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CREMONA TRANSFORMATIONS AND HOMEOMORPHISMS OF SURFACES

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The simplest *Cremona transformation* of projective 3-space is the involution

$$\sigma\colon (x_0 : x_1 : x_2 : x_3) \mapsto \Bigl(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\Bigr),$$

which is a homeomorphism outside the tetrahedron $(x_0x_1x_2x_3 = 0)$. More generally, if $L_i := \sum_j a_{ji}x_j$ are linear forms defining the faces of a tetrahedron, we get the Cremona transformation

$$\sigma_{\mathbf{L}}\colon (x_0 : x_1 : x_2 : x_3) \mapsto \Bigl(\frac{1}{L_0} : \frac{1}{L_1} : \frac{1}{L_2} : \frac{1}{L_3}\Bigr) \cdot \bigl(a_{ij}\bigr)^{-1},$$

which is a homeomorphism outside the tetrahedron $(L_0L_1L_2L_3 = 0)$. The vertices of the tetrahedron are called the *base points*. If Q is a quadric surface in $\mathbb{P}^3$, its image under a Cremona transformation is, in general, a sextic surface. However, if Q passes through the 4 base points, then its image $\sigma_{\mathbf{L}}(Q)$ is again a quadric surface in $\mathbb{P}^3$ passing through the 4 base points. In many cases, we can view $\sigma_{\mathbf{L}}$ as a map of Q to itself.

The aim of this paper is to show that the action of Cremona transformations on the real points of quadrics exhibits the full complexity of the homeomorphisms of S^2 and of all non-orientable surfaces.

Let us start with the sphere $S^2 := (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$ and view this as the set of real points of the quadric $Q := (x^2 + y^2 + z^2 = t^2) \subset \mathbb{P}^3$ in projective 3-space. Pick 2 conjugate point pairs $p, \bar{p}, q, \bar{q}$ on the complex quadric $Q(\mathbb{C})$ and let $\sigma_{p,q}$ denote the Cremona transformation with base points $p, \bar{p}, q, \bar{q}$. As noted above, $\sigma_{p,q}(Q)$ is another quadric surface. The faces of the tetrahedron determined by these 4 points are disjoint from S^2, hence $\sigma_{p,q}$ is a homeomorphism from S^2 to the real part of $\sigma_{p,q}(Q)$. Thus Q and $\sigma_{p,q}(Q)$ are projectively equivalent and the corresponding Cremona transformation $\sigma_{p,q}$ can be viewed as a homeomorphism of S^2 to itself, well defined up to left and right multiplication by $O(3,1)$. Let us call these the Cremona transformations with *imaginary base points*. Our first result says that every homeomorphism of S^2 to itself can be approximated by composites of these transformations.

Theorem 1. *The Cremona transformations with imaginary base points $\sigma_{p,q}$ and $O(3,1)$ generate a dense subgroup of* $\operatorname{Homeo}(S^2)$.

Building on [Biswas-Huisman07], it is proved in [Huisman-Mangolte08a] that $\operatorname{Aut}(S^2)$ is n-transitive for any $n \geq 1$. Using this, it is easy to see (31) that the above density property also holds with assigned fixed points.

Corollary 2. $\operatorname{Aut}(S^2, p_1, \dots, p_n)$ *is dense in* $\operatorname{Homeo}(S^2, p_1, \dots, p_n)$ *for any finite set of distinct points $p_1, \dots, p_n \in S^2$, where* $\operatorname{Aut}(\ )$ *denotes the group of algebraic* *automorphisms of* S^2 *fixing* $p_1, \dots, p_n$ *and* Homeo( ) *the group of homeomorphisms fixing* $p_1, \dots, p_n$.

Note that, for a real algebraic variety X, the semigroup of algebraic diffeomorphisms is usually much bigger than the group of algebraic automorphisms $\mathrm{Aut}(X)$. For instance, $x \mapsto x + \frac{1}{x^2+1}$ is an algebraic diffeomorphism of $\mathbb{R}$ (and also of $\mathbb{RP}^1 \sim S^1$), but its inverse involves square and cube roots. The difference is best seen in the case of the circle $S^1 = (x^2 + y^2 = 1)$.

Essentially by the Weierstrass approximation theorem, any differentiable map $\phi\colon S^1 \to S^1$ can be approximated by polynomial maps $\Phi\colon S^1 \to S^1$. By contrast, the group of algebraic automorphisms of S^1 is the real orthogonal group $O(2,1) \cong PGL(2,\mathbb{R})$, which has real dimension 3. Thus $\mathrm{Aut}(S^1)$ is a very small closed subgroup in the infinite dimensional groups $\mathrm{Homeo}(S^1)$ and $\mathrm{Diff}(S^1)$.

The Cremona transformations with real base points do not give homeomorphisms of S^2; they are not even defined at the real base points. Instead, they give generators of the mapping class groups of non-orientable surfaces.

Let R_g be a non-orientable, compact surface of genus g without boundary. Coming from algebraic geometry, we prefer to think of it as S^2 blown up at g points $p_1, \dots, p_g \in S^2$. Topologically, R_g is obtained from S^2 by replacing g discs centered at the p_i by g Möbius bands. Up to isotopy, a blow-up form of R_g is equivalent to giving g disjoint embedded Möbius bands $M_1, \dots, M_g \subset R_g$.

There are two ways to think of a Cremona transformation with real base points as giving elements of the mapping class group of R_g.

Let us start with the case when there are four real base points $p_1, \dots, p_4$. We can factor the Cremona transformation $\sigma_{p_1,\dots,p_4}$ as

$$\sigma_{p_1,\dots,p_4}\colon Q \stackrel{\pi_1}{\leftarrow} B_{p_1,\dots,p_4}Q \stackrel{\pi_2}{\to} Q$$

where on the left $\pi_1\colon B_{p_1,\dots,p_4}Q \to Q$ is the blow up of Q at the 4 points $p_1, \dots, p_4$ and on the right $\pi_2\colon B_{p_1,\dots,p_4}Q \to Q$ contracts the birational transforms of the circles $Q \cap L_i$ where the $\{L_i\}$ are the faces of the tetrahedron with vertices $\{p_i\}$. In Figure 1, the $\bullet$ represent the 4 base points. On the left hand side, the 4 exceptional curves E_i lie over the four points marked $\bullet$. On the right hand side, the images of the E_i are 4 circles, each passing through 3 of the 4 base points. Since $\sigma_{p_1,\dots,p_4}$ is an involution, dually, the four points marked $\bullet$ on the right hand side map to the 4 circles on the left hand side.

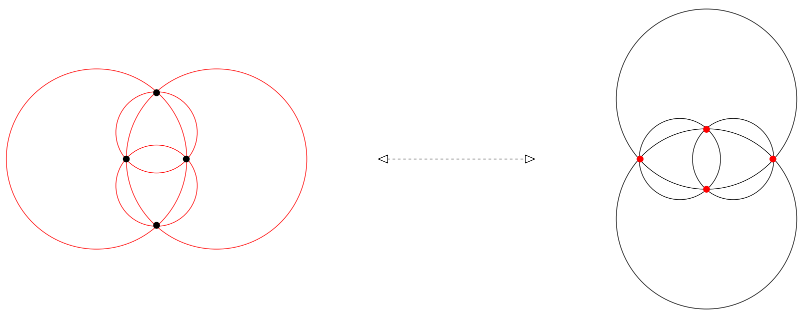

FIGURE 1. Cremona transformation with four real base points.

A Cremona transformation $\sigma_{p_1,p_2,q,\bar{q}}$ with 2 real and a conjugate complex pair of base points act similarly. Here only two Möbius bands are altered.

In general, we can think of the above real Cremona transformation $\sigma_{p_1,\dots,p_4}$ as a topological operation that replaces the set of g Möbius bands $\big(M_1,\dots,M_g\big)$ by a new set $\big(M_1',\dots,M_4',M_5,\dots,M_g\big)$. In this version, $\sigma_{p_1,\dots,p_4}$ is the identity on the surfaces but acts nontrivially on the set of isotopy classes of g disjoint Möbius bands. One version of our result says that the transformations $\sigma_{p_1,\dots,p_4}$ and $\sigma_{p_1,p_2,q,\bar{q}}$ act transitively on the set of isotopy classes of g disjoint Möbius bands.

The other way to view $\sigma_{p_1,\dots,p_4}$ is as follows. First, we obtain an isomorphism

$$\sigma'_{p_1,\dots,p_4} : B_{p_1,\dots,p_g}S^2 \cong B_{q_1,\dots,q_g}S^2$$

for some $q_1,\dots,q_g \in S^2$. Under this isomorphism, the exceptional curve $E(p_i) \subset B_{p_1,\dots,p_g}S^2$ is mapped to the exceptional curve $E(q_i) \subset B_{q_1,\dots,q_g}S^2$ for $i \geq 5$ and to the circle passing through the points $\{q_j : 1 \leq j \leq 4, j \neq i\}$ for $i \leq 4$. As we noted above, there is an automorphism $\Phi \in \operatorname{Aut}(S^2)$ such that $\Phi(q_i) = p_i$ for $1 \leq i \leq n$. Thus

$$\Phi \circ \sigma'_{p_1,\dots,p_4} : B_{p_1,\dots,p_g}S^2 \xrightarrow{\cong} B_{p_1,\dots,p_g}S^2$$

is an automorphism of $B_{p_1,\dots,p_g}S^2$ which maps $E(p_i)$ to $E(q_i)$ for $i \geq 5$ and to a simple closed curve passing through the points $\{p_j : 1 \leq j \leq 4, j \neq i\}$ for $i \leq 4$.

Theorem 3. *For any g, the Cremona transformations with $4, 2$ or 0 real base points generate the (non-orientable) mapping class group $\mathcal{M}_g$.*

Finally, we can put these results together to obtain a general approximation theorem for homeomorphisms of such real algebraic surfaces. We have been using homeomorphisms instead of diffeomorphisms advisedly. Our methods give only C^0-approximations. However, the differentiability problems occur at one point only.

Theorem 4. *Let $R \cong B_{p_1,\dots,p_g}S^2$ be a real algebraic surface birational to $\mathbb{P}^2$ and $q_1,\dots,q_n \in R$ distinct marked points. Let $q \in R$ be another point. Then the group of algebraic automorphisms $\operatorname{Aut}(R,q_1,\dots,q_n)$ is dense in*

(1) $\operatorname{Homeo}(R,q_1,\dots,q_n)$ *in the C^0-topology on R, and in*
(2) $\operatorname{Diff}(R,q_1,\dots,q_n)$ *in the compact-open C^∞-topology on $R \setminus \{q\}$.*

We expect that $\operatorname{Aut}(R,q_1,\dots,q_n)$ is dense in the group of diffeomorphisms $\operatorname{Diff}(R,q_1,\dots,q_n)$, but our method in Section 1 definitely produces only a C^0-approximation.

5 (Other algebraic varieties)**.** Similar assertions definitely fail for most other algebraic varieties. Real algebraic varieties of general type have only finitely many birational automorphisms. (See [Ueno75] for an introduction to these questions.) For varieties whose Kodaira dimension is between 0 and the dimension, every birational automorphism preserves the Iitaka fibration. If the Kodaira dimension is 0 (e.g., Calabi-Yau varieties, Abelian varieties), then every birational automorphism preserves the canonical class, that is, a volume form, up to sign. The automorphism group is finite dimensional but may have infinitely many connected components. In particular, using [Comessatti14], for surfaces we obtain the following.

Proposition 6. *Let S be a smooth real algebraic surface. If $S(\mathbb{R})$ is an orientable surface of genus ≥ 2 then $\operatorname{Aut}(S)$ is not dense in $\operatorname{Homeo}\big(S(\mathbb{R})\big)$.* □

The analogous question for $S^1 \times S^1$ remains open.

If X has Kodaira dimension $-\infty$, then every birational automorphism preserves the MRC fibration [Kollár96, Sec.IV.5]. Thus the main case when density could hold is when the variety is rationally connected [Kollár96, Sec.IV.3]. It is clear that the analog of (1) fails even for most geometrically rational real algebraic surfaces. Consider, for instance, the case when $R \to \mathbb{P}^1$ is a minimal conic bundle with at least 8 singular fibers. Then $\operatorname{Aut}(R)$ is infinite dimensional, but every automorphism of R preserves the conic bundle structure [Iskovskikh96, Thm. 1.6(iii)]. There are only a handful of other surface cases where the analog of (1) may hold. Conic bundles with 4 singular fibers may be the best candidates.

It is less clear to us what happens with higher dimensional spheres. As in (28), for any real algebraic map $M\colon [-1,1] \to O(n)$, the twisting map $\Phi_M\colon S^n \to S^n$ defined by

$$(x_0, x_1, \dots, x_n) \mapsto \big(x_0, (x_1, \dots, x_n) \cdot M(x_0)\big)$$

is an algebraic automorphism of S^n. Thus $\operatorname{Aut}(S^n)$ is infinite dimensional for $n \geq 2$. These automorphisms, however, do not seem to generate a dense subgroup of $\operatorname{Homeo}(S^n)$ for $n \geq 3$. For $n \geq 3$ the generators of $\operatorname{Aut}(S^n)$ are not known and the density of $\operatorname{Aut}(S^n)$ in $\operatorname{Homeo}(S^n)$ is an open problem.

