CREMONA GROUPS OF REAL SURFACES

JÉRÉMY BLANC AND FRÉDÉRIC MANGOLTE

Abstract. We give an explicit set of generators for various natural subgroups of the real Cremona group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. This completes and unifies former results by several authors.

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1. Introduction

1.1. **On the real Cremona group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.** The classical Noether-Castelnuovo Theorem [Cas1901] (see also [A02, Chapter 8] for a modern exposition of the proof) gives generators of the group $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ of birational transformations of the complex projective plane. The group is generated by the biregular automorphisms, which form the group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^2) = \mathrm{PGL}(3,\mathbb{C})$ of projectivities, and by the standard quadratic transformation

$$\sigma_0 \colon (x:y:z) \dashrightarrow (yz:xz:xy).$$

This result does not work over the real numbers. Indeed, recall that a *base point* of a birational transformation is a (possibly infinitely near) point of indeterminacy; and note that two of the base points of the quadratic involution

$$\sigma_1 \colon (x:y:z) \dashrightarrow (y^2+z^2:xy:xz)$$

are not real. Thus σ_1 cannot be generated by projectivities and σ_0. More generally, we cannot generate this way maps having non real base-points. Hence the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of birational transformations of the real projective plane is not generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}(3,\mathbb{R})$ and σ_0.

The first result of this note is that $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ_0, σ_1, and a family of birational maps of degree 5 having only non real base-points.

Theorem 1.1. *The group* $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$*,* σ_0*,* σ_1*, and the* standard quintic *transformations of* $\mathbb{P}^2$ *(defined in Example* 3.1*).*

The proof of this result follows the so-called Sarkisov program, which amounts to decompose a birational map between Mori fibre spaces as a sequence of simple maps, called *Sarkisov links*. The description of all possible links has been done in [Isko96] for perfect fields, and in [Poly97] for real surfaces. We recall it in Section 2 and show how to deduce Theorem 1.1 from the list of Sarkisov links.

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Note that a family of generators of $\mathrm{Bir}_{\mathbb{K}}(\mathbb{P}^2)$ is given in [Isko91], for any perfect field $\mathbb{K}$. When taking $\mathbb{K}=\mathbb{R}$, the list is however longer than the one given in Theorem 1.1.

Let X be an algebraic variety defined over $\mathbb{R}$ (always assumed to be geometrically irreducible), we denote as usual by $X(\mathbb{R})$ the set of real points endowed with the induced algebraic structure. The topological space $\mathbb{P}^2(\mathbb{R})$ is then the real projective plane, letting $\mathbb{F}_0 := \mathbb{P}^1\times\mathbb{P}^1$, the space $\mathbb{F}_0(\mathbb{R})$ is the torus $\mathbb{S}^1\times\mathbb{S}^1$ and letting $Q_{3,1}=\{(w:x:y:z)\in\mathbb{P}^3 \mid w^2=x^2+y^2+z^2\}$, the real locus $Q_{3,1}(\mathbb{R})$ is the sphere $\mathbb{S}^2$.

Recall that an *isomorphism* $X(\mathbb{R})\to Y(\mathbb{R})$ is a birational map $\varphi\colon X \dashrightarrow Y$ defined over $\mathbb{R}$ such that φ is defined at all real points of X and φ^{-1} at all real points of Y. The set of automorphisms of $X(\mathbb{R})$ form a group $\mathrm{Aut}(X(\mathbb{R}))$, and we have natural inclusions

$$\mathrm{Aut}_{\mathbb{R}}(X)\subset \mathrm{Aut}(X(\mathbb{R}))\subset \mathrm{Bir}_{\mathbb{R}}(X).$$

The strategy used to prove Theorem 1.1 allows us to treat similarly the case of natural subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, namely the groups $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $\mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ of three *minimal* real rational surfaces (see Corollary 2.10). This way, we give a unified treatment to prove three theorems on generators, the first two of them already proved in a different way in [RV05] and [KM09].

Observe that $\mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ and $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ are not really subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, but each of them is isomorphic to a subgroup which is determined up to conjugation. Indeed, for any choice of a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow X$ ($X=Q_{3,1}$ or $\mathbb{F}_0$), $\psi^{-1}\,\mathrm{Aut}(X(\mathbb{R}))\psi\subset \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$.

Theorem 1.2 ([RV05])**.** *The group* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)=\mathrm{PGL}(3,\mathbb{R})$ *and by standard quintic transformations.*

Note that up to the action of $\mathrm{PGL}(3,\mathbb{R})$, the standard quintic transformations form an algebraic variety of (real) dimension 4. This is in contrast with the complex case, where the set of standard quadratic transformations is $\{\sigma_0\}$, up to the action of $\mathrm{PGL}(3,\mathbb{C})$.

Theorem 1.3 ([KM09])**.** *The group* $\mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(Q_{3,1})=\mathrm{PO}(3,1)$ *and by standard cubic transformations.*

Here the real dimension of the variety of standard cubic transformations, modulo $\mathrm{PO}(3,1)$, is 2.

Theorem 1.4. *The group* $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)\cong \mathrm{PGL}(2,\mathbb{R})^2\rtimes\mathbb{Z}/2\mathbb{Z}$ *and by the involution*

$$\tau_0\colon ((x_0:x_1),(y_0:y_1)) \dashrightarrow ((x_0:x_1),(x_0y_0+x_1y_1:x_1y_0-x_0y_1)).$$

In each case, we don't know any easy way of computing the relations between the given generators. (See [IKT93] for a description in a more general setting, whose application to the real case does not fit our set of generators.)

The proof of theorems 1.1, 1.2, 1.3, 1.4 is given in Sections 4, 3, 5, 6, respectively. Section 7 is devoted to present some related recent results on birational geometry of real projective surfaces.

In the sequel, surfaces and maps are assumed to be real. In particular if we consider that a real surface is a complex surface endowed with a Galois-action of

$G := \mathrm{Gal}(\mathbb{C}|\mathbb{R})$, a map is G-equivariant. On the contrary, points and curves are not assumed to be real a priori.

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2. Mori theory for real rational surfaces and Sarkisov program

We work with the tools of Mori theory. A good reference in dimension 2, over any perfect field, is [Isko96]. The theory, applied to smooth projective real rational surfaces, becomes really simple. The description of Sarkisov links between real rational surfaces has been done in [Poly97], together with a study of relations between these links. In order to state this classification, we first recall the following classical definitions (which can be found in [Isko96]).

Definition 2.1. A smooth projective real rational surface X is said to be *minimal* if any birational morphism $X \to Y$, where Y is another smooth projective real surface, is an isomorphism.

Definition 2.2. A *Mori fibration* is a morphism $\pi\colon X \to W$ where X is a smooth projective real rational surface and one of the following occurs

(1) $\rho(X) = 1$, W is a point (usually denoted $\{*\}$), and X is a del Pezzo surface;
(2) $\rho(X) = 2$, $W = \mathbb{P}^1$ and the map π is a conic bundle.

Note that for an arbitrary surface, the curve W in the second case should be any smooth curve, but we restrict ourselves to rational surfaces which implies that W is isomorphic to $\mathbb{P}^1$.

Proposition 2.3. *Let X be a smooth projective real rational surface. If X is minimal, then it admits a morphism $\pi\colon X \to W$ which is a Mori fibration.*

Proof. Follows from [Isko79, Theorem 1]. See also [Mori82]. □

Definition 2.4. A *Sarkisov link* between two Mori fibrations $\pi_1\colon X_1 \to W_1$ and $\pi_2\colon X_2 \to W_2$ is a birational map $\varphi\colon X_1 \dashrightarrow X_2$ of one of the following four types, where each of the diagrams is commutative:

(1) Link of Type I

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"'] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"'] \\ \{*\} = W_1 & W_2 = \mathbb{P}^1 \arrow[l, "\tau"'] \end{tikzcd}$$

where $\varphi^{-1}\colon X_2 \to X_1$ is a birational morphism, which is the blow-up of either a real point or two non-real conjugate points of X_1, and where τ is the contraction of $W_2 = \mathbb{P}^1$ to the point W_1.

(2) Link of Type II

$$\begin{tikzcd} X_1 \arrow[d, "\pi_1"'] \arrow[rr, dashed, bend right, "\varphi"'] & Z \arrow[l, "\sigma_1"'] \arrow[r, "\sigma_2"] & X_2 \arrow[d, "\pi_2"] \\ W_1 \arrow[rr, "\simeq", "\tau"'] & & W_2 \end{tikzcd}$$

where $\sigma_i\colon Z \to X_i$ is a birational morphism, which is the blow-up of either a real point or two non-real conjugate points of X_i, and where τ is an isomorphism between W_1 and W_2.

(3) Link of Type III

$$\begin{tikzcd} X_1 \arrow[r, "\varphi"'] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"'] \\ \mathbb{P}^1 = W_1 \arrow[r, "\tau"'] & W_2 = \{*\} \end{tikzcd}$$

where $\varphi\colon X_1 \to X_2$ is a birational morphism, which is the blow-up of either a real point or two non-real conjugate points of X_2, and where τ is the contraction of $W_1 = \mathbb{P}^1$ to the point W_2. (*It is the inverse of a link of type* I.)

(4) Link of Type IV

$$\begin{tikzcd} X_1 \arrow[r, "\simeq", "\varphi"'] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"'] \\ \mathbb{P}^1 = W_1 & W_2 = \mathbb{P}^1 \end{tikzcd}$$

where $\varphi\colon X_1 \to X_2$ is an isomorphism and $\pi_1, \pi_2 \circ \varphi$ are conic bundles on X_1 with distinct fibres.

Remarks 2.5.

(1) The morphism τ is important only for links of type II, between two surfaces with a Picard group of rank 2 (in higher dimension τ is important also for other links).

(2) There is only one possible W_1 and one possible W_2 in cases I, III, IV but a priori several possibilities in case II.

(3) We shall see in Example 2.13(2) that indeed there exists links of type II where $W_1 = W_2 = \{*\}$. This is a feature of the real case that does not arise in the complex case.

Definition 2.6. If $\pi\colon X \to W$ and $\pi'\colon X' \to W'$ are two (Mori) fibrations, an isomorphism $\psi\colon X \to X'$ is called an *isomorphism of fibrations* if there exists an isomorphism $\tau\colon W \to W'$ such that $\pi'\psi = \tau\pi$.

