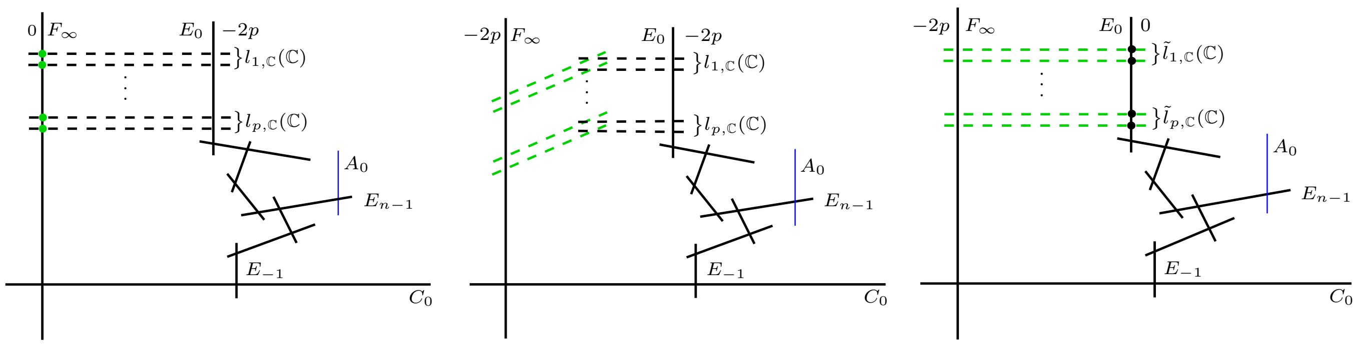

FIGURE 4.4. Elementary link: even case

Since V' is rational, the complete linear system $|E_0|$ generates a $\mathbb{P}^1$-fibration $\overline{\pi}' : V' \to \mathbb{P}^1_{\mathbb{R}}$ having the unique irreducible component E_{i_1} of E intersecting E_0 as a section. Furthermore, since E_0^2 is even, the multiplicity of E_{i_1} as an irreducible component of the degenerate fiber of ξ_{p_0} is even and so, it follows from Lemma 4.11 that E_{i_1} is not a branching component of E. This implies that the closure T in V' of $B \setminus (E_{i_1} \cup E_0)$ is contained in a unique degenerate fiber of $\overline{\pi}'$. After making a sequence of birational transformations consisting on the one hand of elementary transformations with $\mathbb{R}$-rational centers (including infinitely near ones) on $E_0 \setminus E_{i_1}$ and contracting the proper transform of E_0, and on the other hand of contracting all successive (-1)-curves supported on the proper transform of T starting from that of C_0, the total transform of $E_0 \cup E_{i_1} \cup T$ can be re-written in the form $B^{(1)} = F_\infty^{(1)} \cup C_0^{(1)} \cup E^{(1)}$, where $F_\infty^{(1)} \simeq \mathbb{P}^1_{\mathbb{R}}$ is the last exceptional divisor of the sequence of elementary transformations, $C_0^{(1)}$ is the proper transform of E_{i_1} and $E^{(1)}$ is the image of T. We let $V^{(1)}$ be the so constructed smooth projective surface and we let $\eta : (V, B) \dashrightarrow (V^{(1)}, B^{(1)})$ be the corresponding birational map. By construction, $E^{(1)}$ is either empty or has strictly less irreducible components than E.

b) If $E_0^2 = -(2s+1)$ is odd, we choose $s+1$ distinct smooth fibers $\ell_i \simeq \mathbb{P}^1_{\mathbb{C}}$ of $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ over $\mathbb{C}$-rationals point of $\mathbb{P}^1_{\mathbb{R}}$ and we let $\eta' : V \dashrightarrow V'$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of the $\mathbb{C}$-rational points $\ell_i \cap F_\infty$, with respective exceptional divisors $\tilde{\ell}_i \simeq \mathbb{P}^1_{\mathbb{C}}$, followed by the contraction of the proper transforms of the ℓ_i, $i = 1, \ldots, s+1$. The proper transforms of E_0 and F_∞ in V'' have self-intersections 1 and $-2s-2$ respectively, while the self-intersections of the remaining irreducible components of B are left unchanged.

