THE GROUP OF AUTOMORPHISMS OF A REAL RATIONAL SURFACE IS n-TRANSITIVE

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Dedicated to Joost van Hamel in memoriam

Abstract. Let X be a rational nonsingular compact connected real algebraic surface. Denote by $\mathrm{Aut}(X)$ the group of real algebraic automorphisms of X. We show that the group $\mathrm{Aut}(X)$ acts n-transitively on X, for all natural integers n.

As an application we give a new and simpler proof of the fact that two rational nonsingular compact connected real algebraic surfaces are isomorphic if and only if they are homeomorphic as topological surfaces.

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1. Introduction

Let X be a nonsingular compact connected real algebraic manifold, i.e., X is a compact connected submanifold of $\mathbb{R}^n$ defined by real polynomial equations, where n is some natural integer. We study the group of algebraic automorphisms of X. Let us make precise what we mean by an algebraic automorphism.

Let X and Y be real algebraic submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. An *algebraic map* φ of X into Y is a map of the form

$$\varphi(x) = \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_m(x)}{q_m(x)}\right) \tag{1.1}$$

where $p_1, \ldots, p_m, q_1, \ldots, q_m$ are real polynomials in the variables $x_1, \ldots, x_n$, with $q_i(x) \neq 0$ for any $x \in X$ and any i. An algebraic map from X into Y is also called a *regular map* [BCR]. Note that an algebraic map is necessarily of class C^∞. An algebraic map $\varphi\colon X \to Y$ is an *algebraic isomorphism*, or *isomorphism* for short, if φ is algebraic, bijective and if φ^{-1} is algebraic. An algebraic isomorphism from X into Y is also called a *biregular map* [BCR]. Note that an algebraic isomorphism is a diffeomorphism of class C^∞. An algebraic isomorphism from X into itself is called an *algebraic automorphism* of X, or *automorphism* of X for short. We denote by $\mathrm{Aut}(X)$ the group of automorphism of X.

For a general real algebraic manifold, the group $\mathrm{Aut}(X)$ tends to be rather small. For example, if X admits a complexification $\mathcal{X}$ that is of general type then $\mathrm{Aut}(X)$ is finite. Indeed, any automorphism of X is the restriction to X

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of a birational automorphism of $\mathcal{X}$. The group of birational automorphisms of $\mathcal{X}$ is known to be finite [Ma63]. Therefore, $\mathrm{Aut}(X)$ is finite for such real algebraic manifolds.

In the current paper, we study the group $\mathrm{Aut}(X)$ when X is a compact connected real algebraic surface, i.e., a compact connected real algebraic manifold of dimension 2. By what has been said above, the group of automorphisms of such a surface is most interesting when the Kodaira dimension of X is equal to $-\infty$, and, in particular, when X is geometrically rational. By a result of Comessatti, a connected geometrically rational real surface is rational (see Theorem IV of [Co12] and the remarks thereafter, or [Si89, Corollary VI.6.5]). Therefore, we will concentrate our attention to the group $\mathrm{Aut}(X)$ when X is a rational compact connected real algebraic surface.

Recall that a real algebraic surface X is *rational* if there are a nonempty Zariski open subset U of $\mathbb{R}^2$, and a nonempty Zariski open subset V of X, such that U and V are isomorphic real algebraic varieties, in the sens above. In particular, this means that X contains a nonempty Zariski open subset V that admits a parametrization by real rational functions in two variables.

Examples of rational real algebraic surfaces are the following:

- the unit sphere S^2 defined by the equation $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$,
- the real algebraic torus $S^1 \times S^1$, where S^1 is the unit circle defined by the equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$, and
- any real algebraic surface obtained from one of the above ones by repeatedly blowing up a point.