Note that the composition $\alpha\varphi\beta$ of a Sarkisov link φ with some automorphisms of fibrations α and β is again a Sarkisov link. We have the following fundamental result:

Proposition 2.7. *If $\pi\colon X \to W$ and $\pi'\colon X' \to W'$ are two Mori fibrations, then any birational map $\psi\colon X \dashrightarrow X'$ is either an isomorphism of fibrations or admits a decomposition into Sarkisov links $\psi = \varphi_n \dots \varphi_1$ such that*

(i) for $i = 1, \dots, n-1$, the birational map $\varphi_{i+1}\varphi_i$ is not biregular;

(ii) for $i = 1, \dots, n$, every base-point of φ_i is a base-point of $\varphi_n \dots \varphi_i$.

Proof. Follows from [Isko96, Theorem 2.5] (see also the appendix of [Cort95]).

Let us give an idea of the strategy here, and refer to [Isko96] for the details. If ψ is not an isomorphism of fibrations, then one can associate with it a Sarkisov degree, which is a triple of numbers (a, r, m) (see Definition at page 601 of [Isko96]).

The number $a \in \mathbb{Q}$ is given by the degree of the linear system $\mathcal{H}_X$ on X associated with ψ, the number $r \in \mathbb{N}$ is the maximal multiplicity of the base-points of this system and m is the number of base-points that realise this maximum. Then, we have the following dichotomy:

(i) If $r > a$, we denote by $\pi \colon \hat{X} \to X$ the blow-up of one real point or two conjugate non-real points that realise the multiplicity, then find that either $\hat{X}$ admits a structure of Mori fibration and $\varphi_1 = \pi^{-1}$ is a link of type I, or find a contraction $\pi' \colon \hat{X} \to X_1$ such that $\varphi_1 = \pi'\pi^{-1} \colon X \dashrightarrow X_1$ is a link of type III.

(ii) If $r \leq a$, we either find a contraction $\varphi_1 \colon X \to X_1$ which is a link of type III or find a link of type IV, which is an automorphism $\varphi_1 \colon X \to X$.

In each case, it is shown that the Sarkisov degree of $\psi(\varphi_1)^{-1} \colon X_1 \dashrightarrow X'$ is smaller than the one of ψ, for the lexicographical ordering. The set of all possible Sarkisov degrees beeing discrete and bounded from below ([Isko96, last paragraph of page 601]), the procedure ends at some point.

Moreover, the construction of the links implies that the two properties described above hold. □

Remark 2.8. In the above decomposition, if ψ has no real base-point (for instance when ψ induces an isomorphism $X(\mathbb{R}) \to X'(\mathbb{R})$), then φ_1 and φ_2 have no real base-point. However, the maps φ_i for $i \geq 3$ can have some real base-points, which have been artificially created, and correspond in fact to the base-points of $(\varphi_j)^{-1}$ for $j < i$.

This phenomenon happens for any $\psi \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R})) \setminus \mathrm{Aut}(\mathbb{P}^2)$, as the first link φ_1 will blow-up two non-real base-point and contract the line through these two, onto a real point, base-point of $(\varphi_1)^{-1}$ and of $\varphi_n \ldots \varphi_2$ (see Example 2.13(2) below).

Theorem 2.9 ([Com14] (see also [Isko79]))**.** *Let X be a real rational surface, if X is minimal, then it is isomorphic to one of the following:*

1. $\mathbb{P}^2$*,*
2. *the quadric* $Q_{3,1} = \{(w : x : y : z) \in \mathbb{P}^3 \mid w^2 = x^2 + y^2 + z^2\}$*,*
3. *a Hirzebruch surface* $\mathbb{F}_n = \{((x : y : z), (u : v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\}$ *with* $n \neq 1$*.*

By [Mang06], if $n - n' \equiv 0 \mod 2$, $\mathbb{F}_n(\mathbb{R})$ is isomorphic to $\mathbb{F}_{n'}(\mathbb{R})$. We get:

Corollary 2.10. *Let $X(\mathbb{R})$ be the real locus of a real rational surface. If X is minimal, then $X(\mathbb{R})$ is isomorphic to one of the following:*

1. $\mathbb{P}^2(\mathbb{R})$*,*
2. $Q_{3,1}(\mathbb{R})$*, diffeomorphic to* $\mathbb{S}^2$*,*
3. $\mathbb{F}_0(\mathbb{R})$*, diffeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^1$*,*
4. $\mathbb{F}_3(\mathbb{R})$*, diffeomorphic to the Klein bottle.*

Remark 2.11. Note that $\mathbb{F}_3(\mathbb{R})$ and $\mathbb{F}_1(\mathbb{R})$ are isomorphic. However, $\mathbb{F}_1$ is not minimal although $\mathbb{F}_3$ is.

In the same vein, there exists a birational morphism $\mathbb{P}^2(\mathbb{R}) \to Q_{3,1}(\mathbb{R})$, that contracts a real line (the map φ^{-1} in Example 2.13(2)).

We give a list of Mori fibrations on real rational surfaces, and will show that, up to isomorphisms of fibrations, this list is exhaustive.

Example 2.12. The following morphisms $\pi \colon X \to W$ are Mori fibrations on the plane, the sphere, the Hirzebruch surfaces, and a particular Del Pezzo surface of degree 6.

(1) $\mathbb{P}^2 \to \{*\}$;
(2) $Q_{3,1} = \{(w : x : y : z) \in \mathbb{P}^3_{\mathbb{R}} \mid w^2 = x^2 + y^2 + z^2\} \to \{*\}$;
(3) $\mathbb{F}_n = \{((x : y : z), (u : v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\} \to \mathbb{P}^1$ for $n \geq 0$ (the map is the projection on the second factor);
(4) $\mathcal{D}_6 = \{(w : x : y : z), (u : v) \in Q_{3,1} \times \mathbb{P}^1 \mid wv = xu\} \to \mathbb{P}^1$ (the map is the projection on the second factor).

Example 2.13. The following maps between the surfaces of Example 2.12 are Sarkisov links (in the list, fibers refer to those of the Mori fibrations introduced in Example 2.12):

(1) The contraction of the exceptional curve of $\mathbb{F}_1$ (or equivalently the blow-up of a real point of $\mathbb{P}^2$), is a link $\mathbb{F}_1 \to \mathbb{P}^2$ of type III. Note that the inverse of this link is of type I.
(2) The stereographic projection from the North pole $p_N = (1 : 0 : 0 : 1)$, $\varphi\colon Q_{3,1} \dashrightarrow \mathbb{P}^2$ given by
$$\varphi\colon (w : x : y : z) \dashrightarrow (x : y : w - z)$$
and its inverse $\varphi^{-1}\colon \mathbb{P}^2 \dashrightarrow Q_{3,1}$ given by
$$\varphi^{-1}\colon (x : y : z) \dashrightarrow (x^2 + y^2 + z^2 : 2xz : 2yz : x^2 + y^2 - z^2)$$
are both Sarkisov links of type II.

The map φ decomposes into the blow-up of p_N, followed by the contraction of the strict transform of the curve $z = w$ (intersection of $Q_{3,1}$ with the tangent plane at p_N), which is the union of two non-real conjugate lines. The map φ^{-1} decomposes into the blow-up of the two non-real points $(1 : \pm\mathbf{i} : 0)$, followed by the contraction of the strict transform of the line $z = 0$.
(3) The projection on the first factor $\mathcal{D}_6 \to Q_{3,1}$ which contracts the two disjoint conjugate non-real (-1)-curves $(0 : 0 : 1 : \pm\mathbf{i}) \times \mathbb{P}^1 \subset \mathcal{D}_6$ onto the two conjugate non-real points $(0 : 0 : 1 : \pm\mathbf{i}) \in Q_{3,1}$ is a link of type III.
(4) The blow-up of a real point $q \in \mathbb{F}_n$, lying on the exceptional section if $n > 0$ (or any point if $n = 0$), followed by the contraction of the strict transform of the fibre passing through q onto a real point of $\mathbb{F}_{n+1}$ not lying on the exceptional section is a link $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ of type II.
(5) The blow-up of two conjugate non-real points $p, \bar{p} \in \mathbb{F}_n$ lying on the exceptional section if $n > 0$, or on the same section of self-intersection 0 if $n = 0$, followed by the contraction of the strict transform of the fibres passing through $p, \bar{p}$ onto two non-real conjugate points of $\mathbb{F}_{n+2}$ not lying on the exceptional section is a link $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+2}$ of type II.
(6) The blow-up of two conjugate non-real points $p, \bar{p} \in \mathbb{F}_n$, $n \in \{0, 1\}$ not lying on the same fibre (or equivalently not lying on a real fibre) and not on the same section of self-intersection $-n$ (or equivalently not lying on a real section of self-intersection $-n$), followed by the contraction of the fibres passing through $p, \bar{p}$ onto two non-real conjugate points of $\mathbb{F}_n$ having the same properties is a link $\mathbb{F}_n \dashrightarrow \mathbb{F}_n$ of type II.
(7) The exchange of the two components $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a link $\mathbb{F}_0 \to \mathbb{F}_0$ of type IV.
(8) The blow-up of a real point $p \in \mathcal{D}_6$, not lying on a singular fibre (or equivalently $p \neq ((1 : 1 : 0 : 0), (1 : 1))$, $p \neq ((1 : -1 : 0 : 0), (1 : -1))$), followed

by the contraction of the strict transform of the fibre passing through p onto a real point of $\mathcal{D}_6$, is a link $\mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ of type II.

(9) The blow-up of two non-real conjugate points $p, \bar{p} \in \mathcal{D}_6$, not lying on the same fibre (or equivalently not lying on a real fibre), followed by the contraction of the strict transform of the fibres passing through $p, \bar{p}$ onto two non-real points of $\mathcal{D}_6$ is a link $\mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ of type II.

Remark 2.14. Note that in the above list, the three links $\mathbb{F}_n \dashrightarrow \mathbb{F}_m$ of type II can be put in one family, and the same is true for the two links $\mathcal{D}_6 \dashrightarrow \mathcal{D}_6$. We distinguished here the possibilities for the base points to describe more precisely the geometry of each link. The two links $\mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ could also be arranged into extra families, by looking if the base points belong to the two exceptional sections of self-intersection -1, but go in any case from $\mathcal{D}_6$ to $\mathcal{D}_6$ (see Definition 2.16 below).

