Chapter 1 : The Hyperbolic plane

I Möbius transformations

We start with a little bit of geometry on the Riemann sphere using Möbius transformation.

A - Definitions

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. It is the one-point compactification of the usual complex plane. It is homemorphic to the sphere $\mathbb{S}^2$ (embedded in $\mathbb{R}^3$) through the stereographic projection. It's also homeomorphic to $\mathbb{C}P^1$, the complex projective line, which is the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by $\mathbb{C}^*$.

Definition I.1
A Möbius transformation is a map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$\phi(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \in \mathbb{C}, z \neq -\frac{d}{c} \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Remark : If $c = 0$, the last two cases are the same

We denote by $\mathrm{Mob}(\hat{\mathbb{C}})$ the set of all Möbius transformations. We can give basic examples of Möbius transformations :

- Translations of the form $T_b : z \mapsto z + b$ with $b \in \mathbb{C}$.
- Rotations. For example $R_\theta : z \mapsto e^{i\theta} z$ are the rotations around 0.
- Homotheties. For example $S_\lambda : z \mapsto \lambda z$ with $\lambda \in \mathbb{R}_{>0}$. are the homotheties from 0.
- The involution : $I : z \mapsto \frac{1}{z}$

Exercise 1: Prove that the group $\mathrm{Mob}(\hat{\mathbb{C}})$ is generated by the set

$$\{T_b, b \in \mathbb{C}\} \cup \{R_\theta, \theta \in \mathbb{R}\} \cup \{S_\lambda, \lambda \in \mathbb{R}_{>0}\} \cup \{I\}$$

(Try to decompose any map into these elementary elements).

Any similarity $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ is of course a Mobius transformation.

B - Matrix representatives

There is an obvious map $\mathrm{GL}(2, \mathbb{C}) \to \mathrm{Mob}(\hat{\mathbb{C}})$ defined by :

$$\Psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \phi$$

with the notations as above.

Proposition I.2
The map Ψ is a surjective homorphism. And $\ker \Psi$ is the subgroup of homotheties $\{kI_2, k \in \mathbb{C}^*\}$.

Proof : As an exercise. □

From this we have a bijection

$$\mathrm{Mob}(\hat{\mathbb{C}}) \simeq \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* \simeq \mathrm{PGL}(2, \mathbb{C})$$

We see that up to multiplying by a complex number, we can always choose a matrix representative with determinant 1. There are two such representative opposite to each other. So we have

$$\mathrm{PGL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{Id}\} \simeq \mathrm{PSL}(2, \mathbb{C})$$

We have that if M is the matrix representative of a Mobius transformation ϕ, then the quantity

$$\mathrm{tr}^2(M) = \frac{(\mathrm{trace}(M))^2}{\det(M)}$$

is independant of the chosen representative. We will often call this the "trace" of the matrix M with a little abuse of notation.

C - Action on $\hat{\mathbb{C}}$

Proposition I.3
The group $\mathrm{Mob}(\hat{\mathbb{C}})$ acts transitively and freely on ordered triples of distinct points in $\hat{\mathbb{C}}$.

Proof : We first prove that for any ordered triple of distinct points (z_1, z_2, z_3), there exists a Mobius map that sends it on $(0, \infty, 1)$. This map is given by :

$$T_{(z_1,z_2,z_3)}(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

From this, we can compose two of these maps to send the triple (z_1, z_2, z_3) on any other ordered triple (z_1', z_2', z_3').

To prove unicity, it suffices to show that any Mobius map sending $(0, \infty, 1)$ to itself is the identity map. This is a straightforward computation. $\phi(0) = 0$ implies that $b = 0$. $\phi(\infty) = \infty$ implies that $c = 0$. And finally $\phi(1) = 1$ implies that $\frac{a}{d} = 1$ so that $a = d$. □

Definition I.4
The cross-ratio of the ordered quadruple of distinct points (z_0, z_1, z_2, z_3) is defined by

$$[z_0, z_1, z_2, z_3] = \frac{z_0 - z_1}{z_0 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

***Remark* :** We note that the map sending (z_1, z_2, z_3) to $(0, \infty, 1)$ is exactly the map $z \mapsto [z, z_1, z_2, z_3]$.

Exercise 2: Prove that circular permutation of the four points preserve the cross-ratio. Determine how the cross-ratio change after a non-circular permutation.

