Geometrically rational real conic bundles and very transitive actions

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Abstract

In this article we study the transitivity of the group of automorphisms of real algebraic surfaces. We characterize real algebraic surfaces with very transitive automorphism groups. We give applications to the classification of real algebraic models of compact surfaces: these applications yield new insight into the geometry of the real locus, proving several surprising facts on this geometry. This geometry can be thought of as a half-way point between the biregular and birational geometries.

1. Introduction

The group of automorphisms of a complex algebraic variety is small: indeed, it is finite in general. Moreover, the group of automorphisms is 3-transitive only if the variety is $\mathbb{P}^1_{\mathbb{C}}$. On the other hand, it was recently proved that for a surface $X(\mathbb{R})$ birational to $\mathbb{P}^2_{\mathbb{R}}$, its group of automorphisms acts n-transitively on $X(\mathbb{R})$ for any n. The main goal of this paper is to determine all real algebraic surfaces $X(\mathbb{R})$ having a group of automorphisms which acts very transitively on $X(\mathbb{R})$. For precise definitions and statements, see below.

The aim of this paper is to study the action of birational maps on the set of real points of a real algebraic variety. Let us emphasize a common terminological source of confusion about the meaning of what is a *real algebraic variety* (see also the enlightening introduction of [Kol01]). From the point of view of general algebraic geometry, a real variety X is a variety defined over the real numbers, and a morphism is understood as being defined over all the geometric points. In most real algebraic geometry texts however, the algebraic structure considered corresponds to the algebraic structure of a neighbourhood of the real points $X(\mathbb{R})$ in the whole complex variety – or, in other words, the structure of a germ of an algebraic variety defined over $\mathbb{R}$.

From this point of view it is natural to view $X(\mathbb{R})$ as a compact submanifold of $\mathbb{R}^n$ defined by real polynomial equations, where n is some natural integer. Likewise, it is natural to say that a map $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is an *isomorphism* if ψ is induced by a birational map $\Psi\colon X \dashrightarrow Y$ such that Ψ (respectively Ψ^{-1}) is regular at any point of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$). In particular, $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a diffeomorphism. This notion corresponds to the notion of biregular maps defined in [BCR98, 3.2.6] for the structure of real algebraic variety commonly used in the context of real algebraic geometry. To distinguish between the Zariski topology and the topology induced by the embedding of $X(\mathbb{R})$ as a topological submanifold of $\mathbb{R}^n$, we will call the latter the *Euclidean topology.* Throughout what follows, topological notions like connectedness or compactness will always refer to the Euclidean topology.

Recall that a real projective surface is rational if it is birationally equivalent to the real projective plane, and that it is geometrically rational if its complexification is birationally equivalent

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to the complex projective plane. The number of connected components is a birational invariant. In particular, if X is a rational projective surface, $X(\mathbb{R})$ is connected.

The paper [HM09a] proves that the group of automorphisms $\mathrm{Aut}\big(X(\mathbb{R})\big)$ acts n-transitively on $X(\mathbb{R})$ for any n and any rational real algebraic surface X. To study the case where $X(\mathbb{R})$ is not connected, we have to refine the notion of n-transitivity. Indeed, if $X(\mathbb{R})$ has non-homeomorphic connected components, then even the group of self-homeomorphisms does not acts 2-transitively.

Definition 0. Let G be a topological group acting continuously on a topological space M. We say that two n-tuples of distinct points $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ are *compatible* if there exists an homeomorphism $\psi\colon M\to M$ such that $\psi(p_i)=q_i$ for each i. The action of G on M is then said to be *very transitive* if for any pair of compatible n-tuples of points $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ of M, there exists an element $g\in G$ such that $g(p_i)=q_i$ for each i. More generally, the action of G is said to be very transitive *on each connected component* if we require the above condition only in case, for each i, p_i and q_i belong to the same connected component of M.

Up till now, it was not known when the automorphism group of a real algebraic surface is big. We give a complete answer to this question: this is one of the main result of this paper. Let $\#M$ be the number of connected components of a compact manifold M.

Theorem 1. *Let X be a nonsingular real projective surface. The group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is then very transitive on each connected component if and only if X is geometrically rational and $\#X(\mathbb{R})\leqslant 3$.*

In the three component case, Theorem 2 below says that the very transitivity of $\mathrm{Aut}(X(\mathbb{R}))$ can be determined by examining the set of possible permutations of connected components.

Theorem 2. *Let X be a nonsingular real projective surface. The group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ then has a very transitive action on $X(\mathbb{R})$ if and only if the following hold:*

i) *X is geometrically rational, and*

ii) (a) *$\#X(\mathbb{R})\leqslant 2$, or*

  (b) *$\#X(\mathbb{R})=3$, and there is no pair of homeomorphic connected components, or*

  (c) *$\#X(\mathbb{R})=M_1\sqcup M_2\sqcup M_3$, $M_1\sim M_2\not\sim M_3$, and there is a morphism $\pi\colon X\to\mathbb{P}^1_{\mathbb{R}}$ whose general fibres are rational curves, and an automorphism of $\mathbb{P}^1_{\mathbb{R}}$ which fixes $\pi(M_3)$ and exchanges $\pi(M_1),\pi(M_2)$, or*

  (d) *$\#X(\mathbb{R})=M_1\sqcup M_2\sqcup M_3$, $M_1\sim M_2\sim M_3$, and there is a morphism $\pi\colon X\to\mathbb{P}^1_{\mathbb{R}}$ whose general fibres are rational curves, such that any permutation of the set of intervals $\big\{\pi(M_1),\pi(M_2),\pi(M_3)\big\}$ is realised by an automorphism of $\mathbb{P}^1_{\mathbb{R}}$.*

Furthermore, when $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not very transitive, it is not even 2-transitive.

This theorem will be proved in Section 9. Note that when $\#X(\mathbb{R})>3$, either any element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ preserves a conic bundle structure (Theorem 25), or $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is countable (Corollary 11): in either case $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not 1-transitive.

These two theorems apply to the classification of algebraic models of real surfaces. Up to this point in the paper $X(\mathbb{R})$ is considered as a submanifold of some $\mathbb{R}^n$. Conversely, let M be a compact $\mathcal{C}^\infty$-manifold. By the Nash-Tognoli theorem [Tog73], every such M is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^m$ for some m. Taking the Zariski closure in $\mathbb{P}^m$ and applying Hironaka's resolution of singularities [Hir64], it follows that M is in fact diffeomorphic to the set of real points $X(\mathbb{R})$ of a nonsingular projective algebraic variety X defined over $\mathbb{R}$. Such a variety X is called an *algebraic model* of M. A natural question is to classify the algebraic models of M up to isomorphism for a given manifold M.

There are several recent results about algebraic models and their automorphism groups [BH07, HM09a, HM09b, KM09]. For example, when M is 2-dimensional, and admits a real rational algebraic model, this rational algebraic model is unique [BH07]. In other words, if X and Y are two rational real algebraic surfaces, then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic if and only if there are homeomorphic. We extend the classification of real algebraic models to geometrically rational surfaces.

Theorem 3. *Let X, Y be two nonsingular geometrically rational real projective surfaces, and assume that $\#X(\mathbb{R}) \leqslant 2$. The surface $X(\mathbb{R})$ is then isomorphic to $Y(\mathbb{R})$ if and only if X is birational to Y and $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$. This is false in general when $\#X(\mathbb{R}) \geqslant 3$.*

Recall that a nonsingular projective surface is minimal if any birational morphism to a nonsingular surface is an isomorphism. We have the following rigidity result on minimal geometrically rational real surfaces.

Theorem 4. *Let X and Y be two minimal geometrically rational real projective surfaces, and assume that either X or Y is non-rational. The following are then equivalent:*

i) *X and Y are birational.*
ii) *$X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.*

In this work, we classify the birational classes of real conic bundles and correct an error contained in the literature (Theorem 25). It follows that the only geometrically rational surfaces $X(\mathbb{R})$ for which equivalence by homeomorphism implies equivalence by isomorphism are the connected ones. In particular, this yields a converse to [BH07, Corollary 8.1].

Corollary 5. *Let M be a compact $\mathcal{C}^\infty$-surface. The surface M then admits a unique geometrically rational model if and only if the following two conditions hold:*

i) *M is connected, and*
ii) *M is non-orientable or M is orientable with genus $g(M) \leqslant 1$.*

For M orientable with $g(M) > 1$, no uniqueness result – even very weak –holds. We can therefore ask what the simplest algebraic model for such an M should be. This question is studied in the forthcoming paper [HM09c].

Another way of measuring the size of $\mathrm{Aut}(X(\mathbb{R}))$ was used in [KM09], where it is proved that for any rational surface X, $\mathrm{Aut}\big(X(\mathbb{R})\big) \subset \mathrm{Diff}\big(X(\mathbb{R})\big)$ is dense for the strong topology. For non geometrically rational surfaces and for most of the non-rational geometrically rational surfaces, the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ cannot be dense. The above paper left the question of density open only for certain geometrically rational surfaces with 2, 3, 4 or 5 connected components. One by-product of our results is the non-density of $\mathrm{Aut}(X(\mathbb{R}))$ for most surfaces with at least 3 connected components – see Proposition 41.

Let us mention some other papers on automorphisms of real projective surfaces. In [RV05], it is proved that $\mathrm{Aut}\big(\mathbb{P}^2(\mathbb{R})\big)$ is generated by linear automorphisms and certain real algebraic automorphisms of degree 5. The paper [HM09b] is devoted to the study of very transitive actions and uniqueness of models for some kind of singular rational surfaces.

Strategy of the proof

In the proof of Theorem 1, the main part concerns minimal conic bundles. We first prove that two minimal conic bundles are isomorphic if they induce the same intervals on the basis. Given a set of intervals, one choose the most special conic bundle, the so-called exceptional conic bundle, to write explicitly the automorphisms and to obtain a fiberwise transitivity. We then use the most general conic bundles which come with distinct foliations on the same surface. The foliations being transversal, this yields the very transivity of the automorphism group in the minimal case.

Outline of the article

In Section 2 we fix notations and in Section 3 we recall the classification of minimal geometrically rational real surfaces.

Section 4, which constitutes the technical heart of the paper, is devoted to conic bundles, especially minimal ones. We give representative elements of isomorphism classes, and explain the links between the various conic bundles.

In Section 5, we investigate real surfaces which admit two conic bundle structures. In particular, we show that these are del Pezzo surfaces, and give descriptions of the possible conic bundles on these surfaces. Section 6 is devoted to the proof of Theorem 4. We firstly correct an inaccuracy in the literature, by proving that if two surfaces admitting a conic bundle structure are birational, then the birational map may be chosen so that it preserves the conic bundle structures. We then strengthen this result to isomorphisms between real parts when the surfaces are minimal, before proving Theorem 4.

In Section 7, we prove that if the real part of a minimal geometrically rational surface has 2 or 3 connected components, then its automorphism group is very transitive on each connected component. In Section 8, we prove the same result for non-minimal surfaces. We show how to separate infinitely close points, which is certainly one of the most counter-intuitive aspects of our geometry, and was first observed in [BH07] for rational surfaces. We also prove the uniqueness of models in many cases.

In Section 9, we then use all the results of the preceding sections to prove the main results stated in the introduction (except Theorem 4, which is proved in Section 6).

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2. Notation

Throughout what follows, by a variety we will mean an algebraic variety, which may be real or complex (i.e. defined over $\mathbb{R}$ or $\mathbb{C}$). If the converse is not expressly stated all our varieties will be projective and all our surfaces will be nonsingular and geometrically rational (i.e. rational over $\mathbb{C}$).

Recall that a real variety X may be identified with a pair (S,σ), where S is a complex variety and σ is an anti-holomorphic involution on S; by abuse of notation we will write $X = (S,\sigma)$. Then, $S(\mathbb{C}) = X(\mathbb{C})$ denotes the set of complex points of the variety, and $X(\mathbb{R}) = S(\mathbb{C})^\sigma$ is the set of real points. A point $p \in X$ may be real (if it belongs to $X(\mathbb{R})$), or imaginary (if it belongs to $X(\mathbb{C})\backslash X(\mathbb{R})$). If $X(\mathbb{R})$ is non empty (which will be the case for all our surfaces), then $\mathrm{Pic}(X) \cong \mathrm{Pic}(S)^\sigma$, [Sil89, I.(4.5)]. As we only work with regular surfaces (i.e. $q(X) = q(S) = 0$), the Picard group is isomorphic to the Néron-Severi group, and $\rho(S)$ and $\rho(X)$ will denote the rank of $\mathrm{Pic}(S)$ and $\mathrm{Pic}(X)$ respectively. Recall that $\rho(X) \leqslant \rho(S)$. We denote by $K_X \in \mathrm{Pic}(X)$ the canonical class, which may be identified with K_S. The intersection of two divisors of $\mathrm{Pic}(S)$ or $\mathrm{Pic}(X)$ will always denote the usual intersection in $\mathrm{Pic}(S)$.

We will use the classical notions of morphisms, rational maps, isomorphisms and automorphisms between real or complex varieties. Moreover, if X_1 and X_2 are two real varieties, an isomorphism *between real parts* $X_1(\mathbb{R}) \stackrel{\psi}{\to} X_2(\mathbb{R})$ is a birational map $\psi\colon X_1 \dashrightarrow X_2$ such that ψ (respectively ψ^{-1}) is regular at any point of $X_1(\mathbb{R})$ (respectively of $X_2(\mathbb{R})$). This endows $X_1(\mathbb{R})$ with a structure of a germ of algebraic variety defined over $\mathbb{R}$ (as in [BCR98, 3.2.6]), whereas the structure of X_1 is the one of an algebraic variety.