Proposition 2.15. *Any Mori fibration $\pi\colon X \to W$, where X is a smooth projective real rational surface, belongs to the list of Example* 2.12.

Any Sarkisov link between two such Mori fibrations is equal to $\alpha\varphi\beta$, where φ or φ^{-1} belongs to the list described in Example 2.13 *and where α and β are isomorphisms of fibrations.*

Proof. Since any birational map between two surfaces with Mori fibrations decomposes into Sarkisov links and all links of Example 2.13 involve only the Mori fibrations of Example 2.12, it suffices to check that any link starting from one of the Mori fibrations of 2.12 belongs to the list 2.13. This is an easy case-by-case study; here are the steps.

Starting from a Mori fibration $\pi\colon X \to W$ where W is a point, the only links we can perform are links of type I or II centered at a real point or two conjugate non-real points. From Theorem 2.9, the surface X is either $Q_{3,1}$ or $\mathbb{P}^2$, and both are homogeneous under the action of $\mathrm{Aut}(X)$, so the choice of the point is not relevant. Blowing-up a real point in $\mathbb{P}^2$ or two non-real points in $Q_{3,1}$ gives rise to a link of type I to $\mathbb{F}_1$ or $\mathcal{D}_6$. The remaining cases correspond to the stereographic projection $Q_{3,1} \dashrightarrow \mathbb{P}^2$ and its converse.

Starting from a Mori fibration $\pi\colon X \to W$ where $W = \mathbb{P}^1$, we have additional possibilities. If the link is of type IV, then X admits two conic bundle structures and by Theorem 2.9, the only possibility is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. If the link is of type III, then we contract a real (-1)-curve of X or two disjoint conjugate non-real (-1)-curves. The only possibilities for X are respectively $\mathbb{F}_1$ and $\mathcal{D}_6$, and the image is respectively $\mathbb{P}^2$ and $Q_{3,1}$ (these are the inverses of the links described before). The last possibility is to perform a link a type II, by blowing up a real point or two conjugate non-real points, on respectively one or two smooth fibres, and to contract the strict transform. We go from $\mathcal{D}_6$ to $\mathcal{D}_6$ or from $\mathbb{F}_m$ to $\mathbb{F}_{m'}$ where $m' - m \in \{-2, -1, 0, 1, 2\}$. All possibilities are described in Example 2.13. □

We end this section by reducing the number of links of type II needed for the classification. For this, we introduce the notion of standard links.

Definition 2.16. The following links of type II are called *standard*:

(1) links $\mathbb{F}_m \dashrightarrow \mathbb{F}_n$, with $m, n \in \{0, 1\}$;

(2) links $\mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ which do not blow-up any point on the two exceptional section of self-intersection -1.

The other links of type II will be called *special*.

The following result allows us to simplify the set of generators of our groups.

Lemma 2.17. *Any Sarkisov link of type* IV *decomposes into links of type* I, III, *and standard links of type* II.

Proof. Note that a link of type IV is, up to automorphisms preserving the fibrations, equal to the following automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$

$$\tau\colon ((x_1 : x_2), (y_1 : y_2)) \mapsto ((y_1 : y_2), (x_1 : x_2)).$$

We denote by $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the birational map $(x : y : z) \dashrightarrow ((x : y), (x : z))$ and observe that $\tau\psi = \psi\sigma$, where $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. Hence, $\tau = \psi\tau\psi^{-1}$. Observing that ψ decomposes into the blow-up of the point $(0 : 0 : 1)$, which is a link of type III, followed by a standard link of type II, we get the result. $\square$

Lemma 2.18. *Let $\pi\colon X \to \mathbb{P}^1$ and $\pi'\colon X' \to \mathbb{P}^1$ be two Mori fibrations, where X, X' belong to the list $\mathbb{F}_0, \mathbb{F}_1, \mathcal{D}_6$. Let $\psi\colon X \dashrightarrow X'$ be a birational map, such that $\pi'\psi = \alpha\pi$ for some $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$. Then, ψ is either an automorphism or $\psi = \varphi_n \cdots \varphi_1$, where each φ_i is a* standard *link of type* II. *Moreover, if ψ is an isomorphism on the real points (i.e. is an isomorphism $X(\mathbb{R}) \to X'(\mathbb{R})$), the standard links φ_i can also be chosen to be isomorphisms on the real points.*

Proof. We first show that $\psi = \varphi_n \cdots \varphi_1$, where each φ_i is a link of type II, not necessarily standard. This is done by induction on the number of base-points of ψ (recall that we always count infinitely near points as base-points). If ψ has no base-point, it is an isomorphism. If q is a real proper base-point, or $q, \bar{q}$ are two proper non-real base-points (here proper means not infinitely near), we denote by φ_1 a Sarkisov link of type II centered at q (or $q, \bar{q}$). Then, $(\varphi_1)^{-1}\psi$ has less base-points than ψ. The result follows then by induction. Moreover, if ψ is an isomorphism on the real points, i.e. if ψ and ψ^{-1} have no real base-point, then so are all φ_i.

Let $\varphi\colon \mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ be a special link of type II. Then, it is centered at two points $p_1, \bar{p_1}$ lying on the (-1)-curves $E_1, \bar{E_1}$. We choose then two general non-real conjugate points $q_1, \bar{q_1}$, and let $q_2 := \varphi(q_1)$ and $\bar{q_2} := \varphi(\bar{q_1})$. For $i = 1, 2$, we denote by $\varphi_i\colon \mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ a standard link centered at $q_i, \bar{q_i}$. The image by φ_2 of E_1 is a curve of self-intersection 1. Hence, $\varphi_2\varphi(\varphi_1)^{-1}$ is a standard link of type II.

It remains to consider the case where each φ_i is a link $\mathbb{F}_{n_i} \dashrightarrow \mathbb{F}_{n_{i+1}}$. We denote by N the maximum of the integers n_i. If $N \leq 1$, we are done because all links of type II between $\mathbb{F}_j$ and $\mathbb{F}_{j'}$ with $j, j' \leq 1$ are standard. We can thus assume $N \geq 2$, which implies that there exists i such that $n_i = N$, $n_{i-1} < N, n_{i+1} \leq N$. We choose two general non-real points $q_{i-1}, \overline{q_{i-1}} \in \mathbb{F}_{n_{i-1}}$, and write $q_i = \varphi_{i-1}(q_{i-1})$, $q_{i+1} = \varphi_i(q_i)$. For $j \in \{i-1, i, i+1\}$, we denote by $\tau_j\colon \mathbb{F}_{n_j} \dashrightarrow \mathbb{F}_{n'_j}$ a Sarkisov link centered at $q_j, \overline{q_j}$. We obtain then the following commutative diagram

$$\begin{tikzcd}
\mathbb{F}_{n_{i-1}} \arrow[r, dashed, "\varphi_{i-1}"] \arrow[d, dashed, "\tau_{i-1}"] & \mathbb{F}_{n_i} \arrow[r, dashed, "\varphi_i"] \arrow[d, dashed, "\tau_i"] & \mathbb{F}_{n_{i+1}} \arrow[d, dashed, "\tau_{i+1}"] \\
\mathbb{F}_{n'_{i-1}} \arrow[r, dashed, "\varphi'_{i-1}"] & \mathbb{F}_{n'_i} \arrow[r, dashed, "\varphi'_i"] & \mathbb{F}_{n'_{i+1}},
\end{tikzcd}$$

where $\varphi'_{i-1}, \varphi'_i$ are Sarkisov links. By construction, $n'_{i-1}, n'_i, n'_{i+1} < N$, we can then replace $\varphi_i\varphi_{i-1}$ with $(\tau_{i+1})^{-1}\varphi'_i\varphi'_{i-1}\tau_{i-1}$ and "avoid" $\mathbb{F}_N$. Repeating this process if needed, we end up with a sequence of Sarkisov links passing only through $\mathbb{F}_1$ and

$\mathbb{F}_0$. Moreover, since this process does not add any real base-point, it preserves the regularity at real points. □

Corollary 2.19. *Let $\pi\colon X \to W$ and $\pi'\colon X' \to W'$ be two Mori fibrations, where X, X' are either $\mathbb{F}_0, \mathbb{F}_1, \mathcal{D}_6$ or $\mathbb{P}^2$. Any birational map $\psi\colon X \dashrightarrow X'$ is either an isomorphism preserving the fibrations or decomposes into links of type* I, III*, and standard links of type* II.

Proof. Follows from Proposition 2.7, Lemmas 2.17 and 2.18, and the description of Example 2.13. □

3. Generators of the group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$

We start this section by describing three kinds of elements of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, which are birational maps of $\mathbb{P}^2$ of degree 5. These maps are associated to three pairs of conjugate non-real points; the description is then analogue to the description of quadratic maps, which are associated to three points.

Example 3.1. Let $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3} \in \mathbb{P}^2$ be three pairs of non-real points of $\mathbb{P}^2$, not lying on the same conic. Denote by $\pi\colon X \to \mathbb{P}^2$ the blow-up of the six points, which induces an isomorphism $X(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$. Note that X is isomorphic to a smooth cubic of $\mathbb{P}^3$. The set of strict transforms of the conics passing through five of the six points corresponds to three pairs of non-real (-1)-curves (or lines on the cubic), and the six curves are disjoint. The contraction of the six curves gives a birational morphism $\eta\colon X \to \mathbb{P}^2$, inducing an isomorphism $X(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$, which contracts the curves onto three pairs of non-real points $q_1, \bar{q_1}, q_2, \bar{q_2}, q_3, \bar{q_3} \in \mathbb{P}^2$; we choose the order so that q_i is the image of the conic not passing through p_i. The map $\psi = \eta\pi^{-1}$ is a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ inducing an isomorphism $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.

Let $L \subset \mathbb{P}^2$ be a general line of $\mathbb{P}^2$. The strict transform of L on X by π^{-1} has self-intersection 1 and intersects the six curves contracted by η into 2 points (because these are conics). The image $\psi(L)$ has then six singular points of multiplicity 2 and self-intersection 25; it is thus a quintic passing through the q_i with multiplicitiy 2. The construction of ψ^{-1} being symmetric as the one of ψ, the linear system of ψ consists of quintics of $\mathbb{P}^2$ having multiplicity 2 at $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3}$.