Proposition I.5
The cross-ratio is invariant under $\mathrm{Mob}(\hat{\mathbb{C}})$.

Proof : Let $f(z) = \dfrac{az+b}{cz+d}$. And $z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}}$.

$$\left[\frac{az_0+b}{cz_0+d}, \frac{az_1+b}{cz_1+d}, \frac{az_2+b}{cz_2+d}, \frac{az_3+b}{cz_3+d}\right] = \frac{\frac{az_0+b}{cz_0+d} - \frac{az_1+b}{cz_1+d}}{\frac{az_0+b}{cz_0+d} - \frac{az_2+b}{cz_2+d}} \cdot \frac{\frac{az_3+b}{cz_3+d} - \frac{az_2+b}{cz_2+d}}{\frac{az_3+b}{cz_3+d} - \frac{az_1+b}{cz_1+d}}$$
$$= \frac{(az_0+b)(cz_1+d)-(az_1+b)(cz_0+d)}{(az_0+b)(cz_2+d)-(az_2+b)(cz_0+d)} \cdot \frac{(az_3+b)(cz_2+d)-(az_2+b)(cz_3+d)}{(az_3+b)(cz_1+d)-(az_1+b)(cz_3+d)}$$
$$= \frac{(ad-bc)(z_0-z_1)}{(ad-bc)(z_0-z_2)} \cdot \frac{(ad-bc)(z_3-z_2)}{(ad-bc)(z_3-z_1)} = [z_0, z_1, z_2, z_3]$$

□

Let us remind one of the main feature of the cross-ratio :

Proposition I.6
Four points $z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ *are cocyclic or colinear if and only if* $[z_0, z_1, z_2, z_3] \in \mathbb{R}$.

Proof : We compute $\arg[z_0, z_1, z_2, z_3] = \arg \dfrac{z_1 - z_0}{z_2 - z_0} + \arg \dfrac{z_2 - z_3}{z_1 - z_3}$.

The points are aligned if and only if both angles are 0 or π.

Let's assume that the points are not aligned, and that z_0 and z_3 are on the same side of the line (z_1, z_2).

In that case the four points are cocyclic $\Leftrightarrow \widehat{z_1 z_0 z_2} = \widehat{z_1 z_3 z_2} \Leftrightarrow \arg \dfrac{z_1 - z_0}{z_2 - z_0} = +\arg \dfrac{z_1 - z_3}{z_2 - z_3}$. We can do a similar reasoning if z_0 and z_3 are on opposite sides. □

From the two previous Propositions, we see that :

Proposition I.7
Mobius transformations maps circles and lines to circles and lines

Remark : A Mobius transformation does not necessarily send lines to lines (or circles to circles) in general

Definition I.8
A map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is called a conformal map, if it is "angle-preserving" in the following sense. For any smooth curves* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *meeting at a point* P *with a (signed) angle* θ*, then* $f(\mathcal{C}_1)$ *and* $f(\mathcal{C}_2)$ *meet at* $f(P)$ *with the same angle* θ *.*

Proposition I.9
Mobius maps are conformal automorphisms of the Riemann sphere

Proof : In general, this is true for any complex analytic map at any point z_0 such that $f'(z_0) \neq 0$. Indeed, we can see that locally, the map is a similarity with ratio $|f'(z_0)|$ and angle $\mathrm{Arg}(f'(z_0))$. And similarity preserve angles.

So it suffices to show that $f'(z) \neq 0$ for all $z \in \hat{\mathbb{C}}$. For general z and f, we see that $f'(z) = \frac{ad-bc}{cz+d}^2 \neq 0$ when $z \neq \infty$. One can also show the same thing at $z = \infty$ writing in terms of $w = 1/z$. □

A more difficult theorem (butnot relevant to our course) is the reciprocal :

Theorem I.10
We have that $\mathrm{Mob}(\hat{\mathbb{C}}) = \mathrm{Aut}(\hat{\mathbb{C}})$.

Proof : admitted □

D - Anti-Mobius transformations

We can also consider transformations of the Riemann sphere that preserve (unsigned) angles but reverse orientation. An example of such a transformation is the complex conjugation $z \mapsto \bar{z}$. So any composition between a Mobius map and complex conjugation will preserve unsigned angles.

