EDITORIAL

New Zealand volume

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Abstract

This volume originated as a proceedings of the conference on Algebraic Geometry organized by four of us at the University of Auckland (Auckland, New Zealand) back in December 2019. This was the first conference on Algebraic Geometry that was ever organized in New Zealand. The New Zealand conference was a great success. Thirty four mathematicians participated in it:

Valery Alexeev (Athens, Georgia), David Baraglia (Adelaide),
Fedor Bogomolov (New York), Frédéric Campana (Nancy),
Ivan Cheltsov (Edinburgh), Jungkai Chen (Taipei), Will Donovan (Beijing),
Eric Edo (New Caledonia), Kento Fujita (Osaka),
Andrei Gabrielov (West Lafayette), Marco Golla (Nantes),
Yoshinori Gongyo (Tokyo), Liana Heuberger (Bath), DongSeon Hwang (Suwon),
Jun-Muk Hwang (Seoul), Ilia Itenberg (Paris), Maxim Jeffs (Boston),
Shin-Yao Jow (Beijing), Seoung Dal Jung (Jeju island), Seung-Jo Jung (Seoul),
Masayuki Kawakita (Kyoto), Yujiro Kawamata (Tokyo), Ivan Losev (New Haven),
Gaoyan Lyu (Beijing), Frédéric Mangolte (Marseille), Sione Ma'u (Auckland),
Grigory Mikhalkin (Geneva), Takuzo Okada (Saga), Jihun Park (Pohang),

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Justin Sawon (Chapel Hill), Laurent Schadeck (Pisa),
Jarosław Wiśniewski (Warsaw), Joonyeong Won (Seoul), Zhipu Zhao (Cambridge).

Before the New Zealand conference, Sione Ma'u, the local organizer, organized a summer school on Algebraic Geometry for students based in Auckland. Ivan Cheltsov, Liana Heuberger, Frédéric Mangolte and Susanna Zimmermann delivered the following mini-courses at this school:

- *Local intersection inequalities in birational geometry* by Ivan Cheltsov;
- *Toric del Pezzo surfaces and quotient singularities* by Liana Heuberger;
- *Real rational surfaces* by Frédéric Mangolte;
- *The real Cremona group of the plane* by Susanna Zimmermann.

This school helped to raise the profile of algebraic geometry and increase interest in the subject among students in New Zealand. We thank the New Zealand Mathematical Society for a small grant to support the lecturers' accommodation.

Thirty eight mathematicians contributed twenty six papers to this volume:

Vladimir Baranovsky, Lev Birbrair, Fedor Bogomolov, Ugo Bruzzo,
Frédéric Campana, Ivan Cheltsov, Jungkai Chen, Pietro Corvaja, Dominic Foord,
Kento Fujita, Andrei Gabrielov, Sergey Galkin, Liana Heuberger, Dale Husemöller,
Jun-Muk Hwang, DongSeon Hwang, Constantin Loginov, Ivan Losev,
Frédéric Mangolte, Sione Ma'u, Grigory Mikhalkin, William Montoya,
Gianluca Occhetta, Yuji Odaka, Takuzo Okada, Jihun Park, Wojciech Porowski,
Christophe Raffalli, Eleonora Romano, Justin Sawon, Costya Shramov,
Luis Solá Conde, Scott Stetson, Kiyohiko Takeuchi, Jarosław Wiśniewski,
YoungHan Yoon, Susanna Zimmermann, Francesco Zucconi.

Many contributors were participants of the New Zealand conference, while the others helped to expand the research breadth of the volume. Let us briefly describe their contributions.

In the paper *Zeta functions of projective hypersurfaces with ordinary double points*, Vladimir Baranovsky and his former PhD student Scott Stetson provide an algorithm on how to compute the zeta function of a hypersurface with at most isolated ordinary double points that is defined over a finite field. The paper provides many explicit examples.

The paper *Lipschitz geometry of pairs of normally embedded Hölder triangles* by Lev Birbrair and Andrei Gabrielov deals with a special case of the outer bi-Lipschitz classification of real semi-algebraic surface germs, which are obtained as a union of two normally embedded Hölder triangles. The authors define a combinatorial invariant of an equivalence class of such surface germs, and conjecture that it is a combinatorial invariant of outer bi-Lipschitz equivalence.

In the paper *Geometric properties of curves defined over number fields*, Fedor Bogomolov and Dale Husemöller provide a detailed proof of the famous Belyi theorem on geometry of complex algebraic curves defined over number fields. The authors also discuss several constructions and conjectures inspired by Belyi's theorem.