One can moreover check that ψ sends the pencil of conics through $p_1, \bar{p_1}, p_2, \bar{p_2}$ onto the pencil of conics through $q_1, \bar{q_1}, q_2, \bar{q_2}$ (and the same holds for the two other real pencil of conics, through $p_1, \bar{p_1}, p_3, \bar{p_3}$ and through $p_2, \bar{p_2}, p_3, \bar{p_3}$).

Example 3.2. Let $p_1, \bar{p_1}, p_2, \bar{p_2} \in \mathbb{P}^2$ be two pairs of non-real points of $\mathbb{P}^2$, not on the same line. Denote by $\pi_1\colon X_1 \to \mathbb{P}^2$ the blow-up of the four points, and by $E_2, \bar{E_2} \subset X_1$ the curves contracted onto $p_2, \bar{p_2}$ respectively. Let $p_3 \in E_2$ be a point, and $\bar{p_3} \in \bar{E_2}$ its conjugate. We assume that there is no conic of $\mathbb{P}^2$ passing through $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3}$ and let $\pi_2\colon X_2 \to X_1$ be the blow-up of $p_3, \bar{p_3}$.

On X, the strict transforms of the two conics $C, \bar{C}$ of $\mathbb{P}^2$, passing through $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3$ and $p_1, \bar{p_1}, p_2, \bar{p_2}, \bar{p_3}$ respectively, are non-real conjugate disjoint (-1) curves. The contraction of these two curves gives a birational morphism $\eta_2\colon X_2 \to Y_1$, contracting $C, \bar{C}$ onto two points $q_3, \bar{q_3}$. On Y_1, we find two pairs of non-real (-1)-curves, all four curves being disjoint. These are the strict transforms of the exceptional curves associated to $p_2, \bar{p_2}$, and of the conics passing through $p_1, p_2, \bar{p_2}, p_3, \bar{p_3}$ and $\bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3}$ respectively. The contraction of these curves gives a birational morphism $\eta_1\colon Y_1 \to \mathbb{P}^2$, and the images of the four curves are points $q_2, \bar{q_2}, q_1, \bar{q_1}$ respectively. Note that the four maps $\pi_1, \pi_2, \eta_1, \eta_2$ are blow-ups of non-real points, so the birational map $\psi = \eta_1\eta_2(\pi_1\pi_2)^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ induces an isomorphism $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$.

In the same way as in Example 3.1, we find that the linear system of ψ is of degree 5, with multiplicity 2 at the points $p_i, \bar{p_i}$. The situation is similar for ψ^{-1}, with the six points $q_i, \bar{q_i}$ in the same configuration: $q_1, \bar{q_1}, q_2, \bar{q_2}$ lie on the plane and $q_3, \bar{q_3}$ are infinitely near to $q_2, \bar{q_2}$ respectively.

One can moreover check that ψ sends the pencil of conics through $p_1, \bar{p_1}, p_2, \bar{p_2}$ onto the pencil of conics through $q_1, \bar{q_1}, q_2, \bar{q_2}$ and the pencil of conics through $p_2, \bar{p_2}, p_3, \bar{p_3}$ onto the pencil of conics through $q_2, \bar{q_2}, q_3, \bar{q_3}$. But, contrary to Example 3.1, there is no pencil of conics through $q_1, \bar{q_1}, q_3, \bar{q_3}$ (because $q_3, \bar{q_3}$ are infinitely near to q_2, $\bar{q_2}$).

Example 3.3. Let $p_1, \bar{p_1}$ be a pair of two conjugate non-real points of $\mathbb{P}^2$. We choose a point p_2 in the first neighbourhood of p_1, and a point p_3 in the first neighbourhood of p_2, not lying on the exceptional divisor corresponding to p_1. We denote by $\pi \colon X \to \mathbb{P}^2$ the blow-up of $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3\bar{p_3}$. We denote by $E_i, \bar{E_i} \subset X$ the irreducible exceptional curves corresponding to the points $p_i, \bar{p_i}$, for $i = 1, 2, 3$. The strict transforms of the two conics through $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3$ and $p_1, \bar{p_1}, p_2, \bar{p_2}, \bar{p_3}$ respectively are disjoint (-1)-curves on X, intersecting the exceptional curves $E_1, \bar{E_1}, E_2, \bar{E_2}$ similarly as $E_3, \bar{E_3}$. Hence, there exists a birational morphism $\eta \colon X \to \mathbb{P}^2$ contracting the strict transforms of the two conics and the curves $E_1, \bar{E_1}, E_2, \bar{E_2}$.

As in Examples 3.1 and 3.2, the linear system of $\psi = \eta\pi^{-1}$ consists of quintics with multiplicity two at the six points $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3}$.

Definition 3.4. The birational maps of $\mathbb{P}^2$ of degree 5 obtained in Example 3.1 will be called *standard quintic transformations* and those of Example 3.2 and Example 3.3 will be called *special quintic transformations* respectively.

Lemma 3.5. *Let $\psi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map inducing an isomorphism $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$. The following hold:*

1. *The degree of ψ is $4k+1$ for some integer $k \geq 0$.*
2. *Every multiplicity of the linear system of ψ is even.*
3. *Every curve contracted by ψ is of even degree.*
4. *If ψ has degree 1, it belongs to $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2) = \mathrm{PGL}(3, \mathbb{R})$.*
5. *If ψ has degree 5, then it is a standard or special quintic transformation, described in Examples 3.1, 3.2 or 3.3, and has thus exactly 6 base-points.*
6. *If ψ has at most 6 base-points, then ψ has degree 1 or 5.*

Remark 3.6. Part (1) is [RV05, Teorema 1].

Proof. Denote by d the degree of ψ and by $m_1, \ldots, m_k$ the multiplicities of the base-points of ψ. The Noether equalities yield $\sum_{i=1}^{k} m_i = 3(d-1)$ and $\sum_{i=1}^{k} (m_i)^2 = d^2 - 1$.

Let $C, \bar{C}$ be a pair of two curves contracted by ψ. Since $C \cap \bar{C}$ does not contain any real point, the degree of C and $\bar{C}$ is even. This yields (3), and implies that all multiplicities of the linear system of ψ^{-1} are even, giving (2).

In particular, $3(d-1)$ is a multiple of 4 (all multiplicities come by pairs of even integers), which implies that $d = 4k+1$ for some integer k. Hence (1) is proved.

If the number of base-points is at most $k=6$, then by Cauchy-Schwartz we get

$$9(d-1)^2=\left(\sum_{i=1}^{k} m_i\right)^2 \leq k\sum_{i=1}^{k}(m_i)^2=k(d^2-1)=6(d^2-1)$$

This yields $9(d-1)\leq 6(d+1)$, hence $d\leq 5$.

If $d=5$, the Noether equalities yield $k=6$ and $m_1=m_2=\cdots=m_6=2$. Hence, the base-points of ψ consist of three pairs of conjugate non-real points $p_1,\bar{p_1},p_2,\bar{p_2},p_3,\bar{p_3}$. Moreover, if a conic passes through 5 of the six points, its free intersection with the linear system is zero, so it is contracted by ψ, and there is no conic through the six points.

(a) If the six points belong to $\mathbb{P}^2$, the map is a standard quintic transformation, described in Example 3.1.

(b) If two points are infinitely near, the map is a special quintic transformation, described in Example 3.2.

(c) If four points are infinitely near, the map is a special quintic transformation, described in Example 3.3. □

Before proving Theorem 1.2, we will show that all quintic transformations are generated by linear automorphisms and standard quintic transformations:

Lemma 3.7. *Every quintic transformation $\psi\in\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ belongs to the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and standard quintic transformations.*

Proof. By Lemma 3.5, we only need to show the result when ψ is a special quintic transformation as in Example 3.2 or Example 3.3.

We first assume that ψ is a special quintic transformation as in Example 3.2, with base-points $p_1,\bar{p_1},p_2,\bar{p_2},p_3,\bar{p_3}$, where $p_3,\bar{p_3}$ are infinitely near to $p_2,\bar{p_2}$. For $i=1,2$, we denote by $q_i\in\mathbb{P}^2$ the point which is the image by ψ of the conic passing through the five points of $\{p_1,\bar{p_1},p_2,\bar{p_2},p_3,\bar{p_3}\}\setminus\{p_i\}$. Then, the base-points of ψ^{-1} are $q_1,\bar{q_1},q_2,\bar{q_2},q_3,\bar{q_3}$, where q_3, $\bar{q_3}$ are points infinitely near to q_2, $\bar{q_2}$ respectively (see Example 3.2). We choose a general pair of conjugate non-real points $p_4,\bar{p_4}\in\mathbb{P}^2$, and write $q_4=\psi(p_4)$, $\bar{q_4}=\psi(\bar{p_4})$. We denote by φ_1 a standard quintic transformation having base-points at $p_1,\bar{p_1},p_2,\bar{p_2},p_4,\bar{p_4}$, and by φ_2 a standard quintic transformation having base-points at $q_1,\bar{q_1},q_2,\bar{q_2},q_4,\bar{q_4}$. We now prove that $\varphi_2\psi(\varphi_1)^{-1}$ is a standard quintic transformation; this will yield the result. Denote by $p_i',\bar{p_i}'$ the base-points of $(\varphi_1)^{-1}$, with the order associated to the p_i, which means that p_i' is the image by φ_i of a conic not passing through p_i (see Example 3.1). Similary, we denote by $q_i',\bar{q_i}'$ the base-points of $(\varphi_2)^{-1}$. We obtain the following commutative of birational maps, where the arrows indexed by points are blow-ups of these points:

$$\begin{tikzcd}
 & Y_1 \arrow[dl, "p_4'{,}\bar{p_4}'"'] \arrow[dr, "p_4{,}\bar{p_4}"] & & Y_2 \arrow[dl, "p_3{,}\bar{p_3}"'] \arrow[dr, "q_3{,}\bar{q_3}"] & & Y_3 \arrow[dl, "q_4{,}\bar{q_4}"'] \arrow[dr, "q_4'{,}\bar{q_4}'"] & \\
X_1 \arrow[d, "p_1'{,}\bar{p_1}'\ p_2'{,}\bar{p_2}'"'] & & X_2 \arrow[ll, dashed, "\hat{\varphi}_1"'] \arrow[rr, dashed, "\hat{\psi}"] \arrow[d, "p_1{,}\bar{p_1}\ p_2{,}\bar{p_2}"'] & & X_3 \arrow[rr, dashed, "\hat{\varphi}_2"] \arrow[d, "q_1{,}\bar{q_1}\ q_2{,}\bar{q_2}"'] & & X_4 \arrow[d, "q_1'{,}\bar{q_1}'\ q_2{,}\bar{q_2}'"'] \\
\mathbb{P}^2 & & \mathbb{P}^2 \arrow[ll, dashed, "\varphi_1"'] \arrow[rr, dashed, "\psi"] & & \mathbb{P}^2 \arrow[rr, dashed, "\varphi_2"] & & \mathbb{P}^2.
\end{tikzcd}$$