Definition I.11
An Anti-Möbius transformation *is a map* $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *defined by*

$$\phi(z) = \begin{cases} \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } z \in \mathbb{C}, \bar{z} \neq -\frac{d}{c} \\ \infty & \text{if } \bar{z} = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

with $a, b, c, d \in \mathbb{C}$ *such that* $ad - bc \neq 0$.

These tranformation do not preserve the cross-ratio, but send the cross-ratio of four points to its conjugate. Hence these transformations also map circles and lines to circles and lines. The composition of two anti-Möbius maps is a Mobius map. We denote by Mob^* the group of Mobius and anti-Mobius maps.

Inversions are particular anti-Mobius map that leaves invariant a circle (or a line). The inversion along the circle of center ω and radius R is given by $z \mapsto \dfrac{R^2}{\overline{z - \omega}} + \omega$. The inversion along a line is just the usual reflection with respect to that line.

Exercise 3: Any Mobius map can be written as a product of inversions.

Just like Möbius maps are the conformal automorphisms, we have the same result for anti-Möbius map ?

Proposition I.12
Anti-conformal maps are exactly the anti-Möbius transformations.

II Hyperbolic Plane

We now turn into the main object of our course : The hyperbolic plane.

A - Poincaré upper half plane

Definition II.1
We consider the set

$$\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\,(z) > 0\}$$

called the Poincaré upper half-plane.

We always think of $\mathbb{H}$ as a subset of the Riemann sphere $\hat{\mathbb{C}}$. So we can consider the boundary $\partial\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This boundary is homeomorphic to a circle.

This set will play the role of a "world map" of the hyperbolic plane. (Just like the Mercator projection can be used as a map of spherical geometry). There are other models of the hyperbolic plane, but in this course, we will mainly focus on this one for simplicity.

B - Mobius transformation preserving $\mathbb{H}$

We say that a Mobius map ϕ preserves $\mathbb{H}$, if $\phi(\mathbb{H}) = \mathbb{H}$. (Note that the condition is stronger than $\phi(\mathbb{H}) \subset \mathbb{H}$.)

As Mobius maps are continuous, if a Mobius map preserves $\mathbb{H}$, then it also preserves $\partial\mathbb{H}$.

Exercise 4: Reciprocally, what can we say about a map that preserves $\partial\mathbb{H}$?

Proposition II.2
If a Möbius map with matrix $M \in \mathrm{PGL}(2, \mathbb{C})$ preserves $\mathbb{H}$ then it has real coefficients up to the $\mathbb{C}^*$ action, and the determinant of its real representative is positive. Reciprocally, any real matrix with positive determinant corresponds to a Möbius map preserving $\mathbb{H}$.

Proof : As ϕ preserves $\partial\mathbb{H}$, we see that ∞ is sent to to an element of $\mathbb{R} \cup \{\infty\}$.

Up to composing by $z \mapsto \dfrac{-1}{z - \phi(\infty)}$, we can assume that $\phi(\infty) = \infty$. This means that we can assume that $c = 0$.

Up to the action of $\mathbb{C}^*$, we can also assume that $d = 1$. So we only need to show that any map $z \mapsto az + b$ that preserves $\mathbb{H}$ has real coefficients and $a > 0$.

Indeed, $\phi(0) = b \in \mathbb{R}$. And $\phi(1) = a + b \in \mathbb{R}$. Moreover, $\phi(i) = b + ai \in \mathbb{H}$ and hence $a > 0$. □

So the group of Mobius transformation preserving $\mathbb{H}$ (sometimes denoted by $\mathrm{Mob}(\mathbb{H})$) is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$.

The same proposition can be made for anti-Möbius maps.

Proposition II.3
If an anti-Möbius map with matrix $M \in \mathrm{PGL}(2, \mathbb{C})$ preserves $\mathbb{H}$ then it has real coefficients up to the $\mathbb{C}^*$ action, and the determinant of its real representative is negative. Reciprocally, any real matrix with negative determinant corresponds to an anti-Möbius map preserving $\mathbb{H}$.

Proof : As an exercise □

We can consider the full group containing all Möbius and anti-Möbius transformations preserving $\mathbb{H}$, and denote it by $\mathrm{Mob}^*(\mathbb{H})$.

The two above propositions show that this group $\mathrm{Mob}^*(\mathbb{H})$ is isomorphic to $\mathrm{PGL}(2, \mathbb{R})$. This group has two connected components corresponding to positive or negative determinant.