This notion of isomorphism between real parts gives rise to a geometry with rather unexpected properties comparing to those of the biregular geometry or the birational geometry. For example, let $\alpha\colon X_1(\mathbb{R}) \to X_2(\mathbb{R})$ be an isomorphism, and $\varepsilon\colon Y_1 \dashrightarrow X_1$, $\eta\colon Y_2 \dashrightarrow X_2$ be two birational maps; the map $\psi := \varepsilon^{-1}\alpha\eta$ is a well-defined birational map. Then ψ can be an isomorphism $Y_1(\mathbb{R}) \to Y_2(\mathbb{R})$ even if neither ε, nor η is an isomorphism between real parts. In the same vein, let $\alpha\colon X_1(\mathbb{R}) \to X_2(\mathbb{R})$ be an isomorphism, and let $\eta_1\colon Y_1 \to X_1$ and $\eta_2\colon Y_2 \to X_2$ be two birational morphisms which are the blow-ups of only real points (which may be proper or infinitely near points of X_1 and X_2). If α sends the points blown-up by η_1 on the points blown-up by η_2, then $\beta = (\eta_2)^{-1}\alpha\eta_1\colon Y_1(\mathbb{R}) \to Y_2(\mathbb{R})$ is an isomorphism.

Using Aut and Bir to denote respectively the group of automorphisms and birational self-maps of a variety, we have the following inclusions for the groups associated to $X = (S, \sigma)$:

$$\begin{array}{ccccc} \mathrm{Aut}(S) & & \subset & & \mathrm{Bir}(S) \\ \cup & & & & \cup \\ \mathrm{Aut}(X) & \subset & \mathrm{Aut}\big(X(\mathbb{R})\big) & \subset & \mathrm{Bir}(X)\ . \end{array}$$

By $\mathbb{P}^n$ we mean the projective n-space, which may be complex or real depending on the context. It is unique as a complex variety – written $\mathbb{P}^n_{\mathbb{C}}$. However, as a real variety, $\mathbb{P}^n$ may either be $\mathbb{P}^n_{\mathbb{C}}$ endowed with the standard anti-holomorphic involution, written $\mathbb{P}^n_{\mathbb{R}}$, or only when n is odd, $\mathbb{P}^n_{\mathbb{C}}$ with a special involution with no real points, written $(\mathbb{P}^n, \emptyset)$. To lighten notation, and since we never speak about $(\mathbb{P}^1, \emptyset)(\mathbb{R})$ we write $\mathbb{P}^1(\mathbb{R})$ for $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$.

3. Minimal surfaces and minimal conic bundles

The aim of this section is to reduce our study of geometrically rational surfaces to surfaces which admit a minimal conic bundle structure. We first recall the classification of geometrically rational surfaces (see [Sil89] for an introduction). The proofs of Theorems 2 and 4 will then split into three cases: rational, del Pezzo with $\rho = 1$, and minimal conic bundle. The rational case is treated in [HM09a] and Proposition 10 below states the case of a del Pezzo surface with $\rho = 1$.

Definition 6. A *conic bundle* is a pair (X, π) where X is a surface and π is a morphism $X \to \mathbb{P}^1$, where any fibre of π is isomorphic to a plane conic. If (X, π) and (X', π') are two conic bundles, a *birational map of conic bundles* $\psi\colon (X, \pi) \dashrightarrow (X', \pi')$ is a birational map $\psi\colon X \dashrightarrow X'$ such that there exists an automorphism α of $\mathbb{P}^1$ with $\pi' \circ \phi = \pi \circ \alpha$.

We will assume throughout what follows that if X is real, then the basis is $\mathbb{P}^1_{\mathbb{R}}$ (and not $(\mathbb{P}^1, \emptyset)$). This avoids certain conic bundles with no real points. We denote by $\mathrm{Aut}(X, \pi)$ (respectively $\mathrm{Bir}(X, \pi)$) the group of automorphisms (respectively birational self-maps) of the conic bundle (X, π). Observe that $\mathrm{Aut}(X, \pi) = \mathrm{Aut}(X) \cap \mathrm{Bir}(X, \pi)$. Similarly, when (X, π) is real we denote by $\mathrm{Aut}(X(\mathbb{R}), \pi)$ the group $\mathrm{Aut}\big(X(\mathbb{R})\big) \cap \mathrm{Bir}(X, \pi)$.

Recall that a real algebraic surface X is *minimal* if and only if there is no real (-1)-curve and no pair of disjoint conjugate imaginary (-1)-curves on X, and that a real conic bundle (X, π) is minimal if and only if the two irreducible components of any real singular fibre of π are imaginary. Compare to the complex case where (X, π) is minimal if and only if there is no singular fibre.

The following two results follow from the work of Comessatti [Com12], (see also [Mani67], [Isk79], [Sil89, Chap. V], or [Kol97]). Recall that a surface X is a del Pezzo surface if the anti-canonical divisor $-K_X$ is ample. The same definition applies for X real or complex.

THEOREM 7. *If X is a minimal geometrically rational real surface such that $X(\mathbb{R}) \neq \emptyset$, then one and exactly one of the following holds:*

i) *X is rational: it is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$, to the quadric $Q_0 := \{(x : y : z : t) \in \mathbb{P}^3_{\mathbb{R}} \mid x^2+y^2+z^2 = t^2\}$, or to a real Hirzebruch surface $\mathbb{F}_n$, $n \neq 1$;*

ii) *X is a del Pezzo surface of degree 1 or 2 with $\rho(X) = 1$;*

iii) *there exists a minimal conic bundle structure $\pi\colon X \to \mathbb{P}^1$ with an even number of singular fibres $2r \geqslant 4$. Moreover, $\rho(X) = 2$.*

Remark 8. If (S, σ) is a minimal geometrically rational real surface such that $S(\mathbb{C})^\sigma = \emptyset$, then S is an Hirzebruch surface of even index.

PROPOSITION 9 Topology of the real part. *In each case of the former theorem, we have:*

i) *X is rational if and only if $X(\mathbb{R})$ is connected. When X is moreover minimal, then $X(\mathbb{R})$ is homeomorphic to one of the following: the real projective plane, the sphere, the torus, or the Klein bottle.*

ii) *When X is a minimal del Pezzo surface of degree 1, it satisfies $\rho(X) = 1$, and $X(\mathbb{R})$ is the disjoint union of one real projective plane and 4 spheres. If X is a minimal del Pezzo surface of degree 2 with $\rho(X) = 1$, then $X(\mathbb{R})$ is the disjoint union of 4 spheres.*

iii) *If X is non-rational and is endowed with a minimal conic bundle with $2r$ singular fibres, then $X(\mathbb{R})$ is the disjoint union of r spheres, $r \geqslant 2$.*

PROPOSITION 10. *Let X, Y be two minimal geometrically rational real surfaces. Assume that X is not rational and satisfies $\rho(X) = 1$ (but $\rho(Y)$ may be equal to 1 or 2).*

i) *If X is a del Pezzo surface of degree 1, then any birational map $X \dashrightarrow Y$ is an isomorphism. In particular,*
$$\mathrm{Aut}(X) = \mathrm{Aut}\big(X(\mathbb{R})\big) = \mathrm{Bir}(X)\ .$$

ii) *If X is a del Pezzo surface of degree 2, X is birational to Y if and only X is isomorphic to Y. Moreover, all the base-points of the elements of $\mathrm{Bir}(X)$ are real, and*
$$\mathrm{Aut}(X) = \mathrm{Aut}\big(X(\mathbb{R})\big) \subsetneq \mathrm{Bir}(X)\ .$$

Proof. Assume the existence of a birational map $\psi\colon X \dashrightarrow Y$. If ψ is not an isomorphism, we decompose ψ into elementary links
$$X = X_0 \overset{\psi_1}{\dashrightarrow} X_1 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{n-1}}{\dashrightarrow} X_{n-1} \overset{\psi_n}{\dashrightarrow} X_n = Y$$
as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that for any link $\psi_i\colon X_{i-1} \dashrightarrow X_i$, X_{i-1} and X_i are isomorphic del Pezzo surfaces of degree 2, and that ψ_i is equal to $\beta\eta\alpha\eta^{-1}$, where η is the blow-up $X' \to X_{i-1}$ of a real point of X_{i-1}, X' is a del Pezzo surface of degree 1, $\alpha \in \mathrm{Aut}(X')$ is the Bertini involution of the surface, and $\beta\colon X_{i+1} \to X_i$ is an isomorphism.

Therefore, Y is isomorphic to X. Moreover, if X has degree 1, ψ is an isomorphism. If X has degree 2, ψ is decomposed into conjugates of Bertini involutions, so each of its base-points is real. This proves that if $\psi \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ then $\psi \in \mathrm{Aut}(X)$. Furthermore, conjugates of Bertini involutions belong to $\mathrm{Bir}(X)$ but not to $\mathrm{Aut}(X) = \mathrm{Aut}\big(X(\mathbb{R})\big)$. □

COROLLARY 11. *Let X_0 be a minimal non-rational geometrically rational real surface with $\rho(X_0) = 1$, and let $\eta\colon X \to X_0$ be a birational morphism.*

Then, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is finite.

Proof. Without changing the isomorphism class of $X(\mathbb{R})$ we may assume that η is the blow-up of only real points (which may belong to X_0 as proper or infinitely near points). Since any base-point of any element of $\mathrm{Bir}(X_0)$ is real (Proposition 10), the same is true for any element of $\mathrm{Bir}(X)$. In particular, $\mathrm{Aut}\big(X(\mathbb{R})\big) = \mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X) \cong \mathbb{Z}^n$, where $n = \rho(X) \geqslant 1$. This action gives rise to an homomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{GL}(n, \mathbb{Z})$. Let us prove that θ is injective. Indeed, if $\alpha \in \mathrm{Ker}(\theta)$, then α is conjugate by η to an element of $\alpha_0 \in \mathrm{Aut}(X_0)$ which acts trivially on $\mathrm{Pic}(X_0)$. Writing S_0 the complex surface obtaining by forgetting the real structure of X_0, S_0 is the blow-up of 7 or 8 points in general position of $\mathbb{P}^2_{\mathbb{C}}$. Thus $\alpha_0 \in \mathrm{Aut}(X_0) \subset \mathrm{Aut}(S_0)$ is the lift of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ which fixes 7 or 8 points, no 3 collinear, hence is the identity.

The morphism θ is injective, and this shows that $\mathrm{Aut}\big(X(\mathbb{R})\big) = \mathrm{Aut}(X)$ is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then $\mathrm{Bir}(X_0) = \mathrm{Aut}(X_0)$ (by Proposition 10). Since $\mathrm{Aut}(X_0)$ is finite, $\mathrm{Aut}\big(X(\mathbb{R})\big) \subset \mathrm{Bir}(X)$ is also finite. □

4. Minimal and exceptional conic bundles

Definition 12. If (X, π) is a real conic bundle, $I(X, \pi) \subset \mathbb{P}^1(\mathbb{R})$ denotes the image by π of the set $X(\mathbb{R})$ of real points of X.

The set $I(X, \pi)$ is the union of a finite number of intervals (which may be $\emptyset$ or $\mathbb{P}^1(\mathbb{R})$), and it is well-known that it determines the birational class of (X, π). We prove that $I(X, \pi)$ also determines the equivalence class of $(X(\mathbb{R}), \pi)$ among the minimal conic bundles, and give the proof of Theorem 4 in the case of conic bundles (Corollary 20).

Lemma and Definition 13. *Let (X, π) be a real minimal conic bundle. The following conditions are equivalent:*

i) *There exists a section s such that s and $\bar{s}$ do not intersect.*
ii) *There exists a section s such that $s^2 = -r$, where $2r$ is the number of singular fibres.*

If any of these conditions occur, we say that (X, π) is exceptional.

Proof. Let s be a section satisfying one of the two conditions. Denote by (S, π) the complex conic bundle obtained by forgetting the real structure of (X, π), and by $\eta : X \to \mathbb{F}_m$ the birational map which contracts in any singular fibre of π the irreducible component which does not intersect s. If s satisfies condition *i*), $\eta(\bar{s})$ and $\eta(s)$ are two sections of $\mathbb{F}_m$ which do not intersect, so they have self-intersections $-m$ and m. This means that $s^2 = \bar{s}^2 = -m$ and that the number of singular fibres is $2m$, and implies *ii*). Conversely, if s satisfies *ii*), $\eta(s)$ and $\eta(\bar{s})$ are sections of $\mathbb{F}_m$ of self-intersection $-r$ and r. If these two sections are distinct, they do not intersect, which means that s and $\bar{s}$ do not intersect. If $\eta(s) = \eta(\bar{s})$, we have $r = 0$, and $X = (\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \sigma)$ for a certain anti-holomorphic involution σ. We may thus choose another section s' of self-intersection 0 which is imaginary. □

Remark 14. The definition of exceptional conic bundles was introduced in [DI06] and [Bla09b] for *complex* conic bundles endowed with an *holomorphic* involution. If (S, π) is an exceptional complex conic bundle with at least 4 singular fibres, $\mathrm{Aut}(S, \pi) = \mathrm{Aut}(S)$ is a maximal algebraic subgroup of $\mathrm{Bir}(S)$ [Bla09b].

Lemma 15. *Let (Y, π_Y) be a minimal real conic bundle such that π_Y has at least one singular fibre. There exists an* exceptional *real conic bundle (X, π_X) and an isomorphism $\psi\colon Y(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_X \circ \psi = \pi_Y$.*

Remark 16. The result is false without the assumption on the number of singular fibres. Consider for example $Y = \mathbb{F}_3(\mathbb{R})$, whose real part is homeomorphic to the Klein bottle. Indeed, any exceptional conic bundle with no singular fibres is a real form of $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \mathrm{pr}_1)$, and thus has a real part either empty or homeomorphic to the torus $S^1 \times S^1$.