Each of the surfaces X_1,X_2,X_3,X_4 admits a conic bundle structure $\pi_i\colon X_i\to\mathbb{P}^1$, which fibres corresponds to the conics passing through the four points blown-up on $\mathbb{P}^2$ to obtain X_i. Moreover, $\hat{\varphi}_1$, $\hat{\psi}$, $\hat{\varphi}_2$ preserve these conic bundle structures. The map $(\hat{\varphi}_1)^{-1}$ blows-up $p_4, \bar{p_4}'$ and contract the fibres associated to them, then $\hat{\psi}$ blows-up $p_3, \bar{p_3}$ and contract the fibres associated to them. The map $\hat{\varphi}_2$ blow-ups the points $q_4, \bar{q_4}$, which correspond to the image of the curves contracted by $(\hat{\varphi}_1)^{-1}$, and contracts their fibres, corresponding to the exceptional divisors corresponding to the points $p_4, \bar{p_4}'$. Hence, $\hat{\varphi}_2\hat{\psi}\hat{\varphi}_1$ is the blow-up of two conjugate non-real points $p_3', \bar{p_3}' \in X_1$, followed by the contraction of their fibres. We obtain the following commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "{p_3',\bar{p_3}'}"'] \arrow[dr, "{q_3',\bar{q_3}'}"] & \\
X_1 \arrow[rr, dashed, "\hat{\varphi}_2\hat{\psi}(\hat{\varphi}_1)^{-1}"] \arrow[d, "{p_1',\bar{p_1}' \atop p_2',\bar{p_2}'}"'] & & X_4 \arrow[d, "{q_1',\bar{q_1}' \atop q_2,\bar{q_2}'}"'] \\
\mathbb{P}^2 \arrow[rr, dashed, "\varphi_2\psi(\varphi_1)^{-1}"] & & \mathbb{P}^2,
\end{tikzcd}$$

and the points $p_3', \bar{p_3}'$ correspond to point of $\mathbb{P}^2$, hence $\varphi_2\psi(\varphi_1)^{-1}$ is a standard quintic transformation.

The remaining case is when ψ is a special quintic transformation as in Example 3.3, with base-points $p_1, \bar{p_1}, p_2, \bar{p_2}, p_3, \bar{p_3}$, where $p_3, \bar{p_3}$ are infinitely near to $p_2, \bar{p_2}$ and these latter are infinitely near to $p_1, \bar{p_1}$. The map ψ^{-1} has base-points $q_1, \bar{q_1}, q_2, \bar{q_2}, q_3, \bar{q_3}$, having the same configuration (see Example 3.3). We choose a general pair of conjugate non-real points $p_4, \bar{p_4} \in \mathbb{P}^2$, and write $q_4 = \psi(p_4)$, $\bar{q_4} = \psi(\bar{p_4})$. We denote by φ_1 a special quintic transformation having base-points at $p_1, \bar{p_1}, p_2, \bar{p_2}, p_4, \bar{p_4}$, and by φ_2 a special quintic transformation having base-points at $q_1, \bar{q_1}, q_2, \bar{q_2}, q_4, \bar{q_4}$. The maps φ_1, φ_2 have four proper base-points, and are thus given in Example 3.2. The same proof as before implies that $\varphi_2\psi(\varphi_1)^{-1}$ is a special quintic transformation with four base-points. This gives the result. □

Lemma 3.8. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map, that decomposes as $\varphi = \varphi_5 \cdots \varphi_1$, where $\varphi_i\colon X_{i-1} \dashrightarrow X_i$ is a Sarkisov link for each i, where $X_0 = \mathbb{P}^2$, $X_1 = Q_{3,1}$, $X_2 = X_3 = \mathcal{D}_6$, $X_4 = Q_{3,1}$, $X_5 = \mathbb{P}^2$. If φ_2 is an automorphism of $\mathcal{D}_6(\mathbb{R})$ and $\varphi_4\varphi_3\varphi_2$ sends the base-point of $(\varphi_1)^{-1}$ onto the base-point of φ_5, then φ is an automorphism of $\mathbb{P}^2(\mathbb{R})$ of degree 5.*

Proof. We have the following commutative diagram, where each π_i is the blow-up of two conjugate non-real points and each η_i is the blow-up of one real point. The two maps $(\varphi_2)^{-1}$ and φ_4 are also blow-ups of non-real points.

$$\begin{tikzcd}
 & & & Y_2 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_3"] & & & \\
 & Y_1 \arrow[ddl, "\pi_1"'] \arrow[dr, "\eta_1"] & \mathcal{D}_6 \arrow[rr, dashed, "\varphi_3"] \arrow[d, "(\varphi_2)^{-1}"] & & \mathcal{D}_6 \arrow[d, "\varphi_4"] & Y_3 \arrow[dl, "\eta_2"'] \arrow[ddr, "\pi_4"] & \\
 & & Q_{3,1} & & Q_{3,1} \arrow[drr, dashed, "\varphi_5"] & & \\
\mathbb{P}^2 \arrow[urr, dashed, "\varphi_1"] \arrow[rrrrrr, dashed, "\varphi"'] & & & & & & \mathbb{P}^2.
\end{tikzcd}$$

The only real base-points are those blown-up by η_1 and η_2. Since η_2 blows-up the image by $\varphi_4\varphi_3\varphi_2$ of the real point blown-up by η_1, the map φ has at most 6 base-points, all being non-real, and the same holds for φ^{-1}. Hence, φ is an automorphism of $\mathbb{P}^2(\mathbb{R})$ with at most 6 base-points. We can moreover see that $\varphi \notin \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, since the two curves of Y_2 contracted by π_2 are sent by $\varphi_4\pi_3$ onto conics of $Q_{3,1}$, which are therefore not contracted by φ_5.

Lemma 3.5 implies that ψ has degree 5. □

Proposition 3.9. *The group* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ *is generated by* $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ *and by elements of* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ *of degree* 5*.*

Proof. Let us prove that any $\varphi \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and elements of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ of degree 5 . Applying Proposition 2.7, we decompose φ into Sarkisov links $\varphi = \varphi_r \cdots \varphi_1$ such that

(i) for $i = 1, \ldots, r-1$, the map $\varphi_{i+1}\varphi_i$ is not biregular;
(ii) for $i = 1, \ldots, r$, every real base-point of φ_i is a base-point of $\varphi_r \ldots \varphi_i$.

In particular, φ_1 has no real base-point.

We proceed by induction on r, the case $r = 0$, corresponding to $\varphi \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, being obvious. We also observe that the decompositions of smaller length obtained by the induction process still satisfy property (ii) above. We can assume that (i) is also satisfied, because removing a biregular map $\varphi_{i+1}\varphi_i$ produces a decomposition of smaller length, still having the property (ii).

Since φ_1 has no real base-point, the first link φ_1 is then of type II from $\mathbb{P}^2$ to $Q_{3,1}$, and $\varphi_r \cdots \varphi_2$ has a unique real base-point $r \in Q_{3,1}$, which is the base-point of $(\varphi_1)^{-1}$. If φ_2 would blow-up this point, then $\varphi_2\varphi_1$ would be biregular, hence φ_2 is a link of type I from $Q_{3,1}$ to $\mathcal{D}_6$. We can write the map $\varphi_2\varphi_1$ as $\eta\pi^{-1}$, where $\pi\colon X \to \mathbb{P}^2$ is the blow-up of two pairs of non-real points, say $p_1, \bar{p_1}, p_2, \bar{p_2}$ and $\eta\colon X \to \mathcal{D}_6$ is the contraction of the strict transform of the real line passing through $p_1, \bar{p_1}$, onto a real point $q \in \mathcal{D}_6$. Note that $p_1, \bar{p_1}$ are proper points of $\mathbb{P}^2$, blown-up by φ_1 and $p_2, \bar{p_2}$ either are proper base-points or are infinitely near to $p_1, \bar{p_1}$.

The fibration $\mathcal{D}_6 \to \mathbb{P}^1$ corresponds to conics through $p_1, \bar{p_1}, p_2, \bar{p_2}$. If φ_3 was a link of type III, then $\varphi_3\varphi_2$ would be biregular, so φ_3 is of type II.

If q is a base-point of φ_3, then $\varphi_3 = \eta'\eta^{-1}$, where $\eta'\colon X \to \mathcal{D}_6$ is the contraction of the strict transform of the line through $p_2, \bar{p_2}$. We can then write $\varphi_3\varphi_2\varphi_1$ into only two links, exchanging p_1 with p_2 and $\bar{p_1}$ with $\bar{p_2}$, and this decreases r, and preserves property (ii) on real base-points.