7 (History of related questions)**.** There are many results in real algebraic geometry that endow certain topological spaces with a real algebraic structure or approximate smooth maps by real algebraic morphisms. In particular real rational models of surfaces were studied in [Bochnak-Coste-Roy87], [Mangolte06] and approximations of smooth maps to spheres by real algebraic morphisms were investigated in [Bochnak-Kucharz87a, Bochnak-Kucharz87b], [Bochnak-Kucharz-Silhol97], [Kucharz99], [Joglar-Kollár03], [Joglar-Mangolte04], [Mangolte06].

The first indication that $\operatorname{Aut}(S^2)$ is surprisingly large comes from [Biswas-Huisman07], with a more precise version developed in [Huisman-Mangolte08a]. We know, however, of no other results that approximate self-homeomorphisms by algebraic automorphisms.

8 (Plan of the proofs)**.** We start with (1). Let $\phi\colon S^2 \to S^2$ be a homeomorphism. Let $n \in S^2$ be the "north pole." We may assume that $\phi(n) = n$. Thus, by stereographic projection, ϕ gives a homeomorphism $\phi^\pi : \mathbb{R}^2 \to \mathbb{R}^2$. By a result of [Andersén-Lempert92], on any compact set, one can approximate ϕ^π by a product of certain algebraic automorphisms, called *overshears*, $g_s \in \operatorname{Aut}(\mathbb{R}^2)$. We can now lift the g_s to birational maps $G_s\colon S^2 \dashrightarrow S^2$. Their product is a C^0-approximation of ϕ, but it is not an algebraic automorphism. In general, G_s is not even differentiable at the "north pole" $n \in S^2$.

Approximating G_s by automorphisms turns out to be quite subtle. (See (17) for counter examples.) First we lift some of the overshears G_s to automorphisms of a singular conic bundle. We then deform the singular conic bundle to a smooth conic bundle and try to deform the automorphism along. This relies on a careful study of the relative automorphism groups of conic bundles. The general computations of Section 4 are applied to our current question in Section 3. Finally, in Section 5 we prove that the Cremona transformations with imaginary base points generate $\operatorname{Aut}(S^2)$. This completes the proof of (1).

Next, in Section 6, we prove (4) for the identity components. If $\phi\colon R \to R$ is homotopic to the identity, then ϕ can be written as the composite of homeomorphisms $\phi_i\colon R \to R$ such that each ϕ_i is the identity outside a small open set

$W_i \subset R$. Moreover, we can choose the W_i in such a way that for every i there is a morphism $\pi_i : R \to S^2$ that is an isomorphism on W_i. The map ϕ_i then pushes down to a homeomorphism of S^2. We take an approximation there and lift it to R. (Note that this would have been much easier using [ibid., Thm. 5.2]. However, the latter result, which is mentioned without a proof, is erroneous. The only compactly supported overshear is the identity, so no approximation is possible.)

Generators of the mapping class group of non-orientable surfaces have been written down by [Chillingworth69] and [Korkmaz02]. In Section 7 we describe a somewhat different set of generators. We thank M. Korkmaz for his help in proving these results.

Theorem 3 is proved in Section 8. We show by explicit constructions that our generators are given by Cremona transformations.

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1. Approximation via overshears

Notation 9. In the sequel, S^2 will always denote the unit sphere $S^2 := (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$ as a real algebraic variety and Q denotes the corresponding projective quadric $Q := (x^2 + y^2 + z^2 = t^2) \subset \mathbb{P}^3$ in projective 3-space. Technically speaking, we consider Q as an $\mathbb{R}$-scheme. In practice this means that it is a complex quadric $Q(\mathbb{C})$ equipped with complex conjugation. The fixed point set of complex conjugation is the set of real points $Q(\mathbb{R})$. It is identified with the unit sphere S^2.

$\mathrm{Aut}(S^2)$ denotes the group of algebraic automorphisms of S^2. That is, these are those birational self-maps $\Phi : Q \dashrightarrow Q$ that are defined over $\mathbb{R}$ (equivalently, commute with complex conjugation) and that are regular at every point of $S^2 = Q(\mathbb{R})$ (equivalently, induce a C^∞ or real analytic map of S^2 to itself.) It is crucial to remark that, for such a birational self-map Φ, the map Φ^{-1} is also regular at every point of $S^2 = Q(\mathbb{R})$, cf. [Bochnak-Coste-Roy87, (3.2.6)].

Let $\phi \in \mathrm{Homeo}(S^2)$. Any homeomorphism of a $\mathcal{C}^\infty$-surface R onto itself can be approximated by a $\mathcal{C}^\infty$-diffeomorphism. (See, for instance, the books [Moise77] or [Kirby-Siebenmann77] for introductions to such questions.) Thus we can assume that ϕ is a $\mathcal{C}^\infty$-diffeomorphism of S^2. (Strictly speaking, this is not necessary, but several of the references use diffeomorphisms, so it is convenient.) Furthermore, up to multiplication by an element of $\mathrm{O}(3)$, we can assume that ϕ is orientation preserving and $\phi(n) = n$ where $n := (0, 0, 1) \in S^2$ is the "north pole".

Let $\pi : S^2 \dashrightarrow \mathbb{R}^2$ be the projection from the north pole to the $(z = 0)$-plane. In concrete equations

$$\pi(x, y, z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big), \quad \pi^{-1}(x, y) = \Big(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\Big).$$

Hence ϕ induces a $\mathcal{C}^\infty$-diffeomorphism $\pi\circ\phi\circ\pi^{-1}$ of $\mathbb{R}^2$. By [Andersén-Lempert92, Theorem 5.1], this $\mathcal{C}^\infty$-diffeomorphism can be approximated in the $\mathcal{C}^\infty$-compact-open topology (or weak topology, see [Hirsh76, Chap. 2]) by a finite composition $G_k\circ\cdots\circ G_1$ of $\mathcal{C}^\infty$-overshears.

By definition, an *overshear* on $\mathbb{R}^2$ is a diffeomorphism that, up to permuting the coordinates, has the form

$$G\colon (x,y)\mapsto (x, a(x)y+b(x)),$$

where $a(x), b(x)$ are smooth and $a(x)$ is nowhere zero.

Let G be an algebraic automorphism of $\mathbb{R}^2$. Then $\pi^{-1}\circ G\circ\pi\colon S^2 \dashrightarrow S^2$ is a birational map and a homeomorphism of S^2 but usually not an automorphism of S^2. For overshears it is easy to check that, unless $a(x)\equiv\pm 1$ and $b(x)$ is bounded, the lifting $\pi^{-1}\circ G\circ\pi$ is not even C^1 at $n\in S^2$.

For later purposes, we need the following stronger approximation result for overshears.

Lemma 10 (Algebraic approximation of overshears)**.** *For any C^∞-overshear*

$$G\colon (x,y)\mapsto \bigl(x, a(x)y+b(x)\bigr)\ ,$$

there are real algebraic overshears $F_s\colon (x,y)\mapsto \bigl(x, a_s(x)y+b_s(x)\bigr)$ such that

(1) *F_s converges to G in the C^∞-compact-open topology, and*
(2) *for each s, F_s satisfies the following conditions*
   (a) *$a_s(x)$ has no real zeros and poles,*
   (b) $\lim_{x\to\pm\infty} a_s(x)$ *is either* 1 *or* -1*,*
   (c) *$b_s(x)$ has no real poles, and*
   (d) $\lim_{x\to\pm\infty} b_s(x)$ *is finite.*

Proof. Apply the Weierstrass approximation theorem to the C^∞-functions a and b to get two families of polynomials $A_s, B_s\in\mathbb{R}[x]$ approximating a and b in the compact-open topology. By adding a polynomial of the form $(x/r)^{2m}$ to $A_s(x)$ and $B_s(x)$ (with r, m depending on s) we can assume that each A_s is nowhere zero and the A_s, B_s have even degree. Denote by α_s (resp. β_s) the leading coefficient of $A_s(x)$ (resp. $B_s(x)$). Set $m_s=\deg A_s$ and $n_s=\deg B_s$. Set

$$\begin{array}{rcl} a_s(x,\varepsilon,r) & := & A_s(x)\cdot\Bigl(1+\frac{\varepsilon^{2r}x^{2r}}{(1+\varepsilon^2x^2)^r}\cdot|\alpha_s|x^{m_s}\Bigr)^{-1} \quad \text{and}\\ b_s(x,\varepsilon,r) & := & B_s(x)\cdot\Bigl(1+\frac{\varepsilon^{2r}x^{2r}}{(1+\varepsilon^2x^2)^r}\cdot|\beta_s|x^{n_s}\Bigr)^{-1}. \end{array}$$

Note that as $\varepsilon\to 0$ and $r\to\infty$, $a_s(x,\varepsilon,r)$ converges to $A_s(x)$ and $b_s(x,\varepsilon,r)$ converges to $B_s(x)$ in the compact-open topology. Both $a_s(x,\varepsilon,r)$ and $b_s(x,\varepsilon,r)$ have limit ±1 as $x\to\pm\infty$. Thus a suitable subsequence $a_s(x,\varepsilon(s),r(s))$ and $b_s(x,\varepsilon(s),r(s))$ will work. □

11 (Overshears and conic bundles)**.** Let $F\colon\mathbb{R}^2\to\mathbb{R}^2, (x,y)\mapsto(x,a(x)y+b(x))$ be an overshear on the plane. Note that F is also an automorphism of the foliation of the plane by the vertical lines $x=$ constant. The inverse of the stereographic projection transforms the foliation by lines to a singular foliation of S^2 by circles tangent at the north pole $n=(0,0,1)$. The left hand side of Figure 2 shows the resulting family of circles on S^2 tangent to each other at n.

To approximate the lifts of overshears by algebraic automorphisms, we want to resolve the singularity of this family of circles into a conic bundle and study the behavior of its automorphism group.

This family of circles on S^2 is obtained by cutting S^2 with the family of planes through the line $L_1 := (x = z - 1 = 0)$ which is tangent to S^2 at the point n. Projection from L_1 is not defined at n. As we move the line away from the sphere in the family $L_t := (x = z - t = 0)$ for $t > 1$, we get another family of circles on S^2 as in the right hand side of Figure 2. For $t > 1$, the line L_t intersects the sphere in 2 imaginary points, hence projection from the line L_t is defined everywhere along the real points.

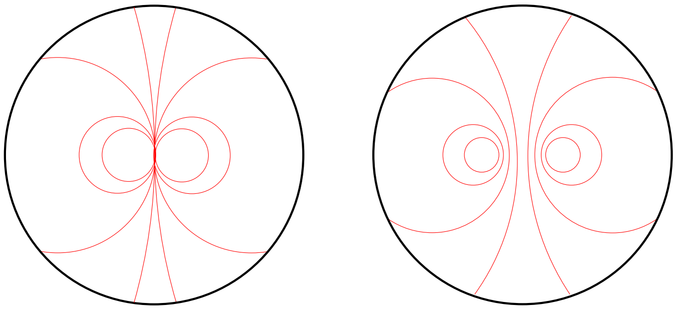

Figure 2. Deformation of the singular foliation.

For $t > 1$, the projection $\pi_t \colon Q \to \mathbb{P}^1$ becomes regular after we blow up the two (imaginary) intersection points of Q and L_t. What is the limit of this blow up as $t \to 1$? The correct answer is, we need to blow up the scheme theoretic intersection $Q \cap L_1$. Let $Z(t) := Q \cap L_t$ denote the scheme theoretic intersection and $\pi_t \colon B_{Z(t)} Q \to \mathbb{P}^1$ the blow up.

One can obtain $B_{Z(1)} S^2$ using ordinary blow ups as follows. We first blow up the north pole $n \in S^2$. We get an exceptional curve $E_1 \subset B_n S^2$. Then we blow up the intersection of E_1 with the birational transform of L_1. Thus we get $B_{2n} S^2 \to B_n S^2 \to S^2$ with two exceptional curves. The birational transform of E_1 is denoted by E and the second exceptional curve is denoted by C. Note that the projection π_1 lifts to a morphism $\pi_{2n} \colon B_{2n} S^2 \to \mathbb{P}^1$. E is contracted by π_{2n} and C is a section of π_{2n}.

We also get a morphism $B_{2n} S^2 \to B_{Z(1)} S^2$ which contracts E to a singular point. The real topology of this and of the $t > 1$ deformation is explained in Section 2.

Next we check which overshears lift to automorphisms of the singular conic bundle $\pi_1 \colon B_{Z(1)} Q \to \mathbb{P}^1$.

Proposition 12. *Let $F \colon (x, y) \mapsto \big(x, a(x)y + b(x)\big)$ be a real algebraic overshear. Then F induces a birational self-map of the complex conic bundle $B_{Z(1)} Q$ which is an automorphism of the real conic bundle $B_{Z(1)} S^2 \to \mathbb{P}^1(\mathbb{R})$ iff $a(x)$ and $b(x)$ satisfy the conditions (10.2.a–d).*

Proof. Let $p \in \mathbb{P}^2$ be the point $(0, 1, 0)$. The lines of our foliation meet there. On $\tau \colon B_p \mathbb{P}^2 \to \mathbb{P}^1$ the foliation becomes smooth, and the overshear preserves the projection τ. Thus the overshear $F \colon (x, y) \mapsto (x, a(x)y + b(x))$ is an automorphism near a fiber $\tau^{-1}(q)$ whenever F induces an actual isomorphism on $\tau^{-1}(q)$. For the fibers $x = \gamma$ in the original affine plane we have the induced map $y \mapsto a(\gamma)y + b(\gamma)$. This is an automorphism iff $a(\gamma) \neq 0, \infty$ and $b(\gamma) \neq \infty$. These are the conditions (10.2.a) and (10.2.c).

