General remark about these exercises : In many cases, the idea is to first use an isometry to place yourself in a nice situation where the computations can be made explicitly without trouble. So, for example, remember that you can use the action of Isom($\mathbb{H}$) to send any geodesic to the geodesic of your choice (and still get an extra degree of freedom).

Exercise 11: (Fellow travelling) Let $(x_n) \in \mathbb{H}^{\mathbb{N}}$ such that $x_n \to x_\infty \in \partial\mathbb{H}$. (Recall that this is the usual convergence in $\hat{\mathbb{C}}$). Let $(y_n) \in \mathbb{H}^{\mathbb{N}}$ such that $d_{\mathbb{H}}(x_n, y_n) \le R$. Then $y_n \to x_\infty$.

Proof : In the upper half-plane model, we have the formula

$$\sinh(d(x_n, y_n)) = \frac{|x_n - y_n|}{2\sqrt{\mathrm{Im}(x_n)\mathrm{Im}(y_n)}} \le \sinh(R)$$

Without loss of generality, we can assume that $x_\infty = 0$ and hence that $x_n \to 0$ in $\widehat{\mathbb{C}}$ and so $\mathrm{Im}(x_n) \to 0$.

As $d(x_n, y_n)$ is bounded, it means that $\dfrac{|x_n - y_n|^2}{\mathrm{Im}(y_n)} \to 0$. From this, we deduce easily that $\dfrac{|y_n|^2}{\mathrm{Im}(y_n)} \to 0$.

As $|y_n| \ge \mathrm{Im}(y_n)$, we get immediately that $|y_n| \to 0$. □

Exercise 12: Let l an hyperbolic line in $\mathbb{H}$.

1. Prove that for any $z \in \mathbb{H}$ there is a unique point $S(z)$ such that $d(z, S(z)) = 2d(z, l) = 2d(S(z), l)$. The map S is called the reflexion about line l.
2. Show that if $z \neq S(z)$ then the line l is the perpendicular bisector of the segment $[zS(z)]$.
3. Prove that reflexions are isometries (but not orientation-preserving).
4. Let $M \in \mathrm{PGL}(2, \mathbb{R})$ be the matrix representative of a reflexion. Prove that $\mathrm{tr}(M) = 0$.

Proof :

1. *(An hypothesis is missing from this question (mea culpa). One has to assume additionally that $S(z)$ and z are not on the same side of the line l)*

   If $z \in l$, then $S(z) = z$ satisfies the hypothesis, and is unique. So we assume that $z \notin l$.

   Up to isometry, we can assume that l is the hyperbolic line represented by the eucliden unit circle, and $z = ia$ with $a > 1$. We denote $R = d(z, l)$ and a simple computation gives $R = \ln(a)$. We see that the point $z' = i/a$ satisfies all the hypothesis of the question, which gives the existence of the point $S(z)$.

   Let's now prove that such a point is unique. A point satisfying the hypothesis should be on the hyperbolic circle of radius $2R$ and center z, which is represented in $\mathbb{H}$ by a euclidean circle centered at a point $i\omega$ which passes through z'.

   Such a point should also be on the hypercycle of points at distance R from l passing through z' (which is where we use the additional hypothesis), which is represented by a euclidean circle centered at a point $i\omega'$ which passes through z'.

   These two euclidean circles are tangent to each other and intersect at a single point which is z', which proves unicity.

   (Remark : A more direct geometric proof is possible using the triangle inequality)
2. In the previous example, the perpendicular bisector of ia and i/a is the hyperbolic line l. This property is invariant by isometry.
3. Up to isometry, we can assume that l is now the vertical imaginary line. In that case, one can show that $S(z) = -\overline{z}$. This is an Anti-Mobius map and hence, an orientation reversing isometry.
4. The previous anti-Mobius map is represented by the matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which has $\mathrm{tr}(M) = 0$.

   A reflexion is conjugated to the previous anti-Mobius map (conjugated by an isometry sending the line l to the imaginary line), and trace is invariant by conjugation.

□

Remark : One can show that if $M \in \mathrm{PGL}(2,\mathbb{R})$ with $\det(M) = -1$ and $\mathrm{tr}(M) = 0$, then M is the matrix representative of a reflexion.

Exercise 13: Isometries as product of reflexions.

1. Prove that any element of $\mathrm{Isom}^+(\mathbb{H})$ can be written as the product of two reflexions about hyperbolic lines. (It suffices to prove that you can do this for any element in K, A and N.)
2. Prove that if ϕ is the product of S_1 and S_2 the reflexions about lines l_1 and l_2, then we have that ϕ is elliptic (resp. hyperbolic, parabolic) if and only if l_1 and l_2 intersect (resp. are ultraparallel, are limiting parallels).
3. Assume that g, h are two elliptic isometries without a common fixed point. Using the previous question, give a geometric proof that the commutator $ghg^{-1}h^{-1}$ is an hyperbolic isometry.