The remaining case is when φ_3 is the blow-up of two non-real points $p_3, \bar{p_3}$ of $\mathcal{D}_6$, followed by the contraction of the strict transforms of their fibres. We denote by $q' \in \mathcal{D}_6(\mathbb{R})$ the image of q by φ_3, consider $\psi_4 = (\varphi_2)^{-1}\colon \mathcal{D}_6 \to Q_{3,1}$, which is a link of type III, and write $\psi_5\colon Q_{3,1} \dashrightarrow \mathbb{P}^2$ the stereographic projection by $\psi_4(q')$, which is a link of type II centered at $\psi_4(q')$. By Lemma 3.8, the map $\chi = \psi_5\psi_4\varphi_3\varphi_2\varphi_1$ is an element of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ of degree 5. Since $\varphi\chi^{-1}$ decomposes into one link less than φ, with a decomposition having still property (ii), this concludes the proof by induction. □

Proof of Theorem 1.2. By Proposition 3.9, $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and by elements of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ of degree 5. Thanks to Lemma 3.7, $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ is indeed generated by projectivities and standard quintic transformations. □

4. Generators of the group $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

Lemma 4.1. *Let $\varphi\colon Q_{3,1} \dashrightarrow Q_{3,1}$ be a birational map, that decomposes as $\varphi = \varphi_3\varphi_2\varphi_1$, where $\varphi_i\colon X_{i-1} \dashrightarrow X_i$ is a Sarkisov link for each i, where $X_0 = Q_{3,1} = X_2$, $X_1 = \mathcal{D}_6$. If φ_2 has a real base-point, then, φ can be written as $\varphi = \psi_2\psi_1$, where $\psi_1, (\psi_2)^{-1}$ are links of type* II *from $Q_{3,1}$ to $\mathbb{P}^2$.*

Proof. We have the following commutative diagram, where each of the maps η_1, η_2 blow-ups a real point, and each of the maps $(\varphi_1)^{-1}, \varphi_3$ is the blow-up of two conjugate non-real points.

$$\begin{tikzcd}
& Y \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
\mathcal{D}_6 \arrow[rr, dashed, "\varphi_2"] \arrow[d, "(\varphi_1)^{-1}"] & & \mathcal{D}_6 \arrow[d, "\varphi_3"] \\
Q_{3,1} \arrow[rr, dashed, "\varphi"] & & Q_{3,1}.
\end{tikzcd}$$

The map φ has thus exactly three base-points, two of them being non-real and one being real; we denote them by $p_1, \bar{p_1}$, q. The fibres of the Mori fibration $\mathcal{D}_6 \to \mathbb{P}^1$ correspond to conics of $Q_{3,1}$ passing through the points $p_1, \bar{p_1}$. The real curve contracted by η_2 is thus the strict transform of the conic C of $Q_{3,1}$ passing through $p_1, \bar{p_1}$ and q. The two curves contracted by φ_3 are the two non-real sections of self-intersection -1, which corresponds to the strict transforms of the two non-real lines L_1, L_2 of $Q_{3,1}$ passing through q.

We can then decompose φ as the blow-up of p_1, p_2, q, followed by the contraction of the strict transforms of C, L_1, L_2. Denote by $\psi_1\colon Q_{3,1} \dashrightarrow \mathbb{P}^2$ the link of type II centered at q, which is the blow-up of q followed by the contraction of the strict transform of L_1, L_2, or equivalenty the stereographic projection centered at q. The curve $\psi_1(C)$ is a real line of $\mathbb{P}^2$, which contains the two points $\psi_1(p_1)$, $\psi_1(\bar{p_1})$. The map $\psi_2 = \varphi(\psi_1)^{-1}\colon \mathbb{P}^2 \dashrightarrow Q_{3,1}$ is then the blow-up of these two points, followed by the contraction of the line passing through both of them. It is then a link of type II. □

Proof of Theorem 1.1. Let us prove that any $\varphi \in \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is in the group generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, σ_0, σ_1, and standard quintic transformations of $\mathbb{P}^2$. We decompose φ into Sarkisov links: $\varphi = \varphi_r \cdots \varphi_1$. By Corollary 2.19, we can assume that all the φ_i are links of type I, III, or standard links of type II.

We proceed by induction on r, the case $r = 0$, corresponding to $\varphi \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, being obvious.

Note that φ_1 is either a link of type I from $\mathbb{P}^2$ to $\mathbb{F}_1$, or a link of type II from $\mathbb{P}^2$ to $Q_{3,1}$. We now study the possibilities for the base-points of φ_1 and the next links:

(1) Suppose that $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ is a link of type I, and that φ_2 is a link $\mathbb{F}_1 \dashrightarrow \mathbb{F}_1$. Then, φ_2 blows-up two non-real base-points of $\mathbb{F}_1$, not lying on the exceptional curve. Hence, $\psi = (\varphi_1)^{-1}\varphi_2\varphi_1$ is a quadratic transformation of $\mathbb{P}^2$ with three proper base-points, one real and two non-real. It is thus equal to $\alpha\sigma_1\beta$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. Replacing φ with $\varphi\psi^{-1}$, we obtain a decomposition with less Sarkisov links, and conclude by induction.

(2) Suppose that $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ is a link of type I, and that φ_2 is a link $\mathbb{F}_1 \dashrightarrow \mathbb{F}_0$. Then, $\varphi_2\varphi_1$ is the blow-up of two real points p_1, p_2 of $\mathbb{P}^2$ followed by the contraction of the line through p_1, p_2. The exceptional divisors corresponding to p_1, p_2 are two (0)-curves of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, intersecting at one real point.

$(2a)$ Suppose first that φ_3 has a base-point which is real, and not lying on E_1, E_2. Then, $\psi = (\varphi_1)^{-1}\varphi_3\varphi_2\varphi_1$ is a quadratic transformation of $\mathbb{P}^2$ with three proper base-points, all real. It is thus equal to $\alpha\sigma_0\beta$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. Replacing φ with $\varphi\psi^{-1}$, we obtain a decomposition with less Sarkisov links, and conclude by induction.

$(2b)$ Suppose now that φ_3 has non-real base-points, which are $q, \bar{q}$. Since φ_3 is a standard link of type II, it goes from $\mathbb{F}_0$ to $\mathbb{F}_0$, so q and $\bar{q}$ do not lie on a (0)-curve, and then do not belong to the curves E_1, E_2. We can then decompose $\varphi_2\varphi_3\colon \mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ into a Sarkisov link centered at two non-real points, followed by a Sarkisov link centered at a real point. This reduces to case (1), already treated before.

$(2c)$ The remaining case (for (2)) is when φ_3 has a base-point p_3 which is real, but lying on E_1 or E_2. We choose a general real point $p_4 \in \mathbb{F}_0$, and denote by $\theta\colon \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ a Sarkisov link centered at p_4. We observe that $\psi = (\varphi_1)^{-1}\theta\varphi_2\varphi_1$ is a quadratic map as in case $(2a)$, and that $\varphi\psi^{-1} = \varphi_n \ldots \varphi_3\theta^{-1}\varphi_1$ admits now a decomposition of the same length, but which is in case $(2a)$.

(3) Suppose now that $\varphi_1\colon \mathbb{P}^2 \dashrightarrow Q_{3,1}$ is a link of type II and that φ_2 is a link of type II from $Q_{3,1}$ to $\mathbb{P}^2$. If φ_2 and $(\varphi_1)^{-1}$ have the same real base-point, the map $\varphi_2\varphi_1$ belongs to $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. Otherwise, $\varphi_2\varphi_1$ is a quadratic map with one unique real base-point q and two non-real base-points. It is then equal to $\alpha\sigma_0\beta$ for some $\alpha, \beta \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. We conclude as before by induction hypothesis.

(4) Suppose that $\varphi_1\colon \mathbb{P}^2 \dashrightarrow Q_{3,1}$ is a link of type II and φ_2 is a link of type I from $Q_{3,1}$ to $\mathcal{D}_6$. If φ_3 is a Sarkisov link of type III, then $\varphi_3\varphi_2$ is an automorphism of $Q_{3,1}$, so we can decrease the length. We only need to consider the case where φ_3 is a link of type II from $\mathcal{D}_6$ to $\mathcal{D}_6$. If φ_3 has a real base-point, we apply Lemma 4.1 to write $(\varphi_2)^{-1}\varphi_3\varphi_2 = \psi_2\psi_1$ where $\psi_1, (\psi_2)^{-1}$ are links $Q_{3,1} \dashrightarrow \mathbb{P}^2$. By (3), the map $\chi = \psi_1\varphi_1$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and σ_0. We can then replace φ with $\varphi\chi^{-1} = \varphi_r \cdots \varphi_3\varphi_2(\psi_1)^{-1} = \varphi_r \cdots \varphi_4\varphi_2\psi_2$, which has a shorter decomposition. The last case is when φ_3 has two non-real base-points. We denote by $q \in Q_{3,1}$ the real base-point of $(\varphi_1)^{-1}$, write $q' = (\varphi_2)^{-1}\varphi_3\varphi_2(q) \in Q_{3,1}$ and denote by $\psi\colon Q_{3,1} \dashrightarrow \mathbb{P}^2$ the stereographic projection centered at q'. By Lemma 3.8, the map $\chi = \psi(\varphi_2)^{-1}\varphi_3\varphi_2\varphi_1$ is an automorphism of $\mathbb{P}^2(\mathbb{R})$ of degree 5, which is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and standard automorphisms of $\mathbb{P}^2(\mathbb{R})$ of degree 5 (Lemma 3.7). We can thus replace φ with $\varphi\chi^{-1}$, which has a decomposition of shorter length. $\square$

5. Generators of the group $\mathrm{Aut}(Q_{3,1}(\mathbb{R}))$

Example 5.1. Let $p_1, \bar{p_1}, p_2, \bar{p_2} \in Q_{3,1} \subset \mathbb{P}^3$ be two pairs of conjugate non-real points, not on the same plane of $\mathbb{P}^3$. Let $\pi\colon X \to Q_{3,1}$ be the blow-up of these points. The non-real plane of $\mathbb{P}^3$ passing through $p_1, \bar{p_2}, \bar{p_2}$ intersects $Q_{3,1}$ onto a conic, having self-intersection 2: two general different conics on $Q_{3,1}$ are the trace of hyperplanes, and intersect then into two points, being on the line of intersection of the two planes. The strict transform of this conic on X is thus a (-1)-curve. Doing the same for the other conics passing through 3 of the points $p_1, \bar{p_1}, p_2, \bar{p_2}$, we obtain four disjoint (-1)-curves on X, that we can contract in order to obtain a birational morphism $\eta\colon X \to Q_{3,1}$; note that the target is $Q_{3,1}$ because it is a smooth projective rational surface of Picard rank 1. We obtain then a birational map $\psi = \eta\pi^{-1}\colon Q_{3,1} \dashrightarrow Q_{3,1}$ inducing an isomorphism $Q_{3,1}(\mathbb{R}) \to Q_{3,1}(\mathbb{R})$.