This is a complete list of rational real algebraic surfaces, as was probably known already to Comessatti. A modern proof may use the Minimal Model Program for real algebraic surfaces [Ko97, Ko01] (cf. [BH07, Theorem 3.1]). For example, the real projective plane $\mathbb{P}^2(\mathbb{R})$—of which an explicit realization as a rational real algebraic surface can be found in [BCR, Theorem 3.4.4]—is isomorphic to the real algebraic surface obtained from S^2 by blowing up 1 point.

The following conjecture has attracted our attention.

Conjecture 1.2 ([BH07, Conjecture 1.4]). *Let X be a rational nonsingular compact connected real algebraic surface. Let n be a natural integer. Then the group $\mathrm{Aut}(X)$ acts n-transitively on X.*

The conjecture seems known to be true only in the case when X is isomorphic to $S^1 \times S^1$:

Theorem 1.3 ([BH07, Theorem 1.3]). *The group $\mathrm{Aut}(S^1 \times S^1)$ acts n-transitively on $S^1 \times S^1$, for any natural integer n.* □

The object of the paper is to prove Conjecture 1.2:

Theorem 1.4. *The group $\mathrm{Aut}(X)$ acts n-transitively on X, whenever X is a rational nonsingular compact connected real algebraic surface, and n is a natural integer.*

Our proof goes as follows. We first prove n-transitivity of $\mathrm{Aut}(S^2)$ (see Theorem 2.3). For this, we need a large class of automorphisms of S^2.

Lemma 2.1 constructs such a large class. Once n-transitivity of $\mathrm{Aut}(S^2)$ is established, we prove n-transitivity of $\mathrm{Aut}(X)$, for any other rational surface X, by the following argument.

If X is isomorphic to $S^1 \times S^1$ then the n-transitivity has been proved in [BH07, Theorem 1.3]. Therefore, we may assume that X is not isomorphic to $S^1 \times S^1$. We prove that X is isomorphic to a blowing-up of S^2 in m distinct points, for some natural integer m (see Theorem 3.1 for a precise statement). The n-transitivity of $\mathrm{Aut}(X)$ will then follow from the $(m+n)$-transitivity of $\mathrm{Aut}(S^2)$.

Theorem 1.4 shows that the group of automorphisms of a rational real algebraic surface is big. It would, therefore, be particularly interesting to study the dynamics of automorphisms of rational real surfaces, as is done for K3-surfaces in [Ca01], for example.

Using the results of the current paper, we were able, in a forthcoming paper [HM08], to generalize Theorem 1.4 and prove n-transitivity of $\mathrm{Aut}(X)$ for curvilinear infinitely near points on a rational surface X.

We also pass to the reader the following interesting question of the referee.

Question 1.5. Let X be a rational nonsingular compact connected real algebraic surface. Is the subgroup $\mathrm{Aut}(X)$ dense in the group $\mathrm{Diff}(X)$ of all C^∞ diffeomorphisms of X into itself?[1]

As an application of Theorem 1.4, we present in Section 4 a simplified proof of the following result.

Theorem 1.6 ([BH07, Theorem 1.2]). *Let X and Y be rational nonsingular compact connected real algebraic surfaces. Then the following statements are equivalent.*

(1) *The real algebraic surfaces X and Y are isomorphic.*
(2) *The topological surfaces X and Y are homeomorphic.*

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2. n-Transitivity of $\mathrm{Aut}(S^2)$

We need to slightly extend the notion of an algebraic map between real algebraic manifolds. Let X and Y be real algebraic submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let A be any subset of X. An *algebraic map* from A into Y is a map φ as in (1.1), where $p_1, \ldots, p_m, q_1, \ldots, q_m$ are real polynomials in the variables $x_1, \ldots, x_n$, with $q_i(x) \neq 0$ for any $x \in A$ and any i. To put it otherwise, a map φ from A into Y is *algebraic* if there is a Zariski open subset U of X containing A such that φ is the restriction of an algebraic map from U into Y.

We will consider algebraic maps from a subset A of X into Y, in the special case where X is isomorphic to the real algebraic line $\mathbb{R}$, the subset A of X is a closed interval, and Y is isomorphic to the real algebraic group $\mathrm{SO}_2(\mathbb{R})$.