One can lift $\pi\colon Q \dashrightarrow \mathbb{P}^2$ to a morphism

$$\tilde{\pi}\colon B_{2n}Q \cong B_{(1,\pm i,0)}B_p\mathbb{P}^2 \to B_p\mathbb{P}^2,$$

where $\tilde{\pi}(E)$ is the line at infinity $(z = 0)$, $\tilde{\pi}(C)$ is the exceptional curve E_p of τ and the conjugate pair of lines $\ell + \bar{\ell} := Q \cap (z = 1)$ is contracted to the conjugate point pair $(1, \pm i, 0)$ on the line at infinity.

Therefore, an overshear F lifts to a birational map F' of $B_{2n}Q$ and F' is a morphism along the line at infinity iff F stabilizes the point pair $\{(1, \pm i, 0)\}$.

As local coordinates at $(1 : 0 : 0)$ take $u = \frac{y}{x}, v = \frac{1}{x}$. Then $v = 0$ is the line at infinity. In these coordinates F is given as

$$F\colon (u, v) \mapsto \Bigl(a\bigl(\tfrac{1}{v}\bigr)u + vb\bigl(\tfrac{1}{v}\bigr), v\Bigr).$$

Our condition about stabilizing the point pair $\{(1, \pm i, 0)\}$ is equivalent to

$$\bigl(\lim_{x\to\infty} a(x)\bigr) \cdot i + \bigl(\lim_{x\to\infty} \tfrac{1}{x}b(x)\bigr) = \pm i.$$

This holds iff $\lim_{x\to\infty} a(x) = \pm 1$ and $\lim_{x\to\infty} b(x)$ is finite. These are the conditions (10.2.b) and (10.2.d). $\square$

2. Topological description of the smoothing

We use the notation of (11).

$B_{2n}S^2$ is obtained from S^2 by 2 blow ups, it is thus a Klein bottle. In Figure 3 we arranged the non-contractible pre-image of the self-intersection set to be exactly $E(\mathbb{R})$. The other exceptional curve is $C(\mathbb{R})$. The foliation by circles is also shown. $E(\mathbb{R})$ is one of the leaves and $C(\mathbb{R})$ is a section of this foliation.

In Figures 3–5 we have also added an extra "neck" on the left hand side. Left of the neck is the compact set $K \subsetneq S^2$ over which we want a C^∞-approximation. In order to get a C^0-approximation on the whole S^2, the parts to the right of the neck should be very small.

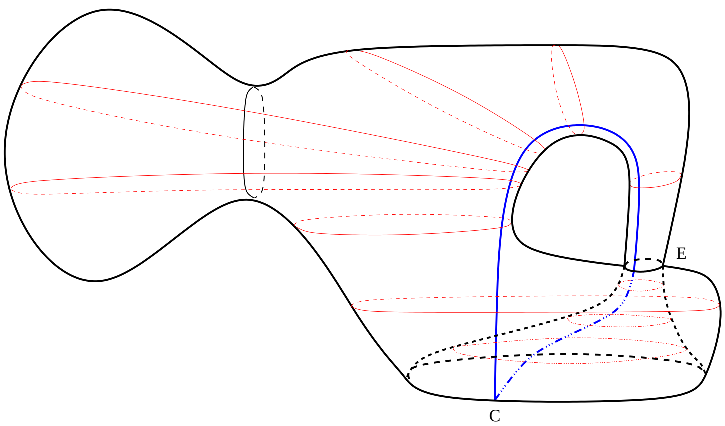

Figure 3. The sphere blown-up at 2 consecutive points.

In order to obtain $B_{Z(1)}S^2$, we need to contract $E(\mathbb{R})$. This is shown in Figure 4. The foliation by circles acquires a singular leaf (the singular point of the surface). The foliation is a conic bundle structure with C as a section. The fiber through the singular point is a conjugate pair of complex lines intersecting there.

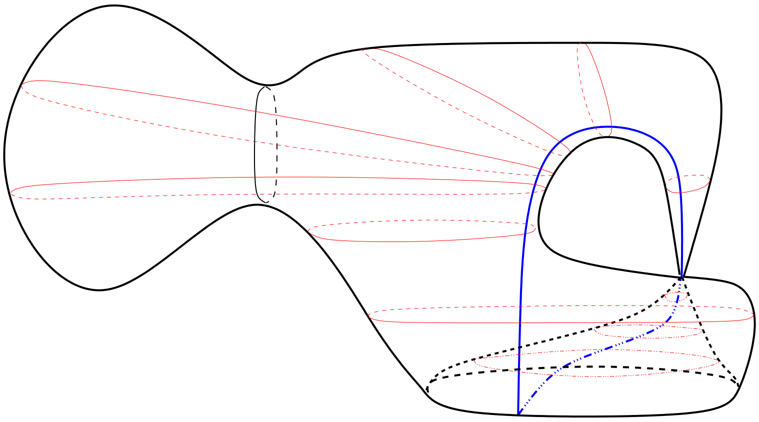

FIGURE 4. The singular blow up $B_{Z(1)}S^2$ (the sewing machine).

The singular blow up $B_{Z(1)}S^2$ has a single conical singularity of the form $(w^2 = u^2+v^2)$. The $t > 1$ direction corresponds to the smoothing $(w^2 = u^2+v^2+(t-1))$. After smoothing, we get Figure 5. Note that the curve $C \subset B_{Z(1)}S^2$ can not be deformed to a curve in $B_{Z(t)}S^2$ for $t \neq 1$. We obtain a topological sphere and a

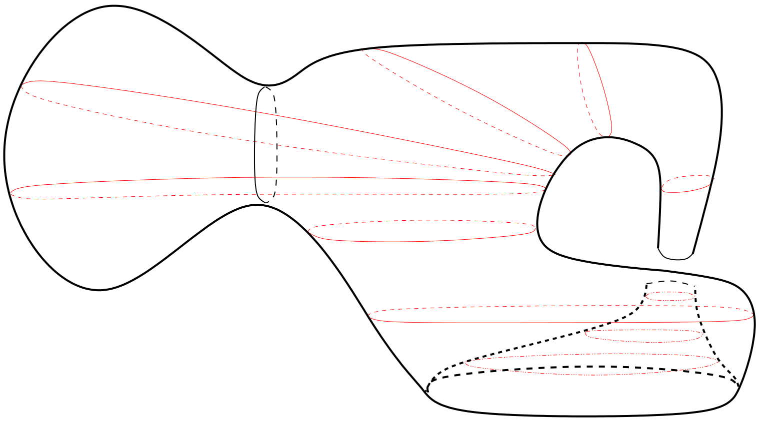

FIGURE 5. $B_{Z(t)}S^2$ for $t > 1$ is a topological sphere.

family of circles on it as in Figure 2.

3. DEFORMATION OF CONIC BUNDLES

In the previous section we lifted certain overshears to automorphisms of the singular conic bundle $\pi_1 \colon B_{Z(1)}Q \to \mathbb{P}^1$. Next we increase t and attempt to move this automorphism along with t to get an automorphism of the smooth conic bundle

$\pi_t \colon B_{Z(t)}Q \to \mathbb{P}^1$. Note that for $t > 1$, the set of real points of $B_{Z(t)}Q$ is identical to $S^2 = Q(\mathbb{R})$, hence this deformation will produce automorphisms of S^2.

The first step in this direction is to answer the following

Question. If we vary a conic bundle continuously, will the group of birational automorphisms also vary contiuously?

To make this more precise, let $\mathbb{A}^1$ denote the affine line with coordinate t. The conic bundles $\pi_t \colon B_{Z(t)}Q \to \mathbb{P}^1$ can be viewed as (t = constant) sections of a 3-dimensional conic bundle

$$\pi\colon \mathbf{Q} := B_Z\bigl(Q \times \mathbb{A}^1\bigr) \to \mathbb{P}^1 \times \mathbb{A}^1 \quad \text{where } Z := \bigl(Q \times \mathbb{A}^1\bigr) \cap (x = z - t = 0).$$

Next we need the following consequence of the study of automorphisms of conic bundles established in Section 4. (Note that this holds only for conic bundles where, as in our case, every fiber is a quadric of rank ≥ 2.)

Proposition 13. *There is a smooth group scheme* $\mathrm{Aut}\bigl(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1\bigr) \to \mathbb{P}^1 \times \mathbb{A}^1$ *such that for* $(b, t) \in \mathbb{P}^1 \times \mathbb{A}^1$ *the following holds, where a subscript* (b, t) *denotes the fiber over the point* (b, t)*.*

(1) *If* $\mathbf{Q}_{(b,t)}$ *is smooth then* $\mathrm{Aut}(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1)_{(b,t)}$ *is* $\mathrm{Aut}(Q_{(b,t)}) \cong PGL(2, \mathbb{R})$*.*
(2) *If* $\mathbf{Q}_{(b,t)}$ *is singular then* $\mathrm{Aut}(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1)_{(b,t)}$ *is a certain subgroup of* $\mathrm{Aut}(Q_{(b,t)})$ *(whose precise definition we do not need right now, cf. (18))*
(3) *Moreover, for each* t*, there is a one-to-one correspondence between*
   (a) *real algebraic sections* $\mathbb{P}^1(\mathbb{R}) \to \mathrm{Aut}(B_{Z(t)}S^2/\mathbb{P}^1)$*, and*
   (b) *real algebraic automorphisms of* $B_{Z(t)}S^2$ *that preserve* π_t*.* □

Using (13), there are three ways to obtain the necessary deformation of overshears.

14 (Topological deformation)**.** Let G be an overshear on $\mathbb{R}^2$ satisfying the conditions (10.2.a–d). By (12), it lifts to an automorphism of the real conic bundle $B_{Z(1)}S^2$, hence it corresponds to a real algebraic section

$$\sigma_1 \colon \mathbb{P}^1(\mathbb{R}) \to \mathrm{Aut}(B_{Z(1)}Q/\mathbb{P}^1)(\mathbb{R}) \ .$$

Since $\mathrm{Aut}\bigl(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1\bigr) \to \mathbb{P}^1 \times \mathbb{A}^1$ is smooth, this section can be deformed to a C^∞-section $\sigma_t \colon \mathbb{P}^1(\mathbb{R}) \to \mathrm{Aut}(B_{Z(t)}Q/\mathbb{P}^1)(\mathbb{R})$ for t near 1. Since $\mathrm{Aut}(B_{Z(t)}Q/\mathbb{P}^1)$ is a group scheme with general fiber $PGL(2)$, we see that $\mathrm{Aut}(B_{Z(t)}Q/\mathbb{P}^1)$ is birational to $\mathbb{P}^1 \times \mathbb{P}^3$. Thus any C^∞-section can be approximated by real algebraic sections, essentially by the Weierstrass theorem. See [Bochnak-Kucharz99] for details.

This takes care of the deformation problem in our situation, but the following algebraic approaches are also of interest. The first one suggests that it should be possible to write down these deformations explicitly and the second applies in much more general circumstances.

15 (Algebraic deformation using Hilbert schemes)**.** The overshear defines a rational section $\sigma_1 \colon \mathbb{P}^1 \dashrightarrow \mathrm{Aut}(B_{Z(t)}Q/\mathbb{P}^1)$ which is an actual section over the real points. We want to use Grothendieck's theory of the Hilbert scheme to conclude that σ_1 deforms with t.

To this end, we introduce in (19) a partial compactification of $\mathrm{Aut}\bigl(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1\bigr) \to \mathbb{P}^1 \times \mathbb{A}^1$ denoted by

$$\bar\pi \colon \overline{\mathrm{Aut}}^{sm}\bigl(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1\bigr) \to \mathbb{P}^1 \times \mathbb{A}^1.$$

For a smooth fiber $\mathbf{Q}_{b,t}$, the fiber of $\bar{\pi}$ is $\operatorname{End}(\mathbf{Q}_{b,t}) \cong \mathbb{P}^3$. For a singular conic, the fiber is the same as before. Then σ_1 becomes an actual section $\sigma_1 \colon \mathbb{P}^1 \to \overline{\operatorname{Aut}}^{sm}(B_{Z(1)}Q/\mathbb{P}^1)$. Let $C_1 \subset \overline{\operatorname{Aut}}^{sm}(B_{Z(1)}Q/\mathbb{P}^1)$ denote its image and N_1 its normal bundle.

If $H^1(C_1, N_1) = 0$, then the general theory of Hilbert schemes shows that C_1 deforms in a family of sections $\{C_t\}$ as we vary t (see, eg. [Kollár96, Sec. I.2]). (Note that this notation somewhat hides the fact that there are many choices for the $\{C_t\}$, thus it may be rather hard to write down these deformations explicitly.)

The normal bundle N_1 is the restriction of the relative tangent bundle T_1 of $\overline{\operatorname{Aut}}^{sm}(B_{Z(1)}Q/\mathbb{P}^1) \to \mathbb{P}^1$ of C_1.