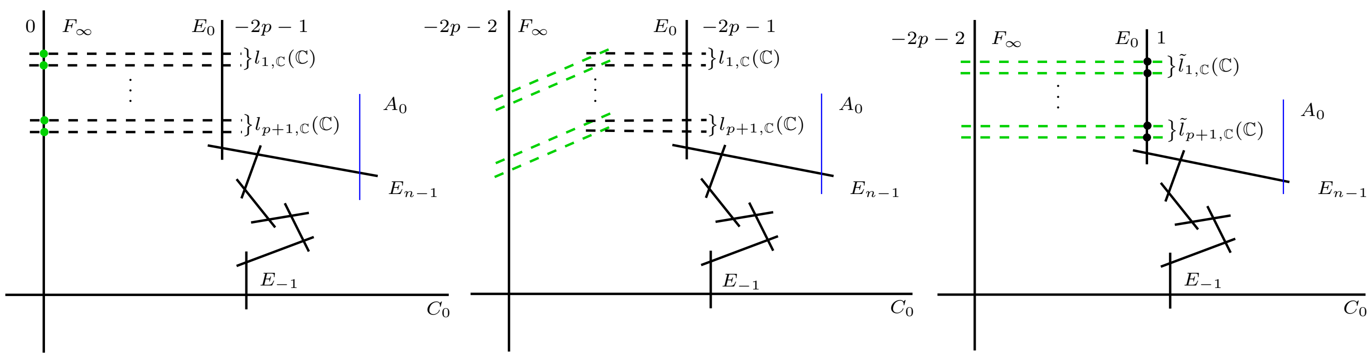

FIGURE 4.5. Elementary link: odd case

Let $V'' \to V'$ be the surface obtained by blowing-up the intersection point of E_0 with the closure T of the strict transform of $B \setminus E_0$ and let $C_0^{(1)}$ be the exceptional divisor. Then T is contained in a unique degenerate fiber of the $\mathbb{P}^1$-fibration $\overline{\pi}'' : V'' \to \mathbb{P}^1_{\mathbb{R}}$ generated by the complete linear system $|E_0|$ on V'' while $C_0^{(1)}$ is a section of this fibration. After contracting all successive (-1)-curves supported in T, starting from that of C_0 and continuing with that of the successive proper transforms of the irreducible components of the nonempty chain of (-2)-curves joining C_0 to the branching component of E contained in the chain L as in Lemma 4.11, the total transform of $E_0 \cup C_0^{(1)} \cup T$ can be rewritten in the form $B^{(1)} = F_\infty^{(1)} \cup C_0^{(1)} \cup E^{(1)}$ where $F_\infty^{(1)}$ and $E^{(1)}$ are the proper transforms of E_0 and T respectively. We let $V^{(1)}$ be the so constructed smooth projective surface and we let $\eta : (V, B) \dashrightarrow (V^{(1)}, B^{(1)})$ be the corresponding birational map defined over $\mathbb{R}$. The description given in Lemma 4.11 implies again that if not empty, $E^{(1)}$ has strictly less irreducible components than E.

Proposition 4.15. *Let $(V, B = F_\infty \cup C_0 \cup E)$ be an r-standard pair such that E is not empty and let $\eta : (V, B) \dashrightarrow (V^{(1)}, B^{(1)})$ be an elementary link as in Definition 4.14 above. Then $(V^{(1)}, B^{(1)})$ is an r-standard pair and the induced birational map $S = V \setminus B \dashrightarrow S^{(1)} = V^{(1)} \setminus B^{(1)}$ is $\mathbb{R}$-biregular.*

Proof. By construction, the birational map $S \dashrightarrow S^{(1)}$ induces a diffeomorphism $S(\mathbb{R}) \overset{\simeq}{\to} S^{(1)}(\mathbb{R})$. So it is enough to show that $S^{(1)}_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic. Indeed, if so, then $S^{(1)}$ is affine whence in particular does not contain any complete curve. As a consequence, the $\mathbb{A}^1$-fibration $\pi^{(1)} : S^{(1)} \to \mathbb{A}^1_{\mathbb{R}}$ induced by the restriction of the $\mathbb{P}^1$-fibration $\overline{\pi}^{(1)} : V^{(1)} \to \mathbb{P}^1_{\mathbb{R}}$ defined by the complete linear system $|F_\infty^{(1)}|$ has at most one degenerate fiber, whose closure is contained in the fiber of $\overline{\pi}^{(1)}$ over the $\mathbb{R}$-rational point $\overline{\pi}^{(1)}(E^{(1)}) \in \mathbb{A}^1_{\mathbb{R}}$. Together with Proposition 4.3, the $\mathbb{Q}$-acyclicity of $S^{(1)}_{\mathbb{C}}(\mathbb{C})$ and the fact that $S^{(1)}(\mathbb{R}) \approx \mathbb{R}^2$ imply that the unique possible degenerate fiber of $\pi^{(1)}$ is isomorphic to $\mathbb{A}^1_{\mathbb{R}}$ when equipped with its reduced structure and has odd multiplicity. So $\pi^{(1)} : S^{(1)} \to \mathbb{A}^1_{\mathbb{R}}$ is an r-standard $\mathbb{A}^1$-fibered surface.