Before proving Lemma 15, we associate to any given exceptional conic bundle X an explicit circle bundle isomorphic to it. The following improves [Sil89, Cor.VI.3.1] where the model is only assumed birational to X.

Lemma 17. *Let (X, π) be an exceptional real conic bundle. Then, there exists an affine real variety $A \subset X$ isomorphic to the affine surface of $\mathbb{R}^3$ given by*

$$y^2 + z^2 = Q(x),$$

where Q is a real polynomial with only simple roots, all real. Moreover, $\pi|_A \colon A \to \mathbb{P}^1_{\mathbb{R}}$ is the projection $(x, y, z) \mapsto (x : 1)$, and $I(X, \pi)$ is the closure of $\{(x : 1) \in \mathbb{P}^1_{\mathbb{R}} \mid Q(x) \geqslant 0\}$.

Furthermore, if $f = \pi^{-1}((1 : 0)) \subset X$ is a nonsingular fibre, the singular fibres of π are those of the points $\{(x : 1) \mid Q(x) = 0\}$ and the inclusion $A \to X$ is an isomorphism $A(\mathbb{R}) \to (X \backslash f)(\mathbb{R})$. In particular, if $(1 : 0) \notin I(X, \pi)$, the inclusion yields an isomorphism $A(\mathbb{R}) \to X(\mathbb{R})$.

Proof. Denote by $2r$ the number of singular fibres of π (which is even, see Lemma 13).

Assume first that $r = 0$, which implies that (X, π) is a real form of $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \mathrm{pr}_1)$, hence is isomorphic to $(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}, \mathrm{pr}_1)$ or to $(\mathbb{P}^1_{\mathbb{R}} \times (\mathbb{P}^1, \emptyset), \mathrm{pr}_1)$, see convention after Definition 6. Taking $Q(x) = 1$ or $Q(x) = -1$ gives the result.

Assume now that $r > 0$, and denote by s and $\bar{s}$ two conjugate imaginary sections of π of self-intersection $-r$. Changing π by an automorphism of $\mathbb{P}^1$, we can assume that $(1 : 0)$. The singular fibres of π are above the points $(a_1 : 1), \dots, (a_{2r} : 1)$, where the a_i are distinct real numbers. Let $J = (J_1, J_2)$ be a partition of $\{a_1, \dots, a_{2r}\}$ into two sets of r points. Let η be the birational morphism (not defined over $\mathbb{R}$) which contracts the irreducible component of $\pi^{-1}((a_i : 1))$ which intersects s if $a_i \in J_1$ and the component which intersects $\bar{s}$ if $a_i \in J_2$. Then, the images of s and $\bar{s}$ are two sections of self-intersection 0. Thus we may assume that η is a birational morphism of conic bundles $(S, \pi) \to (\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \mathrm{pr}_1)$, where S is the complex surface obtained by forgetting the real structure of X, pr_1 is the projection on the first factor, and where $\eta(s)$ and $\eta(\bar{s})$ are equal to $\mathbb{P}^1_{\mathbb{C}} \times (0 : 1)$ and $\mathbb{P}^1_{\mathbb{C}} \times (1 : 0)$.

We write $P_1(x_1, x_2) = \prod_{a \in J_1}(x_1 - a x_2)$ and $P_2(x_1, x_2) = \prod_{a \in J_2}(x_1 - a x_2)$, and denote by α and σ the self-maps of S, which are the lifts by η of the following self-maps of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$:

$$\begin{aligned}
\alpha' \colon \big((x_1 : x_2), (y_1 : y_2)\big) &\dashrightarrow \big((x_1 : x_2), (-y_2 \cdot P_1(x_1, x_2) : y_1 \cdot P_2(x_1, x_2)\big), \\
\sigma' \colon \big((x_1 : x_2), (y_1 : y_2)\big) &\dashrightarrow \big((\overline{x_1} : \overline{x_2}), (-\overline{y_2} \cdot P_1(\overline{x_1}, \overline{x_2}) : \overline{y_1} \cdot P_2(\overline{x_1}, \overline{x_2})\big).
\end{aligned}$$

The map α' is a birational involution of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, which is defined over $\mathbb{R}$, and whose base-points are precisely the $2r$ points $\{((x : 1), (0 : 1)) \mid x \in J_1\} \cup \{((x : 1), (1 : 0)) \mid x \in J_2\}$ blown-up by η. Since α' is an involution and η is the blow-up of all of its base-points, $\alpha = \eta^{-1}\alpha'\eta$ is an automorphism of S, which belongs to $\mathrm{Aut}(S, \pi)$. In consequence, σ is an anti-holomorphic involution of S.

Denote by σ_X the anti-holomorphic involution on S which gives the real structure of X. The map $\sigma_X \circ \sigma^{-1}$ belongs to $\mathrm{Aut}(S, \pi)$ and acts trivially on the basis, since σ and σ_X have the same action on the basis. Moreover, since both σ_X and σ exchange the irreducible components of each singular fibre, $\sigma_X \circ \sigma^{-1}$ preserves any curve contracted by η and is therefore the lift by η of $\beta \colon \big((x_1 : x_2), (y_1 : y_2)\big) \mapsto \big((x_1 : x_2), (\mu y_1 : y_2)\big)$ for some $\mu \in \mathbb{C}^*$. It follows that $\sigma'_X = \eta \circ \sigma_X \circ \eta^{-1} = \beta \circ \sigma'$ is the map

$$\sigma'_X \colon \big((x_1 : x_2), (y_1 : y_2)\big) \dashrightarrow \big((\overline{x_1} : \overline{x_2}), (-\mu \cdot \overline{y_2} P_1(\overline{x_1}, \overline{x_2}) : \overline{y_1} P_2(\overline{x_1}, \overline{x_2}))\big).$$

Let us write $Q(x) = -\mu P_1(x, 1) P_2(x, 1)$, denote by $B \subset \mathbb{C}^3$ the affine hypersurface of equation $y^2 + z^2 = Q(x)$, and by $\pi_B \colon B \to \mathbb{P}^1$ the map $(x, y, z) \mapsto (x : 1)$. Let $A = (B, \sigma_B)$, where

σ_B sends (x,y,z) onto $(\bar{x},\bar{y},\bar{z})$. Denote by $\theta\colon B \dashrightarrow \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ the map that sends (x,y,z) onto $\big((x:1),(y-\mathbf{i}z:P_2(x,1))\big)$ if $P_2(x,1)\neq 0$ and onto $\big((x:1),(-\mu P_1(x,1):y+\mathbf{i}z)\big)$ if $P_1(x,1)\neq 0$. Then θ is a birational morphism, and θ^{-1} sends $\big((x_1:x_2),(y_1:y_2)\big)$ on

$$\left(\frac{x_1}{x_2},\frac{1}{2}\left(\frac{y_1}{y_2}P_2(x_1,x_2)-\frac{y_2}{y_1}\mu P_1(x_1,x_2)\right),\frac{\mathbf{i}}{2}\left(\frac{y_1}{y_2}P_2(x_1,x_2)+\frac{y_2}{y_1}\mu P_1(x_1,x_2)\right)\right).$$

Observe that $\sigma'_X\theta=\sigma_B\theta$. In consequence, $\psi=\eta^{-1}\circ\theta$ is a real birational map $A\dashrightarrow X$.

Moreover, ψ is an isomorphism from B to the complement in S of the union of $\pi^{-1}((1:0))$ and the pull-back by η of $\mathbb{P}^1\times(0:1)$ and $\mathbb{P}^1\times(1:0)$. Indeed let $x_0\in\mathbb{C}$. If $x_0\in\mathbb{C}$ is such that $Q(x_0)\neq 0$, then θ restricts to an isomorphism from $\pi_B^{-1}((x_0:1))$ to $\{((x_0:1),(y_1:y_2))\in\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\mid y_1y_2\neq 0\}\cong\mathbb{C}^*$. If $Q(x_0)=0$, then $x_0\in J_1\cup J_2$, and the fibre $\pi_B^{-1}((x_0:1))$ consists of two lines of $\mathbb{C}^2$ which intersect, given by $y=\mathbf{i}z$ and $y=-\mathbf{i}z$. If $x_0\in J_1$, then the line $y+\mathbf{i}z=0$ is sent isomorphically by θ onto the fibre $\{((x_0:1),(y_1:y_2))\in\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\mid y_2\neq 0\}\cong\mathbb{C}^*$, and the line $y-\mathbf{i}z$ is contracted on the point $((x_0:1),(0:1))$. The map ψ sends thus isomorphically $\pi_B^{-1}((x_0:1))$ onto the fibre $\pi^{-1}((x_0:1))$ minus the two points corresponding to the two sections of self-intersection $-r$. The situation when $x_0\in J_2$ is similar.

The map ψ is therefore an inclusion $A\to X$ and, by construction, it satisfies all the properties stated in the lemma. ☐

Proof of Lemma 15. Take a section s of π_Y. If s intersects its conjugate $\bar{s}$ into a real point p (respectively into a pair of imaginary points q_1 and q_2), then blow-up the point p (respectively q_1 and q_2), and contract the strict transform of the fibre of the blown-up point(s). Repeating this process, we obtain a minimal real conic bundle (Z,π_Z) and a birational map $\phi\colon Y\dashrightarrow Z$ such that $\pi_Z\circ\phi=\pi_Y$ and $\phi(s)$ does not intersect its conjugate.

If all the base-points of ϕ are imaginary, we set $\psi=\phi$ and $(X,\pi_X)=(Z,\pi_Z)$. Otherwise, by induction on the number of real base-points of ϕ, it suffices to prove the existence of ψ when ϕ is an elementary link centered at only one real point.

Denote by $q\in Z$ the real point which is the base-point of ϕ^{-1}. Since π_Y has at least one singular fibre, this is also the case for π_Z, and thus $I(Z,\pi_Z)$ is not the whole $\mathbb{P}^1(\mathbb{R})$ (By Lemma 17). We may thus assume that $(1:0)\notin I(Z,\pi_Z)$, that $\pi_Z(q)=(1:1)$, and that the interval of $I(Z,\pi_Z)$ which contains $\pi_Z(q)$ is $\{(x:1)\in\mathbb{P}^1_{\mathbb{R}}\mid 0\leqslant x\leqslant a\}$ for some $a\in\mathbb{R}$, $a>1$. Take the affine surface $A\subset Z$ given by Lemma 17, which is isomorphic to $y^2+z^2=Q(x)$ for some polynomial Q. Then, $Q(0)=Q(a)=0$ and $Q(x)>0$ for $0<x<a$, and we may assume that $Q(1)=1$. Denote by s the section of $\pi_Z\colon Z\to\mathbb{P}^1_{\mathbb{R}}$ given locally by $y+\mathbf{i}z=\mathbf{i}x^n$, for some positive integer n. Its conjugate is given by $y-\mathbf{i}z=-\mathbf{i}x^n$, or $y+\mathbf{i}z=Q(x)/(-\mathbf{i}x^n)$. Thus, s intersects $\bar{s}$ at some real point $p\in Z$, its image $x=\pi_Z(p)$ satisfies $Q(x)/(-\mathbf{i}x^n)=\mathbf{i}x^n$, or $Q(x)=x^{2n}$. Taking n large enough, this can only happen when $x=0$ or $x=1$. The first possibility cannot occur since a section does not pass through the singular point of a singular fibre. Thus, s intersects $\bar{s}$ at only one real point, which is q. In consequence, the strict pull-back by ϕ of s is a section of Y which intersects its conjugate at only imaginary points. This shows that $(Y(\mathbb{R}),\pi_Y)$ is isomorphic to an exceptional real conic bundle (X,π_X). ☐

Corollary 18. *Let (X,π_X) and (Y,π_Y) be two minimal real conic bundles, and assume that either π_X or π_Y has at least one singular fibre. Then, the following are equivalent:*

i) *$I(X,\pi_X)=I(Y,\pi_Y)$;*

ii) *there exists an isomorphism $\varphi\colon X(\mathbb{R})\to Y(\mathbb{R})$ such that $\pi_Y\circ\varphi=\pi_X$.*

Proof. It suffices to prove *i)* $\Rightarrow$ *ii)*. By Lemma 15, we may assume that both (X,π_X) and (Y,π_Y) are exceptional. We may now assume that the fibre over $(1:0)$ is not singular and use Lemma 17: let

$A_X \subset X$ and $B_X \subset Y$ be the affine surfaces given by the lemma, with equations $y^2+z^2 = Q_X(x)$ and $y^2+z^2 = Q_Y(x)$ respectively. Since $I(X,\pi_X) = I(Y,\pi_Y)$, $Q_Y(x) = \lambda Q_X(x)$ for some positive $\lambda \in \mathbb{R}$. The map $(x,y,z) \mapsto (x, \sqrt{\lambda}y, \sqrt{\lambda}z)$ then yields an isomorphism $(X(\mathbb{R}),\pi_X) \to (Y(\mathbb{R}),\pi_Y)$. □

The above result implies the next two corollaries. The first one strengthen a result of Comessatti [Com12] (see also [Kol97, Theorem 4.5]).

Corollary 19. *Let (X,π) and (X',π') be two real conic bundles. Assume that (X,π) and (X,π') are minimal. Then $(X(\mathbb{R}),\pi)$ and $(X'(\mathbb{R}),\pi')$ are isomorphic if and only if there exists an automorphism of $\mathbb{P}^1_{\mathbb{R}}$ that sends $I(X,\pi)$ on $I(X',\pi')$.* □

Corollary 20. *Let (X,π_X) and (Y,π_Y) be two minimal conic bundles. Then, the following are equivalent:*

i) *$(X(\mathbb{R}),\pi_X)$ and $(Y(\mathbb{R}),\pi_Y)$ are isomorphic;*
ii) *(X,π_X) is birational to (Y,π_Y) and $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.*

Proof. The implication $i) \Rightarrow ii)$ is evident. Let us prove the converse.