As noted in (12), $B_{Z(1)}Q$ can also be obtained from $\rho \colon B_p\mathbb{P}^2 \to \mathbb{P}^1$ by blowing up a pair of conjugate points in the fiber over infinity $(1:0) \in \mathbb{P}^1$ and then contracting the birational transform of that fiber. We can thus view our overshear as a section $C_2 \subset \overline{\operatorname{Aut}}^{sm}(B_p\mathbb{P}^2/\mathbb{P}^1) \to \mathbb{P}^1$ with normal bundle N_2. By the above construction, a deformation of C_2 that is the identity over $(1:0) \in \mathbb{P}^1$ corresponds to a deformation of C_1. This gives a map $N_2(-1) \to N_1$ and a surjection

$$H^1\big(C_2, N_2(-1)\big) \twoheadrightarrow H^1(C_1, N_1).$$

Let T_2 denote the relative tangent bundle of $\overline{\operatorname{Aut}}^{sm}(B_p\mathbb{P}^2/\mathbb{P}^1) \to \mathbb{P}^1$. Since $B_p\mathbb{P}^2 \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle corresponding to the vector bundle $\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-1)$, we see that $\overline{\operatorname{Aut}}^{sm}(B_p\mathbb{P}^2/\mathbb{P}^1) \to \mathbb{P}^1$ is the $\mathbb{P}^3$-bundle

$$\mathbf{P} := \mathbb{P}\operatorname{End}\big(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-1)\big) = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}^2 + \mathcal{O}_{\mathbb{P}^1}(1)\big)$$

with projection $\rho \colon \mathbf{P} \to \mathbb{P}^1$. Thus T_2 sits in the standard exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}} \to \rho^*\operatorname{End}\big(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-1)\big) \otimes \mathcal{O}_{\mathbf{P}}(1) \to T_2 = T_{\mathbf{P}/\mathbb{P}^1} \to 0.$$

(Depending on which convention one uses for $\mathbb{P}$ of a vector bundle, one may need the dual of the $\rho^*(\ )$ term. An endomorphism bundle is self dual, so this does not matter for us.) With the classical convention, the section C_2 corresponds to a sublinebundle

$$L_2^* \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}^2 + \mathcal{O}_{\mathbb{P}^1}(1),$$

and so the normal bundle N_2 is a quotient

$$\operatorname{Hom}\Big(\mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}^2 + \mathcal{O}_{\mathbb{P}^1}(1), L_2\Big) \twoheadrightarrow N_2.$$

Thus we see that if $\deg L_2 \geq 1$ then N_2 is semi-positive and hence

$$H^1\big(C_2, N_2(-1)\big) = H^1(C_1, N_1) = 0\ .$$

Let us write the overshear $(x,y) \mapsto (x, a(x)y + b(x))$ in homogeneous form as

$$(x:y:z) \mapsto \big(\gamma(x:z)\cdot x, \alpha(x:z)y + \beta(x:z)\big).$$

Its degree is the value $d := \deg\alpha = \deg\gamma = \deg\beta - 1$. Thus $a(x) = \alpha(x:1)/\gamma(x:1)$ and $b(x) = \beta(x:1)/\gamma(x:1)$. As an element of $\operatorname{End}\big(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-1)\big)$ it corresponds to the sublinebundle

$$\mathcal{O}_{\mathbb{P}^1}(-d) \stackrel{(0,\alpha,\gamma,\beta)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}^2 + \mathcal{O}_{\mathbb{P}^1}(1).$$

Thus $H^1(C_1, N_1) = 0$ unless $d \leq 0$, that is, when $a(x)$ is constant and $b(x)$ is linear. Such an overshear satisfies the conditions (10.2.a–d) iff $b(x)$ is also constant. In this case the lifting of the overshear is in $O(3,1)$, hence no smoothing is necessary. $\square$

The above Hilbert scheme approach has problems in general since not all conic bundles are obtained from a $\mathbb{P}^1$-bundle by a birational transformation. So our reduction to the $\mathbb{P}^1$-bundle computation would not work directly, but it should work after taking a ramified double cover of $\mathbb{P}^1$. The end result should be that all overshears with high enough degree do deform.

16 (Algebraic deformation, general case)**.** This approach is technically harder but it always gives an algebraic deformation of the overshear.

We use that $\operatorname{Aut}(B_{Z(t)}Q/\mathbb{P}^1)$ is birational to $\mathbb{P}^1 \times \mathbb{P}^3$ and the overshear defines a rational section $\sigma_1 \colon \mathbb{P}^1 \dashrightarrow \operatorname{Aut}(B_{Z(t)}Q/\mathbb{P}^1)$ which is an actual section over the real points. We want to use the deformation theory of sections of rationally connected fibrations to conclude that σ_1 deforms with t. (See [Araujo-Kollár02] for an introduction to such techniques.)

As before, we use the partial compactification (19.4)

$$\bar{\pi} \colon \overline{\operatorname{Aut}}^{sm}\big(\mathbf{Q}/\mathbb{P}^1 \times \mathbb{A}^1\big) \to \mathbb{P}^1 \times \mathbb{A}^1.$$

Let $C_1 \subset \overline{\operatorname{Aut}}^{sm}(B_{Z(1)}Q/\mathbb{P}^1)$ denote the image of a section.

Next we note that deformation theory is local near the section that we want to deform. Thus it does not matter that $\overline{\operatorname{Aut}}^{sm}(B_{Z(1)}Q/\mathbb{P}^1) \to \mathbb{P}^1$ is not proper. All we need is that σ_1 is an actual section and that there is at least one smooth, proper fiber.

Thus, by adding many conjugate pairs of vertical rational curves to C_1, the resulting 1-cycle will deform with t, cf. [Araujo-Kollár02, Sec.6]. Since we added only curves without real points to C_1, the real points of the complex deformation give a C^∞-deformation of $C_1(\mathbb{R})$. This gives an algebraic lifting of σ_1 to sections $\sigma_t \colon \mathbb{P}^1 \to \overline{\operatorname{Aut}}^{sm}(B_{Z(t)}Q/\mathbb{P}^1)$. $\square$

It is natural to wonder if the above approach using conic bundles is truly necessary. The following example shows that even for linear automorphism of $\mathbb{R}^2$, the corresponding birational self-map of S^2 is, in general, not algebraically smoothable.

17 (Non-smoothable birational homeomorphisms)**.** Let $\Phi_0 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be an algebraic automorphism. Using the stereographic projection, we can lift Φ_0 to a birational map $\Phi_0' \colon S^2 \to S^2$ which is also a homeomorphism. In general, Φ_0' is not an automorphism since it is not a regular map at the north pole $n \in S^2$. We have shown that in some cases Φ_0' has an algebraic deformation Φ_t' where Φ_t' is an algebraic automorphism of S^2 for $t \neq 0$. It is natural to ask if this is always possible or not. Here we show that, even in some quite simple cases, the answer is negative.

Consider the simplest case when Φ_0 is a linear transformation

$$(u,v) \mapsto \big(\ell_1(u,v) := a_1u + b_1v + c_1, \ell_2(u,v) := a_2u + b_2v + c_2\big).$$

Then Φ_0' is given by

$$(x,y,z,t) \mapsto \Big(2(t-z)L_1, 2(t-z)L_2, L_1^2 + L_2^2 - (t-z)^2, L_1^2 + L_2^2 + (t-z)^2\Big)$$

where $L_i = a_ix + b_iy + c_i(t-z)$.

Assume now that $A(\Phi) \colon (u,v) \mapsto (a_1u + b_1v, a_2u + b_2v)$ has complex conjugate eigenvectors different from $(1, \pm i)$. Then one can factor the map $\Phi_0' \colon Q \to Q$ through blow-ups and downs as follows.

(1) Blow up the north pole $n \in Q$ to get the exceptional curve $E_n \subset B_nS^2$.

(2) Blow up a conjugate point pair on E_n (corresponding to the eigenvectors of $A(\Phi)$) to get $E'_n \subset Q_2$. Note that E'_n has self intersection -3.
(3) Contract the birational transforms of the (complex conjugate) lines on Q passing through n to get $E''_n \subset Q_3$. Note that E''_n has self intersection -1.
(4) Contract E''_n to get again Q.

Let $\Gamma_0 \subset Q \times Q$ be the closure of the graph of Φ'_0. We see that Γ_0 is obtained from Q_2 by contracting E'_n. In particular, Γ_0 has a single triple point, and the projections $p_i \colon \Gamma_0(\mathbb{R}) \to S^2$ are homeomorphisms for $i = 1, 2$.

An algebraic smoothing of Φ'_0 would be a deformation $\Gamma_t \subset Q \times Q$ such that Γ_t is the graph of a birational map that is an automorphism of S^2 for $t \neq 0$.

We claim that there is no such deformation.

By an example of [Artin74], every deformation of our rational triple point is simultaneously resolvable. (In order to assert this over $\mathbb{R}$ and not just over $\mathbb{C}$, we need that $Q_2 \to \Gamma_0$ is a minimal resolution without exceptional -2-curves.) That is, there is a base change $t = s^n$ after which the Γ_s for $s \neq 0$ are also deformations of the minimal resolution $Q_2 \to \Gamma_0$. Since $Q_2(\mathbb{R}) \sim B_n S^2 \sim \mathbb{RP}^2$, we conclude that $\Gamma_t(\mathbb{R}) \sim \mathbb{RP}^2$ for $t \neq 0$.

In particular, $\Gamma_t(\mathbb{R})$ can not be the graph of a homeomorphism $S^2 \to S^2$.

4. Automorphism groups of conic bundles

Let k be a field, $\operatorname{char} k \neq 2$, Z a normal variety over k and $X \to Z$ a conic bundle. That is, there is a $\mathbb{P}^2$-bundle $\mathbf{P}_Z \to Z$ and a closed embedding $X \hookrightarrow \mathbf{P}_Z$ such that every fiber of $X \to Z$ becomes a conic in $\mathbb{P}^2$. We assume that the generic fiber is a smooth conic. The singular fibers are either a pair of intersecting lines or a double line.

For our applications we need the case when Z is a smooth surface over $\mathbb{R}$, but the above generality poses no extra problems.

Definition 18 (Automorphisms)**.** There are several ways to define the relative automorphisms of a conic bundle.

Since $X \to Z$ is flat, the scheme of relative automorphisms $\operatorname{Aut}_Z(X)$ exists (cf. [Kollár96, I.1.10]). Note that $\operatorname{Aut}_Z(X) \to Z$ is not flat, not even equidimensional. A smooth conic has a 3-dimensional automorphism group but a singular conic has a 4-dimensional automorphism group.

Every automorphism of a conic $Q \subset \mathbb{P}^2$ extends uniquely to an automorphism of $\mathbb{P}^2$. (This holds even for the double line with scheme theoretic automorphisms. This is, however, not important for us.) For conic bundles $X \to Z$, this gives an embedding

$$\operatorname{Aut}_Z(X) \hookrightarrow \operatorname{Aut}_Z(\mathbf{P}_Z) \hookrightarrow \operatorname{End}_Z(\mathbf{P}_Z).$$

Since $\mathbf{P}_Z \to Z$ is a $\mathbb{P}^2$-bundle, $\operatorname{End}_Z(\mathbf{P}_Z) \to Z$ is naturally a $\mathbb{P}^8$-bundle.

Let $Z^0 \subset Z$ be the open set corresponding to smooth conics and $X^0 \subset X$ its pre-image. Set

$$\overline{\operatorname{Aut}}(X/Z) := \text{closure of } \operatorname{Aut}_{Z^0}(X^0) \text{ in } \operatorname{End}_Z(\mathbf{P}_Z). \tag{18.1}$$

We prove in (19) that for conic bundles without double fibers, $\overline{\operatorname{Aut}}(X/Z) \to Z$ is equidimensional. Furthermore

$$\operatorname{Aut}(X/Z) := \overline{\operatorname{Aut}}(X/Z) \cap \operatorname{Aut}_Z(X)$$

is a smooth group scheme over Z which can be viewed as a Néron model of $\mathrm{Aut}_{Z^0}(X^0)$. (Rather, only part of a Néron model since the universality condition for extending sections is not always satisfied.)

The fibers of $\mathrm{Aut}(X/Z) \to Z$ are described as follows. At a singular fiber, we have a singular conic isomorphic to $Q_a := (a(x,y) = 0) \subset \mathbb{P}^2$ where $a(x,y)$ is a homogeneous quadric. Since $\mathrm{Aut}(Q_a)$ is 4-dimensional, it can not be the fiber of $\overline{\mathrm{Aut}}(X/Z) \to Z$.

Let $p \in Q_a$ be the singular point. We have a canonical representation

$$\rho \colon \mathrm{Aut}(Q_a) \to GL\bigl(T_p Q_a\bigr)$$

where $T_p Q_a$ denotes the Zariski tangent space. Let $\mathrm{Aut}^0(Q_a) \subset \mathrm{Aut}(Q_a)$ be the subgroup of those elements $\sigma \in \mathrm{Aut}(Q_a)$ such that $\det \rho(\sigma) = 1$. We will show that $\mathrm{Aut}^0(Q_a)$ is the fiber of $\mathrm{Aut}(X/Z) \to Z$ at a singular conic.