Denote by $H \subset Q_{3,1}$ a general hyperplane section. The strict transform of H on X by π^{-1} has self-intersection 2 and has intersection 2 with the 4 curves contracted. The image $\psi(H)$ has thus multiplicity 2 and self-intersection 18; it is then the trace of a cubic section. The construction of ψ and ψ^{-1} being similar, the linear system of ψ consists of cubic sections with multiplicity 2 at $p_1, \bar{p_1}, p_2, \bar{p_2}$.

Example 5.2. Let $p_1, \bar{p_1} \in Q_{3,1} \subset \mathbb{P}^3$ be two conjugate non-real points and let $\pi_1\colon X_1 \to Q_{3,1}$ be the blow-up of the two points. Denote by $E_1, \bar{E_1} \subset X_1$ the curves contracted onto $p_1, \bar{p_1}$ respectively. Let $p_2 \in E_1$ be a point, and $\bar{p_2} \in \bar{E_1}$ its conjugate. We assume that there is no conic of $Q_{3,1} \subset \mathbb{P}^3$ passing through $p_1, \bar{p_1}, p_2, \bar{p_2}$ and let $\pi_2\colon X_2 \to X_1$ be the blow-up of $p_2, \bar{p_2}$.

On X, the strict transforms of the two conics $C, \bar{C}$ of $\mathbb{P}^2$, passing through $p_1, \bar{p_1}, p_2$ and $p_1, \bar{p_1}, \bar{p_2}$ respectively, are non-real conjugate disjoint (-1) curves. The contraction of these two curves gives a birational morphism $\eta_2\colon X_2 \to Y_1$. On this latter surface, we find two disjoint conjugate non-real (-1)-curves. These are the strict transforms of the exceptional curves associated to $p_1, \bar{p_1}$. The contraction of these curves gives a birational morphism $\eta_1\colon Y_1 \to Q_{3,1}$. The birational map $\psi = \eta_1\eta_2(\pi_1\pi_2)^{-1}\colon Q_{3,1} \dashrightarrow Q_{3,1}$ induces an isomorphism $Q_{3,1}(\mathbb{R}) \to Q_{3,1}(\mathbb{R})$.

Definition 5.3. The birational maps of $Q_{3,1}$ of degree 3 obtained in Example 5.1 will be called *standard cubic transformations* and those of Example 5.2 will be called *special cubic transformations*.

Note that since $\mathrm{Pic}(Q_{3,1}) = \mathbb{Z}H$, where H is an hyperplane section, we can associate to any birational map $Q_{3,1} \dashrightarrow Q_{3,1}$, an integer d, which is the *degree of the map*, such that $\psi^{-1}(H) = dH$.

Lemma 5.4. *Let $\psi\colon Q_{3,1} \dashrightarrow Q_{3,1}$ be a birational map inducing an isomorphism $Q_{3,1}(\mathbb{R}) \to Q_{3,1}(\mathbb{R})$. The following hold:*

(1) *The degree of ψ is $2k+1$ for some integer $k \geq 0$.*
(2) *If ψ has degree 1, it belongs to $\mathrm{Aut}_{\mathbb{R}}(Q_{3,1}) = \mathrm{PO}(3,1)$.*
(3) *If ψ has degree 3, then it is a standard or special cubic transformation, described in Examples 5.1 and 5.2, and has thus exactly 4 base-points.*
(4) *If ψ has at most 4 base-points, then ψ has degree 1 or 3.*

Proof. Denote by d the degree of ψ and by $a_1, \dots, a_n$ the multiplicities of the base-points of ψ. Denote by $\pi\colon X \to Q_{3,1}$ the blow-up of the base-points, and by $E_1, \dots, E_n \in \mathrm{Pic}(X)$ the divisors being the total pull-back of the exceptional (-1)-curves obtained after blowing-up the points. Writing $\eta\colon X \to Q_{3,1}$ the birational morphism $\psi\pi$, we obtain

$$\begin{array}{rcl} \eta^*(H) & = & d\pi^*(H) - \sum_{i=1}^n a_i E_i \\ K_X & = & \pi^*(-2H) + \sum_{i=1}^n E_i. \end{array}$$

Since H corresponds to a smooth rational curve of self-intersection 2, we have $(\eta^*(H))^2$ and $\eta^*(H) \cdot K_X = -4$. We find then

$$\begin{array}{rcrcl} 2 & = & (\eta^*(H))^2 & = & 2d^2 - \sum_{i=1}^n (a_i)^2 \\ 4 & = & -K_X \cdot \eta^*(H) & = & 4d - \sum_{i=1}^n a_i. \end{array}$$

Since multiplicities come by pairs, $n = 2m$ for some integer m and we can order the a_i so that $a_i = a_{n+1-i}$ for $i = 1, \dots, m$. This yields

$$\begin{array}{rcl} d^2 - 1 & = & \sum_{i=1}^m (a_i)^2 \\ 2(d-1) & = & \sum_{i=1}^m a_i \end{array}$$

Since $(a_i)^2 \equiv a_i \pmod 2$, we find $d^2 - 1 \equiv 2(d-1) \equiv 0 \pmod 2$, hence d is odd. This gives (1).

If the number of base-points is at most 4, we can choose $m = 2$, and obtain by Cauchy-Schwartz

$$4(d-1)^2 = \left(\sum_{i=1}^m a_i\right)^2 \leq m \sum_{i=1}^m (a_i)^2 = m(d^2-1) = 2(d^2-1).$$

This yields $2(d-1) \leq d+1$, hence $d \leq 3$.

If $d = 1$, all a_i are zero, and $\psi \in \mathrm{Aut}_{\mathbb{R}}(Q_{3,1})$.

If $d = 3$, we get $\sum_{i=1}^m (a_i)^2 = 8$, $\sum_{i=1}^m a_i = 4$, so $m = 2$ and $a_1 = a_2 = 2$. Hence, the base-points of ψ consist of two pairs of conjugate non-real points $p_1, \bar{p_1}, p_2, \bar{p_2}$. Moreover, if a conic passes through 3 of the points, its free intersection with the linear system is zero, so it is contracted by ψ, and there is no conic through the four points.

(a) If the four points belong to $Q_{3,1}$, the map is a standard cubic transformation, described in Example 5.1.

(b) If two points are infinitely near, the map is a special cubic transformation, described in Example 5.2. □

Lemma 5.5. *Let $\varphi \colon Q_{3,1} \dashrightarrow Q_{3,1}$ be a birational map, that decomposes as $\varphi = \varphi_3\varphi_2\varphi_1$, where $\varphi_i \colon X_{i-1} \dashrightarrow X_i$ is a Sarkisov link for each i, where $X_0 = Q_{3,1} = X_2$, $X_1 = \mathcal{D}_6$. If φ_2 is an automorphism of $\mathcal{D}_6(\mathbb{R})$ then φ is a cubic automorphism of $Q_{3,1}(\mathbb{R})$ of degree* 3 *described in in Examples* 5.1 *and* 5.2. *Moreover, φ is a standard cubic transformation if and only if the link φ_2 of type* II *is a standard link of type* II.

Proof. We have the following commutative diagram, where each of the maps π_1, π_2, $(\varphi_1)^{-1}$, φ_3 is the blow-up of two conjugate non-real points.

$$\begin{tikzcd}
& Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{D}_6 \arrow[rr, dashed, "\varphi_2"] \arrow[d, "(\varphi_1)^{-1}"] & & \mathcal{D}_6 \arrow[d, "\varphi_3"] \\
Q_{3,1} \arrow[rr, dashed, "\varphi"] & & Q_{3,1}.
\end{tikzcd}$$

Hence, φ is an automorphism of $\mathbb{P}^2(\mathbb{R})$ with at most 4 base-points. We can moreover see that $\varphi \notin \mathrm{Aut}_{\mathbb{R}}(Q_{3,1})$, since the two curves of Y contracted by π_2 are sent by $\varphi_3\pi_2$ onto conics of $Q_{3,1}$, contracted by φ^{-1}.

Lemma 3.5 implies that φ is cubic automorphism of $Q_{3,1}(\mathbb{R})$ of degree 3 described in Examples 5.1 and 5.2. In particular, φ has exactly four base-points, blown-up by $(\varphi_1)^{-1}\pi_1$. Moreover, φ is a standard cubic transformation if and only these four points are proper base-points of $Q_{3,1}$. This corresponds to saying that the two base-points of φ_2 do not belong to the exceptional curves contracted by $(\varphi_1)^{-1}$, and is thus the case exactly when φ_2 is a standard link of type II. □

Proof of Theorem 1.3. Let us prove that any $\varphi \in \mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(Q_{3,1})$ and standard cubic transformations of $\mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ of degree 3. Applying Proposition 2.7, we decompose φ into Sarkisov links: $\varphi = \varphi_r \cdots \varphi_1$, and assume that every real base-point of φ_i is a base-point of $\varphi_r \dots \varphi_i$. This property implies that all links are either of type I, from $Q_{3,1}$ to $\mathcal{D}_6$, of type II from $\mathcal{D}_6$ to $\mathcal{D}_6$ with non-real base-points, or of type III from $\mathcal{D}_6$ to $Q_{3,1}$. In particular, all base-points of the φ_i and their inverses are non-real. (Note that here the situation is easier than in the case of $\mathbb{P}^2$, since no link produces "artificial" real base-points).

By Lemma 2.18, we can also assume that all links of type II are standard.

We proceed by induction on r. The first link φ_1 is of type I from $Q_{3,1}$ to $\mathcal{D}_6$. If φ_2 is of type III, then $\varphi_2\varphi_1 \in \mathrm{Aut}_{\mathbb{R}}(Q_{3,1})$. We replace these two links and conclude by induction. If φ_2 is a standard link of type II, then $\psi = (\varphi_1)^{-1}\varphi_2\varphi_1$ is a standard cubic transformation. Replacing φ with $\varphi\psi^{-1}$ decreases the number of links, so we conclude by induction. □

Twisting maps and factorisation. Choose a real line $L \subset \mathbb{P}^3$, which does not meet $Q_{3,1}(\mathbb{R})$. The projection from L gives a morphism $\pi_L \colon Q_{3,1}(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$, which induces a conic bundle structure on the blow-up $\tau_L \colon \mathcal{D}_6 \to Q_{3,1}$ of the two non-real points of $L \cap Q_{3,1}$.