Since (X,π_X) is birational to (Y,π_Y) and both of them are minimal, the number of singular fibres of π_X and π_Y is the same, equal to $2r$ for some non-negative integer r.

Assume that $r=0$, which means that X is an Hirzebruch surfaces $\mathbb{F}_m$ for some m and that $Y = \mathbb{F}_n$ for some n. Since $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$, we have $m \equiv n \mod 2$. It is easy to prove that $(X(\mathbb{R}),\pi)$ and $(Y(\mathbb{R}),\pi)$ are isomorphic, by taking elementary links at two imaginary distinct fibres (see for example [Mang06, Proof of Theorem 6.1]).

When $r>0$, already the fact that (X,π_X) is birational to (Y,π_Y) implies that $(X(\mathbb{R}),\pi_X)$ is isomorphic to $(Y(\mathbb{R}),\pi_Y)$ (Proposition 18). □

5. Conic bundles on del Pezzo surfaces

In this section, we focus on surfaces admitting distinct minimal conic bundles. We will see that these surfaces are necessarily del Pezzo surfaces (Lemma 23). We begin by the description of all possible minimal real conic bundles occurring on del Pezzo surfaces.

Lemma 21. *Let V be is a subset of $\mathbb{P}^1(\mathbb{R})$, then the following are equivalent:*

i) *there exists a minimal real conic bundle (X,π) with $I(X,\pi) = V$ such that X is a del Pezzo surface;*
ii) *the set V is a union of closed intervals, and $\#V \leqslant 3$.*

Proof. The part $i) \Rightarrow ii)$ is easy. Indeed, if (X,π) is minimal, it is well-know that the number of singular fibres of π is even, denoted $2r$, and that $2r = 8-(K_X)^2$. Since $-K_X$ is ample, $K_X^2 \geqslant 1$, thus $r \leqslant 3$. The conclusion follows as $I(X,\pi)$ is the union of r closed intervals.

Let us prove the converse. If $V = \mathbb{P}^1(\mathbb{R})$ or $V = \emptyset$, we take (X,π) to be $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \mathrm{pr}_1)$, where pr_1 is the projection on the first factor, endowed with the anti-holomorphic map that sends $\big((x_1:x_2),(y_1:y_2)\big)$ onto $\big((\overline{x_1}:\overline{x_2}),(\pm\overline{y_2}:\overline{y_1})\big)$.

Now we can assume that V consists of k closed intervals $I_1,\dots,I_k$, with $1 \leqslant k \leqslant 3$. For $j = 1,\dots,3$, we denote by m_j an homogenous form of degree 2. If $j \leqslant k$, we choose that m_j vanishes at the boundary of the interval I_j, and is non-negative on I_j. If $j > k$, we choose m_j such that m_j is positive on $\mathbb{P}^1(\mathbb{R})$. In any case, we choose that $m_1 \cdot m_2 \cdot m_3$ has 6 distinct roots. We consider the real surface given by

$$X := \left\{ ((x:y:z),(a:b)) \in \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \mid x^2 m_1(a,b) + y^2 m_2(a,b) + z^2 m_3(a,b) = 0 \right\}.$$

The projection on $\mathbb{P}^2_{\mathbb{R}}$ is a double covering. A straightforward calculation shows that this covering is ramified over a smooth quartic. In consequence, X is a smooth surface, and precisely a del Pezzo surface of degree 2. Taking $\pi\colon X \to \mathbb{P}^1_{\mathbb{R}}$ as the second projection, we obtain a conic bundle (X, π) on the del Pezzo surface X such that $I(X, \pi) = V$. If $k = 3$, the conic bundle is minimal. Otherwise, we contract components in the imaginary singular fibres (corresponding to the roots of m_j for $j > k$) to obtain the result. ☐

Recall the following classical result, that will be useful throughout what follows.

Lemma 22. *Let $\pi\colon S \to \mathbb{P}^1_{\mathbb{C}}$ be a complex conic bundle, and assume that S is a del Pezzo surface, with $(K_S)^2 = 9-m \leqslant 7$. Then, there exists a birational morphism $\eta\colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is a blow-up of m points $p_1, \dots, p_m$ and which sends the fibres of π onto the lines passing through p_1. The curves of self-intersection -1 of S are*

- *the exceptional curves $\eta^{-1}(p_1), \dots, \eta^{-1}(p_m)$;*
- *the strict transforms of the lines passing through 2 of the p_i;*
- *the conics passing through 5 of the p_i;*
- *the cubics passing through 7 of the p_i and being singular at one of these.*

Proof. Denote by ε the contraction of one component in each singular fibre of π. Then, ε is a birational morphism of conic bundles – not defined over $\mathbb{R}$ – from S to a del Pezzo surface which is also an Hirzebruch surface. Changing the contracted components, we may assume that ε is a map $S \to \mathbb{F}_1$. Contracting the exceptional section onto a point $p_1 \in \mathbb{P}^2_{\mathbb{C}}$, we get a birational map $\eta\colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is the blow-up of m points $p_1, \dots, p_m$ of $\mathbb{P}^2_{\mathbb{C}}$, and which sends the fibres of π_1 onto the lines passing through p_1. The description of the (-1)-curves is well-known and may be found for example in [Dem76]. ☐

Lemma 23. *Let $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ be a minimal real conic bundle. Then, the following conditions are equivalent:*

i) *There exist a real conic bundle $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$, such that π_1 and π_2 induce distinct foliations on $X(\mathbb{C})$.*
ii) *Either X is isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, or X is a del Pezzo surface of degree 2 or 4.*

Moreover, if the conditions are satisfied, then the following occur:

a) *The map π_2 is unique, up to an automorphism of $\mathbb{P}^1_{\mathbb{R}}$.*
b) *There exist $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\pi_1\alpha = \beta\pi_2$. Moreover, if X is a del Pezzo surface of degree 2, α may be chosen to be the Geiser involution.*
c) *Denoting by $f_1, f_2 \subset \mathrm{Pic}(X)$ the divisors of the general fibre of respectively π_1 and π_2, we have $f_1 + f_2 = -cK_X$ where $c = 4/(K_X)^2 \in \mathbb{N} \cdot \frac{1}{2}$.*

Proof. We now prove that *i*) implies *ii*), *a*), and *c*). Assuming the existence of π_2, we denote by f_i the divisor of the fibre of π_i for $i = 1, 2$. We have $(f_1)^2 = (f_2)^2 = 0$ and by adjunction formula $f_1 \cdot K_X = f_2 \cdot K_X = -2$, where K_X is the canonical divisor. Let us write $d = (K_X)^2$.

Since (X, π_1) is minimal, $\mathrm{Pic}(X)$ has rank 2, hence $f_1 = aK_X + bf_2$, for some $a, b \in \mathbb{Q}$. Computing $(f_1)^2$ and $f_1 \cdot K_X$ we find respectively $0 = a^2d - 4ab = a(ad - 4b)$ and $-2 = ad - 2b$. If $a = 0$, we find $f_1 = f_2$, a contradiction. Thus, $4b = ad$ and $2b = ad + 2$, which yields $b = -1$ and $ad = -4$, so $f_1 + f_2 = -4/d \cdot K_X$. This shows that f_2 is uniquely determined by f_1, which is the assertion *a*).

Denote as usual by S the complex surface associated to X. Let $C \in \mathrm{Pic}(S)$ be an effective divisor, with reduced support, and let us prove that $C \cdot (f_1 + f_2) > 0$. Since C is effective, $C \cdot f_1 \geqslant 0$ and $C \cdot f_2 \geqslant 0$. If $C \cdot f_1 = 0$, then the support of C is contained in one fibre of π_1. If C is a multiple of f_1, then $C \cdot f_2 > 0$; otherwise, C is a multiple of a (-1)-curve contained in a singular fibre of f_1, and the orbit of C by the anti-holomorphic involution is equal to a multiple of f_1, whence $C \cdot f_2 > 0$.

Since $f_1 + f_2$ is ample, and $f_1 + f_2 = -4/d \cdot K_X$ either K_X or $-K_X$ is ample. The surface X being geometrically rational, the former cannot occur, whence $d > 0$.

If S is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, the existence of π_1, π_2 shows that X is isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. Otherwise, K_X is not a multiple in $\mathrm{Pic}(X_{\mathbb{C}})$ and thus d is equal to 1, 2 or 4. The number of singular fibres being even and equal to $8 - (K_X)^2$, the only possibilities are then 2 and 4.

We have proved that *i*) implies *ii*), *a*), and *c*).

Assume now that $X = (S, \sigma)$ is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ or a del Pezzo surface of degree 2 or 4. We construct an automorphism α of X which does not belong to $\mathrm{Aut}(X, \pi)$. Then, by taking $\pi_2 = \pi_1 \alpha$ we get assertion *i*). Taking into account the unicity of π_2, we get *b*).

If X is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, the two conic bundles are given by the projections on each factor, and we can get for α the swap of the factors.

If X is a del Pezzo surface of degree 2, the anti-canonical map $\zeta \colon X \to \mathbb{P}^2$ is a double covering ramified along a smooth quartic, cf. e.g. [Dem76]. Let α be the involution associated to the double covering – α is classically called the *Geiser involution*. It fixes a smooth quartic, hence cannot preserve any conic bundle.

The remaining case is when X is a del Pezzo surface of degree 4. By Lemma 22, there is a birational map $\eta \colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is the blow-up of five points $p_1, \ldots, p_5$ of $\mathbb{P}^2_{\mathbb{C}}$, no three being collinear and which sends the fibres of π_1 on the lines passing through p_1 . There are 16 exceptional curves (curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ of self-intersection (-1)) on S:

- $E_1 = \eta^{-1}(p_1), ..., E_5 = \eta^{-1}(p_5)$ (5 curves);
- the strict transforms of the lines passing through p_i and p_j, denoted by L_{ij} (10 curves);
- the strict transform of the conic passing through the five points, denoted by Γ.

Note that the four singular fibres of π_1 are $E_i \cup L_{ij}$, $i = 2, \ldots, 5$, and that σ exchanges thus E_i and L_{ij} for $i = 1, \ldots, 5$. The intersection form being preserved, this implies that σ acts on the 16 exceptional curves as

$$(E_2\ L_{12})(E_3\ L_{13})(E_4\ L_{14})(E_5\ L_{15})(E_1\ \Gamma)(L_{23}\ L_{45})(L_{24}\ L_{35})(L_{25}\ L_{34}).$$

After a linear change of coordinates, we may assume that $p_1 = (1:1:1)$, $p_2 = (1:0:0)$, $p_3 = (0:1:0)$, $p_4 = (0:0:1)$ and $p_5 = (a:b:c)$ for some $a, b, c \in \mathbb{C}^*$. Denote by ϕ the birational involution $(x:y:z) \dashrightarrow (ayz : bxz : cxy)$ of $\mathbb{P}^2_{\mathbb{C}}$. Since the base-points of ϕ are p_2, p_3, p_4 and since ϕ exchanges p_1 and p_5, the map $\alpha = \eta^{-1}\phi\eta$ is an automorphism of S. Its action on the 16 exceptional curves is given by the permutation

$$(L_{23}\ E_4)(L_{24}\ E_3)(L_{34}\ E_2)(L_{12}\ L_{25})(L_{13}\ L_{35})(L_{14}\ L_{45})(\Gamma\ L_{15})(E_1\ E_5).$$

Observe that the actions of α and σ on the set of 16 exceptional curves commute. This means that $\alpha\sigma\alpha^{-1}\sigma^{-1}$ is an holomorphic automorphism of S which preserves any of the 16 curves. It is the lift of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ that fixes the 5 points $p_1, \ldots, p_5$ and hence is the identity. Consequently, α and σ commute, so $\alpha \in \mathrm{Aut}(X)$. Since ϕ sends a general line passing though p_1 onto a conic passing through $p_2, \ldots, p_5$, α belongs to $\mathrm{Aut}(X) \setminus \mathrm{Aut}(X, \pi)$. □

COROLLARY 24. *Let X be a minimal geometrically rational real surface, which is not rational. Then, the following are equivalent:*

i) $\#X(\mathbb{R}) = 2$ *or* $\#X(\mathbb{R}) = 3$*;*

ii) *There exists a geometrically rational real surface $Y(\mathbb{R})$ isomorphic to $X(\mathbb{R})$, and such that Y admits two minimal conic bundles $\pi_1 \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2 \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $Y(\mathbb{C})$.*

Proof. [$ii) \Rightarrow i)$] By Lemma 23, Y is then a del Pezzo surface, which has degree 2 or 4 since Y is not rational. This implies that $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$ by Proposition 9.

[$i) \Rightarrow ii)$]. According to Theorem 7 and Proposition 9, (1) implies the existence of a minimal real conic bundle structure $\pi_X \colon X \to \mathbb{P}^1_{\mathbb{R}}$ with 4 or 6 singular fibres. This condition is equivalent to the fact that $I(X, \pi_X)$ is the union of 2 or 3 intervals. According to Lemma 21, there exists a minimal real conic bundle (Y, π_1) such that Y is a del Pezzo surface and $I(Y, \pi_1) = I(X, \pi_X)$. Corollary 19 shows that $(X(\mathbb{R}), \pi_X)$ and (Y, π_1) are isomorphic. Moreover Lemma 23 yields the existence of π_2. □

6. Equivalence of surfaces versus equivalence of conic bundles

This section is devoted to the proof of Theorem 4. From Theorem 7 and Proposition 10, it remains to solve the conic bundle case, which is done in Theorem 27. First of all, we correct an existing inaccuracy in the literature; in [Kol97, Exercice 5.8] or [Sil89, VI.3.5], it is asserted that all minimal real conic bundles with four singular fibres belong to a unique birational equivalence class. To the contrary, the following general result, which includes the case with four singular fibres, occurs:

Theorem 25. *Let $\pi_X \colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_Y \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ be two real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

i) *The two real surfaces X and Y are birational.*

ii) *The two real conic bundles (X, π_X) and (Y, π_Y) are birational.*

iii) *There exists an automorphism of $\mathbb{P}^1$ which sends $I(X, \pi_X)$ onto $I(Y, \pi_Y)$.*

Moreover, if the number of singular fibres of π_X is at least 8, then $\mathrm{Bir}(X) = \mathrm{Bir}(X, \pi_X)$.

