Exercice 1 (Hyperbolic Geometry)**.**

In this exercise, we will always abuse notation and use the same symbol for an isometry $g \in \mathrm{Isom}^+(\mathbb{H})$ and one of its matrix representatives $g \in \mathrm{SL}(2,\mathbb{R})$.

(1) (a) Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{S}L(2,\mathbb{R})$.

Prove that $ad > 1$ *if and only if* f is an hyperbolic isometry whose axis intersects the imaginary line $i\mathbb{R}$.

(b) Let f and g be two hyperbolic isometries. Show that the axis of f and g intersect if and only if :

$$\mathrm{tr}^2(f) + \mathrm{tr}^2(g) + \mathrm{tr}^2(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) - 4 < 0.$$

(2) (a) Let F be a quadrilateral in $\mathbb{H}$ with vertices $P, Q, R, S \in \mathbb{H}$ and respective angles $\alpha, \beta, \gamma, \delta$. We assume that there are side pairings $g, h \in \mathrm{Isom}(\mathbb{H})$, such that g sends side $[SR]$ on $[PQ]$ and h sends $[QR]$ on $[PS]$.

Write the (reduced) vertex cycle associated to vertex P.

(b) Write conditions on angles $\alpha, \beta, \gamma, \delta$, so that the group $\Gamma = \langle g, h\rangle$ generated by g, h is Fuchsian of signature $(1; 3)$.

(Recall that signature $(1; 3)$ means genus 1 with a single period 3.)

(c) In that case, what is the area of the quotient $\mathbb{H}/\Gamma$?

(3) Let $h = \begin{pmatrix} \cosh(\mu) & \sinh(\mu) \\ \sinh(\mu) & \cosh(\mu) \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, with $\mu \in \mathbb{R}^*$.

(a) Justify that h is an hyperbolic isometry. Determine the axis of h and its translation length in terms of μ.

(b) (*Bonus : Hard*) Let $g = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$, with $\lambda > 0$.

Show that if $4\sinh^2(\lambda)\sinh^2(\mu) = 3$ then the group $\Gamma = \langle g, h\rangle$ is Fuchsian of signature $(1; 3)$.

Solution

(1) (a) Assume $ad > 1$, and compute the fixed points of M. Let $z \in \mathbb{C}$ such that $\dfrac{az+b}{cz+d} = z$. So we have $cz^2 + (d-a)z - b = 0$.

$$\Delta = (d-a)^2 + 4bc = (d-a)^2 + 4(ad-1) > 0$$

So M has two fixed points that are in $\partial\mathbb{H}$, and hence is hyperbolic. The two roots are :

$$z_1 = \frac{(a-d) + \sqrt{(a-d)^2 + 4(ad-1)}}{2c} \qquad z_2 = \frac{(a-d) - \sqrt{(a-d)^2 + 4(ad-1)}}{2c}$$

As $\sqrt{(a-d)^2 + 4(ad-1)} > (a-d)$ we see that z_1 and z_2 are of opposite sign. The axis of M is the geodesic line joining z_1 to z_2, and hence it will intersect the vertical line $i\mathbb{R}$.

Reciprocally, if M is an hyperbolic isometry whose axis intersect the imaginary line, then the two endpoints of that axis are of opposite sign. These fixed points are the solutions of $cz^2 + (d-a)z - b = 0$. Hence we have $z_1 z_2 = \frac{-b}{c} < 0$. So b and c have the same sign (and are both non zero). Which means that $ad = 1 + bc > 1$.

(b) Up to isometry, we can assume that ϕ is an hyperbolic geometry whose axis is the imaginary line, of the form $\phi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, and $\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In that case we have that $\mathrm{tr}(\phi) = \lambda + \lambda^{-1}$, $\mathrm{tr}(\psi) = a + d$ and $\mathrm{tr}(\phi\psi) = \lambda a + \lambda^{-1} d$. So we get :