We denote by $T(Q_{3,1}, \pi_L) \subset \mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ the group of elements $\varphi \in \mathrm{Aut}(Q_{3,1}(\mathbb{R}))$ such that $\pi_L\varphi = \pi_L$ and such that the lift $(\tau_L)^{-1}\varphi\tau_L \in \mathrm{Aut}(\mathcal{D}_6(\mathbb{R}))$ preserves the set of two non-real (-1)-curves which are sections of the conic bundle $\pi_L\tau_L$.

Any element $\varphi \in T(Q_{3,1}, \pi_L)$ is called a *twisting map of $Q_{3,1}$ with axis L*.

Choosing the line $w = x = 0$ for L, we can get the more precise description given in [HM09, KM09]: the twisting maps corresponds in local coordinates $(x, y, z) \mapsto (1 : x : y : z)$ to

$$\varphi_M \colon (x, y, z) \mapsto \big(x, (y, z) \cdot M(x)\big)$$

where $M \colon [-1, 1] \to O(2) \subset \mathrm{PGL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{P}^1)$ is a real algebraic map.

Proposition 5.6. *Any twisting map with axis L is a composition of twisting maps with axis L, of degree* 1 *and* 3.

Proof. We can assume that L is the line $y = z = 0$.

The blow-up $\tau_L \colon \mathcal{D}_6 \to Q_{3,1}$ is a link of type III, described in Example 2.13(3), which blows-up two non-real points of $Q_{3,1}$. The fibres of the Mori Fibration $\pi \colon \mathcal{D}_6 \to \mathbb{P}^1$ correspond then, via τ_L, to the fibres of $\pi_L \colon Q_{3,1}(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. Hence, a twisting map of $Q_{3,1}$ corresponds to a map of the form $\tau\varphi\tau^{-1}$, where $\varphi \colon \mathcal{D}_6 \dashrightarrow \mathcal{D}_6$ is a birational map such that $\pi\varphi = \pi$, and which preserves the set of two (-1)-curves. This implies that φ has all its base-points on the two (-1)-curves. It remains to argue as in Lemma 2.18, and decompose φ into links that have only base-points on the set of two (-1)-curves. □

6. Generators of the group $\mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$

Proof of Theorem 1.4. Let us prove that any $\varphi \in \mathrm{Aut}(\mathbb{F}_0(\mathbb{R}))$ is generated by $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ and by the the involution

$$\tau_0 \colon ((x_0 : x_1), (y_0 : y_1)) \dashrightarrow ((x_0 : x_1), (x_0y_0 + x_1y_1 : x_1y_0 - x_0y_1)).$$

Observe that τ_0 is a Sarkisov link $\mathbb{F}_0 \dashrightarrow \mathbb{F}_0$ that is the blow-up of the two non-real points $p = ((\mathbf{i} : 1), (\mathbf{i} : 1))$, $\bar{p} = ((-\mathbf{i} : 1), (-\mathbf{i} : 1))$, followed by the contraction of the two fibres of the first projection $\mathbb{F}_0 \to \mathbb{P}^1$ passing through $p, \bar{p}$.

Applying Proposition 2.7, we decompose φ into Sarkisov links: $\varphi = \varphi_r \cdots \varphi_1$, and assume that every real base-point of φ_i is a base-point of $\varphi_r \dots \varphi_i$. This property implies that all links are either of type IV from $\mathbb{F}_0$ to $\mathbb{F}_0$, or of type II, from $\mathbb{F}_{2d}$ to $\mathbb{F}_{2d'}$, with exactly two non-real base-points. In particular, as for the case of $Q_{3,1}$, there is no real base-point which is artificially created.

By Lemma 2.18, we can also assume that all links of type II are standard, so all go from $\mathbb{F}_0$ to $\mathbb{F}_0$.

Each link of type IV is an element of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)$.

Each link φ_i of type II consists of the blow-up of two non-real points $q, \bar{q}$, followed by the contraction of the fibres of the first projection $\mathbb{F}_0 \to \mathbb{P}^1$ passing through $q, \bar{q}$. Since the two points do not belong to the same fibre by any projection, we have $q = ((a + \mathbf{i}b : 1), (c + \mathbf{i}d : 1))$, for some $a, b, c, d \in \mathbb{R}$, $bd \neq 0$. There exists thus an element $\alpha \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ that sends q onto p and then $\bar{q}$ onto $\bar{p}$. In consequence, $\tau_0\alpha(\varphi_i)^{-1} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$. This yields the result. □

7. Other results

Infinite transitivity on surfaces. The group of automorphisms of a complex projective algebraic variety is small: in most of the cases it is a finite dimensional algebraic group. Moreover, the group of automorphisms is 3-transitive only if the variety is $\mathbb{P}^1$. On the other hand, it was proved in [HM09] that for a real rational surface X, the group of automorphisms $\mathrm{Aut}(X(\mathbb{R}))$ acts n-transitively on $X(\mathbb{R})$ for any n. The next theorem determines all real algebraic surfaces X having a group of automorphisms which acts infinitely transitively on $X(\mathbb{R})$.

Definition 7.1. Let G be a topological group acting continuously on a topological space M. We say that two n-tuples of distinct points $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ are *compatible* if there exists an homeomorphism $\psi \colon M \to M$ such that $\psi(p_i) = q_i$ for each i. The action of G on M is then said to be *infinitely transitive* if for any pair of compatible n-tuples of points $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ of M, there exists an element $g \in G$ such that $g(p_i) = q_i$ for each i. More generally, the action of G is said to be infinitely transitive *on each connected component* if we require the above condition only in case, for each i, p_i and q_i belong to the same connected component of M.

Theorem 7.2. [BM10] *Let X be a nonsingular real projective surface. The group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is then infinitely transitive on each connected component if and only if X is geometrically rational and $\#X(\mathbb{R}) \leq 3$.*

Density of automorphisms in diffeomorphisms. In [KM09], it is proved that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is dense in $\mathrm{Diff}\big(X(\mathbb{R})\big)$ for the $\mathcal{C}^\infty$-topology when X is a geometrically rational surface with $\#X(\mathbb{R}) = 1$ (or equivalently when X is rational). In the cited paper, it is said that $\#X(\mathbb{R}) = 2$ is probably the only other case where the density holds. The following collect the known results in this direction.

Theorem 7.3. [KM09, BM10]

Let X be a smooth real projective surface.

- *If X is not a geometrically rational surface, then $\overline{\mathrm{Aut}\big(X(\mathbb{R})\big)} \neq \mathrm{Diff}\big(X(\mathbb{R})\big)$;*

- *If X is a geometrically rational surface, then*
  - *If $\#X(\mathbb{R}) \geq 5$, then $\overline{\mathrm{Aut}\big(X(\mathbb{R})\big)} \neq \mathrm{Diff}\big(X(\mathbb{R})\big)$;*
  - *if $\#X(\mathbb{R}) = 1$, then $\overline{\mathrm{Aut}\big(X(\mathbb{R})\big)} = \mathrm{Diff}\big(X(\mathbb{R})\big)$.*

For $i = 3, 4$, there exists smooth real projective surfaces X with $\#X(\mathbb{R}) = i$ such that $\overline{\mathrm{Aut}\big(X(\mathbb{R})\big)} \neq \mathrm{Diff}\big(X(\mathbb{R})\big)$.

In the above statements, the closure is taken in the $\mathcal{C}^\infty$-topology.

References

[A02] M. Alberich-Carramiñana, *Geometry of the plane Cremona maps*, Lecture Notes in Math., 1769, Springer, Berlin (2002).

[BM10] J. Blanc, F. Mangolte, *Geometrically rational real conic bundles and very transitive actions*, Compos. Math. **147** (2011), no. **1**, 161–187.

[Cas1901] G. Castelnuovo. *Le trasformazioni generatrici del gruppo cremoniano nel piano.* Atti della R. Accad. delle Scienze di Torino, **36**, 861–874, 1901.

[Com14] A. Comessatti, *Sulla connessione delle superficie razionali reali*, Ann. Mat. Pura Appl. **23** (1914), 215–283.

[Cort95] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.

[HM09] J. Huisman, F. Mangolte, *The group of automorphisms of a real rational surface is n-transitive*, Bull. London Math. Soc. **41**, 563–568 (2009).

[Isko79] V.A. Iskovskikh, *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no 1, 19-43, 237.

[Isko91] V.A. Iskovskikh, *Generators of the two-dimensional Cremona group over a nonclosed field.* Trudy Mat. Inst. Steklov. **200** (1991), 157–170; translation in Proc. Steklov Inst. Math. 1993, no. 2 (200), 173188.

[Isko96] V.A. Iskovskikh, *Factorization of birational mappings of rational surfaces from the point of view of Mori theory.* Uspekhi Mat. Nauk **51** (1996) no 4 (310), 3-72.

[IKT93] V.A. Iskovskikh, F.K. Kabdykairov, S.L. Tregub, *Relations in a two-dimensional Cremona group over a perfect field.* Izv. Ross. Akad. Nauk Ser. Mat. 57 (1993), no. 3, 3–69.

[KM09] J. Kollár, F. Mangolte, *Cremona transformations and diffeomorphisms of surfaces*, Adv. in Math. **222**, 44-61 (2009).

[Mang06] F. Mangolte, *Real algebraic morphisms on 2-dimensional conic bundles*, Adv. Geom. **6** (2006), 199–213.

[Mori82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), 133-176.

[Poly97] Yu. Polyakova, *Factorization of birational mappings of rational surfaces over the field of real numbers*, Fundam. Prikl. Mat. **3** (1997), no. 2, 519–547.

[RV05] F. Ronga, T. Vust, *Diffeomorfismi birazionali del piano proiettivo reale*, Comm. Math. Helv. **80** (2005), 517–540.

Jérémy Blanc, Mathematisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel, Schweiz

E-mail address: `Jeremy.Blanc@unibas.ch`

Frédéric Mangolte, LUNAM Université, LAREMA, Université d'Angers, Bd. Lavoisier, 49045 Angers Cedex 01, France

E-mail address: `frederic.mangolte@univ-angers.fr`