Denote by S^2 the 2-dimensional sphere defined in $\mathbb{R}^3$ by the equation

$$x^2 + y^2 + z^2 = 1.$$

[1] This question has now been answered, see [KM08].

Lemma 2.1. *Let L be a line through the origin of $\mathbb{R}^3$ and denote by $I \subset L$ the closed interval whose boundary is $L \cap S^2$. Denote by $L^\perp$ the plane orthogonal to L containing the origin. Let $f\colon I \to \mathrm{SO}(L^\perp)$ be an algebraic map. Define $\varphi_f\colon S^2 \to S^2$ by*

$$\varphi_f(z,x) = (f(x)z, x)$$

where $(z,x) \in (L^\perp \oplus L) \cap S^2$. Then φ_f is an automorphism of S^2.

Proof. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we may assume that $S^2 \subset \mathbb{C} \times \mathbb{R}$ is given by the equation $|z|^2 + x^2 = 1$, and that L is the line $\{0\} \times \mathbb{R}$. Then $L^\perp = \mathbb{C} \times \{0\}$ and $\mathrm{SO}(L^\perp) = S^1$. It is clear that the map φ_f is an algebraic map from S^2 into itself. If $f^{-1}(x)$ denotes the inverse of $f(x)$ then $\varphi_{f^{-1}}$ is the inverse of φ_f. (We use that since SO is an algebraic group the inverse is a regular map to itself.) Therefore φ_f is an automorphism of S^2.

□

Lemma 2.2. *Let $x_1, \ldots, x_n$ be n distinct points of the closed interval $[-1,1]$, and let $\alpha_1, \ldots, \alpha_n$ be elements of $\mathrm{SO}_2(\mathbb{R})$. Then there is an algebraic map $f\colon [-1,1] \to \mathrm{SO}_2(\mathbb{R})$ such that $f(x_j) = \alpha_j$ for $j = 1, \ldots, n$.*

Proof. Since $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the unit circle S^1, it suffices to prove the statement for S^1 instead of $\mathrm{SO}_2(\mathbb{R})$. Let P be a point of S^1 distinct from $\alpha_1, \ldots, \alpha_n$. Since $S^1 \setminus \{P\}$ is isomorphic to $\mathbb{R}$, it suffices, finally, to prove the statement for $\mathbb{R}$ instead of $\mathrm{SO}_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation. □

Theorem 2.3. *Let n be a natural integer. The group $\mathrm{Aut}(S^2)$ acts n-transitively on S^2.*

Proof. We will need the following terminology. Let W be a point of S^2, let L be the line in $\mathbb{R}^3$ passing through W and the origin. The intersection of S^2 with any plane in $\mathbb{R}^3$ that is orthogonal to L is called a *parallel of S^2 with respect to W*.

Let $P_1, \ldots, P_n$ be n distinct points of S^2, and let $Q_1, \ldots, Q_n$ be n distinct points of S^2. We need to show that there is an automorphism φ of S^2 such that $\varphi(P_j) = Q_j$, for all j.

Up to a projective linear automorphism of $\mathbb{P}^3(\mathbb{R})$ fixing S^2, we may assume that all the points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ are in a sufficiently small neighborhood of the north pole $N = (0,0,1)$ of S^2. Indeed, we may first assume that none of these points is contained in a small spherical disk D centered at N. Then the images of the points by the inversion with respect to the boundary of D are all contained in D.

We can choose two points W and W' of S^2 in the xy-plane such that the angle WOW' is equal to $\pi/2$ and such that the following property holds. Any parallel with respect to W contains at most one of the points $P_1, \ldots, P_n$, and any parallel with respect to W' contains at most one of $Q_1, \ldots, Q_n$. Denote by Γ_j the parallel with respect to W that contains P_j, and by Γ'_j the one with respect to W' that contains Q_j.