Note that since (V, B) is an r-standard pair, $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic by virtue of Proposition 4.3. Furthermore, it follows from the proof of this proposition that $H_1(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$, where m is the multiplicity of the unique degenerate fiber of the $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$. By the description given in § 4.2.1.1 and Figure 4.3, we may view the pair (V, B) as being constructed from the surface $\tau_0 : V_1 \to \mathbb{F}_1$ obtained by blowing-up the point $p_0 \in E_{-1} \setminus C_0$ by blowing-up further a sequence of $\mathbb{R}$-rational points supported on the successive total transforms on E_{-1}. Contracting C_0 from V_1, we may also view (V, B) as being obtained from another Hirzebruch surface $\rho_1' : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ having E_0 and F_∞ as sections with self-intersections -1

and $+1$ respectively by a sequence $\alpha : V \to \mathbb{F}_1$ of blow-ups of $\mathbb{R}$-rational points in in such a way that the $\mathbb{P}^1$-fibration $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ coincides with $\rho_1' \circ \tau'$. The image $D = \alpha_*(B)$ of B consists of the union of E_0, F_∞ and E_{-1}, which is a fiber of τ'. With the notation of 2.2.1, the kernel R of the surjective map $j_{\mathbb{C}} : \mathbb{Z}\langle D_{\mathbb{C}}\rangle \to \mathrm{Cl}(\mathbb{F}_1)$ is generated by $F_\infty - E_0 - E_{-1}$ while $\mathbb{Z}\langle \mathcal{E}_0\rangle$ is the free abelian group generated by $A_{0,\mathbb{C}}$. Letting f_∞, e_0 and e_{-1} be the coefficients of A_0 in the total transforms in V of F_∞, E_0 and E_{-1} respectively, the homomorphism $\varphi : R \to \mathbb{Z}\langle \mathcal{E}_0\rangle$ with respect to the chosen bases is simply the multiplication by $f_\infty - e_0 - e_{-1}$, and since the diagram chasing in the proof of Lemma 2.5 (see also Remark 2.5) provides an isomorphism $H_1(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}\langle \mathcal{E}_0\rangle/\mathrm{Im}\varphi$, we have $f_\infty - e_0 - e_{-1} = \pm m$.

On the other hand, with the notation of Definition 4.14, $S^{(1)} = V^{(1)} \setminus B^{(1)}$ is isomorphic to the complement in the projective surface V' of the proper transform B' of B. Since by construction V' is obtained from V by performing $r = -E_0^2$ elementary birational transformations along $\mathbb{C}$-rational smooth fibers of $\xi_{p_0} : V \to \mathbb{P}^1_{\mathbb{R}}$ with centers on F_∞, we have a commutative diagram

$$\begin{array}{ccc} V & \stackrel{\eta'}{\dashrightarrow} & V' \\ \alpha \downarrow & & \downarrow \alpha' \\ \mathbb{F}_1 & \dashrightarrow & \mathbb{F}_{1-2r} \end{array}$$

where $\mathbb{F}_1 \dashrightarrow \mathbb{F}_{1-2r}$ consists of r elementary birational transformations along $\mathbb{C}$-rational smooth fibers of $\rho_1' : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ with centers on F_∞, η' restricts to an isomorphism in a open neighborhood of $B \setminus (E_0 \cup F_\infty)$ and $\alpha' : V' \to \mathbb{F}_{1-2r}$ is a sequence of blow-ups of $\mathbb{R}$-rational points. It follows in particular that the coefficients of A_0 in the total transforms in V' of F_∞, E_0 and E_{-1} are again equal to f_∞, e_0 and e_{-1} respectively. On the other hand, the proper transforms of E_{-1}, E_0 and F_∞ in $\mathbb{F}_{1-2r}$ are respectively a fiber of the $\mathbb{P}^1$-bundle structure and a pair of sections with self-intersections $-1+2r$ and $1-2r$. Thus E_0 is linearly equivalent in $\mathbb{F}_{1-2r}$ to $F_\infty + (2r-1)E_{-1}$ and, letting $D' = \alpha'_* B'$, a basis of the kernel R' of the surjective map $j'_{\mathbb{C}} : \mathbb{Z}\langle D'_{\mathbb{C}}\rangle \to \mathrm{Cl}(\mathbb{F}_1)$ is generated by $F_\infty + (2r-1)E_{-1} - E_0$. As a consequence, the homomorphism $\varphi' : R' \to \mathbb{Z}\langle \mathcal{E}_0\rangle$ coincides with the multiplication by $f_\infty + (2r-1)e_{-1} - e_0$ and we deduce from the generalization of Lemma 2.5 given in Remark 2.5 that $S^{(1)}_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic unless $f_\infty + (2r-1)e_{-1} - e_0 = 0$. But this second possibility never occurs because $f_\infty - e_0 - e_{-1} = \pm m$ is odd by hypothesis. This completes the proof. ☐