***Remark* :** The map from $\mathrm{GL}(2, \mathbb{R})$ to $\mathrm{Mob}^*(\mathbb{H})$ is not the restriction of the map $\mathrm{GL}(2, \mathbb{C}) \to \mathrm{Mob}(\hat{\mathbb{C}})$ to matrices with real coefficients. The maps is not defined in the same way for matrices with positive or negative determinant.

The groups $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PGL}(2, \mathbb{R})$ both act on $\mathbb{H} \cup \partial\mathbb{H}$. From now on, we will always confuse the Mobius map (or anti-mobius map) and its real representative with determinant ± 1, up to sign.

Proposition II.4
The action of $\mathrm{PSL}(2, \mathbb{R})$ is transitive on $\mathbb{H}$.

Proof : Let $x = a + ib$ and $y = c + id$. Then the map $z \mapsto \frac{b}{d}(z - c) + a$ corresponds to an element of $\mathrm{PSL}(2, \mathbb{R})$ □

In geometrical term, this means that $\mathbb{H}$ is a homogeneous G-space, with $G = \mathrm{PSL}(2, \mathbb{R})$ which means that no point in $\mathbb{H}$ is special.
***Remark* :** The action of $\mathrm{PGL}(2, \mathbb{R})$ is also transitive.

The action is no longer 2-transitive on $\mathbb{H}$ which means that we can define invariant properties of pair of points. (In fact, this is how one would define what a geometry is in the sense of Klein. It's the study of invariant properties for the action of a group on an homogeneous space.) In the sense of Klein, hyperbolic geometry corresponds to the pair $(\mathbb{H}, \mathrm{PGL}(2, \mathbb{R}))$.

For example, we could say that two segments are *congruent* if we can find a element of $\mathrm{Mob}(\mathbb{H})$ that sends one to the other. We will see that we can define an invariant metric such that "congruent" means "having the same length".

Proposition II.5
The $\mathrm{PGL}(2, \mathbb{R})$ *action is 3-transitive on* $\partial\mathbb{H}$*.*

Proof : Use the cross-ratio $\square$

Exercise 5: Is the action of $\mathrm{PSL}(2, \mathbb{R})$ still 3-transitive on $\partial\mathbb{H}$?

C - Metric on $\mathbb{H}$

We can define the Riemannian metric given by $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ on $\mathbb{H}$.

Proposition II.6
With the above metric, the space $\mathbb{H}$ *is a Riemannian manifold of constant curvature* -1*.*

Proof : This is a relatively straightforward differential geometry exercise, but outside the scope of these lectures. $\square$

In a sense the geometry of the hyperbolic plane is the "opposite" of spherical geometry, which is the geometry of an homogeneous Riemannian metric space of curvature $+1$.

Definition II.7
The length of a path parametrized by a C^1 *map* $\gamma : [0, 1] \to \mathbb{H}$ *is given by :*

$$l_{\mathbb{H}}(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(\mathrm{t}))} dt$$

This allows us to define a distance on $\mathbb{H}$.

Definition II.8
The distance between two points, $a, b \in \mathbb{H}$ *is*

$$d_{\mathbb{H}}(a, b) = \inf\{l_{\mathbb{H}}(\gamma) \mid \gamma(0) = a, \gamma(1) = b\}$$

Exercise 6: Prove that this is a distance

Theorem II.9
The group Mob($\mathbb{H}$) *preserves the distance.*

Proof : Let γ be a path in $\mathbb{H}$. And let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$. Let $\gamma(t) = x(t) + iy(t)$ and $T \circ \gamma(t) = u(t) + iv(t)$. We can see that $v(t) = \dfrac{y(t)}{|c\gamma(t)+d|^2}$ A simple computation gives

$$\frac{dT(z)}{dz} = \frac{a(cz+d) - c(az+d)}{(cz+d)^2} = \frac{1}{cz+d}^{2}$$

So we infer that $\left|\frac{dT}{dz}(\gamma(t))\right| = \frac{v(t)}{y(t)}$. Hence we have

$$l_{\mathbb{H}}(T \circ \gamma) = \int_0^1 \frac{|dT\gamma(t)/dt|}{y(t)} dt = \int_0^1 \frac{|\gamma'(t)| \times |dT/dz|(\gamma(t)}{y(t)} dt = \int_0^1 \frac{|\gamma'(t)|}{y(t)} dt = l_{\mathbb{H}}(\gamma)$$

□

Exercise 7: Do the same thing for anti-Möbius transformations.