Theorem 19. *Let k be a field, $\operatorname{char} k \neq 2$, Z a normal variety over k and $X \to Z$ a conic bundle without double line fibers. Then:*

(1) *Every fiber of $\overline{\mathrm{Aut}}(X/Z) \to Z$ has dimension 3.*
(2) *At a smooth conic Q, the fiber is $\mathrm{End}(Q) \cong \mathbb{P}^3$.*
(3) *At a singular conic Q, the fiber has 4 irreducible components. $\mathrm{Aut}^0(Q_a)$ is a dense open subset of the union of two of them.*
(4) *Let $\overline{\mathrm{Aut}}^{sm}(X/Z) \subset \overline{\mathrm{Aut}}(X/Z)$ be the open set of points where $\overline{\mathrm{Aut}}(X/Z) \to Z$ is smooth. Then:*
    (a) *At a smooth conic Q, the fiber of $\overline{\mathrm{Aut}}^{sm}(X/Z) \to Z$ is $\mathrm{End}(Q) \cong \mathbb{P}^3$.*
    (b) *At a singular conic Q, the fiber of $\overline{\mathrm{Aut}}^{sm}(X/Z) \to Z$ is $\mathrm{Aut}^0(Q_a)$.*

Proof. Pick any point $z \in Z \setminus Z^0$ and let $g \colon (0 \in C) \to (z \in Z)$ be a smooth curve mapping to Z such that $g(0) = z$ and $g(C)$ intersects Z^0 nontrivially. By pull-back, we obtain a conic bundle $X_C \to C$ and so we can construct $\overline{\mathrm{Aut}}(X_C/C) \to C$. Since C is a smooth curve, $\overline{\mathrm{Aut}}(X_C/C) \to C$ is automatically flat. Let $A(z,g) \subset \mathrm{End}(\mathbb{P}^2_z)$ be its fiber.

We check below that the reduced structure of $A(z,g)$ does not depend on g. Let us denote this reduced structure by $A(z)$. (It is possible that $A(z,g)$ itself does not depend on g, but we have not checked this.)

This means that the fiber of $\overline{\mathrm{Aut}}(X/Z) \to Z$ over z has the same support as $A(z)$. Hence, by the theory of Chow varieties, $z \mapsto A(z)$ is a well defined 3-dimensional family of proper algebraic cycles over Z. (See [Kollár96, Sec.I.3] especially [Kollár96, I.3.17] for the relevant definitions and results.) Finally, by [Kollár96, I.6.5], $\overline{\mathrm{Aut}}(X/Z) \to Z$ is actually smooth at the smooth points of $A(z)$. Because of the group structure, this holds at every point that corresponds to an automorphism of the fiber.

It remains to check the above assertions about $A(z,g)$.

As before, let k be a field, $\operatorname{char} k \neq 2$, C a smooth curve over k and $S \to C$ a conic bundle without double fibers whose generic fiber is smooth.

Let $C^0 \subset C$ be the open set corresponding to smooth conics. If $c \in C^0$ then $\mathrm{Aut}(S_c)$ is a $k(c)$-form of PGL_2 and $\mathrm{Aut}(S^0/C^0) \to C^0$ is smooth. Moreover, we have an embedding $\mathrm{Aut}(S_c) \hookrightarrow \mathbb{P}^3$ which makes $\mathrm{Aut}(S^0/C^0) \to C^0$ into an open subset of a $\mathbb{P}^3$-bundle $\overline{\mathrm{Aut}}(S^0/C^0) \to C^0$. Note that the identity element gives a canonical section of $\overline{\mathrm{Aut}}(S^0/C^0) \to C^0$, in particular $\overline{\mathrm{Aut}}(S^0/C^0)$ is birational (over C^0) to $\mathbb{P}^3 \times C^0$.

Next let us consider a singular fiber. We have a singular conic isomorphic to $Q_a := (a(x,y)=0) \subset \mathbb{P}^2$ where $a(x,y)$ is a homogeneous quadric. We claim that $\operatorname{Aut}^0(Q_a)$ is the fiber of $\operatorname{Aut}(S/C) \to C$ at a singular fiber.

This assertion can be checked after a field extension, hence we may assume that k is algebraically closed. We can then write down the family as

$$(xz - ty^2 = 0) \subset \mathbb{P}^2_{xyz} \times \mathbb{A}^1_t.$$

Write an element of PGL_2 as $(u,v) \mapsto (au+bv, cu+dv)$. We can write $(xz-ty^2)$ as the image of $\mathbb{P}^1$ under the map

$$x = \tau u^2,\ y = uv,\ z = \tau v^2 \quad \text{where} \quad \tau^2 = t.$$

The corresponding matrix in PGL_3 is given by

$$M(a,b,c,d;\tau) := \begin{pmatrix} a^2 & 2\tau ab & b^2 \\ \tau^{-1}ac & ad+bc & \tau^{-1}bd \\ c^2 & 2\tau cd & d^2 \end{pmatrix}.$$

For $\tau \neq 0$ we have the standard Veronese embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}^9$ projected to $\mathbb{P}^8$. In particular, the image has dimension 3 and degree 8.

We need to compute the flat closure of the set of these matrices as $t \to 0$. Write $M(a,b,c,d;\tau) = (a_{ij})$. It is easy to write down the 10 quadratic equations satisfied by the a_{ij}.

(5) 4 equations coming from rows 1,3 and columns 1,3. The first one is

$$4\tau^2 \cdot a^2 \cdot b^2 = \bigl(2\tau ab\bigr)^2 \quad \text{giving} \quad 4\tau^2 a_{11}a_{13} = a_{12}^2.$$

(6) 4 equations coming from multiplying a_{22} by the 4 non-corner entries. The first one is

$$(ad+bc)(2\tau ab) = 2\tau^2(a^2)(\tau^{-1}bd) + 2\tau^2(b^2)(\tau^{-1}ac)$$

which gives

$$a_{22}a_{12} = \tau^2 a_{11}a_{23} + \tau^2 a_{13}a_{21}.$$

(7) Squaring a_{22}, giving

$$a_{22}^2 = a_{11}a_{33} + a_{13}a_{31} + \tau^2 a_{21}a_{23}.$$

(8) The 4 non-corner elements give

$$a_{12}a_{32} = \tau^4 a_{21}a_{23}.$$

Note that these equations depend only on $t = \tau^2$. Setting $t = 0$ we get equations for the central fiber of the flat closure. It is not a priori clear that these are all the equations. However, as we see below, these equations define a subscheme of the correct dimension and degree. Thus, aside from some possible embedded components, these equations do define the flat limit. We get 4 irreducible components as follows.

(9) Matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{where} \quad a_{11}a_{33} = a_{22}^2,$$

giving a 3-dimensional subvariety of degree 2. These are obtained as

$$\lim_{\tau\to 0} M(a,\tau b,\tau c,d;\tau)=\begin{pmatrix} a^2 & 0 & 0\\ ac & ad & bd\\ 0 & 0 & d^2\end{pmatrix}.$$

Those with nonzero determinant give the identity component of $\mathrm{Aut}^0(Q_0)$. The singular point of the conic is $(0:1:0)$. The representation on the Zariski tangent space $\langle x/y, z/y\rangle$ is

$$\begin{pmatrix} a^2/ad & 0\\ 0 & d^2/ad\end{pmatrix}$$

which has determinant 1.

(10) Matrices of the form

$$\begin{pmatrix} 0 & 0 & *\\ * & * & *\\ * & 0 & 0\end{pmatrix}\quad \text{where}\quad a_{13}a_{31}=a_{22}^2,$$

giving a 3-dimensional subvariety of degree 2. Those with nonzero determinant give the non-identity component of $\mathrm{Aut}^0(Q_0)$. These can be obtained as $\lim_{\tau\to 0} M(\tau a,b,c,\tau d;\tau)$.

(11) Matrices of the form

$$\begin{pmatrix} * & 0 & *\\ * & 0 & *\\ 0 & 0 & 0\end{pmatrix}$$

giving a 3-dimensional subvariety of degree 1. These matrices are always singular and this component appears with multiplicity 2. These can be obtained as $\lim_{\tau\to 0} M(a,b,\tau c,\tau d,\tau)$.

(12) Matrices of the form

$$\begin{pmatrix} 0 & 0 & 0\\ * & 0 & *\\ * & 0 & *\end{pmatrix}$$

giving a 3-dimensional subvariety of degree 1. These matrices are always singular and this component appears with multiplicity 2. These can be obtained as $\lim_{\tau\to 0} M(\tau a,\tau b,c,d,\tau)$.

Geometrically, the last 2 components correspond to maps that map the components of Q_a to a point on them.

Taking into account the multiplicities, we have described a 3-dimensional cycle of degree $8=2+2+2\cdot 1+2\cdot 1$. This is the same degree as the generic fiber, hence this is the whole limit cycle. From the above descriptions it is also clear that the support depends only on Q_a. (We have not checked that the double structure along the two extra components does not depend on the deformation, nor have we tried to prove that there are no other nilpotents in the scheme theoretic fiber. These are, however, not needed.) $\square$

20 (Remarks on double fibers)**.** Although we do not need it, it is interesting to check what happens for conic bundles with a double line as a fiber. A typical example is

$$(txz-y^2=0)\subset \mathbb{P}^2_{xyz}\times \mathbb{A}^1_t.$$

We can write $(txz - y^2 = 0)$ as the image of $\mathbb{P}^1$ under the map

$$x = u^2, \ y = \tau uv, \ z = v^2 \quad \text{where} \quad \tau^2 = t.$$

The corresponding matrix in PGL_3 is given by

$$N(a, b, c, d; \tau) := \begin{pmatrix} a^2 & 2\tau^{-1}ab & b^2 \\ \tau ac & ad + bc & \tau bd \\ c^2 & 2\tau^{-1}cd & d^2 \end{pmatrix}.$$

The flat closure can be computed as before. Let us look for example at the first component, consisting of matrices of the form

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \quad \text{where} \quad a_{11}a_{33} = a_{22}^2.$$

These are obtained as $\lim_{\tau \to 0} N(a, \tau b, \tau c, d; \tau)$.

This looks very much like (19.9), but the geometric picture is completely different. The above matrices give those automorphisms of the double line $(y^2 = 0)$ that fix the two points $(y = xz = 0)$. Thus, this group depends not only on the fiber $(y^2 = 0)$ but also on the smoothing $(y^2 + txz = 0)$. In particular, if $S \to C$ is a conic bundle over a 1-dimensional smooth curve C, then $\overline{\mathrm{Aut}}(S/C) \to C$ behaves as in (19), even if there are double fibers. However, for a conic bundle over a higher dimensional base, $\overline{\mathrm{Aut}}(X/Z) \to Z$ is usually not equidimensional at the double fibers.

The following provides a more general approach to the computation in (19) in all dimensions.

Let $Q \subset \mathbb{P}^n$ be a singular quadric of rank n. Q has a unique singular point $p \in Q$. Let T_pQ be the Zariski tangent space. We can view Q as a quadratic form q on T_pQ, defined up to multiplicative scalar. The image of the representation of $\mathrm{Aut}(Q)$ on T_pQ fixes q, but only up to a scalar. Let $\mathrm{Aut}^0(Q) \subset \mathrm{Aut}(Q)$ be the subgroup whose image under ρ fixes q.

Theorem 21. *Let k be a field, $\mathrm{char}\, k \neq 2$, Z a normal variety over k, $\mathbf{P} \to Z$ a $\mathbb{P}^n$-bundle and $Q \subset \mathbf{P}$ a quadric bundle such that each fiber has rank $\geq n$ and the generic fiber has rank $n+1$. Then there is a smooth group scheme $\mathrm{Aut}(Q/Z) \subset \mathrm{Aut}_Z(\mathbf{P})$ such that*

1. *if Q_z is smooth then $\mathrm{Aut}(Q/C)_z$ is $\mathrm{Aut}(Q_z)$, and*
2. *if Q_z is singular then $\mathrm{Aut}(Q/C)_z$ is $\mathrm{Aut}^0(Q_z)$.*

Proof. As before, it is enough to check this when $Z = C$ is a curve over an algebraically closed field.

We can trivialize $\mathbf{P}$ and let Q be the family of quadrics $(x_0^2 + \cdots + x_{n-1}^2 + tx_n^2 = 0)$. Write a matrix in block form as

$$\begin{pmatrix} A & \mathbf{b} \\ \mathbf{c}^t & d \end{pmatrix},$$

where A is an $n \times n$ matrix, $\mathbf{b}, \mathbf{c}$ column vectors and all their entries are functions of t. This is in $\mathrm{Aut}_C(Q)$ iff

$$\begin{aligned} A^tA + t\mathbf{c}\mathbf{c}^t &= \lambda \mathbf{1}_n, \\ A^t\mathbf{b} + td\mathbf{c} &= 0, \\ \mathbf{b}^t\mathbf{b} + td^2 &= \lambda t, \end{aligned} \tag{21.4}$$

where λ is also a function of t. (Because of the presence of λ, these are not honest equations. To get the actual equations, we need to eliminate λ. Since λ occurs only linearly, this is easy to do, but we lose the simple form of the above equations. Alternatively, we can consider λ to be a new variable of degree 2 and work in the corresponding weighted projective space.) Setting $t = 0$ we get the equations of $\operatorname{Aut}(Q_0)$:

$$A(0)^t A(0) = \lambda(0)\mathbf{1}_n \quad \text{and} \quad \mathbf{b}(0) = 0.$$

This group, however, has dimension $\dim \operatorname{Aut}(Q_t) + 1$ which is too big. In order to get $\overline{\operatorname{Aut}}(Q/Z)$, we need one more equation. Multiplying (21.4.ii) on the left by $A^{t,adj}$ (the determinant theoretic adjoint of A^t) we obtain

$$\det A \cdot \mathbf{b} = -tdA^{t,adj}\mathbf{c}$$

which gives that

$$\det^2 A \cdot \mathbf{b}^t\mathbf{b} = t^2 d^2 \mathbf{c}^t A^{adj} A^{t,adj}\mathbf{c}$$

Substituting into (21.4.iii) and canceling t we get

$$\det^2 A \cdot (\lambda - d^2) = td^2\mathbf{c}^t A^{adj} A^{t,adj}\mathbf{c}.$$

From (21.4.i) we see that

$$\det^2 A = \det\bigl(\lambda\mathbf{1}_n - t\mathbf{c}\mathbf{c}^t\bigr)$$

and the right hand side can be expanded as $\lambda^n + t \cdot (\text{polynomial})$. Thus setting $t = 0$ gives the new equation

$$\lambda(0)^n \cdot \bigl(\lambda(0) - d(0)^2\bigr) = 0. \tag{21.5}$$

The case $\lambda(0) = 0$ gives singular limits.