4.2.3. *Proof of Theorem 4.9.*

By hypothesis S is a smooth affine geometrically integral surface defined over $\mathbb{R}$, with $S(\mathbb{R}) \approx \mathbb{R}^2$ and equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ such that all but at most one fibers of π over $\mathbb{R}$-rational points of $\mathbb{A}^1_{\mathbb{R}}$ contain a reduced $\mathbb{R}$-rational irreducible component. By virtue of Proposition 4.13, there exists an $\mathbb{R}$-biregular birational map $\varphi_0 : S \dashrightarrow S^{(0)}$ onto an r-standard surface $\pi^{(0)} : S^{(0)} \dashrightarrow \mathbb{A}^1_{\mathbb{R}}$ such that $\pi = \pi^{(0)} \circ \varphi_0$. Letting $(V^{(0)}, B^{(0)} = F^{(0)}_\infty \cup C^{(0)}_0 \cup E^{(0)})$ be a smooth completion of $S^{(0)}$ defined over $\mathbb{R}$ as in § 4.2.1.1, we have the following alternative: either $E^{(0)}$ is empty and then $V^{(0)} \simeq \mathbb{F}_1$ and $S^{(0)} = V^{(0)} \setminus B^{(0)} \simeq \mathbb{A}^2_{\mathbb{R}}$ or, by virtue of Proposition 4.15, there exists an elementary link $\eta_0 : (V^{(0)}, B^{(0)}) \dashrightarrow (V^{(1)}, B^{(1)} = F^{(1)}_\infty \cup C^{(1)}_0 \cup E^{(1)})$ to an r-standard pair $(V^{(1)}, B^{(1)})$ restricting to an $\mathbb{R}$-biregular birational map $\eta_0 : S^{(0)} \dashrightarrow S^{(1)} = V^{(1)} \setminus B^{(1)}$ between r-standard surfaces. Since $E^{(1)}$ is either empty or has strictly less irreducible component then $E^{(0)}$ we conclude by induction that there exists a finite sequence of elementary links

$$(V^{(0)}, B^{(0)}) \stackrel{\eta_0}{\dashrightarrow} (V^{(1)}, B^{(1)}) \stackrel{\eta_1}{\dashrightarrow} \cdots \stackrel{\eta_{m-1}}{\dashrightarrow} (V^{(m)}, B^{(m)}) \stackrel{\eta_m}{\dashrightarrow} (V^{(m+1)}, B^{(m+1)})$$

terminating with an r-standard pair $(V^{(m+1)}, B^{(m+1)})$ for which $E^{(m+1)}$ is empty and such that the composition $\eta_m \circ \cdots \circ \eta_0 : S^{(0)} \dashrightarrow S^{(m+1)} = V^{(m+1)} \setminus B^{(m+1)} \simeq \mathbb{A}^2_{\mathbb{R}}$ is an $\mathbb{R}$-biregular birational map. This completes the proof of Theorem 4.9.

Example 4.16. Let $D \subset \mathbb{P}^2_{\mathbb{R}}$ be a geometrically integral $\mathbb{R}$-rational curve of degree $d = 2n+1 \geq 1$ with a unique singular point p of multiplicity $2n$, such that $D_{\mathbb{C}}$ has a unique analytic branch at p, and let $S = \mathbb{P}^2_{\mathbb{R}} \setminus D$. Then $D(\mathbb{R})$ is connected and since d is odd, $S(\mathbb{R})$ is connected, hence homeomorphic to $\mathbb{R}^2$. On the other hand, $S_{\mathbb{C}}(\mathbb{C})$ is $\mathbb{Q}$-acyclic with $H_1(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathrm{Cl}(S_{\mathbb{C}}) \simeq \mathbb{Z}_d$ and so, S is not isomorphic to

$\mathbb{A}^2_{\mathbb{R}}$ as a scheme over $\mathbb{R}$. But it follows from Theorem 4.9 that S is $\mathbb{R}$-biregularly birationally isomorphic to $\mathbb{A}^2_{\mathbb{R}}$. Indeed, the pencil $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$ generated by D and d times its tangent line $L \simeq \mathbb{P}^1_{\mathbb{R}}$ at p restricts on S to an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ with a unique degenerate fiber isomorphic to $\mathbb{A}^1_{\mathbb{R}} = L \cap S$, of multiplicity d.

5. Complements

In the note [12], we exhibit examples of real fake planes of every Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$ which are birationally diffeomorphic to $\mathbb{R}^2$.

5.1. **Exceptional $\mathbb{Q}$-homology euclidean planes of Kodaira dimension 0.** By virtue of [13, (8.64)] (see also [27, Lemma 4.4.2]), a smooth affine complex $\mathbb{Q}$-acyclic surface of Kodaira dimension 0 is either $\mathbb{A}^1_*$-ruled over a base curve isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ or $\mathbb{P}^1_{\mathbb{C}}$, or is isomorphic to one of the so-called exceptional surfaces $Y(3,3,3)$, $Y(2,4,4)$ or $Y(2,3,6)$. Hereafter, we construct all real models of these exceptional surfaces and characterize $\mathbb{Q}$-acyclic euclidean planes among them. We show in particular that $Y(2,4,4)$ admits two real forms of very different nature: one whose real locus is not diffeomorphic to $\mathbb{R}^2$ and a second one which is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$. This illustrates the fact that neither the topology of the complex model nor the Kodaira dimension are invariant under $\mathbb{R}$-biregular birational equivalence.