So we have a large group that acts as isometries of $\mathbb{H}$ with its metric. We now have to understand in more details the lines in this metric.

D - Geodesics

We give the general definition of a geodesic in a metric space.

Definition II.10
A geodesic is a path $\gamma : I \to \mathbb{H}$ that locally minimizes distances, in other words if there exists $v \geq 0$ such that for any $t \in I$, there exists a neighborhood U around t, such that for any $t_1, t_2 \in U$ we have

$$d(\gamma(t_1), \gamma(t_2)) = v|t_1 - t_2|$$

We can reparametrize such a path by a simple scaling to get $v = 1$.

If γ is a geodesic and the interval $I = \mathbb{R}$ we call the image of γ a geodesic line, or a $\mathbb{H}$-line (and eventually just a line when there will be no ambiguity anymore). If $I = [0, +\infty[$ then the image of γ is called a geodesic ray, or a $\mathbb{H}$-ray.

We can give the first example of a geodesic

Proposition II.11
Let z and z' be two points on the same vertical line. Then the vertical path between them is the unique geodesic from z to z', up to reprarametrization.

Proof : Let $\gamma(t) = x(t) + iy(t)$ be any path between ai and bi, with $a, b \in \mathbb{R}_{>0}$.

$$l_{\mathbb{H}}(\gamma) = \int_0^1 \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2} dt \geq \int_0^1 \frac{1}{y(t)} \sqrt{y'(t)^2} dt = \int_a^b \frac{dy}{y} = \ln \frac{b}{a} = l_{\mathbb{H}}(\gamma_0)$$

where γ_0 is the obvious vertical path at unit speed. □

***Remark* :** We see that a consequence of the previous proposition is that if $b > a > 0$ then $d_{\mathbb{H}}(ai, bi) = \ln \frac{b}{a}$.

We can now determine all the geodesics of $\mathbb{H}$, using the group Mob($\mathbb{H}$).

Proposition II.12
Given $a \neq b \in \mathbb{H}$ there exists an isometry ϕ such that $\phi(a) = i$ and $\phi(b) \in i\mathbb{R}$. Consequently, there exists a unique geodesic between a and b.

Proof : There exists a first isometry ψ such that $\psi(a) = i$ because $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}$. LEt $b' = \psi(b)$. Now we just need to prove that there exists $\theta \in \mathbb{R}$ such that $f(\theta) = \mathrm{Re}(R_\theta(b')) = \mathrm{Re}\frac{\cos(\theta)b' - \sin(\theta)}{\sin(\theta)b' + \cos(\theta)} = 0$. One can just use the intermediate value theorem as $f(0) = \mathrm{Re}(b')$ and $f(\frac{\pi}{2}) = \mathrm{Re}\frac{-1}{b'}$ have opposite sign. □

Remark : Knowing this, we can give an alternative definition of a geodesic in $\mathbb{H}$. Namely that a geodesic is a path γ from a to b such that $l(\gamma) = d_{\mathbb{H}}(a, b)$.

Theorem II.13
The $\mathbb{H}$-lines of $\mathbb{H}$ are euclidean vertical lines and euclidean half-circles orthogonal to the boundary.

Proof : The image of a geodesic line by an isometry is a geodesic line. Möbius transformations send the vertical line on circles orthogonal to the boundary or vertical lines.

If two points are not on the same vertical line, then there exists a unique circle orthogonal to the boundary that passes through these two points (that's an euclidean construction). We can then find a Möbius transformation sending this circle to the vertical line. Hence the circle orthogonal to the boundary is a geodesic line.

There are no other geodesic lines due to the uniqueness of the geodesic for two points on the same vertical line. □

Let L be a geodesic line. Then L can be extended in $\hat{C}$ and there are exactly two intersection points between $\hat{L}$ and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$. A geodesic line is uniquely determined by its two end points on $\partial\mathbb{H}$, which means that the space of oriented geodesics is in bijection with $(\partial\mathbb{H})^{(2)}$.

A consequence of the previous proof is the following :

Proposition II.14
Given two points, there exists a unique geodesic line joining them

Proposition II.15
The $\mathrm{PSL}(2, \mathbb{R})$ action is transitive on pair of points at equal distance

Proof : Exercise □

There are various formulas to compute the distance between two points in $\mathbb{H}$.