Let V be a complex toric variety of dimension $d \geqslant 3$, and let X be an irreducible normal prime divisor in it. The corresponding Noether–Lefschetz locus is the locus

consisting of all irreducible normal prime divisors in V that are rationally equivalent to X such that $H^d(V, \mathbb{C}) \cong H^d(X, \mathbb{C})$. In the paper *Codimension bounds for the Noether–Lefschetz components for toric varieties*, Ugo Bruzzo and his former PhD student William Montoya found a bound for the codimension of the Noether–Lefschetz locus assuming some natural conditions on X and V.

A complex function field version of the famous Lang's conjecture describes the qualitative distribution of sections of a non-isotrivial fibration over an algebraic curve when the generic fibre is of general type. In the paper *Rational points over complex function fields: remarks on isotriviality and dominatedness*, Frédéric Campana extends this conjecture to a general setting.

Let C_4 be a smooth twisted quartic curve in $\mathbb{P}^3$. Then C_4 is contained in a unique smooth quadric surface $Q \subset \mathbb{P}^3$. We may assume that $Q = \{x_0 x_3 = x_1 x_2\}$, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$. Fix the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$([u : v], [x : y]) \mapsto [xu : xv : yu : yv],$$

where $([u : v], [x : y])$ are coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$. After an appropriate change of coordinates, we may assume that $C_4 = \{u(x^3 + ax^2 y) = v(y^3 + by^2 x)\} \subset Q$ for some complex numbers a and b. Moreover, if $a \neq 0$ and $b \neq 0$, then, scaling the coordinates, we may assume that C_4 is given by

$$u(x^3 + \lambda x^2 y) = v(y^3 + \lambda y^2 x) \qquad (\bigstar)$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$. In the paper *K-stable Fano threefolds of rank 2 and degree 30*, Ivan Cheltsov (an organizer of this volume) and Jihun Park prove that the smooth Fano threefold obtained as the blow up of the projective space $\mathbb{P}^3$ along the twisted quartic curve C_4 is K-stable if and only if the curve C_4 can be given by $(\bigstar)$ with $\lambda \notin \{0, \pm 1, \pm 3\}$.

Let X be a normal projective threefold of general type. Then it is expected that

$$\mathrm{vol}(X) \geqslant \frac{4 p_g(X) - 10}{3},$$

where $p_g(X) = h^0(\mathscr{O}_X(K_X))$. In the short paper *Threefolds of general type on the Noether line*, Jungkai Chen studies the case when $\mathrm{vol}(X) = \frac{4 p_g(X) - 10}{3}$.

In the paper *The surface of Gauss double points*, Pietro Corvaja and Francesco Zucconi study the surface of Gauss double points associated to a very general quartic surface in $\mathbb{P}^3$.

In the paper *Birationally rigid Fano cyclic covers over a hypersurface containing a singular point*, Dominic Foord studies birational superrigidity of higher-dimensional Fano varieties that are cyclic covers of hypersurfaces containing one singular point. The author proves that many such singular Fano varieties are birational superrigidit and, in particular, they are irrational.

Motivated by a conjecture of Mukai, Kento Fujita classifies smooth Fano varieties admitting many nontrivial free divisors in his paper *Fano manifolds with many free divisors*.

Sergey Galkin and Grigory Mikhalkin set up a topological framework for degenerations of symplectic manifolds into singular spaces paying special attention to the behavior of Lagrangian manifolds and their (holomorphic) membranes in their joint paper *Singular symplectic spaces and holomorphic membranes*. In this paper, the authors show that degenerations into singular toric varieties provide a source of exotic Lagrangian tori.

Liana Heuberger contributed the paper *A primer on toric varieties* to this volume, which contains the lecture notes of her mini-course delivered at the summer school mentioned above. This paper is aimed at graduate students or busy mathematicians in need either of a quick reminder, or of examples illustrating specific properties of toric varieties.

Jun-Muk Hwang contributed his short paper *Lagrangian loci in moduli of abelian surfaces*. In this paper, Jun-Muk shows that any smooth surface germ in the moduli of abelian surfaces arises from a Lagrangian fibration of abelian surfaces.

Recall that a toric Fano variety is K-polystable $\iff$ the barycenter of the corresponding moment polytope is the origin. In the paper *On Kähler–Einstein fake weighted projective spaces*, DongSeon Hwang and YoungHan Yoon prove that K-polystable $\mathbb{Q}$-factorial projective toric varieties of Picard number must be of type B_∞. They also show there exist non-K-polystable $\mathbb{Q}$-factorial projective toric varieties of Picard number one that are of type B_∞.