5.1.1. *Real model of* $Y(3,3,3)$. Let D be the union of four general lines $\ell_i \simeq \mathbb{P}^1_{\mathbb{R}}$, $i = 0,1,2,3$ in $\mathbb{P}^2_{\mathbb{R}}$ and let $\tau\colon V \to \mathbb{P}^2_{\mathbb{R}}$ be the projective surface obtained by first blowing-up the points $p_{ij} = \ell_i \cap \ell_j$ with exceptional divisors E_{ij}, $i,j = 1,2,3$, $i \neq j$ and then blowing-up the points $\ell_1 \cap E_{12}$, $\ell_2 \cap E_{23}$ and $\ell_3 \cap E_{1,3}$ with respective exceptional divisors E_1, E_2 and E_3. We let $B = \ell_0 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{12} \cup E_{23} \cup E_{13}$. The dual graphs of D, its total transform $\tau^{-1}(D)$ in V and B are depicted in Figure 5.1.

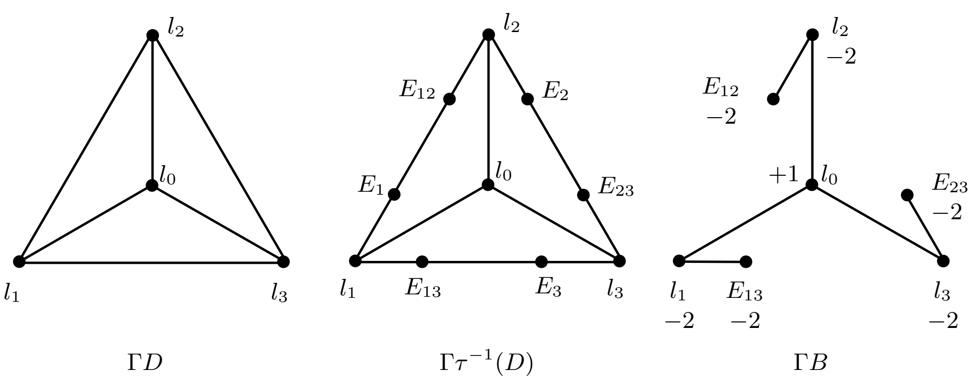

Figure 5.1. Construction of $Y(3,3,3)$

We let $Y(3,3,3) = V \setminus B$. With the notation of § 2.2.1, the kernel R of the surjective map $j_{\mathbb{C}} : \mathbb{Z}\langle D_{\mathbb{C}}\rangle \to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{C}})$ is generated by the classes $\ell_{i,\mathbb{C}} - \ell_{0,\mathbb{C}}$, $i = 1,2,3$ while $\mathbb{Z}\langle \mathcal{E}_0\rangle$ is the free abelian group generated by $E_{1,\mathbb{C}}$, $E_{2,\mathbb{C}}$ and $E_{3,\mathbb{C}}$. With this choice of basis, the induced homomorphism $\varphi : R \to \mathbb{Z}\langle\mathcal{E}_0\rangle$ is represented by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

which has determinant $\det A = 9$. So by virtue of Lemma 2.5, $Y(3,3,3)_{\mathbb{C}}$ is $\mathbb{Q}$-acyclic with $H_2(Y(3,3,3)_{\mathbb{C}};\mathbb{Z}) = 0$ and $H_1(Y(3,3,3)_{\mathbb{C}};\mathbb{Z}) \simeq \mathbb{Z}_9$. Furthermore, since $j : \mathbb{Z}\langle D\rangle \to \mathrm{Cl}(\mathbb{P}^2_{\mathbb{R}})$ is surjective and V is obtained from $\mathbb{P}^2_{\mathbb{R}}$ be blowing-up $\mathbb{R}$-rational points only, we deduce from c) in the same Lemma that $Y(3,3,3)(\mathbb{R}) \approx \mathbb{R}^2$.

Proposition 5.1. *Let S be a smooth surface defined over $\mathbb{R}$ such that $S(\mathbb{R}) \approx \mathbb{R}^2$ and $S_{\mathbb{C}} \simeq Y(3,3,3)_{\mathbb{C}}$. Then S is isomorphic to $Y(3,3,3)$ as a scheme over $\mathbb{R}$.*

Proof. Since the automorphism group of $Y(3,3,3)_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}_3$, it follows that $Y(3,3,3)$ has no nontrivial $\mathbb{R}$-form: indeed isomorphy classes of $\mathbb{R}$-forms of $Y(3,3,3)$ are in one-to-one correspondence with elements of the cohomology group $H^1(\mathbb{Z}_2, \mathrm{Aut}(Y(3,3,3)_{\mathbb{C}})) \simeq H^1(\mathbb{Z}_2,\mathbb{Z}_3) = 0$, as every element in $\mathbb{Z}_3$ is a multiple of 2. ☐