Remark 26. It is well-known that this result is false when X and Y are rational. Indeed, consider $(X, \pi_X) = (\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}, \mathrm{pr}_1)$ and (Y, π_Y) be a real conic bundle with two singular fibres. The surfaces X and Y are birational, but the conic bundles (X, π_X) and (Y, π_Y) are not.

Proof. The equivalence $(iii) \Leftrightarrow (ii)$ was proved in Corollary 19 and $(ii) \Rightarrow (i)$ is evident.

Let us prove now $(i) \Leftrightarrow (ii)$. We may assume that (X, π_X) and (Y, π_Y) are minimal and that X is not rational, hence π_X has at least 4 singular fibres. Let $\psi \colon X \dashrightarrow Y$ a birational map, and decompose ψ into elementary links: $\psi = \psi_n \circ \cdots \circ \psi_1$ (see [Isk96, Theorem 2.5]). Consider $\psi_1 \colon X \dashrightarrow X_1$ the first link, which may be of type II or IV only by [Isk96, Theorem 2.6]. If ψ_1 is of type II, then ψ_1 is a birational map of conic bundles $(X, \pi_X) \dashrightarrow (X_1, \pi_1)$ for some conic bundle structure $\pi_1 \colon X_1 \to \mathbb{P}^1$. If ψ_1 is of type IV, then ψ_1 is an isomorphism $X \to X_1$ and the link is precisely a change of conic bundle structure from π_X to $\pi_1 \colon X_1 \to \mathbb{P}^1$, which induce distinct foliations on $X(\mathbb{R})$. Applying Lemma 23, X is a del Pezzo surfaces of degree 2 or 4, and there exist automorphisms $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\pi_1 \psi_1 \alpha = \beta \pi_2$, whence (X, π) is isomorphic to (X_1, π_1). We proceed by induction on the number of elementary links to conclude that (X, π_X) is birational to (Y, π_Y). Moreover, if π_X has at least 8 singular fibres, then no link of type IV may occur, so ψ is a birational map of conic bundles $(X, \pi_X) \dashrightarrow (Y, \pi_Y)$. □

When the conic bundles are minimal, we can strengthen Theorem 25 to get an isomorphism between the real parts.

Theorem 27. *Let $\pi_X \colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_Y \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ be two* minimal *real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

i) *X and Y are birational.*

ii) *$X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.*

iii) *$(X(\mathbb{R}),\pi_X)$ and $(Y(\mathbb{R}),\pi_Y)$ are isomorphic.*

Proof. The implications iii) $\Rightarrow$ ii) $\Rightarrow$ i) being evident, it suffices to prove i) $\Rightarrow$ iii). Since X and Y are not rational, both π_X and π_Y have at least one singular fibre. Applying Lemma 15, we may assume that both (X,π_X) and (Y,π_Y) are exceptional real conic bundles. Then, since (X,π_X) and (Y,π_Y) are birational (Theorem 25), we may assume that $I(X,\pi_X)=I(Y,\pi_Y)$, up to an automorphism of $\mathbb{P}^1_\mathbb{R}$. Then Corollary 19 shows that (X,π_X) is isomorphic to (Y,π_Y). $\square$

We are now able to prove Theorem 4 concerning minimal surfaces.

Proof of Theorem 4. Let X and Y be two minimal geometrically rational real surfaces, and assume that either X or Y is non-rational.

If $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic, it is clear that X and Y are birational. Let us prove the converse.

Theorem 7 lists all the possibilities for X. If $\rho(X)=1$ or $\rho(Y)=1$, Proposition 10 shows that X is isomorphic to Y. Otherwise, since neither X nor Y is rational, there exist minimal conic bundle structures on X and on Y. From Theorem 27, we conclude that $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$. $\square$

To go further with non-minimal surfaces, we need to know when the group $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is very transitive for X minimal. This is done in the next sections.

7. Very transitive actions

Thanks to the work done in Section 4, it is easy to apply the techniques of [HM09a] to prove that $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is fiberwise very transitive on a real conic bundle. After describing the transitivity of $\operatorname{Aut}\big(X(\mathbb{R})\big)$ on the tangent space of a general point, we set the main result of that section: $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component when X is minimal and admits two conic bundle structures (Proposition 33). We end the section by giving a characterisation of surfaces X for which $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is able to mix the connected components of $X(\mathbb{R})$.

Lemma 28. *Let (X,π) be a minimal real conic bundle over $\mathbb{P}^1_\mathbb{R}$ with at least one singular fibre. Let $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ be two n-tuples of distinct points of $X(\mathbb{R})$, and let $(b_1,\dots,b_m)$ be m points of $I(X,\pi)$. Assume that $\pi(p_i)=\pi(q_i)$ for each i, that $\pi(p_i)\neq\pi(p_j)$ for $i\neq j$ and that $\pi(p_i)\neq b_j$ for any i and any j.*

Then, there exists $\alpha\in\operatorname{Aut}\big(X(\mathbb{R})\big)$ such that $\alpha(p_i)=q_i$ for every i, $\pi\alpha=\pi$ and $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i.

Remark 29. The same result holds for minimal real conic bundles with no singular fibre, see [BH07, 5.4]. The following proof uses *twisting maps*, see below, which were introduced in [HM09a] to prove that the action of the group of automorphisms $\operatorname{Aut}(S^2)$ on the quadric sphere $S^2:=\{(x:y:z)\in\mathbb{R}^3\mid x^2+y^2+z^2=1\}$ is very transitive.

Proof. By Lemma 15, we may assume that (X,π) is exceptional. Moreover, Lemma 17 yields the existence of an affine real surface $A\subset X$ isomorphic to the hypersurface of $\mathbb{R}^3$ given by

$$y^2+z^2=-\prod_{i=1}^{2r}(x-a_i),$$

for some $a_1,\dots,a_{2r}\in\mathbb{R}$ with $a_1<a_2<\cdots<a_{2r}$, where $\pi|_A$ corresponds to the projection $(x,y,z)\mapsto x$, and where the inclusion $A\subset X$ induces an isomorphism $A(\mathbb{R})\to X(\mathbb{R})$.

For $i = 1, \dots, n$, let us denote by (x_i, y_i, z_i) the coordinates of p_i in $A \subset \mathbb{R}^3$ and by (u_i, v_i, w_i) the ones of q_i. From hypothesis, we have $x_i = u_i$ for all i, thus we get $y_i^2 + z_i^2 = v_i^2 + w_i^2$ for all i. Let $\Phi_i \in \mathrm{SO}_2(\mathbb{R})$ be the rotation sending (x_i, y_i) to (u_i, v_i). Then by [HM09a, Lemma 2.2], there exists an algebraic map $\Phi\colon [a_1, a_{2r}] \to \mathrm{SO}_2(\mathbb{R})$ such that $\Phi(x_i) = \Phi_i$ for $i = 1, \dots, n$ and $\Phi(b_i)$ is the identity for $i = 1, \dots, m$. Let us recall the proof; since $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the unit circle $S^1 := \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 + y^2 = z^2\}$, it suffices to prove the statement for S^1 instead of $\mathrm{SO}_2(\mathbb{R})$. Let Φ_0 be a point of S^1 distinct from $\Phi_1, \dots, \Phi_n$ and from the identity. Since $S^1 \setminus \{\Phi_0\}$ is isomorphic to $\mathbb{R}$, it suffices, finally, to prove the statement for $\mathbb{R}$ instead of $\mathrm{SO}_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation.

Then the map defined by $\alpha\colon (x, y, z) \mapsto \big(x, (y, z) \cdot \Phi(x)\big)$ induces an automorphism $A(\mathbb{R}) \to A(\mathbb{R})$ called the *twisting map* of π associated to Φ. Moreover, $\alpha(p_i) = q_i$, for all i, $\pi\alpha = \pi$, $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i, and π induces an automorphism $X(\mathbb{R}) \to X(\mathbb{R})$. □

Lemma 30. *Let (X, π) be a minimal real conic bundle over $\mathbb{P}^1_{\mathbb{R}}$ with at least one singular fibre. Let $p \in X$ be a real point in a nonsingular fibre of π, and let $\Sigma \subset I(X, \pi)$ be a finite subset, with $\pi(p) \in \Sigma$. Denote by $\eta\colon Y \to X$ the blow-up of p, and by $E \subset Y$ the exceptional curve. Let $q \in E$ the point corresponding to the direction of the fibre of π passing through p.*

Then, the lift of the group

$$G = \Big\{\alpha \in \mathrm{Aut}\big(X(\mathbb{R})\big), \pi\alpha = \pi \;\Big|\; \alpha|_{\pi^{-1}(\Sigma)} \text{ is the identity}\Big\}$$

by η is a subgroup $\eta^{-1}G\eta \subset \mathrm{Aut}\big(Y(\mathbb{R})\big)$ which fixes the point q, and acts transitively on $E \backslash q \cong \mathbb{A}^1_{\mathbb{R}}$.

Proof. Since G acts identically on $\pi^{-1}(\Sigma)$, it fixes p, and therefore lifts to $H = \eta^{-1}G\eta \subset \mathrm{Aut}(Y(\mathbb{R}), \pi\eta)$, which preserves E. Moreover, G preserves the fibre of π passing through p, so H preserves its strict transform, which intersects transversally E at q, so q is fixed.

Let us prove now that the action of $\eta^{-1}G\eta$ on $E \backslash q$ is transitive. By Lemma 15, we may assume that (X, π) is exceptional. Then, we take an affine surface $A \subset X$, isomorphic to the hypersurface $y^2 + z^2 = P(x)$ of $\mathbb{R}^3$ for some polynomial P, such that $A|_\pi$ is the projection $\mathrm{pr}_x\colon (x, y, z) \mapsto x$ and the inclusion $A \subset X$ gives an isomorphism $A(\mathbb{R}) \to X(\mathbb{R})$ (Lemma 17). Let us write $(x_0, y_0, z_0) \in \mathbb{R}^3$ the coordinates of p. Since x is on a nonsingular fibre of π, then $P(x_0) > 0$. Up to an affine automorphism of $\mathbb{R}^3$, and up to multiplication of P by some constant, we may assume that $x_0 = 0$, $P(0) = 1$, $y_0 = 0$, and $z_0 = 0$.

To any real polynomial $\lambda \in \mathbb{R}[X]$, we associate the matrix

$$\begin{pmatrix} \alpha(X) & \beta(X) \\ -\beta(X) & \alpha(X) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}(X)) ,$$

where $\alpha = \frac{1-\lambda^2}{1+\lambda^2} \in \mathbb{R}(X)$ and $\beta = \frac{2\lambda}{1+\lambda^2} \in \mathbb{R}(X)$. And corresponding to this matrix, we associate the map

$$\psi_\lambda\colon (x, y, z) \mapsto (x, \alpha(x) \cdot y - \beta(x) \cdot z, \beta(x) \cdot y + \alpha(x) \cdot z),$$

which belongs to $\mathrm{Aut}(A(\mathbb{R}), \mathrm{pr}_x)$. To impose that ψ_λ is the identity on $(\mathrm{pr}_x)^{-1}(\Sigma)$ is the same to ask that $\lambda(x) = 0$ for each $(x : 1) \in \Sigma \subset \mathbb{P}^1(\mathbb{R})$, and in particular for $x = 0$.

Denote by $\mathcal{O} = \mathbb{R}[x, y, z]/(y^2 + z^2 - P(x))$ the ring of functions of A, by $\mathfrak{p} \subset \mathcal{O}$ the ideal of functions vanishing at p, by $\mathcal{O}_\mathfrak{p}$ the localisation, and by $\mathfrak{m} \subset \mathcal{O}_\mathfrak{p}$ the maximal ideal of $\mathcal{O}_\mathfrak{p}$. Then, the cotangent ring $T^*_{p,A}$ of p in A is equal to $\mathfrak{m}/\mathfrak{m}^2$, and is generated by the images $[x]$, $[y]$, $[z - 1]$ of $x, y, z - 1 \in \mathbb{R}[x, y, z]$. Since $P(0) = 1$, we may write $P(x) = 1 + xQ(x)$, for some real polynomial Q. We compute

$$[0] = [y^2 + z^2 - P(x)] = [y^2 + (z - 1)^2 + 2(z - 1) - xQ(x)] = [2(z - 1) - xQ(0)] \in \mathfrak{m}/\mathfrak{m}^2 .$$

We see that $[z-1]=[xQ(0)/2]$, thus $\mathfrak{m}/\mathfrak{m}^2$ is generated by $[x]$ and $[y]$ as a $\mathbb{R}$-module. Since $\lambda(0)=0$, we can write $\lambda(x)=x\mu(x)$, for some real polynomial μ. The linear action of ψ_λ on the cotangent space $T^*_{p,A}$ fixes $[x]$ and sends $[y]$ onto

$$\begin{array}{rcl}[\alpha(x)\cdot y-\beta(x)\cdot z] & = & \left[\frac{(1-\lambda(x)^2)y-2\lambda(x)z}{\lambda(x)^2+1}\right]=[y-2\lambda(x)(1+xQ(0)/2)]\\ & = & [y-2\mu(0)x]\ .\end{array}$$

It suffices to change the derivative of λ at 0 (which is equal to $\mu(0)$), which may be any real number. Therefore, the action of G on the projectivisation of $T^*_{p,A}$, fixes a point (corresponding to $[x]$) but acts transitively on the complement of this point. Since E corresponds to the projectivisation of $T_{p,A}$, G acts transitively on $E\backslash q$. $\square$

Lemma 31. *Let X be a real projective surface endowed with two minimal conic bundles $\pi_1\colon X\to\mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X\to\mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$. There exists a real projective surface X' such that $X'(\mathbb{R})$ and $X(\mathbb{R})$ are isomorphic, X is endowed with two minimal conic bundles $\pi_1'\colon X\to\mathbb{P}^1_{\mathbb{R}}$ and $\pi_2'\colon X\to\mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X'(\mathbb{C})$ and the following condition holds:*

($\star$) Let F_j be a real fibre of π_j', $j=1,2$. If $F_1(\mathbb{R})\cap F_2(\mathbb{R})\neq\emptyset$, then at most one of the curves F_j can be singular.