Since the disk D has been chosen sufficiently small, $\Gamma_j \cap \Gamma'_j$ is nonempty for all $j = 1, \ldots, n$. Let R_j be one of the intersection points of Γ_j and Γ'_j

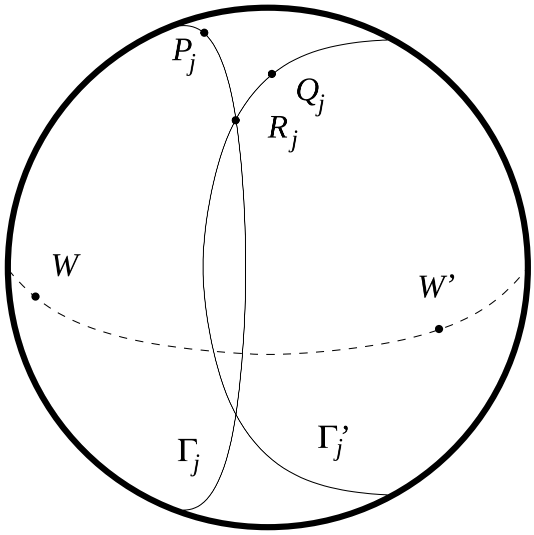

FIGURE 1. The sphere S^2 with the parallels Γ_j and Γ'_j.

(see Figure 1). It is now sufficient to show that there is an automorphism φ of S^2 such that $\varphi(P_j) = R_j$.

Let again L be the line in $\mathbb{R}^3$ passing through W and the origin. Denote by $I \subset L$ the closed interval whose boundary is $L \cap S^2$. Let x_j be the unique element of I such that $\Gamma_j = (x_j + L^\perp) \cap S^2$. Let $\alpha_j \in \mathrm{SO}(L^\perp)$ be such that $\alpha_j(P_j - x_j) = R_j - x_j$. According to Lemma 2.2, there is an algebraic map $f\colon I \to \mathrm{SO}(L^\perp)$ such that $f(x_j) = \alpha_j$. Let $\varphi := \varphi_f$ as in Lemma 2.1. By construction, $\varphi(P_j) = R_j$, for all $j = 1, \dots, n$. $\square$

3. n-TRANSITIVITY OF $\mathrm{Aut}(X)$

Theorem 3.1. *Let X be a rational nonsingular compact connected real algebraic surface and let S be a finite subset of X. Then,*

(1) *X is either isomorphic to $S^1 \times S^1$, or*
(2) *there are distinct points $R_1, \dots, R_m$ of S^2 and a finite subset S' of S^2 such that*
   (a) *$R_1, \dots, R_m \notin S'$, and*
   (b) *there is an isomorphism $\varphi\colon X \to B_{R_1,\dots,R_m}(S^2)$ such that $\varphi(S) = S'$.*

Proof. By what has been said in the introduction, X is either isomorphic to $S^1 \times S^1$, in which case there is nothing to prove, or X is isomorphic to a real algebraic surface obtained from S^2 by successive blow-up. Therefore, we may assume that there is a sequence

$$X = X_m \xrightarrow{f_m} X_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} X_0 = S^2 ,$$

where f_i is the blow-up of X_{i-1} at a point P_i of X_{i-1}.

Let $\widetilde{S}$ be the union of S and the set of centers $P_1, \dots, P_m$. Since the elements of $\widetilde{S}$ can be seen as infinitely near points of S^2, there is a natural partial ordering on $\widetilde{S}$. The partially ordered set $\widetilde{S}$ is a finite forest with respect to that ordering.

The statement that we need to prove is that there is a sequence of blow-ups as above such that all trees of the corresponding forest have height 0. We prove that statement by induction on the sum h of heights of the trees of the forest $\widetilde{S}$. If $h = 0$ there is nothing to prove. Suppose, therefore, that $h \neq 0$. We may then assume, renumbering the P_i if necessary, that either $P_2 \leq P_1$ or that a point $P \in S$ is mapped onto P_1 by the composition $f_1 \circ \cdots \circ f_m$.