$$\begin{aligned}
&\mathrm{tr}^2(\phi) + \mathrm{tr}^2(\psi) + \mathrm{tr}^2(\psi\phi) - \mathrm{tr}(\psi)\mathrm{tr}(\phi)\mathrm{tr}(\phi\psi) - 4 \\
=&(\lambda + \lambda^{-1})^2 + (a+d)^2 + (\lambda a + \lambda^{-1} d)^2 - (\lambda + \lambda^{-1})(a+d)(\lambda a + \lambda^{-1} d) - 4 \\
=&\lambda^2 + \lambda^{-2} + 2 + a^2 + d^2 + 2ad + \lambda^2 a^2 + \lambda^{-2} d^2 + 2ad \\
&- (\lambda^2 a^2 + ad + \lambda^2 ad + d^2 + a^2 + \lambda^{-2} ad + ad + \lambda^{-2} d^2) - 4 \\
=&(\lambda - \lambda^{-1})^2 - ad(\lambda^2 + \lambda^{-2} - 2) \\
=&(\lambda - \lambda^{-1})^2 (1 - ad)
\end{aligned}$$

So we see that $(\lambda - \lambda^{-1})^2(1 - ad) < 0$ if and only if $ad > 1$.

(2) (a) The vertex cycle is given by

$$P \xrightarrow{g^{-1}} S \xrightarrow{h^{-1}} R \xrightarrow{g} Q \xrightarrow{h} P$$

(b) The genus of the surface obtained by identification of the side of the quadrilateral is g such that $2 - 2g = F - E + V = 1 - 2 + 1 = 0$. Hence $g = 1$.

For the period to be 3, we need to have the sum of the angles of the vertex cycle to be equal to $\dfrac{2\pi}{3}$. As there is a unique vertex cycle, it means that

$$\alpha+\beta+\gamma+\delta=\frac{2\pi}{3}$$

By Poincare Polygon Theorem, this condition is sufficient for Γ to be a Fuchsian group.

(c) We have $\mathrm{Area}(\mathbb{H}/\Gamma)=\mathrm{Area}(F)$, which is given by

$$A(F)=2\pi-(\alpha+\beta+\gamma+\delta)=\frac{4\pi}{3}$$

Note that we can also use the signature of the Fuchsian group

$$A(\mathbb{H}/\Gamma)=2\pi\left((2g-2)+\sum_{i=1}^{r}(1-\frac{1}{m_i})\right)=2\pi(0+(1-\frac{1}{3}))=\frac{4\pi}{3}$$

(3) Let $h=\begin{pmatrix}\cosh(\mu) & \sinh(\mu)\\ \sinh(\mu) & \cosh(\mu)\end{pmatrix}\in\mathrm{PSL}(2,\mathbb{R})$, with $\mu\in\mathbb{R}^*$.

(a) We have $\mathrm{tr}(h)=2\cosh(\mu)>2$ so h is an hyperbolic isometry. A simple computation shows that the fixed points are $\{1,-1\}$, and hence the axis of h is the unit circle.
If l is the translation length, then the formula gives $2\cosh\left(\frac{l}{2}\right)=\mathrm{tr}(h)=2\cosh(\mu)$. So we get that the translation length is $l=2\mu$.

(b) (*Bonus : Hard*) An explicit computation gives that

$$\begin{aligned}\mathrm{tr}(ghg^{-1}h^{-1})&=(\cosh^2(\mu)-2\sinh^2(\mu)e^{2\lambda})+(\cosh^2(\mu)-\sinh^2(\mu)e^{-2\lambda})\\&=2\cosh^2\mu-\sinh^2(\mu)(\cosh(2\lambda))\\&=2+2\sinh^2(\mu)(1-\cosh 2\lambda)=2-2\sinh^2(\mu)(2\sinh^2(\lambda))=-1\end{aligned}$$

So $ghg^{-1}h^{-1}$ is an elliptic element of angle $\frac{4\pi}{3}$.
Let P be its unique fixed point in $\mathbb{H}$. And we denote $Q=h^{-1}(P)$, $R=g^{-1}h^{-1}(P)$ and $S=hg^{-1}h^{-1}(P)$. We see that the quadrilateral $PQRS$ satisfies the side pairings of question 2.
From the vertex cycle, we see that the sum of the angle around vertex P is going in the opposite direction compared to the angle of $ghg^{-1}h^{-1}$, and hence the sum of the angle is $-\frac{4\pi}{3}=\frac{2\pi}{3}$. So Γ is Fuchsian of signature $(1;3)$.