Otherwise $\lambda(0) \neq 0$ and such limit automorphisms all satisfy $\lambda(0) = d(0)^2$.

The representation on the tangent space T_pQ_0 is

$$\begin{pmatrix} A(0) & \mathbf{b}(0) \\ \mathbf{c}(0)^t & d(0) \end{pmatrix} \;\mapsto\; \tfrac{1}{d(0)}A(0).$$

Thus

$$\bigl(\tfrac{1}{d(0)}A(0)\bigr)^t\bigl(\tfrac{1}{d(0)}A(0)\bigr) = \tfrac{\lambda(0)}{d(0)^2}\mathbf{1}_n = \mathbf{1}_n. \quad \square$$

5. Generators of $\operatorname{Aut}(S^2)$

Noether proved that the involution $(x, y, z) \mapsto \bigl(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\bigr)$ and PGL_3 generate the group of birational self-maps $\operatorname{Bir}(\mathbb{P}^2)$ over $\mathbb{C}$. Using similar ideas, [Ronga-Vust05] proved that $\operatorname{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by linear automorphisms and certain real algebraic automorphisms of degree 5. In this section, we prove that $\operatorname{Aut}(S^2)$ is generated by linear automorphisms and by the $\sigma_{p.q}$. The latter are real algebraic automorphisms of degree 3.

Example 22 (Cubic involutions of $\mathbb{P}^3$)**.** On $\mathbb{P}^3$ take coordinates (x, y, z, t). We need two types of cubic involutions of $\mathbb{P}^3$. Let us start with the Cremona transformation

$$(x, y, z, t) \mapsto \bigl(\tfrac{1}{x}, \tfrac{1}{y}, \tfrac{1}{z}, \tfrac{1}{t}\bigr) = \tfrac{1}{xyzt}(yzt, ztx, txy, xyz)$$

whose base points are the 4 "coordinate vertices". We will need to put the base points at complex conjugate point pairs, say $(1, \pm i, 0, 0), (0, 0, 1, \pm i)$. Then the above involution becomes

$$\tau\colon (x, y, z, t) \mapsto \bigl((x^2 + y^2)z, (x^2 + y^2)t, (z^2 + t^2)x, (z^2 + t^2)y\bigr).$$

Check that

$$\tau^2(x,y,z,t) = (x^2+y^2)^2(z^2+t^2)^2 \cdot (x,y,z,t),$$

thus τ is indeed a rational involution on $\mathbb{P}^3$.

Consider a general quadric passing through the points $(1,\pm i,0,0), (0,0,1,\pm i)$. It is of the form

$$Q = Q_{abcdef}(x,y,z,t) := a(x^2+y^2) + b(z^2+t^2) + cxz + dyt + ext + fyz.$$

By direct computation,

$$Q_{abcdef}\big(\tau(x,y,z,t)\big) = (x^2+y^2)(z^2+t^2) \cdot Q_{abcdfe}(x,y,z,t).$$

(Note that ef changes to fe. Thus, if $e = f$, then τ restricts to an involution of the quadric $(Q = 0)$ but not in general.)

Assume now that we are over $\mathbb{R}$. We claim that τ is regular on the real points if $a, b \neq 0$. The only possible problem is with points where $(x^2+y^2)(z^2+t^2) = 0$. Assume that $(x^2+y^2) = 0$. Then $x = y = 0$ and so $Q(x,y,z,t) = 0$ gives that $b(z^2+t^2) = 0$ hence $z = t = 0$, a contradiction.

Whenever Q has signature $(3,1)$, we can view $(Q = 0)$ as a sphere and then τ gives a real algebraic automorphism of the sphere S^2, which is well defined up to left and right multiplication by $O(3,1)$. A priori the automorphisms depend on a,b,c,d,e,f, so let us denote them by τ_{abcdef}.

Given S^2, the above τ_{abcdef} depends on the choice of the base points, that is, 2 conjugate pairs of points on the complex quadric $S^2(\mathbb{C})$. The group $O(3,1)$ has real dimension 6. Picking 2 complex points has real dimension 8. So the τ_{abcdef} should give a real 2-dimensional family of automorphisms modulo $O(3,1)$.

We also need a degenerate version of the Cremona transformation when the 4 base points come together to a pair of points. With base points $(1,0,0,0)$ and $(0,1,0,0)$, we get

$$(x,y,z,t) \mapsto (xz^2, yt^2, zt^2, z^2t).$$

If we put the base points at $(1,\pm i,0,0)$ then we get the transformation

$$\sigma' \colon (x,y,z,t) \mapsto \big(y(z^2-t^2)+2xzt, x(t^2-z^2)+2yzt, t(z^2+t^2), z(z^2+t^2)\big).$$

Take any quadric of the form

$$Q = Q'_{abcdef}(x,y,z,t) := a(x^2+y^2) + bz^2 + czt + dt^2 + e(xt+yz) + f(xz-yt).$$

By direct computation,

$$Q'_{abcdef}\big(\sigma'(x,y,z,t)\big) = (z^2+t^2)^2 \cdot Q'_{adcbef}(x,y,z,t).$$

As before, if Q' has signature $(3,1)$, we can view $(Q' = 0)$ as a sphere and then σ' gives a real algebraic automorphism of the sphere S^2, which is well defined up to left and right multiplication by $O(3,1)$. Let us denote them by σ_{abcdef}. Despite the dimension count, the group $O(3,1)$ does not act with a dense orbit on the set of complex conjugate point pairs and complex conjugate directions. Indeed, after complexification, the quadric becomes $\mathbb{P}^1 \times \mathbb{P}^1$ and we can chose the two points to be $p_1 := (0,0)$ and $p_2 := (\infty,\infty)$. The subgroup fixing these two points is $\mathbb{C}^* \times \mathbb{C}^*$ and the diagonal acts trivially on the tangent directions at both of the points p_i. Thus the σ_{abcdef} form a 1-dimensional family.

Theorem 23. *The group of algebraic automorphisms of S^2 is generated by $O(3,1)$, the τ_{abcdef} and σ_{abcdef}.*

Remark 24. It is possible that the τ_{abcdef} alone generate $\mathrm{Aut}(S^2)$. In any case, as the 4 base points come together to form 2 pairs, the τ_{abcdef} converge to the corresponding σ_{abcdef}. Thus the τ_{abcdef} generate a dense subgroup of $\mathrm{Aut}(S^2)$ (in the C^∞-topology.)

One reason to use the σ_{abcdef} is that, as the proof shows, the τ_{abcdef} and σ_{abcdef} together generate $\mathrm{Aut}(S^2)$ in an "effective manner." By this we mean the following.

Any rational map $\Phi\colon S^2 \dashrightarrow S^2$ can be given by 4 polynomials

$$\Phi(x,y,z,t) = \bigl(\Phi_1, \Phi_2, \Phi_3, \Phi_4\bigr).$$

Note that Φ does not determine the Φ_i uniquely, but there is a "minimal" choice. We can add any multiple of $x^2+y^2+z^2-t^2$ to the Φ_i and we can cancel common factors. We choose $\max_i\{\deg \Phi_i\}$ to be minimal and call it the *degree* of Φ. It is denoted by $\deg \Phi$. (It is easy to see that these minimal Φ_i are unique up to a multiplicative constant.) Note that $\deg \Phi = 1$ iff $\Phi \in O(3,1)$.

By "effective" generation we mean that given any $\Phi \in \mathrm{Aut}(S^2)$ with $\deg \Phi > 1$, there is a ρ which is either of the form τ_{abcdef} or σ_{abcdef} such that

$$\deg\bigl(\Phi \circ \rho\bigr) < \deg \Phi.$$

25 (Proof of (23))**.** The proof is an application of the Noether-Fano method. See [Kollár-Smith-Corti04, Secs. 2.2–3] for details.

Let k be a field and $Q \subset \mathbb{P}^3$ a quadric defined over k. Assume that $\mathrm{Pic}(Q) = \mathbb{Z}[H]$ where H is the hyperplane class. Let Q' be any other quadric and $\Phi\colon Q \dashrightarrow Q'$ a birational map. Then $\Gamma := \Phi^*|H_{Q'}|$ is a 3-dimensional linear system on Q and $\Gamma \subset |dH_Q|$ for some d. Let p_i be the (possibly infinitely near) base points of Γ (over $\bar{k}$) and m_i their multiplicities. As in [Kollár-Smith-Corti04, 2.8], we have the equalities

$$\Gamma^2 - \sum m_i^2 = \deg Q' \quad \text{and} \quad \Gamma \cdot K_Q + \sum m_i = \deg K_{Q'}.$$

In our case, these become

$$\sum m_i^2 = 2d^2 - 2 \quad \text{and} \quad \sum m_i = 4d - 4.$$

Next we see how the transformations τ_{abcdef} and σ_{abcdef} change the degree of a linear system Γ.

Example 26 (Cremona transformation on a quadric)**.** For the τ_{abcdef} series, pick 4 distinct points $p_1, \dots, p_4 \in Q$ such that no two are on a line in Q, not all 4 on a conic and assume that $s := m_1 + \cdots + m_4 > 2d$. Blow up the 4 points and contract the 4 conics that pass through any 3 of them. The p_i are replaced by 4 base points of multiplicities $2d - s + m_i$. Their sum is $8d - 4s + s = 8d - 3s$. Thus $4d - 4 = \sum m_i$ is replaced by $\sum m_i - s + (8d - s)$, hence d becomes $d - (s - 2d) < d$.

For σ_{abcdef}, pick 2 distinct points $p_1, p_2 \in Q$ and 2 infinitely near points $p_3 \to p_1$ and $p_4 \to p_2$ such that no two are on a line in Q, not all 4 on a conic and assume that $s := m_1 + \cdots + m_4 > 2d$. Blow up the points p_1, p_2 and then the points p_3, p_4. After this, we can contract the two conics that pass through $p_1 + p_2 + p_3$ (resp. $p_1 + p_2 + p_4$) and we can also contract the birational transforms of the exceptional curves over p_1 and p_2. The rest of the computation is the same. The p_i are replaced by 4 base points of multiplicities $2d - s + m_i$. Their sum is $8d - 4s + s = 8d - 3s$. Thus $4d - 4 = \sum m_i$ is replaced by $\sum m_i - s + (8d - s)$ hence d becomes $d - (s - 2d) < d$.

Thus, as long as we can find $p_1, \dots, p_4 \in Q$ (or infinitely near) such that $m_1 + \dots + m_4 > 2d$, we can lower $\deg \Phi$ by a suitable degree 3 Cremona transformation.

In order to find such p_i, assume first to the contrary that $m_i \le d/2$ for every i. Then

$$2d^2 - 2 = \sum m_i^2 \le \tfrac{d}{2} \sum m_i = \tfrac{d}{2}(4d-4) = 2d^2 - 2d.$$

This is a contradiction, unless $d = 1$ and Φ is a linear isomorphism.

If we work over $\mathbb{R}$ and we assume that there are no real base points, then we have at least one complex conjugate pair of base points with multiplicity $m_i > d/2$. We are done if we have found 2 such pairs.

In any case, up to renumbering the points, we have $m_1 = m_2 = \frac{d}{2} + a$ for some $d/2 \ge a > 0$. Assume next that all the other $m_j \le \frac{d}{2} - a$. Then

$$\begin{aligned} 2d^2 - 2 = \textstyle\sum m_i^2 &\le 2\big(\tfrac{d}{2} - a\big)^2 + \big(\tfrac{d}{2} - a\big)\big(\textstyle\sum m_i - d + 2a\big) \\ &= 2\big(\tfrac{d}{2} - a\big)^2 + \big(\tfrac{d}{2} - a\big)\big(4d - 4 - d + 2a\big). \end{aligned}$$

By expanding, this becomes

$$(a+2)(d-4) \le -6.$$

Thus $d \in \{1, 2, 3\}$. If $d = 3$ then $a + 2 \ge 6$ so $d/2 \ge a \ge 4$ gives a contradiction. If $d = 2$ then we get $a = 1$. Thus Γ consists of quadric sections with singular points at p_1, p_2. These are necessarily reducible (they have $p_a = 1$ with 2 singular points), again impossible.

We also need to show that no two of the points lie on a line and not all 4 are on a conic. For any line $L \subset Q(\mathbb{C})$, $(L \cdot \Gamma) = d$ gives that

$$\sum_{i: p_i \in L} m_i \le d.$$

In particular, $m_i \le d$ for every i and if p_i, p_j are on a line then $m_i + m_j \le d$. Thus out of $p_1, \dots, p_4$ only p_3, p_4 could be on a line. But p_3, p_4 are conjugates, thus they would be on a real line. There is, however, no real line on S^2.

Similarly, for any conic $C \subset Q(\mathbb{C})$, $(C \cdot \Gamma) = 2d$ gives that $\sum_{i: p_i \in C} m_i \le 2d$. Thus not all 4 points are on a conic.

