Algebraic models of the line in the real affine plane

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(joint work with Adrien Dubouloz)

It is a standard consequence of the Jordan–Schoenflies Theorem that every two smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ are ambient diffeotopic. Every algebraic closed embedding of the real affine line $\mathbb{A}^1_{\mathbb{R}}$ into the real affine plane $\mathbb{A}^2_{\mathbb{R}}$ induces a smooth embedding of the real locus $\mathbb{R}$ of $\mathbb{A}^1_{\mathbb{R}}$ into the real locus $\mathbb{R}^2$ of $\mathbb{A}^2_{\mathbb{R}}$. Given two such algebraic embeddings $f, g : \mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$, the famous Abhyankar-Moh Theorem [1], which is valid over any field of characteristic zero [23, § 5.4], asserts the existence of a polynomial automorphism ϕ of $\mathbb{A}^2_{\mathbb{R}}$ such that $f = \phi \circ g$. This implies in particular that the smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by f and g are equivalent under composition by a polynomial diffeomorphism of $\mathbb{R}^2$.

In this talk, we consider a similar problem in a natural category intermediate between the real algebraic and the smooth ones. Our main object of study consists of smooth embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by rational algebraic maps $\mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ defined on the real locus of $\mathbb{A}^1_{\mathbb{R}}$ and whose restrictions to this locus induce smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$. We call these maps *rational smooth embeddings*, and the question is the classification of these embeddings up to *birational diffeomorphisms* of $\mathbb{A}^2_{\mathbb{R}}$, that is, diffeomorphisms of $\mathbb{R}^2$ which are induced by birational algebraic endomorphisms of $\mathbb{A}^2_{\mathbb{R}}$ containing $\mathbb{R}^2$ in their domains of definition and admitting rational inverses of the same type.

A first natural working question in this context is to decide whether any rational smooth embedding is equivalent up to birational diffeomorphism to the standard regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ as a linear subspace. Since every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ uniquely extends to a morphism $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ birational onto its image, a rational smooth embedding which can be rectified to a linear embedding by a birational diffeomorphism defines in particular a rational plane curve C that can be mapped onto a line by a birational automorphism of $\mathbb{P}^2_{\mathbb{R}}$. By classical results of Coolidge [6], Iitaka [14] and Kumar-Murthy [17], complex curves with this property are characterized by the negativity of the logarithmic Kodaira dimension of the complement of their proper transform in a minimal resolution of their singularities. Building on these ideas and techniques, we show the existence of *non-rectifiable* rational smooth embedding. In particular, we obtain the following result:

Theorem. *For every integer $d \geq 5$ there exists a non-rectifiable rational smooth embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a rational nodal curve of degree d.*

The existence of non-rectifiable rational smooth embeddings motivates the search for weaker properties which can be satisfied by rational smooth embeddings. To this end we observe that the Abhyankar-Moh Theorem implies that the image of a regular closed embedding $\mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a real fiber of a structure of trivial $\mathbb{A}^1$-bundle $\rho : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ on $\mathbb{A}^2_{\mathbb{R}}$. In the complex case, this naturally leads to a "generalized

Abhyankar-Moh property" for closed embeddings of the affine line in affine surfaces S equipped with $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{\mathbb{C}}$, i.e. morphisms $\pi : S \to \mathbb{A}^1_{\mathbb{C}}$ whose general fibers are affine lines, which was studied for certain classes of surfaces in [11]: the question there is whether the image of every regular closed embedding of $\mathbb{A}^1_{\mathbb{C}}$ in such a surface is an irreducible component of a fiber of an $\mathbb{A}^1$-fibration. The natural counterpart in our real birational setting consists in shifting the focus to the question whether the image of a rational smooth embedding is actually a fiber of an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ on a suitable real affine surface S birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$, but possibly non biregularly isomorphic to it. A rational smooth embedding with this property is said to be *biddable*.

Being a fiber of an $\mathbb{A}^1$-fibration on a surface birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$ imposes strong restrictions on the scheme-theoretic image $f_*(\mathbb{A}^1_{\mathbb{R}})$ of a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. We show in particular that the *real Kodaira dimension* [4] $\kappa_{\mathbb{R}}(\mathbb{A}^2_{\mathbb{R}} \setminus f_*(\mathbb{A}^1_{\mathbb{R}}))$ of the complement of the image has to be negative, with the consequence for instance that none of the rational smooth embedding mentioned in the theorem above is actually biddable. In contrast, a systematic study of small degree embeddings reveals the existence of non-rectifiable biddable rational smooth embeddings whose images are in a natural way smooth fibers of $\mathbb{A}^1$-fibrations on some *fake real planes*, a class of real birational models of $\mathbb{A}^2_{\mathbb{R}}$ recently introduced and studied in the series of papers [7, 8]. These are smooth real surfaces S non isomorphic to $\mathbb{A}^2_{\mathbb{R}}$ whose real loci are diffeomorphic to $\mathbb{R}^2$ and whose complexifications have trivial reduced rational singular homology groups.

We therefore develop a collection of geometric techniques to tackle the classification of equivalence classes of biddable rational smooth embeddings up to birational diffeomorphisms. As a result, we obtain in particular the following synthetic criterion:

Theorem. *For $i = 1, 2$, let $f_i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$, be a biddable rational smooth embedding and let $\alpha_i : \mathbb{A}^2_{\mathbb{R}} \dashrightarrow S_i$ be a birational diffeomorphism onto an $\mathbb{A}^1$-fibered fake real plane $\pi_i : S_i \to \mathbb{A}^1_{\mathbb{R}}$ such that $\alpha_i \circ f_i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow S_i$ is a closed immersion as the support of a* smooth *fiber of π_i.*

Then f_1 and f_2 are not rectifiable. Furthermore, the following conditions are equivalent:

a) $f_1 : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ and $f_2 : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ are equivalent rational smooth embeddings

b) There exists a birational diffeomorphism $\beta : S_1 \dashrightarrow S_2$ and an automorphism γ of $\mathbb{A}^1_{\mathbb{R}}$ such that $\gamma \circ \pi_1 = \pi_2 \circ \beta$.

As an application of this characterization, we derive in particular the existence of infinitely many equivalence classes of biddable rational smooth embeddings.

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