Remark 32. It is possible that the condition ($\star$) above does not hold for (X,π_1,π_2), taking for example for X the del Pezzo surface of degree 2 given in the proof of Lemma 21 for $k=3$:

$$X:=\left\{((x:y:z),(a:b))\in\mathbb{P}^2_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}}\mid x^2m_1(a,b)+y^2m_2(a,b)+z^2m_3(a,b)=0\right\}.$$

The map $\pi_1\colon X\to\mathbb{P}^1_{\mathbb{R}}$ is given by the second projection, and the 6 singular points of its singular fibres correspond to only three points of $\mathbb{P}^2_{\mathbb{R}}$, namely $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. This shows that the Geiser involution preserves the set of the 6 points, so each of these points is the singular point of a singular fibre of π_2.

Proof. Suppose that the condition ($\star$) does not hold for (X,π_1,π_2) (otherwise, the result is obvious). Then F_i is the union of two (-1)-curves $E_{i,1}$ and $E_{i,2}$, intersecting transversally at some point p_i. Since p_i is the only real point of F_i, we have $p_1=p_2$. Hence, $F_1\cdot E_{2,i}\geqslant 2$ for $i=1,2$, which implies that $F_1\cdot F_2\geqslant 4$. According to Lemma 23, X is a del Pezzo surface of degree 2 or 4, and we have $F_1+F_2=-cK_X$ with $c=4/(K_X)^2$. Computing $16/(K_X)^2=(F_1+F_2)^2=2F_1\cdot F_2\geqslant 8$, we see that $(K_X)^2=2$.

Let $q\in X(\mathbb{R})$ be a real point, let $\eta\colon Y\to X$ be the blow-up of q, and let $\varepsilon\colon Y\to X'$ be the contraction of the strict transform of the fibre of π_1 passing through q, and write $\psi\colon X\dashrightarrow X'$ the composition $\psi=\varepsilon\circ\eta^{-1}$. We prove now that if q is general enough, then X' is a del Pezzo surface of degree 2, and $\pi_1'=\pi_1\circ\psi$ and $\pi_2'=\sigma_{X'}\circ\pi_1'$ (where $\sigma_{X'}\in\mathrm{Aut}(X')$ is the Geiser involution of X') satisfy the condition ($\star$).

Firstly, it is well-known that blowing-up a general point of a del Pezzo surface of degree 2 yields a del Pezzo surface of degree 1 (it suffices that q does not belong to any of the (-1)-curves of X and to the ramification curve of the double covering $X\to\mathbb{P}^2$); then a contraction from a del Pezzo surface of degree 1 yields a del Pezzo surface of degree 2.

Secondly, we denote respectively by S,S',T the complex surfaces obtained by forgetting the real structures of X,X',Y and study condition ($\star$) by working now in the Picard groups of these surfaces, identifying a curve with its equivalence class. We choose a (-1)-curve (not defined over $\mathbb{R}$) in any of the six singular fibres of π_1', and denote these by $C_1,\dots,C_6$, and denote by $p_1,\dots,p_6$ the singular points of the six singular fibres, so that $p_i\in C_i$. Condition ($\star$) amounts to prove that $D_i:=\psi^{-1}\sigma_{X'}\psi(C_i)\subset S$ does not pass through p_j for any i and any j. Fixing i and j, we will see that this yields a curve of X where q should not lie. Note that the action of the Geiser involution $\sigma_{X'} \in \mathrm{Aut}(X') \subset \mathrm{Aut}(S')$ on $\mathrm{Pic}(S')$ is given $\sigma_{X'}(D) = (D \cdot K_{X'})K_{X'} - D$ (follows directly from the fact that the invariant part of $\mathrm{Pic}(S')$ has rank 1). In consequence, the (-1)-curve $D_i' := \sigma_{X'}\psi(C_i) \subset S'$ is equal to $-K_{X'} - \psi(C_i)$, and thus $\varepsilon^*(D_i) = -\varepsilon^*(K_{X'}) - \eta^*(C_i)$. Writing E_q the (-1)-curve contracted by η, and f a general fibre of π_1, the (-1)-curve contracted by ε is equivalent to $\eta^*(f) - E_q$. We have $K_Y = \eta^*(K_X) + E_q = \varepsilon^*(K_{X'}) + \eta^*(f) - E_q$ in $\mathrm{Pic}(Y)$. This implies that

$$\eta^*(D_i) = \varepsilon^*(D_i') = -\eta^*(K_X) + \eta^*(f) - \eta^*(C_i) - 2E_q \in \mathrm{Pic}(Y).$$

This means that D_i is a curve with a double point at q, is equivalent to $-K_X+f-C_i \in \mathrm{Pic}(S)$ and has self-intersection 3. Moreover, the linear system Λ_i of curves in $\mathrm{Pic}(S)$ equivalent to $-K_X+f-C_i$ has dimension 3. Note that Λ_i does not depend on q, but only on i. Denote by $\Lambda_{i,j} \subset \Lambda_i$ the sublinear system of curves of Λ_i passing through p_j. This system has dimension 2; after blowing-up p_j, the system $\Lambda_{i,j}$ yields a ramified double covering of $\mathbb{P}^2$. If D_i passes through p_j, then D_i corresponds to a member of $\Lambda_{i,j}$, singular at q and this implies that q belongs to the ramified locus of the double covering induced by $\Lambda_{i,j}$. It suffices to choose q outside of all these locus to obtain condition $(\star)$. $\square$

We now use the above lemmas to show that the action of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component when X is a surface with two conic bundles.

PROPOSITION 33. *Let X be a real projective surface, which admits two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$.*

Let $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ be two n-tuples of distinct points of $X(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i. Then, there exists an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ which sends p_i on q_i for each i, and which sends each connected component of $X(\mathbb{R})$ on itself.

Proof. When X is rational, the result follows from [HM09a, Theorem 1.4]. Thus we assume that X is non-rational, and in particular that $X(\mathbb{R})$ is non-connected.

From Lemma 31, we can assume that any real point which is critical for one fibration is not critical for the second fibration. Otherwise speaking (recall that the fibrations are minimal) a real intersection point of a fibre F_1 of π_1 with a fibre F_2 of π_2 cannot be a singular point of F_1 and of F_2 at the same time. By Lemma 28 applied to (X, π_1), and to (X, π_2), we may assume without loss of generality that all points $p_1, \dots, p_n, q_1, \dots, q_n$ belong to smooth fibres of π_1 and to smooth fibres of π_2. We now use Lemma 28 to obtain an automorphism α of $(X(\mathbb{R}), \pi_1)$ such that $\pi_2(\alpha(p_i)) \neq \pi_2(\alpha(p_j))$ and $\pi_2(\alpha(q_i)) \neq \pi_2(\alpha(q_j))$ for $i \neq j$. Hence, we may suppose that $\pi_2(p_i) \neq \pi_2(p_j)$ and $\pi_2(q_i) \neq \pi_2(q_j)$ for $i \neq j$.

Likewise, using an automorphism of $(X(\mathbb{R}), \pi_2)$ we may suppose that $\pi_1(p_i) \neq \pi_1(p_j)$ and $\pi_1(q_i) \neq \pi_1(q_j)$ for $i \neq j$.

We now show that for $i = 1, \dots, m$, there exists an element $\alpha_i \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ that sends p_i on q_i and that restricts to the identity on the sets $\cup_{j\neq i}\{p_j\}$ and $\cup_{j\neq i}\{q_j\}$. Then, the composition of the α_i will achieve the proof. Observe that $\zeta = \pi_1 \times \pi_2$ gives a finite surjective morphism $X \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ which is 2-to-1 or 4-to-1 depending of the degree of X (follows from assertion (c) of Lemma 23). Denote by W the image of $X(\mathbb{R})$. The map $X(\mathbb{R}) \to W$ is a differential map, which has topological finite degree. Denote by W_i the connected component of W which contains both $\zeta(p_i)$ and $\zeta(q_i)$. Observe that W_i is contained in the square $I(X, \pi_1) \times I(X, \pi_2)$, and that for each point $x \in W_i$, the intersection of the horizontal and vertical lines (fibres of the two projections of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$) passing through x with W_i is either only $\{x\}$, when x is on the boundary of W_i, or is a bounded interval. Moreover, W_i is connected. Then, there exists a path from $\zeta(p_i)$ to $\zeta(q_i)$ which is a sequence of vertical or horizontal segments contained in W_i. We may furthermore assume that none of the segments is contained in $(\mathrm{pr}_1)^{-1}(\pi_1(a))$ or $(\mathrm{pr}_2)^{-1}(\pi_2(a))$ for any $a \in (\cup_{j\neq i}\{p_j\}) \cup (\cup_{j\neq i}\{q_j\})$. Denote by $r_1, ..., r_l$ the points of U that are sent on the singular points or ending points of the path, and by $s_1, \dots, s_l$ some points of $X(\mathbb{R})$ which are sent by ζ on $r_1, \dots, r_l$ respectively. Up to renumbering, $s_1 = p_i, s_l = q_i$ and two consecutive points s_j and s_{j+1} are such that $\pi_1(s_j) = \pi_1(s_{j+1})$ or $\pi_2(s_j) = \pi_2(s_{j+1})$. We construct then α_i as a composition of $l-1$ maps, each one belonging either to $\mathrm{Aut}(X(\mathbb{R}), \pi_1)$ or $\mathrm{Aut}(X(\mathbb{R}), \pi_2)$ and sending s_j on s_{j+1}, and fixing the points $(\cup_{j\neq i}\{p_j\}) \cup (\cup_{j\neq i}\{q_j\})$. ☐

The following proposition describes the possible mixes of connected components.

PROPOSITION 34. *Let (X, π) be a minimal real conic bundle. Denote by $I_1, \dots, I_r$ the r connected components of $I(X, \pi)$, and by $M_1, \dots, M_r$ the r connected components of $X(\mathbb{R})$, where $I_i = \pi(M_i)$, $M_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$. If $\nu \in \mathrm{Sym}_r$ is a permutation of $\{1, \dots, r\}$, the following are equivalent:*

- i) *there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\alpha(I_i) = I_{\nu(i)}$ for each i;*
- ii) *there exists $\beta \in \mathrm{Aut}(X(\mathbb{R}), \pi)$ such that $\beta(M_i) = M_{\nu(i)}$ for each i;*
- iii) *there exists $\beta \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ such that $\beta(M_i) = M_{\nu(i)}$ for each i;*
- iv) *there exist two real Zariski open sets $V, W \subset X$, and $\beta \in \mathrm{Bir}(X)$, inducing an isomorphism $V \to W$, such that $\beta(V(\mathbb{R}) \cap M_i) = W(\mathbb{R}) \cap M_{\nu(i)}$ for each i.*

Moreover, the conditions are always satisfied when $r \leqslant 2$, and are in general not satisfied when $r \geqslant 3$.

Proof. The implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious.

The implication $(i) \Rightarrow (ii)$ is a direct consequence of Corollary 19).

We prove now that if $r \leqslant 2$, Assertion (i) is always satisfied, hence all the conditions are equivalent (since all are true). When $r \leqslant 1$, take α to be the identity. When $r = 2$, we make a linear change of coordinates to the effect that $I_1 = \{(x : 1) \mid 0 \leqslant x \leqslant 1\}$ and I_2 is bounded by $(1 : 0)$ and $(\lambda : 1)$, for some $\lambda \in \mathbb{R}$, $\lambda > 1$ or $\lambda < 0$. Then, $\alpha\colon (x_1 : x_2) \mapsto (\lambda x_2 : x_1)$ is an involution which exchanges I_1 and I_2.

It remains to prove the implication $(iv) \Rightarrow (i)$ for $r \geqslant 3$. We decompose β into elementary links

$$X = X_0 \overset{\beta_1}{\dashrightarrow} X_1 \overset{\beta_2}{\dashrightarrow} \cdots \overset{\beta_{n-1}}{\dashrightarrow} X_{n-1} \overset{\beta_n}{\dashrightarrow} X_n = X$$

as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that each of the links is of type II or IV, and that the links of type II are birational maps of conic bundles and the links of type IV occur on del Pezzo surfaces of degree 2.

In consequence, each of the X_i admits a conic bundle structure given by $\pi_i\colon X_i \to \mathbb{P}^1_{\mathbb{R}}$, where $\pi_0 = \pi_n = \pi$, and if β_i has type II, it is a birational map of conic bundles $(X_{i-1}, \pi_{i-1}) \dashrightarrow (X_i, \pi_i)$, and if it has type IV, it is an isomorphism $X_{i-1} \to X_i$ which does not send the general fibre of π_{i-1} on those of π_i. In this latter case, since π_i and $\pi_{i-1}\beta_i$ have distinct general fibres, X_{i-1} and X_i are del Pezzo surfaces of degree 2, and the Geiser involution $\iota_{i-1} \in \mathrm{Aut}(X_{i-1})$ exchanges the two general fibres (follows from [Isk96, Theorem 2.6], but also from Lemma 23). This means that the map $\beta_i \circ \iota_{i-1}$, that we denote by γ_i, is an isomorphism of conic bundles $(X_{i-1}, \pi_{i-1}) \to (X_i, \pi_i)$.