Let C be a smooth curve, let $Y \to C$ be a fibration whose general fiber is a projective smooth Fano variety, and let F be a fiber of this fibration. Suppose, in addition, that Y is smooth, and F has simple normal crossings. In the paper *On semistable degenerations of Fano varieties*, Constantin Loginov shows that the dual complex of F is a simplex of dimension $\leqslant \dim(F)$.

A Procesi bundle is a special vector bundle on the Hilbert scheme of points in $\mathbb{C}^2$ that has been constructed by Mark Haiman. In the paper *On inductive construction of Procesi bundles*, Ivan Losev gives a new conceptual proof of the inductive formula for the Procesi bundle that plays an important role in Haiman's construction. Ivan uses the formula to prove several nice results.

In the short paper *On a question of supports*, Frédéric Mangolte and Christophe Raffalli give a sufficient condition for n closed connected subsets in the n-dimensional real projective space to admit a common multitangent hyperplane.

As we already mentioned, Sione Ma'u was a local organizer of our conference in Auckland. Without his energy and support, we would not be able to organize our New Zealand conference. Sione contributed the research paper *Transfinite diameter on the graph of a polynomial mapping and the DeMarco–Rumely formula* to this volume. In this paper, he studies transfinite diameters on the graph of polynomial selfmaps of the complex plane $\mathbb{C}^2$. Sione shows that two transfinite diameters of a compact subset of the graph are equal if the set has some symmetries.

In the paper *Small modifications of Mori dream spaces arising from $\mathbb{C}^*$-actions*, Gianluca Occhetta, Eleonora Romano, Luis Solá Conde, and Jarosław Wiśniewski study a relation between small modifications of projective algebraic varieties with a $\mathbb{C}^*$-action and their GIT quotients. To be precise, using flips with centers in closures of Białynicki-Birula cells, the authors produce a system of birational equivariant modifi-

cations of the original variety, which includes those on which a quotient map extends from a set of semistable points to a regular morphism.

In the paper *Degenerated Calabi–Yau varieties with infinite components, moduli compactifications, and limit toroidal structures*, Yuji Odaka introduces a new limit space for any degenerating Calabi–Yau family whose dense subspace is the disjoint union of countably infinite open Calabi–Yau varieties, parametrized by the rational points of the Kontsevich–Soibelman's essential skeleton. This long paper includes other important topics.

In the paper *Stable rationality of index one Fano hypersurfaces containing a linear space*, Takuzo Okada shows that a general hypersurface of degree n in $\mathbb{P}^n$ containing an r-plane with multiplicity m is not stably rational under some mild assumptions on n, m and r.

In the paper *Anabelian reconstruction of the Néron–Tate local height function*, Wojciech Porowski shows that one can recover the local Néron–Tate height of a rational point from the data of two homomorphisms of topological groups.

Let M be an irreducible compact complex hyperkähler manifold of dimension six such that its cohomology satisfies the Looijenga–Lunts–Verbitsky decomposition. Then $h_2(M, \mathbb{C}) \leqslant 23$. This is proved by Justin Sawon in his paper *A bound on the second Betti number of hyperkähler manifolds of complex dimension six* contributed to this volume.

In the paper *Finite groups acting on elliptic surfaces*, Costya Shramov shows that automorphism groups of Hopf and Kodaira surfaces have unbounded finite subgroups.

In the paper *Weak Fano threefolds with del Pezzo fibration*, Kiyohiko Takeuchi studies smooth weak Fano 3-folds having an extremal ray of type D.

Susanna Zimmermann delivered an excellent mini-course on the Cremona group at the summer school for PhD students at the University of Auckland that preceded the New Zealand conference. Unfortunately, due to other commitments, she had to leave Auckland before the conference. Nevertheless, Susanna was able to contribute the paper *A remark on Geiser involutions* to this volume. In this paper, Susanna constructs a morphism from the Cremona group over a perfect field to the free product generated by classes of Geiser involutions, and indicates certain conditions under which the constructed morphism is non-trivial.

Organizing a conference on Algebraic Geometry in New Zealand was a new experience for us. For many of us it was an antipodal travel, which is a real involution. But it was a great success. The beautiful landscapes of the country as the kindness of the people created an unforgettable atmosphere of creativity we all felt there. This volume is born from this incredible ambience. For most of us, this conference was the last in-person conference before the COVID-19 pandemic, and it was a very nice memory to remember during endless online meetings in the next two years. We cannot conclude without quoting the great Henri Miller

> "One's destination is never a place, but a new way of seeing things."