Question 5.2. *Is $Y(3,3,3)$ $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$?*

5.1.2. *Real forms of $Y(2,4,4)$.* Starting from $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, we can construct two non isomorphic real forms $Y_r(2,4,4)$ and $Y_c(2,4,4)$ of the same complex surface as follows:

a) The surface $Y_r(2,4,4)$. We let D_r be the union of three distinct $\mathbb{R}$-rational fibers $\ell_j \simeq \mathbb{P}^1_{\mathbb{R}}$, $j=1,2,3$, of the first projection and of three distinct $\mathbb{R}$-rational fibers $M_i \simeq \mathbb{P}^1_{\mathbb{R}}$, $i=1,2,3$, of the second projection. We let $\pi_r : V_r \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ be the surface obtained by first blowing-up the $\mathbb{R}$-rational points $p_{12} = M_1 \cap \ell_2$, $p_{13} = M_1 \cap \ell_3$, $p_{23} = M_2 \cap \ell_3$ and $p_{32} = M_3 \cap \ell_2$ with respective exceptional divisors E_{12}, E_{13}, E_{23} and E_{32}, and then blowing-up the $\mathbb{R}$-rational points $M_2 \cap E_{23}$ and $M_3 \cap E_{32}$ with respective exceptional divisors F_{23} and F_{32}. We let $B_r = M_1 \cup M_2 \cup M_3 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{23} \cup E_{32}$ and we let $Y_r(2,4,4) = V_r \setminus B_r$.

b) The surface $Y_c(2,4,4)$. We let D_c be the union of a $\mathbb{R}$-rational fibers $\ell_1 \simeq \mathbb{P}^1_{\mathbb{R}}$ and $M_1 \simeq \mathbb{P}^1_{\mathbb{R}}$ of the first and second projection and of a pair of conjugate $\mathbb{C}$-rational fibers $\ell \simeq \mathbb{P}^1_{\mathbb{C}}$ and $M \simeq \mathbb{P}^1_{\mathbb{C}}$ of the first and second projection respectively. We let $\pi_c : V_c \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ be the surface defined over $\mathbb{R}$ obtained by blowing-up the $\mathbb{C}$-rational points $M_1 \cap \ell$ and $M \cap \ell$ with respective exceptional divisors E_1 and E and then blowing-up the $\mathbb{C}$-rational point $M \cap E$ with exceptional divisor F. We let $B_c = M_1 \cup M \cup \ell_1 \cup \ell \cup E$ and we let $Y_c(2,4,4) = V_c \setminus B_c$.

By construction, the surfaces $Y_r(2,4,4)$ and $Y_c(2,4,4)$ are not isomorphic over $\mathbb{R}$, but their complexifications $Y_r(2,4,4)_{\mathbb{C}}$ and $Y_c(2,4,4)_{\mathbb{C}}$ are isomorphic over $\mathbb{C}$. With the notation of §2.2.1, the kernel R of the homomorphism $j_{\mathbb{C}} : \mathbb{Z}\langle D_{r,\mathbb{C}}\rangle \to \mathrm{Cl}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$ is generated by the classes $\ell_{2,\mathbb{C}} - \ell_{1,\mathbb{C}}$, $\ell_{3,\mathbb{C}} - \ell_{1,\mathbb{C}}$, $M_{2,\mathbb{C}} - M_{1,\mathbb{C}}$ and $M_{3,\mathbb{C}} - M_{1,\mathbb{C}}$ and letting $\mathbb{Z}\langle \mathcal{E}_0\rangle$be the free abelian group generated by $E_{12,\mathbb{C}}$, $E_{13,\mathbb{C}}$, $F_{23,\mathbb{C}}$ and $F_{32,\mathbb{C}}$, the induced homomorphism $\varphi : R \to \mathbb{Z}\langle \mathcal{E}_0\rangle$ is represented by the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

which has determinant $\det A = 8$. A similar argument as in the proof of Lemma 2.5, see also Remark 2.5, shows that $Y_r(2,4,4)_{\mathbb{C}}$ is $\mathbb{Q}$-acyclic with $H_2(Y_r(2,4,4)_{\mathbb{C}};\mathbb{Z}) = 0$ and $H_1(Y_r(2,4,4)_{\mathbb{C}};\mathbb{Z}) \simeq \mathbb{Z}_8$.