Remark 27. Note that we started the proof over an arbitrary field, but at the end we had to assume that that we worked over $\mathbb{R}$. For a quadric surface Q with Picard number one, the above method should give generators for the group $\mathrm{Bir}^*(Q)$ of those birational self-maps that are regular along $Q(k)$. However, for other fields k, other generators also appear if there are more than 2 conjugate base points.

6. The identity component

The purpose of this section is to prove (4) for the identity components. We start with the proof of (2) which settles the case $R = S^2$. Next we prove (4) for the identity components in the case R is the non-orientable surface R_g.

Definition 28. Let X and Y be real algebraic manifolds and let I be any subset of X. A map f from I into Y is *algebraic* if there is a Zariski open subset U of X containing I such that f is the restriction of an algebraic map from U into Y.

Consider the standard sphere $S^2 \subset \mathbb{R}^3$ and let L be a line through the origin. Choose coordinates such that L is the x-axis and $S^2 := (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$.

Let $M\colon [-1,1] \to O(2)$ be a real algebraic map. Then

$$\Phi_M\colon S^2 \to S^2 \quad \text{given by} \quad (x,y,z) \mapsto \bigl(x, (y,z)\cdot M(x)\bigr)$$

is an automorphism of S^2, called the *twisting map* with axis L and associated to M. A conjugate of a twisting map by an element of $O(3,1)$ is also called a twisting map.

The following results are proved in [Huisman-Mangolte08a].

Theorem 29. *Notation as above.*

(1) *Any C^∞ map $M_0 : [-1,1] \to O(2)$ can be approximated by real algebraic maps $M_s : [-1,1] \to O(2)$. Moreover, given finitely many points $t_i \in [-1,1]$, we can choose the M_s such that $M_s(t_i) = M_0(t_i)$ for every i.*

(2) *Given distinct points $p_1, \dots, p_m$ and $q_1, \dots, q_m$ there are two twisting maps (with different axes) Φ_1 and Φ_2 such that $\Phi_1 \circ \Phi_2(q_i) = p_i$ for every i. Moreover,*
   (a) *if $p_j = q_j$ for some values of j then we can assume that $\Phi_1(q_j) = \Phi_2(q_j) = q_j$ for these values of j, and*
   (b) *if p_i is near q_i for every i then we can assume that the Φ_1, Φ_2 are near the identity.*

(3) *Let R be any real algebraic surface that is obtained from S^2 by repeatedly blowing up m real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in R. Then there are (nonunique) distinct points $p_1, \dots, p_m$ and $q_1, \dots, q_n$ and an isomorphism $\phi : R \to B_{p_1,\dots,p_m}S^2$ such that $\phi(r_i) = q_i$.*

By adding more points in (29.3) and compactness, we obtain the following stronger version:

Corollary 30. *Let R be any real algebraic surface that is obtained from S^2 by repeatedly blowing up m real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in R. There is a finite open cover $R = \cup_j W_j$ such that for every j there are distinct points $p_{1j}, \dots, p_{mj}, q_{1j}, \dots, q_{nj} \in S^2$ and an isomorphism $\phi_j : R \to B_{p_{1j},\dots,p_{mj}}S^2$ such that $\phi_j(r_i) = q_{ij}$ and $\phi_j(W_j) \subset S^2 \setminus \{p_{1j}, \dots, p_{mj}\}$.* □

31 (Proof of (2))**.** Let $p_1, \dots, p_n, q \in S^2$ be any finite set of distinct points, and let $\phi \in \mathrm{Diff}(S^2, p_1, \dots, p_n)$. In Section 1 we proved that there are automorphisms $\psi_s \in \mathrm{Aut}(S^2)$ such that ψ_s converges to ϕ in the compact-open C^∞-topology on $S^2 \setminus \{q\}$, and in the C^0-topology on S^2.

For any s and i, set $q_i^s := \psi_s(p_i)$. As ψ_s converges to ϕ, the q_i^s converge to p_i for every i. By (29.2.b) there are autmorphisms Φ_s such that $\Phi_s(q_i^s) = p_i$ and Φ_s converges to the identity. Thus the composites $\Phi_s \circ \psi_s$ are in $\mathrm{Aut}(S^2, p_1, \dots, p_n)$ and they converge to ϕ in the compact-open C^∞-topology on $S^2 \setminus \{q\}$, and in the C^0-topology on S^2.

Proposition 32. *Let R be any real algebraic surface that is obtained from S^2 by repeatedly blowing up g real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in R. Let $r_0 \in R$ be another point. Then the group $\mathrm{Aut}^0(R, r_1, \dots, r_n)$ of algebraic automorphisms homotopic to identity is dense in*

(1) $\mathrm{Homeo}^0(R, r_1, \dots, r_n)$ *in the C^0-topology on R, and in*
(2) $\mathrm{Diff}^0(R, r_1, \dots, r_n)$ *in the compact-open C^∞-topology on $R \setminus \{r_0\}$.*

Proof. Let $\phi\colon R \to R$ be a C^∞-diffeomorphism fixing $r_1, \dots, r_n$, and homotopic to the identity. Choose $R = \cup_j W_j$ as in (30). By a partition of unity argument, ϕ can be written as the composite of diffeomorphisms $\phi_\ell\colon R \to R$ fixing $r_1, \dots, r_n$ such that each ϕ_ℓ is the identity outside some $W_j \subset R$.

In particular, each ϕ_ℓ descends to a diffeomorphism ϕ'_ℓ of S^2 which fixes the points $p_{1j}, \dots, p_{gj}$ and $q_{1j}, \dots, q_{nj}$. By (2), we can approximate ϕ'_ℓ by algebraic automorphisms $\Phi'_{\ell,s}$ fixing all the points $p_{1j}, \dots, p_{gj}$ and $q_{1j}, \dots, q_{nj}$. Since the $\Phi'_{\ell,s}$ fix $p_{1j}, \dots, p_{gj}$, they lift to algebraic automorphisms $\Phi_{\ell,s}$ of $R \cong B_{p_{1j},\dots,p_{gj}}S^2$ fixing the points $r_1, \dots, r_n$. The composite of the $\Phi_{\ell,s}$ then converges to ϕ.

Note that as we blow up we loose one derivative, so even for the case of homeomorphisms it is better to use a C^1-approximation on S^2. $\square$

7. Generators of the mapping class group

Definition 33. Let R be a compact, closed surface and $p_1, \dots, p_n$ distinct points on R. The *mapping class group* is the group of connected components of those diffeomorphisms $\phi\colon R \to R$ such that $\phi(p_i) = p_i$ for $i = 1, \dots, n$.

$$\mathcal{M}(R, p_1, \dots, p_n) := \pi_0\big(\mathrm{Diff}(R, p_1, \dots, p_n)\big).$$

Up to isomorphism, this group depends only on the orientability and the genus of R. The orientable case has been intensely studied. Recent important results about the non-orientable case are in [Korkmaz02, Wahl08].

(In the literature, $\mathcal{M}_{g,n}$ is used to denote both the mapping class group of an orientable genus g (hence with Euler characteristic $2-2g$) surface with n marked points and the mapping class group of a non-orientable genus g (hence with Euler characteristic $2-g$) surface with n marked points.)

In preparation for the next section, we establish a somewhat new explicit set of generators in the non-orientable case.

Write R as $B_{p_1,\dots,p_g}S^2$, the blow up of S^2 at g points. We start by describing some elements of the mapping class group. For more details see [Lickorish65, Chillingworth69, Korkmaz02].

Definition 34 (Dehn twist)**.** Let R be any surface and $C \subset R$ a simple closed smooth curve such that R is orientable along C. Cut R along C, rotate one side around once completely and glue the pieces back together. This defines a diffeomorphism t_C of R, see Figure 6. The inverse t_C^{-1} corresponds to rotating one side

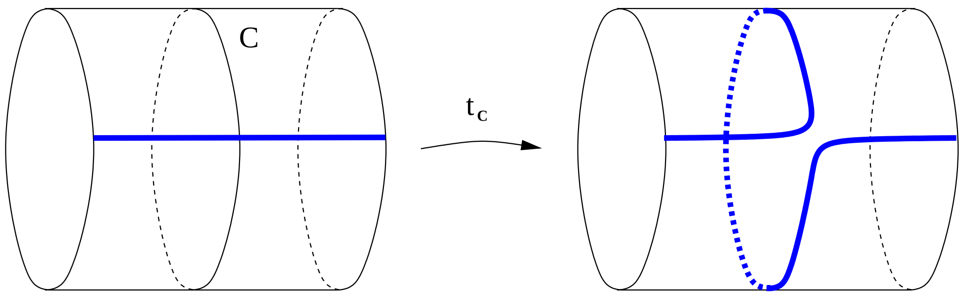

Figure 6. The effect of the Dehn twist around C on a curve.

the other way. Up to isotopy, the pair $\{t_C, t_C^{-1}\}$ does not depend on the choice of

C or the rotation. Either of t_C and t_C^{-1} is called a *Dehn twist* using C. On an oriented surface, with C oriented, one can make a sensible distinction between t_C and t_C^{-1}. This is less useful in the non-orientable case.

Definition 35 (Crosscap slide)**.** Let D be a closed disc and $p, q \in D$ two points. Take a simple closed curve C in D passing through p, q and let C' denote the corresponding curve in $B_q D$. Let M_p be a small disc around p. Let $\{\phi_t : t \in [0,1]\}$ be a continuous family of diffeomorphisms of $B_q D$ such that ϕ_0 is the identity, each ϕ_t is the identity near the boundary and as t increases, the ϕ_t slide M_p once around C. At $t = 1$, M_p returns to itself with its orientation reversed, as in Figure 7. In particular, $\phi_1(p) = p$. Thus ϕ_1 can be lifted to a diffeomorphism of $B_{p,q}D$ which is not isotopic to the identity but is the identity near the boundary.

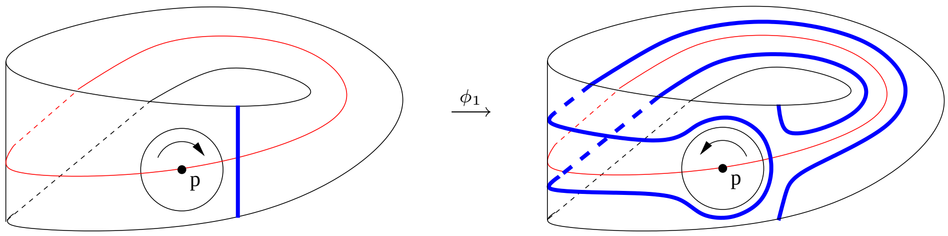

FIGURE 7. Cross-cap slide.

Let R be any surface, $U \subset R$ a closed subset with C^∞ boundary and $\tau : U \to B_{p,q}D$ a diffeomorphism. Then $\tau^{-1}\phi_1\tau : U \to U$ is the identity near the boundary of U, hence it can be extended by the identity on $R \setminus U$ to a diffeomorphism of R. Up to isotopy, this diffeomorphism does not depend on the choice of C, ϕ_t and τ. It is called a *cross-cap slide* or a *Y-homeomorphism* using U. Note that for a cross-cap slide to exist, R must be non-orientable and of genus at least 2.

36 (Generators of the mapping class group)**.** Let R_g be a non-orientable surface of genus $g \geq 1$. We write $R_g := B_{p_1,\dots,p_g} S^2$ with exceptional curves $E_i \subset R_g$ and let $\pi : R_g \to S^2$ be the blow down map.

The map π gives a one-to-one correspondence between

- simple closed smooth curves $C_R \subset R_g$ whose intersection with any exceptional curve E_i is transversal, and
- immersed curves $C = \pi(C_R) \subset S^2$ whose only self-intersections are at the p_i and no two branches are tangent.

Generators of the mapping class group were first established by [Lickorish65] and simplified by [Chillingworth69]. The case with marked points was settled by [Korkmaz02].

The generators are the following

(1) Dehn twists along C_R for certain smooth curves $C \subset S^2$ that pass through an even number of the $p_1, \dots, p_g$. (No self-intersections at the p_i.)
(2) Cross-cap slides using a disc $D \subset S^2$ that contains exactly 2 of the $p_1, \dots, p_g$.

The results of [Chillingworth69] and of [Korkmaz02] are more precise in that only very few of these generators are used. In the unmarked case, the above formulation is established in the course of the proof and stated on [Chillingworth69, p.427].

We will need somewhat different generators. We thank M. Korkmaz for answering many questions and especially for pointing out that one should use the lantern relation (38) to establish the following.

Proposition 37. *The following elements generate the mapping class group of the marked surface* $(B_{p_1,\dots,p_g}S^2, q_1,\dots,q_n)$.

(1) *Dehn twists along* C_R *for certain smooth curves* $C \subset S^2$ *that pass through* $0, 2$ *or* 4 *of the points* $p_1,\dots,p_g$. *(No self-intersections at the* p_i.*)*

(2) *Cross-cap slides using a disc* $D \subset S^2$ *that contains exactly 2 of the points* $p_1,\dots,p_g$.

Proof. We have included all the cross-cap slides from (36). Thus we need to deal with Dehn twists along C_R where $C \subset S^2$ is a simple closed curve passing through m of the points $p_1,\dots,p_g$ with $m > 4$.

Using induction, it is enough to show that the Dehn twist along C_R can be written as the product of Dehn twists along curves C'_R where each $C' \subset S^2$ is a simple closed curve passing through fewer than m of the points $p_1,\dots,p_g$.