Now, we prove by induction on the number of links of type IV that β may be decomposed into compositions of elements of $\mathrm{Bir}(X, \pi)$ and maps of the form $\psi\iota\psi^{-1}$ where ψ is a birational map of conic bundles $(X, \pi) \dashrightarrow (X', \pi')$, (X', π') is a del Pezzo surface of degree 2 and $\iota \in \mathrm{Aut}(X')$ is the Geiser involution. If there is no link of type IV, β preserves the conic bundle structure given by π. Otherwise, denote by β_i the first link of type IV, which is an isomorphism $\beta_i\colon X_i \to X_{i+1}$, and write $\beta_i = \gamma_i \circ \iota_{i-1}$ as before. We write $\psi = \beta_{i-1} \circ \cdots \circ \beta_1$, which is a birational map of conic bundles $\psi\colon (X, \pi) \dashrightarrow (X_i, \pi_i)$. Then, $\beta = (\beta_n \circ \cdots \circ \beta_{i+1} \circ \gamma_i \circ \psi)(\psi^{-1}\iota_{i-1}\psi)$. Applying the induction hypothesis on the map $(\beta_n \circ \cdots \circ \beta_{i+1} \circ \gamma_i \circ \psi) \in \mathrm{Bir}(X)$, we are done.

Now, observe that when (X', π') is a minimal real conic bundle and X' is a del Pezzo surface of degree 2, the map $\zeta\colon X' \to \mathbb{P}^2_{\mathbb{R}}$ given by $|-K_{X'}|$ is a double covering, ramified over a smooth quartic curve $\Gamma \subset \mathbb{P}^2_{\mathbb{R}}$ (see e.g. [Dem76]). Since (X,π) is minimal, $(K_X)^2 = 8-2r$ thus π has $r=6$ singular fibres , so $I(X,\pi)$ is the union of three intervals and $X(\mathbb{R})$ is the union of 3 connected components. This implies that $\Gamma(\mathbb{R})$ is the union of three disjoint ovals. A connected component M of $X(\mathbb{R})$ is homeomorphic to a sphere, and surjects by ζ to the interior of one of the three ovals. The Geiser involution (induced by the double covering) induces an involution on M, which fixes the preimage of the oval. This means that the Geiser involution sends any connected component of $X(\mathbb{R})$ on itself. Thus, in the decomposition of β into elements of $\mathrm{Bir}(X,\pi)$ and conjugate elements of Geiser involutions, the only relevant elements are those of $\mathrm{Bir}(X,\pi)$. There exists thus $\beta' \in \mathrm{Bir}(X,\pi)$ which acts on the connected components of $X(\mathbb{R})$ in the same way as β. This shows that (iv) implies (i).

We finish by proving that (i) is false in general, when $r \geqslant 3$. This follows from the fact that if Σ is a general finite subset of $2r$ distinct points of $\mathbb{P}^1_{\mathbb{R}}$, the group $\{\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}}) \mid \alpha(\Sigma)=\Sigma\}$ is trivial. Supposing this fact true, we obtain the result by applying it to the $2r$ boundary points of $I(X,\pi)$. Let us prove the fact. The set of $2r$-tuples of $\mathbb{P}^1_{\mathbb{R}}$ is an open subset W of $(\mathbb{P}^1_{\mathbb{R}})^{2r}$. For any non-trivial permutation $\upsilon \in \mathrm{Sym}_{2r}$, we denote by $W_\upsilon \subset W$ the set of points $a=(a_1,\ldots,a_{2r}) \in W$ such that there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ with $\alpha(a_i)=a_{\upsilon(i)}$ for each i. Let $a \in W_\upsilon$, and take two 4-tuples Σ_1,Σ_2 of a_i's with $\Sigma_1 \neq \Sigma_2$ and $\Sigma_2 = \upsilon(\Sigma_1)$ (this is possible since υ is non-trivial). Then, the cross-ratio of the a_i's in Σ_1 and in Σ_2 are the same. This implies a non-trivial condition on W. Consequently, W_υ is contained in a closed subset of W. Doing this for all non-trivial permutations υ, we obtain the result. $\square$

8. Real algebraic models

The aim of this section is to go further with non-minimal surfaces with 2 or 3 connected components. We begin to show how *to separate* infinitely near points to the effect that any such a surface $Y(\mathbb{R})$ is isomorphic to a blow-up $B_{a_1,\ldots,a_m}X(\mathbb{R})$ where X is minimal and $a_1,\ldots,a_m$ are distinct proper points of $X(\mathbb{R})$. Then, we replace $X(\mathbb{R})$ by an isomorphic del Pezzo model (Corollary 24) and we use the fact that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component for such an X (Proposition 33) to prove that in many cases, if two birational surfaces Y and Z have homeomorphic real parts then $Y(\mathbb{R})$ and $Z(\mathbb{R})$ are isomorphic. As a corollary, we get that in any cases, $\mathrm{Aut}\big(Y(\mathbb{R})\big)$ is very transitive on each connected component.

Proposition 35. *Let X be a minimal geometrically rational real surface, with $\#X(\mathbb{R})=2$ or $\#X(\mathbb{R})=3$, and let $\eta\colon Y \to X$ be a birational morphism.*

Then there exists a blow-up $\eta'\colon Y' \to X$, whose centre is a finite number of distinct real proper points of X, and such that $Y'(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Moreover, we can assume that the isomorphism $Y(\mathbb{R}) \to Y'(\mathbb{R})$ induces an homeomorphism $\eta^{-1}(M) \to (\eta')^{-1}(M)$ for each connected component M of $X(\mathbb{R})$.

Proof. According to Corollary 24, we may assume that X admits two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$. Preserving the isomorphism class of $Y(\mathbb{R})$, we may assume that the points in the centre of η are all real (such a point may be a proper point of $X(\mathbb{R})$ or an infinitely near point). Let us denote by m $(=K_X^2-K_Y^2)$ the number of those points. We prove the result by induction on m.

The cases $m=0$ and $m=1$ being obvious (take $\eta'=\eta$), we assume that $m \geqslant 2$. We decompose η as $\eta=\theta\circ\varepsilon$, where $\varepsilon\colon Y \to Z$ is the blow-up of one real point $q \in Z$, and $\theta\colon Z \to Y$ is the blow-up of $m-1$ real points. By induction hypothesis, we may assume that θ is the blow-up of $m-1$ proper points of X, namely $a_1,\cdots,a_{m-1} \in X(\mathbb{R})$. Moreover, applying Proposition 33, we may move the points by an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$, and assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i, for each i.

If $\theta(q) \notin \{a_1, \ldots, a_{m-1}\}$, then η is the blow-up of m distinct proper points of X, hence we are done. Otherwise, assume that $\theta(q) = a_1$. We write $E = \theta^{-1}(a_1) \subset Z$, and denote by $F_i \subset Z$ the strict pull-back by η of the fibre of π_i passing through a_1, for $i = 1, 2$. Then, F_1 and F_2 are two (-1)-curves which do not intersect. Hence, the point $q \in E$ belongs to at most one of the two curves, so we may assume that $q \notin F_1$. Denote by $\theta_2 \colon Z \to X_2$ the contraction of the $m-1$ disjoint (-1)-curves $F_1, \theta^{-1}(a_2), \ldots, \theta^{-1}(a_{m-1})$. Since q does not belong to any of these curves, $\eta_2 = \theta_2 \circ \varepsilon$ is the blow-up of $m-1$ distinct proper points of X_2. It remains to find an isomorphism $\gamma \colon X_2(\mathbb{R}) \to X(\mathbb{R})$ such that for each connected component M of $X(\mathbb{R})$, $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M.

Denoting $\pi' = \pi_1 \circ \theta \circ \theta_2^{-1}$, the map $\psi = \theta_2 \circ \theta^{-1}$ is a birational map of conic bundles $(X, \pi_1) \dashrightarrow (X_2, \pi')$, which factorizes as the blow-up of a_1, followed by the contraction of the strict transform of the fibre passing through a_1. Therefore, the conic bundle (X_2, π') is minimal. Since $\#X(\mathbb{R}) > 1$ and $\pi'\psi = \pi_1$, Proposition 18 yields the existence of an isomorphism $\gamma \colon X_2(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_1\gamma = \pi'$. Observe that $\gamma\eta_2 \circ \eta^{-1} = \gamma\theta_2 \circ \theta^{-1} = \gamma\psi$ is a birational map $X \dashrightarrow X$ which satisfies $\pi \circ (\gamma\eta_2 \circ \eta^{-1}) = \pi$. Consequently, for any connected component M of $X(\mathbb{R})$, which corresponds to $\pi^{-1}(V) \cap X(\mathbb{R})$, for some interval $V \subset \mathbb{P}^1_{\mathbb{R}}$, we find $\pi(\gamma\eta_2\eta^{-1}(M)) = \pi(M) = V$, thus $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M. ☐

Corollary 36. *Let X be a minimal geometrically rational real surface, such that $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$, and let $\eta \colon Y \to X$, $\varepsilon \colon Z \to X$ be two birational morphisms. Denote by $M_1, \ldots, M_r$ the connected components of $X(\mathbb{R})$ $(r = 2, 3)$. Then, the following are equivalent:*

i) *$\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic for each i;*
ii) *there exists an isomorphism $Y(\mathbb{R}) \to Z(\mathbb{R})$ which induces an homeomorphism $\eta^{-1}(M_i) \to \varepsilon^{-1}(M_i)$ for each i.*

Proof. (2) $\Rightarrow$ (1) being obvious, let us prove the converse. According to Proposition 35, we may assume that η and ε are the blow-ups of a finite number of distinct real proper points of X. Denote by Σ_η and Σ_ε these two finite sets. For each i, the fact that $\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic implies that the numbers of points of $\Sigma_\eta \cap M_i$ and $\Sigma_\varepsilon \cap M_i$ coincide.

By Corollary 24 and Proposition 33, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component of $X(\mathbb{R})$. In particular, there exists an element $\alpha \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ such that $\alpha(M_i) = M_i$ for each i and $\alpha(\Sigma_\eta) = \Sigma_\varepsilon$. Then, $\psi = \varepsilon^{-1}\alpha\eta \colon Y(\mathbb{R}) \to Z(\mathbb{R})$ is the wanted isomorphism. ☐

Corollary 37. *Let Y be a geometrically rational real surface with $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$. Let $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ be two n-tuples of distinct points of $Y(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i.*

Then, there exists an element $\alpha \in \mathrm{Aut}\big(Y(\mathbb{R})\big)$, which leaves each connected component of $Y(\mathbb{R})$ invariant and such that $\alpha(p_i) = q_i$ for each i.

Proof. Let $\eta \colon Y \to X$ be a birational morphism to a minimal real surface X; observe that $\#X(\mathbb{R}) = \#Y(\mathbb{R})$. According to Corollary 24, we may assume that X admits two minimal conic bundles $\pi_1 \colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2 \colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$. By Proposition 35, we can suppose that η is the blow-up of m distinct real proper points $a_1, \ldots, a_m \in X$. We prove the result by induction on m.

If $m = 0$, which means that $X = Y$, the result follows from Proposition 33.

If $m > 0$, denote by $\eta_0 \colon Z \to X$ the blow-up of $a_1, \ldots, a_{m-1}$ (η_0 is the identity if $m = 1$), and by $\eta_1 \colon Y \to Z$ the blow-up of $b = \eta_0^{-1}(a_r)$.

Applying Proposition 33, we may assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i, for each i. Let us denote by $E \subset Y$ the exceptional curve $\eta_1^{-1}(b)$ of η_1 and by F_i the strict transform on Y of the fibre of π_i passing through a_m, for $i = 1, 2$. Then E, F_1 and F_2 are three (-1)-curves, F_1 and F_2 do not intersect, and E intersect transversally each of the F_i. By induction hypothesis, we may use the lift of an element of $\mathrm{Aut}\big(Z(\mathbb{R})\big)$ which fixes b to assume that no one of the points p_i belongs to $F_1 \backslash E$, $F_2 \backslash E$ or to $\eta^{-1}(a_i)$ for $i = 1, \dots, m-1$. Then the group $G = \{\alpha \in \mathrm{Aut}\big(X(\mathbb{R})\big) \mid \pi_1 \alpha = \pi_1, \alpha \text{ fixes } a_1, \dots, a_m, \eta(p_1), \dots, \eta(p_n)\}$, acts transitively on $E \backslash F_1$ (Lemma 30). Lifting a well-chosen element of this group in $\mathrm{Aut}\big(Y(\mathbb{R})\big)$, we may move the points p_i and assume that no one of the p_i belongs to F_2 (i.e. we can avoid $F_2 \cap E$). Denote by $\eta' \colon Y \to X'$ the contraction of the disjoint (-1)-curves $F_2, \eta^{-1}(a_1), \dots \eta^{-1}(a_{m-1})$.