Proposition 5.3. *Let S be a smooth surface defined over $\mathbb{R}$ such that $S(\mathbb{R}) \approx \mathbb{R}^2$ and $S_{\mathbb{C}} \simeq Y_r(2,4,4)_{\mathbb{C}} \simeq Y_c(2,4,4)_{\mathbb{C}}$. Then S is isomorphic to $Y_c(2,4,4)$ as a scheme over $\mathbb{R}$ and is $\mathbb{R}$-regularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$.*

Proof. The automorphism group of $Y_r(2,4,4)_{\mathbb{C}}$ being isomorphic to $\mathbb{Z}_2$, $Y_r(2,4,4)$ and $Y_c(2,4,4)$ are the only two $\mathbb{R}$-forms of $Y_r(2,4,4)$. Since $j : \mathbb{Z}\langle D_r\rangle \to \mathrm{Cl}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$ is surjective and $\pi_r : V_r \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ consists of blow-up of $\mathbb{R}$-rational points only, we infer similarly as in the proof of c) in Lemma 2.5 that $Y_r(2,4,4)(\mathbb{R}) \approx \mathbb{R}^2$ if and only if $\varphi \otimes \mathrm{id} : R \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathbb{Z}\langle \mathcal{E}_0\rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is an isomorphism, which is not the case. Since the reduction of A modulo 2 is not invertible, we conclude that $Y_r(2,4,4)(\mathbb{R}) \not\approx \mathbb{R}^2$. On the other hand, since $\pi_c : V_c \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ consists only of blow-ups of $\mathbb{C}$-rational points, the pair (V_c, B_c) is $\mathbb{R}$-regularly birationally equivalent to $(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}, M_1 \cup \ell_1)$ and so $Y_c(2,4,4)$ is $\mathbb{R}$-regularly birationally equivalent to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \setminus (M_1 \cup \ell_1) \simeq \mathbb{A}^2_{\mathbb{R}}$. ☐

5.1.3. *Real model of $Y(2,3,6)$.* Staring again, with two triples of $\mathbb{R}$-rational fibers M_i and ℓ_j, $i,j = 1,2,3$ in $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ of the second and first projection respectively, we let $\pi : V \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ be the smooth projective surface obtained by first blowing-up the points $p_{13} = M_1 \cap \ell_3$, $p_{31} = M_3 \cap \ell_1$, $p_{22} = M_2 \cap \ell_2$ and $p_{23} = M_2 \cap \ell_3$ with respective exceptional divisors E_{13}, E_{31}, E_{22} and E_{23} and then blowing-up the points $p'_{22} = E_{22} \cap \ell_2$, $p'_{13} = M_1 \cap E_{13}$ and $p'_{31} = E_{31} = M_3 \cap E_{31}$ with respective exceptional divisors F_{22}, F_{13} and F_{31}. We let $B = M_1 \cup M_2 \cup M_3 \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup E_{22} \cup E_{13} \cup E_{31}$ and we let $Y(2,3,6) = V \setminus B$. Then $Y(2,3,6)$ is a smooth affine surface defined over $\mathbb{R}$ such that $H_1(Y(2,3,6)_{\mathbb{C}};\mathbb{Z}) \simeq \mathbb{Z}_6$ and $H_2(Y(2,3,6)_{\mathbb{C}};\mathbb{Z}) = 0$. Since V is obtained from $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ by blowing-up $\mathbb{R}$-rational points only, we deduce in a similar way as in the previous case that $Y(2,3,6)(\mathbb{R}) \not\approx \mathbb{R}^2$. Furthermore, since the automorphism group of $Y(2,3,6)_{\mathbb{C}}$ is trivial, there is no nontrivial $\mathbb{R}$-form of $Y(2,3,6)$. Summing up, there is no smooth affine surface S defined over $\mathbb{R}$ with $S(\mathbb{R}) \approx \mathbb{R}^2$ and $S_{\mathbb{C}} \simeq Y(2,3,6)_{\mathbb{C}}$.

5.2. **Moduli of $\mathbb{R}$-biregularly birationally rectifiable surfaces of negative Kodaira dimension.** As seen in the introduction, in the rational projective case, there is a unique minimal complexification or at most one family of pairwise non isomorphic but $\mathbb{R}$-biregularly birationally and deformation equivalent minimal complexifications. Non minimal complexifications are obtained from these models by blowing-up sequences of pairs of non-real conjugate points. It is natural to expect an affine counterpart of this type of results in the form of continuous moduli of $\mathbb{Q}$-acyclic euclidean planes of negative Kodaira dimension all $\mathbb{R}$-biregularly birationally equivalent to each others. For instance, starting with the standard open embedding of $\mathbb{A}^2_{\mathbb{R}}$ in $\mathbb{P}^2_{\mathbb{R}}$ as the complement of a line $L_\infty \simeq \mathbb{P}^1_{\mathbb{R}}$ and performing a sequence of blow-ups $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ defined over $\mathbb{R}$ whose centers lie over L_∞, one obtains open embeddings $\mathbb{A}^2_{\mathbb{R}} \hookrightarrow V$ into various smooth projective surfaces defined over $\mathbb{R}$, which, in restriction to the real loci correspond to smooth open embeddings of $\mathbb{R}^2$ into smooth compact non-orientable surfaces of arbitrary genus $g \geq 1$. For a fixed number $g-1 \geq 0$ of $\mathbb{R}$-rational points blown-up, the isomorphy type as real algebraic varieties of the so-constructed surfaces V with $g(V(\mathbb{R})) = g$ depend on the choice of the points, giving rise in general to a continuous moduli of such algebraic surfaces. In contrast, it follows from [4] that their equivalence classes up to $\mathbb{R}$-biregular birational isomorphisms depend only on g, which in this particular case coincides simply with the number of $\mathbb{R}$-rational irreducible components of the boundary $B = V \setminus \mathbb{A}^2_{\mathbb{R}}$.