Assume for simplicity that C passes through $p_1,\dots,p_m$ with $m > 4$ (and even). For $I \subset \{1,\dots,m\}$ let t_I be a Dehn twist using a simple closed curve C_I passing through the $\{p_i : i \in I\}$ but none of the others. The precise choice of the curve will be made later. We show that, with a suitable choice of the curves, $t_{12345\dots m}$ is a product of the Dehn twists $t_{125\dots m}, t_{345\dots m}, t_{1234}, t_{5\dots m}, t_{12}, t_{34}$.

This is best shown by a picture for $m = 8$. In Figure 8, $t_{12345678}$ is a product of the Dehn twists $t_{125678}, t_{345678}, t_{1234}, t_{5678}, t_{12}, t_{34}$. The shaded region is a sphere with four holes, and corresponds to a neighborhood of the lift to R_8 of $C_{12345678}$. On each side of the picture are drawn the curves corresponding to the Dehn twists of the same side in (38.1):

a) C_{12}, C_{34}, C_{5678}, $C_{12345678}$, b) C_{1234}, C_{125678}, C_{345678}. □

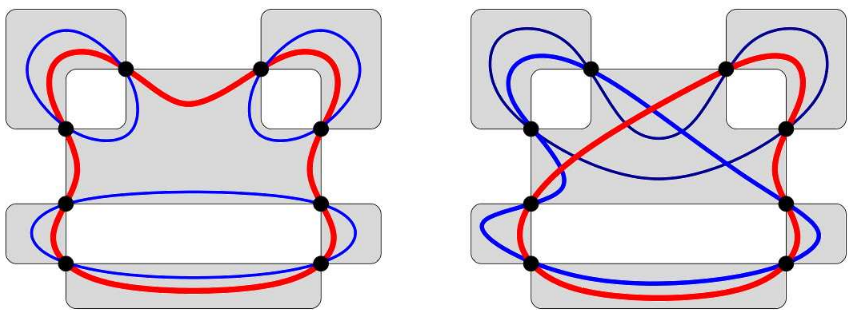

FIGURE 8. Lantern relation for $m = 8$.

38 (Lantern relation of Dehn). [Dehn38, Johnson79] Fix 4 points $q_0,\dots,q_3 \in S^2$. Let t_i be the Dehn twist using a small circle around q_i and for $i, j \in \{1,2,3\}$, let t_{ij} be the Dehn twist using a simple closed curve that separates q_i, q_j from the other 2 points. Then, with suitable orientations,

$$t_0t_1t_2t_3 = t_{12}t_{13}t_{23}\ , \tag{38.1}$$

where the equality is understood to hold in $\mathcal{M}(S^2, q_0,\dots,q_3)$.

8. Automorphisms and the mapping class group

The main result of this section is the following.

Theorem 39. *Let R be a real algebraic surface that is obtained from S^2 by blowing up points and $p_1, \dots, p_n \in R$ distinct marked points. Then the natural map*

$$\operatorname{Aut}(R, p_1, \dots, p_n) \twoheadrightarrow \mathcal{M}(R, p_1, \dots, p_n) \quad \textit{is surjective.}$$

Proof. We prove that all the generators of the mapping class group listed in (37) can be realized algebraically. There are 4 cases to consider:

(1) Dehn twists along $C_R \subset R$ for smooth curves $C \subset S^2$ that pass through either
  (a) none of the points p_i,
  (b) exactly 2 of the points p_i, or
  (c) exactly 4 of the points p_i.
(2) Cross-cap slides using a disc $D \subset S^2$ that contains exactly 2 of the points p_i.

We start with the easiest case (39.1.a).

40 (Algebraic realization of Dehn twists)**.** Let $C \subset S^2$ be a smooth curve passing through none of the points p_i. After applying a suitable automorphism of S^2, we may assume that C is the big circle $(x = 0)$.

Consider the map $g\colon [-1, 1] \to O(2)$ where $g(t) = \mathbf{1}$ for $t \in [-1, -\epsilon] \cup [\epsilon, 1]$ and $g(t)$ is the rotation by angle $\pi(1 + t/\epsilon)$ for $t \in [-\epsilon, \epsilon]$. Let $M\colon [-1, 1] \to O(2)$ be an algebraic approximation of g such that the corresponding twisting (28) Φ_M is the identity at the points p_i. Then Φ_M is an algebraic realization of the Dehn twist around C.

On the torus, the same argument works for either of the S^1-factors. Up to isotopy and the natural $GL(2, \mathbb{Z})$-action, this takes care of all simple closed curves.

Next we deal with the hardest case (39.1.c).

41 (4 pt case)**.** After applying a suitable automorphism of S^2, we may assume that C is close to a circle in S^2 but the 4 points do not lie on a circle.

Let us take an annular neighborhood of C and blow up the 4 points $p_1, \dots, p_4$. The resulting open surface is denoted by $W \subset B_{p_1,\dots,p_4} S^2$. It contains the curve C_R and the 4 exceptional curves $E_1, \dots, E_4$.

If we cut the blown-up annulus W along the 5 curves $A_1, \dots, A_4, D$ as indicated of the left hand side of Figure 9, we get the contractible surface U indicated on the right hand side of Figure 9. The left and right hand sides of U are identified to form a cylinder, giving a neighborhood of the curve $C_R \subset B_{p_1,\dots,p_4} S^2$. The big rectangle with lighter shading in U on the right corresponds to the lighter shaded are in W on the left. The 4 top and 4 bottom line segments of U are identified to form 4 Möbius bands.

Next, in Figure 10 we show the 4 exceptional curves.

Figure 11 shows the images of the curves E_i after the Dehn twist around C_R.

These images can be deformed to obtain a configuration as in Figure 12. Note that now E_i intersects E'_j iff $i \neq j$.

Next we convert this back to the annulus model W on the left hand side of Figure 9. We obtain Figure 13.

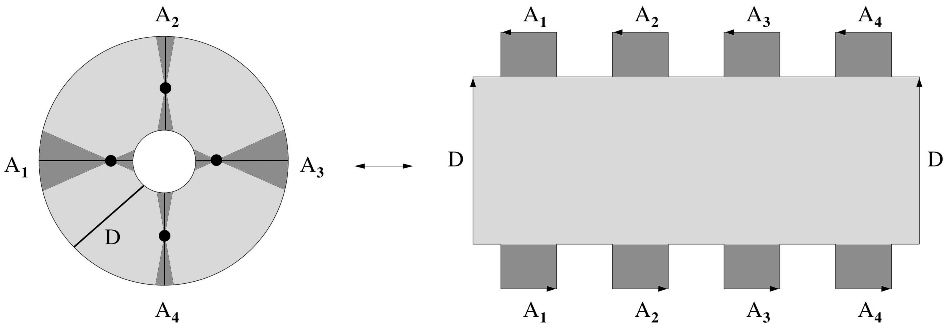

FIGURE 9. Two models of the annulus blown up in 4 points.

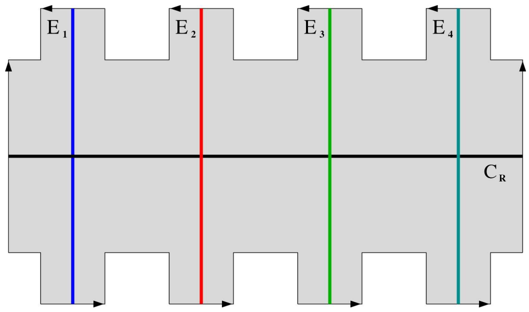

FIGURE 10. The 4 exceptional curves.

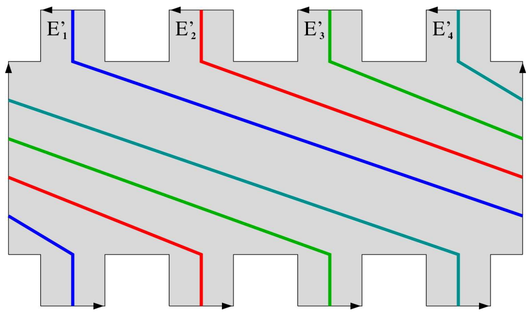

FIGURE 11. Effect of the Dehn twist around C_R.

The images of the exceptional curves $E_1, \dots, E_4$ under the standard Cremona transformation with base points $p_1, \dots, p_4$ are shown in Figure 1.

We see by direct inspection that the two quartets of curves in Figures 1 and 13 are isotopic. Thus, if we first apply the Dehn twist and then the (inverse) Cremona transformation and a suitable isotopy, we get a diffeomorphism $\phi\colon R_n \to R_n$ such that $\phi(E_i) = E_i$. That is, ϕ is lifted from a diffeomorphism of the g-pointed sphere $(S^2, p_1, \dots, p_n)$. By (2), any such diffeomorphism is isotopic to an algebraic automorphism. Hence the Dehn twist along C_R is also algebraic.

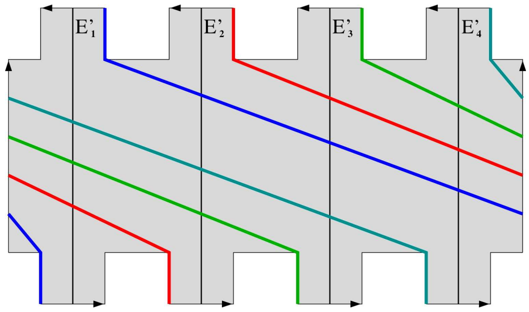

FIGURE 12. Deformation of Figure 11.

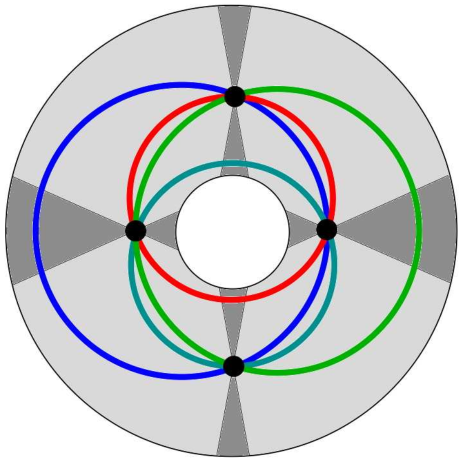

FIGURE 13. Images of the four exceptional curves.

42 (2 pt case)**.** The proof is the same as in the 4 point case but the description is easier.

A neighborhood of C gives an annulus with 2 blown-up points. After the Dehn twist we get two curves E_1', E_2' as in Figure 14.

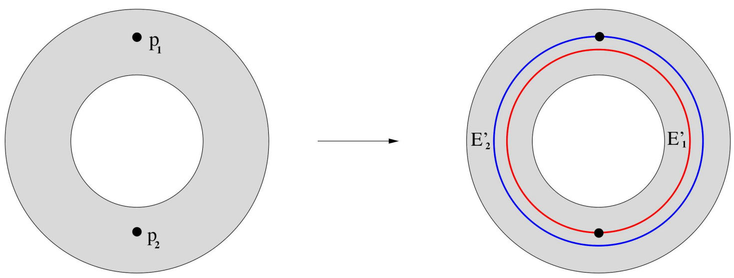

FIGURE 14. Cremona transformation with 2 real base points.

We can assume that the two curves E_1', E_2' are close to being circles, that is, close to the intersections $S^2 \cap H_i$ for some planes for $i = 1, 2$. Let $q, \bar{q}$ be the

2 (complex conjugate) points where these 2 planes H_i intersect the complexified sphere Q. Then the Cremona transformation with base points $p_1, p_2, q, \bar{q}$ is the inverse of the Dehn twist, again up to a diffeomorphism of S^2.

43 (Crosscap slides)**.** Here the topological picture is given by Figure 15. Note that

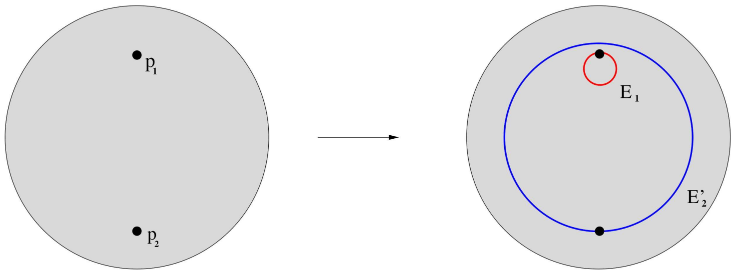

FIGURE 15. Cross-cap slides.

E_1 is mapped to itself and E_2 is mapped to the (almost) circle E_2'. Up to isotopy, we can replace E_1 with a small circle E_1' passing through p_1.

As in (42), we obtain $q, \bar{q}$ such that the Cremona transformation with base points $p_1, p_2, q, \bar{q}$ is the inverse of the Dehn twist, up to a diffeomorphism of S^2.

44 (Proof of (4))**.** Let $\phi : (R, q_1, \dots, q_n) \to (R, q_1, \dots, q_n)$ be any diffeomorphism. By (32), there is an automorphism $\Phi_1 \in \mathrm{Aut}(R, q_1, \dots, q_n)$ such that $\Phi_1^{-1} \circ \phi$ is homotopic to the identity.

By (39), we can approximate $\Phi_1^{-1} \circ \phi$ by a sequence of automorphisms $\Psi_s \in \mathrm{Aut}(R, q_1, \dots, q_n)$. Thus $\Phi_1 \circ \Psi_s \in \mathrm{Aut}(R, q_1, \dots, q_n)$ converges to ϕ. $\square$

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