As we have mentioned in the introduction, the real algebraic surface obtained from S^2 by blowing up at P_1 is isomorphic to the real projective plane $\mathbb{P}^2(\mathbb{R})$. Moreover, the exceptional divisor in $\mathbb{P}^2(\mathbb{R})$ is a real projective line L. We identify $B_{P_1}(S^2)$ with $\mathbb{P}^2(\mathbb{R})$. Choose a real projective line L' in $\mathbb{P}^2(\mathbb{R})$ such that no element of $\widetilde{S} \setminus \{P_1\}$ is mapped into L' by a suitable composition of some of the maps $f_2, \ldots, f_m$. Since the group of linear automorphisms of $\mathbb{P}^2(\mathbb{R})$ acts transitively on the set of projective lines, the line L' is an exceptional divisor for a blow-up $f_1' \colon \mathbb{P}^2(\mathbb{R}) \to S^2$ at a point of S^2. It is clear that the sum of heights of the trees of the corresponding forest is equal to $h - 1$. The statement of the theorem follows by induction. $\square$

Corollary 3.2. *Let X be a rational nonsingular compact connected real algebraic surface. Then,*

(1) *X is either isomorphic to $S^1 \times S^1$, or*
(2) *there are distinct points $R_1, \ldots, R_m$ of S^2 such that X is isomorphic to the real algebraic surface obtained from S^2 by blowing up the points $R_1, \ldots, R_m$.* $\square$

Proof of Theorem 1.4. Let X be a rational surface and let $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ by two n-tuples of disctinct points of X. By Theorem 3.1, X is either isomorphic to $S^1 \times S^1$ or to the blow-up of S^2 at a finite number of distinct points $R_1, \ldots, R_m$. If X is isomorphic to $S^1 \times S^1$ then $\mathrm{Aut}(X)$ acts n-transitively by [BH07, Theorem 1.3]. Therefore, we may assume that X is the blow-up $B_{R_1,\ldots,R_m}(S^2)$ of S^2 at $R_1, \ldots, R_m$. Moreover, we may assume that the points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ do not belong to any of the exceptional divisors. This means that these points are elements of S^2, and that, $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ are two n-tuples of distinct points of S^2. It follows that $(R_1, \ldots, R_m, P_1, \ldots, P_n)$ and $(R_1, \ldots, R_m, Q_1, \ldots, Q_n)$ are two $(m+n)$-tuples of distinct points of S^2. By Theorem 2.3, there is an automorphism ψ of S^2 such that $\psi(R_i) = R_i$, for all i, and $\psi(P_j) = Q_j$, for all j. The induced automorphism φ of X has the property that $\psi(P_j) = Q_j$, for all j. $\square$

4. Classification of rational real algebraic surfaces

Proof of Theorem 1.6. Let X and Y be a rational nonsingular compact connected real algebraic surfaces. Of course, if X and Y are isomorphic then X and Y are homeomorphic. In order to prove the converse, suppose that X and Y are homeomorphic. We show that there is an isomorphism from X onto Y.

By Corollary 3.2, we may assume that X and Y are not homeomorphic to $S^1 \times S^1$. Then, again by Corollary 3.2, X and Y are both isomorphic to a real algebraic surface obtained from S^2 by blowing up a finite number of distinct points. Hence, there are distinct points $P_1, \ldots, P_n$ of S^2 and distinct points $Q_1, \ldots, Q_m$ of S^2 such that

$$X \cong B_{P_1,\ldots,P_n}(S^2) \quad \text{and} \quad Y \cong B_{Q_1,\ldots,Q_m}(S^2).$$

Since X and Y are homeomorphic, $m = n$. By Theorem 2.3, there is an automorphism φ from S^2 into S^2 such that $\varphi(P_i) = Q_i$ for all i. It follows that φ induces an algebraic isomorphism from X onto Y. □

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