The next proposition illustrates the existence of infinitely many deformation equivalence classes of pairwise non isomorphic $\mathbb{Q}$-acyclic euclidean planes all $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$, each deformation equivalence class having further a moduli of arbitrary positive dimension $n \geq 3$.

Proposition 5.4. *Let $Y = \mathrm{Spec}(\mathbb{R}[a_1, \ldots, a_n])$, $n \geq 3$, let $r \geq 3$ be an odd integer, and let $\mathfrak{X} \subset Y \times \mathbb{A}^3_{\mathbb{R}}$ be the subvariety with equation $x^{n+1}z = y^r + \sum_{i=2}^n a_i x^{i+1} + x^2 + x$. Then the following hold:*

1) The projection $\mathrm{pr}_Y : \mathfrak{X} \to Y$ is smooth and $\mathrm{pr}_Y(\mathbb{R}) : \mathfrak{X}(\mathbb{R}) \to Y(\mathbb{R})$ is a trivial $\mathbb{R}^2$-bundle over $Y(\mathbb{R}) \approx \mathbb{R}^n$.

2) For every $\mathbb{R}$-rational point $p \in Y$, the scheme theoretic fiber $S = \mathfrak{X}_p$ is a smooth connected affine surface defined of $\mathbb{R}$, of negative Kodaira dimension with $S(\mathbb{R}) \approx \mathbb{R}^2$ and $H_1(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z}_r$, $H_2(S_{\mathbb{C}}(\mathbb{C}); \mathbb{Z}) = 0$. The restriction of S of the projection pr_x is an $\mathbb{A}^1$-fibration $q \colon S \to \mathbb{A}^1_{\mathbb{R}}$ with $q^{-1}(0)$ as a unique geometrically irreducible degenerate fiber, of multiplicity r. In particular, S is $\mathbb{R}$-biregularly birationally equivalent to $\mathbb{A}^2_{\mathbb{R}}$.

3) Every $S = \mathfrak{X}_p$ is deformation equivalent to $\mathfrak{X}_0$ via the retraction $Y \to \{0\}$, $(a_2, \ldots, a_n) \in \mathbb{R}^{n-1} \mapsto (ta_2, \ldots, ta_n)$, $t \in \mathbb{R}$.

4) Let $p = (a_1, \ldots, a_n) \in Y(\mathbb{R})$ and $p' = (a'_1, \ldots, a'_n) \in Y(\mathbb{R})$. Then $\mathfrak{X}_p$ is isomorphic to $\mathfrak{X}_{p'}$ if and only if $p = p'$.

Proof. The first assertion follows from the Jacobian criterion and the observation that the map

$$\psi : Y(\mathbb{R}) \times \mathbb{R}^2 \to \mathfrak{X}(\mathbb{R}) \qquad (a_1, \ldots, a_n, x, z) \mapsto (a_1, \ldots, a_n, x, \sqrt[r]{x^{n+1}z - \sum_{i=2}^n a_i x^{i+1} - x^2 - x}, z)$$

is an homeomorphism. For every $p \in Y(\mathbb{R})$, $q : S = \mathfrak{X}_p \to \mathbb{A}^1_{\mathbb{R}}$ is an r-standard $\mathbb{A}^1$-fibered surface with $q^{-1}(\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}) \simeq \mathrm{Spec}(\mathbb{R}[x^{\pm 1}, y]) \simeq (\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}) \times \mathbb{A}^1_{\mathbb{R}}$ and $q^*(\{0\}) \simeq \mathrm{Spec}(\mathbb{R}[y]/(y^r)[z])$. So 2) follows from Proposition 4.3 and Theorem 4.9. The third assertion is clear. For the last assertion, letting $S = \mathfrak{X}_p$ and $S' = \mathfrak{X}_{p'}$, it follows from Theorem 6.1 in [31] that $S_{\mathbb{C}}$ and $S'_{\mathbb{C}}$ are isomorphic if and only if there exists $\lambda, \alpha, \mu \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$ such that

$$(\alpha y + \beta)^r + \sum_{i=2}^n a_i (\lambda x)^{i+1} + (\lambda x)^2 + \lambda x = \mu (y^r + \sum_{i=2}^n a'_i x^{i+1} + x^2 + x).$$

The previous identity implies that $\beta = 0$, $\mu = \alpha^r = \lambda^2 = \lambda$ and $a_i \lambda^{i+1} = \mu a'_i$ for $i = 2, \ldots, n$. Thus $\lambda = \mu = 1$ necessarily and so $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$. ☐

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