Proof :

1. Let $\phi \in \mathrm{Isom}^+(\mathbb{H})$ and P the matrix representative of ϕ. Then P is conjugated in $\mathrm{PSL}(2,\mathbb{R})$ to an element M in K, A or N.
   - If $M = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in K$. Then
   $$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$
   - If $M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N$. Then
   $$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & -1 \end{pmatrix}$$
   - If $M = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} inA$ Then
   $$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a^{-1} \\ a & 0 \end{pmatrix}$$
2. If l_1 and l_2 intersect at a point z, then $\phi(z) = S_1(S_2(z)) = S_1(z) = z$. Hence ϕ has a fixed point in $\mathbb{H}$ and hence is elliptic.

   If l_1 and l_2 are ultraparallel, then there exists a unique line d perpendicular to l_1 and l_2. The line d is stable by S_1 and S_2, and hence the line d is stable by ϕ. Which means that the two endpoints d^+ and d^- in $\partial\mathbb{H}$ are fixed points of ϕ. Hence ϕ has two fixed points in $\partial\mathbb{H}$ and is hyperbolic.

   If l_1 and l_2 are limiting parallel, then they share an endpoint $x_\infty \in \partial\mathbb{H}$. As before, we have $\phi(x_\infty) = x_\infty$. Hence ϕ has a fixed point in $\partial\mathbb{H}$ and hence is parabolic.
3. Let $\phi = ghg^{-1}h^{-1}$. Assume that g is an elliptic element with oriented angle θ and center ω. The element $hg^{-1}h^{-1}$ is conjugated to g^{-1} and hence is an elliptic element with angle $-\theta$ and center $\omega' = h(\omega)$. As g and h do not share fixed point, we have $\omega' \neq \omega$. Let d be the line joining ω and ω', and S the reflexion along d.

   One can write g as a product $g = S_1 S$, with S_1 the reflection along line d_1 passing through ω where the oriented angle between d and d_1 is $\frac{\theta}{2}$. Similarly one can write $hg^{-1}h^{-1}$ as a product SS_2 where the oriented angle between d_2 and d is equal to $\frac{-\theta}{2}$. So $ghg^{-1}h^{-1} = S_1 S_2$.

   But d_1 and d_2 intersect d with the same angle θ (in the same orientation), and hence they are ultraparallel (if d, d_1 and d_2 formed a triangle, we would have that the sum of angles is $\theta/2 + (\pi - \theta/2) + \alpha \geq \pi$.) And the product of two reflections along ultraparallel lines is hyperbolic.

□

Exercise 14: Let g, h be two hyperbolic elements without common fixed points.

1. Prove that for all large $n, m > 0$, the element $g^n h^m$ is hyperbolic.
2. Show that the fixed points of $g^n h^m$ are disjoint with those of g, h.
3. Same questions with g parabolic and h hyperbolic.

Proof :

1. Up to isometry, we can consider that $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$, and $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d > 2$. As g and h do not share a fixed point, we can see that $bc \neq 0$.
$$\mathrm{tr}(g^n h) = \mathrm{tr}\left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \lambda^n a + \lambda^{-n} d$$
If $a \neq 0$, then $\lim |\lambda^n a + \lambda^{-n} d| = +\infty$ and hence for n large enough we have that $\lambda^n a + \lambda^{-n} d > 2$. If $a = 0$, then $h^2 = \begin{pmatrix} bc & bd \\ cd & d^2 + bc \end{pmatrix}$. We can then apply the previous reasoning to $g^n h^2$ because $bc \neq 0$.
2. We keep the same notation as before. So $h^m = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $b'c' \neq 0$ because h^m and h have the same fixed point (which are disjoint from 0 and ∞). Hence
$$g^n h^m = \begin{pmatrix} \lambda^n a' & \lambda^n b' \\ \lambda^{-b} c' & \lambda^{-n} d' \end{pmatrix}$$
We can see that $(\lambda^n b')(\lambda^{-n} c') = b'c' \neq 0$ which means $g^n h^m$ does not fixes 0 nor ∞.
3. A similar reasoning holds. Up to isometry, one can assume $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. In that case
$$g^n h = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}$$
So $\mathrm{tr}(g^n h) = a + d + nc \to \pm\infty$ and $g^n h$ does not fixes ∞.

$\square$