Then, the birational map $\psi = \eta' \eta^{-1} \colon X \dashrightarrow X'$ is a birational map of conic bundles $(X, \pi_2) \dashrightarrow (X', \pi')$, where $\pi' = \pi_2 \psi^{-1}$, which consists of the blow-up of a_m, followed by the contraction of the strict transform of the fibre passing through a_m. Therefore, the conic bundle (X', π') is minimal. Since $\# X(\mathbb{R}) > 1$, Proposition 18 yields the existence of an isomorphism $\gamma \colon X'(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_2 \gamma = \pi'$. Therefore, there exists an element $\beta \in \mathrm{Aut}\big(X'(\mathbb{R})\big)$ which fixes all the points blown-up by η', which fixes all the points $\{\eta'(p_i), p_i \notin E\}$, and which sends the points $\{\eta'(p_i), p_i \in E\}$ outside of $\eta'(E)$. Applying the lift of β on $\mathrm{Aut}\big(Y(\mathbb{R})\big)$, we may assume that none of the points p_i belongs to E. Doing the same manipulation with the q_i, it remains to use the lift of an element of $\mathrm{Aut}\big(Z(\mathbb{R})\big)$ which fixes b and sends $\eta_1(p_i)$ on $\eta_1(q_i)$ for each i. □

9. Proof of the main results

The proof of Theorem 4 was given at the end of Section 5. Now, we deduce the others results stated in the introduction from the results of Sections 7 and 8. The following lemma serves to prove most of them.

Lemma 38. *Let (X, π) be a minimal real conic bundle, such that $I(X, \pi)$ is the union of r intervals $I_1, \dots, I_r$, with $r = 2$ or $r = 3$.*

Let $\eta_Y \colon Y \to X$ and $\eta_Z \colon Z \to X$ be two birational morphisms. For $i = 1, \dots, r$, we write $X_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$, $Y_i = \eta_Y^{-1}(X_i) \cap Y(\mathbb{R})$ and $Z_i = \eta_Z^{-1}(X_i) \cap Z(\mathbb{R})$.

Let $p_1, \dots, p_n \in Y(\mathbb{R})$, $q_1, \dots, q_n \in Z(\mathbb{R})$ be two n-tuples of distinct points, and assume the existence of an homeomorphism $h \colon Y(\mathbb{R}) \to Z(\mathbb{R})$ which sends p_i on q_i for each i, and sends Y_i on $Z_{\nu(i)}$, where $\nu \in \mathrm{Sym}_r$ is a permutation of $\{1, \dots, r\}$. Then, the following are equivalent:

i) *There exists an isomorphism $\beta \colon Y(\mathbb{R}) \to Z(\mathbb{R})$ which sends Y_i on $Z_{\nu(i)}$ for each $i \in \{1, \dots, r\}$ and sends p_j on q_j for each $j \in \{1, \dots, n\}$.*

ii) *There exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ which sends I_i on $I_{\nu(i)}$ for each $i \in \{1, \dots, r\}$.*

Moreover, both assertions are true if $r = 2$, and false in general when $r = 3$.

Proof. Observe that the X_i (respectively the Y_i, Z_i) are the connected components of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$, $Z(\mathbb{R})$).

[*i*) $\Rightarrow$ *ii*)] The map $\eta_Z \beta \eta_Y^{-1}$ is a birational self-map of X, which restricts to an isomorphism $\varphi \colon V \to W$, where V and W are two real Zariski open subsets of X. Moreover, the hypothesis on β implies that $\varphi(V(\mathbb{R}) \cap X_i) = W(\mathbb{R}) \cap X_{\nu(i)}$. The existence of α is provided by Proposition 34.

[*ii*) $\Rightarrow$ *i*)] Proposition 34 yields the existence of $\gamma \in \mathrm{Aut}(X(\mathbb{R}), \pi)$ such that $\gamma(X_i) = X_{\nu(i)}$. We may thus assume that ν is the identity. According to Proposition 35, we may moreover suppose that η_Y and η_Z are the blow-ups of a finite set of disjoint real proper points of X. Since Y_i is homeomorphic to Z_i for each i, η_Y is the blow-up of $a_1, \dots, a_m$ and η_Z is the blow-up of $b_1, \dots, b_m$, where a_j and b_j belong to the same connected component of $X(\mathbb{R})$ for each j. Then, there exists an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ which preserves each connected component of X and sends a_j on b_j for each j (Corollary 37). We may thus assume that $Y = Z$, and conclude by applying Corollary 37 to Y.

The fact that *ii*) is true when $r = 2$ and false in general when $r = 3$ was proved in Proposition 34. $\square$

The following case shares many features with the rational case.

THEOREM 39. *Let X be a nonsingular geometrically rational real projective surface, and assume that $\#X(\mathbb{R}) = 2$. Then the action of the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ on $X(\mathbb{R})$ is very transitive.*

Proof. Let Y be a nonsingular geometrically rational real projective surface, with $\#Y(\mathbb{R}) = 2$. Let $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ be two n-tuples of points which are compatible. We want to prove the existence of $\alpha \in \mathrm{Aut}\big(Y(\mathbb{R})\big)$ such that $\alpha(p_i) = q_i$ for each i.

If p_i and q_i are in the same connected component of $Y(\mathbb{R})$, the result follows from Corollary 37.

Otherwise, the compatibility means that the two components of $X(\mathbb{R})$ are homeomorphic and that p_i and q_i are in a distinct component for each i. Lemma 38 provides the existence of an element of $\mathrm{Aut}\big(Y(\mathbb{R})\big)$ which permutes the two connected components of $Y(\mathbb{R})$. This reduces the situation to the previous case. $\square$

Before proving Theorems 1 and 2, we describe the cases where the group of automorphisms is not very transitive.

LEMMA 40. *Let X be a nonsingular real projective surface, and assume that either X is* not *geometrically rational or $\#X(\mathbb{R}) > 3$. The group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is then not very transitive on each connected component, and is neither 2-transitive.*

Proof. If X has Kodaira dimension 2, (surface of general type), it has only finitely many birational self-maps (see e.g. [Uen75].) If X has Kodaira dimension 1, every birational self-map of X preserves the elliptic fibration induced by $|K_X|$. If X has Kodaira dimension 0, and X is minimal, then $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)$ is an algebraic group of dimension 1 or 2 (its neutral component is an elliptic curve or an Abelian surface). Thus, $\mathrm{Bir}(X)$ can not be 2-transitive. The case when X is not minimal is deduced from this case.

If X is a surface with Kodaira dimension $-\infty$, then X is uniruled. If furthermore, X is not geometrically rational and $X(\mathbb{R})$ is non-empty, then the Albanese map $X \to C$ is a real ruling over a curve with genus $g(C) > 0$, see e.g. [Sil89, V.(1.8)], and the Albanese map is preserved by any birational self-map.

The remaining case is when X is geometrically rational and $\#X(\mathbb{R}) > 3$; we prove now that the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not transitive. Denote by $\eta\colon X \to X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) > 3$. Let us discuss the two cases for X_0 given by Theorem 7. If X_0 is a del Pezzo surface with $\rho(X_0) = 1$, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is countable (Corollary 11), thus $\mathrm{Aut}\big(X(\mathbb{R})\big)$ cannot be transitive. The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0), and $\mathrm{Bir}(X_0) = \mathrm{Bir}(X_0, \pi_0)$ (Theorem 25). Since the action of $\mathrm{Bir}(X_0, \pi_0)$ on the basis of the conic bundle is finite (there are too much boundary points), neither $\mathrm{Aut}\big(X_0(\mathbb{R})\big)$ nor $\mathrm{Aut}\big(X(\mathbb{R})\big)$ may be transitive. $\square$

Proof of Theorem 1. When X is not geometrically rational or $\#X(\mathbb{R}) > 3$, $\mathrm{Aut}(X(\mathbb{R}))$ is not very transitive on connected components by Lemma 40. In the remaining cases, $\mathrm{Aut}(X(\mathbb{R}))$ is very transitive on connected components. When $\#X(\mathbb{R}) = 2, 3$, this is Corollary 37. When $\#X(\mathbb{R}) = 1$, this is the main result of [HM09a]. $\square$

Proof of Theorem 2. According to Lemma 40, we can assume from now on that X is a geometrically rational surface with $\#X(\mathbb{R}) \leqslant 3$. When $\#X(\mathbb{R}) = 1$, X is rational; the fact that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is n-transitive for every n (and thus very transitive) is the main result of [HM09a]. When $\#X(\mathbb{R}) = 2$, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive by Theorem 39.

When $\#X(\mathbb{R}) = 3$, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component (Theorem 1). Thus, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive if and only if for any homeomorphism $h\colon X(\mathbb{R}) \to X(\mathbb{R})$, there exists $\beta \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ which permutes the components of $X(\mathbb{R})$ in the same way that h does. When these conditions are not satisfied, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not 2-transitive.

Let $X(\mathbb{R}) = M_1 \sqcup M_2 \sqcup M_3$ be the decomposition into connected components. If there is no pair (i,j) such that $M_i \sim M_j$, then there is no nontrivial such h, hence $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive. If $M_1 \sim M_2 \not\sim M_3$ or $M_1 \sim M_2 \sim M_3$, the possibilities when this occur follow from Lemma 38.

For example, when X is minimal (therefore $M_1 \sim M_2 \sim M_3 \sim S^2$), it admits a minimal real conic bundle structure (X,π) (Theorem 7 and Proposition 9), where π has 6 singular fibres. Then, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive if and only if $\{\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}}) \mid \alpha(I(X,\pi)) = I(X,\pi)\}$ acts transitively on the three intervals of $I(X,\pi)$. This is true in some special cases, but false in general. When X is not minimal, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive for example when there is no pair of homeomorphic connected components of $X(\mathbb{R})$, or when X is the blow-up of a minimal surface Y with a very transitive group $\mathrm{Aut}\big(Y(\mathbb{R})\big)$. □

Proof of Theorem 3. Let X, Y be two geometrically rational real surfaces, and assume that $\#X(\mathbb{R}) \leqslant 2$. We assume that X is birational to Y and that $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$, and prove that $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Remark that all geometrically rational surfaces with connected real part are birational to each others, thus in this case the statement follows from the unicity of rational models [BH07]. We may thus assume that $\#X(\mathbb{R}) = 2$. Denote by $\eta_X\colon X \to X_0$ and $\eta_Y\colon Y \to Y_0$ birational morphisms to minimal real surfaces.

Since X_0 and Y_0 are birational, $X_0(\mathbb{R})$ and $Y_0(\mathbb{R})$ are isomorphic (Theorem 4), so we may assume that $X_0 = Y_0$. The result now follows from Lemma 38. □

Proof of Corollary 5. If M is connected, and M is non-orientable or M is orientable with genus $g(M) \leqslant 1$, then it admits a unique geometrically rational model by [BH07, Corollary 8.1]. Moreover, this model is in fact rational.

Conversely let M be a compact $\mathcal{C}^\infty$-surface and assume that M admits a unique geometrically rational model X. The existence of such a model implies, by Comessatti's theorem [Com14], that any connected component of M is non-orientable or is orientable with genus $g \leqslant 1$. The unicity means that for any geometrically rational model Y of M, then $Y(\mathbb{R})$ is isomorphic to $X(\mathbb{R})$. In particular, this implies that all geometrically rational models of M belong to a unique birational class. From Theorem 25 and Proposition 10, this means that X is rational. It remains to observe that when X is rational, $X(\mathbb{R})$ is connected, and is either non-orientable or orientable of genus $\leqslant 1$. When X is minimal, this follows from Proposition 9. Then, blowing-up points on a surface either does nothing on the topology of the real part (if the points blown-up are imaginary), or it gives a non-orientable real part (if the points blown-up ar real). □

We finish by a result on non-density. In [KM09], it is proved that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$ when X is a geometrically rational surface with $\#X(\mathbb{R}) = 1$ (or equivalently when X is rational). In the cited paper, it is said that $\#X(\mathbb{R}) = 2$ is probably the only other case where the density holds. The following collect the known results in this direction. The first two of them are new.

Proposition 41. *Let X be a geometrically rational surface.*

- *If $\#X(\mathbb{R}) \geqslant 5$, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$;*
- *if $\#X(\mathbb{R}) = 4$, and either X is the blow-up of a minimal conic bundle or $\rho(X) = 1$, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$;*
- *if $\#X(\mathbb{R}) = 3$ and X is minimal, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$ for a general X;*
- *if $\#X(\mathbb{R}) = 1$, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$.*

Proof. The case $\#X(\mathbb{R}) = 1$ is the main result of [KM09]. Assume from now on that $\#X(\mathbb{R}) \geqslant 3$, and denote by $\eta\colon X \to X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) \geqslant 3$. Let us discuss the two cases for X_0 given by Theorem 7.

Assume that X_0 is a del Pezzo surface with $\rho(X_0) = 1$. If the degree of X_0 is 1 then $\mathrm{Bir}(X_0)$ is finite (Corollary 11), thus $\mathrm{Aut}\big(X(\mathbb{R})\big)$ cannot be dense. If X_0 has degree 2, then $\#X_0(\mathbb{R}) = 4$ (Proposition 9), so $\#X(\mathbb{R}) = 4$ too. Since $\mathrm{Aut}\big(X_0(\mathbb{R})\big) = \mathrm{Aut}(X_0)$ is finite, $\mathrm{Aut}\big(X_0(\mathbb{R})\big)$ cannot be dense (but maybe $\mathrm{Aut}\big(X(\mathbb{R})\big)$ could be).

The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0). If $\#X(\mathbb{R}) = \#X_0(\mathbb{R}) \geqslant 4$, then $\mathrm{Bir}(X_0) = \mathrm{Bir}(X_0, \pi_0)$ (Theorem 25), so $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not dense. If $\#X_0(\mathbb{R}) = 3$, then in general $\mathrm{Aut}\big(X_0(\mathbb{R})\big)$ does not exchanges the connected component of $X_0(\mathbb{R})$. Consequently, $\mathrm{Aut}\big(X_0(\mathbb{R})\big)$ is not dense (but maybe $\mathrm{Aut}\big(X(\mathbb{R})\big)$ could be, if the connected components of $X(\mathbb{R})$ are not homeomorphic). ☐

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