Proposition II.16
Let z and w be two points in $\mathbb{H}$, and let z^* and w^* be the end points in $\partial\mathbb{H}$ of the unique geodesic passing through z and w. Then
$$d_{\mathbb{H}}(z, w) = \ln[z^*, z, w, w^*]$$

Proof : Up to composition with some isometry, we can consider that $z = i$ and $z^* = 0$, $w^* = \infty$ so that $w = \lambda i$ with $\lambda > 1$. In that case $d(i\lambda i) = \ln(\lambda)$. And on the other hand

$$[w^*, z, w, , z^*] = \frac{(w^* - z)(z^* - w)}{(w^* - w)(z^* - z)} = \frac{(\infty - i)(0 - \lambda i)}{(\infty - \lambda i)(0 - i)} = \lambda$$

□

From this we infer various expression of the distance (Exercise).

Proposition II.17

We have

$$\cosh(d(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}$$

$$\sinh(d(z,w)) = \frac{|z-w|}{2\sqrt{\mathrm{Im}(z)\mathrm{Im}(w)}}$$

$$d(z,w) = \ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$$

End of Lecture 1

To learn a little bit about the Thurston-Weeks triple linkage, one can read the original article in the Scientific American

https://pi.math.cornell.edu/~mathclub/Media/math-of-3-dimensional-manifolds.pdf

E - Circles

Given a point z and a radius $R > 0$ we can consider the $\mathbb{H}$-circle around z at distance R.

Proposition II.18
Hyperbolic circles are the Euclidean circles included in $\mathbb{H}$

Proof : First we prove that the $\mathbb{H}$-circle of center i and radius R is a Euclidean circle. Indeed the equation

$$\cosh^2(R) = 1 + \frac{|z-i|^2}{2\mathrm{Im}(z)} \Leftrightarrow \sinh^2(R) = \frac{x^2 + (y-1)^2}{2y}$$

which is the equation of the Euclidean circle with center $i\cosh(R)$ and radius $\sinh R$.

Now, let $\mathcal{C}$ be a $\mathbb{H}$-circle of center ω and radius R. As the $\mathrm{PSL}(2,\mathbb{R})$ is transitive on $\mathbb{H}$, there exists an isometry sending $\mathcal{C}$ on the $\mathbb{H}$-circle of center i and radius R, which is represented by a Euclidean circle. As Mobius transformations send (euclidean) circles and lines to (euclidean) circles and lines, we infer that the preimage of the Euclidean circle is either a euclidean circle or a line. But this preimage is compact in $\mathbb{H}$ and hence is a Euclidean circle. $\square$

From this discussion on circles, we see that the topology of $\mathbb{H}$ is not different from the Euclidean one. The topology induced by the metric on $\mathbb{H}$ is the same as the usual topology as the hyperbolic disks and the euclidean discs coincide. So $\mathbb{H}$ is not compact, but is complete and proper.

In the Euclidean plane, if you fix a point P and consider the the circles of center O passing through P, and you let O escape to infinity on a line, then the "limit circle" is simply a line. The same procedure in the hyperbolic plane gives rise to a new object called an horocycle.

Definition II.19
An horocycle is the set of point in $\mathbb{H}$ *which are located on an Euclidean circle tangent to the real line, or on an horizontal line. The intersection of that circle with* $\partial\mathbb{H}$ *is the base of the horocycle.*

This will play an important role in the next chapters.

III Other models

A - The Poincaré disk model

The Poincaré model is the unit disc $\mathbb{D}$ endowed with the metric $ds = \frac{2|dz|}{1-|z|^2}$.
We consider the Cayley transform

$$\begin{aligned} C : \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{z-i}{z+i} \end{aligned}$$

This map is an isometry between the two models.

Hyperbolic lines in this model are represented by circles orthogonal to the boundary and diameters.

This model is very useful for visualisation. It respects the homogeneicity of the boundary at infinity, and placing the figures in the center makes them more symmetrical.

The group of Mobius transformation preserving $\mathbb{D}$ corresponds to the isometries of this model. A simple computation shows that this group is isomorphic to $\mathrm{PSU}(1,1)$

B - The hyperboloid model

We consider $\mathbb{R}^{2+1}$ with the quadratic form $q(x) = x_1^2 + x_2^2 - x_3^2$, associated to the symmetric bilinear form $[\cdot,\cdot]$ and we define

$$X = \{x \in \mathbb{R}^3, q(x) = -1, x_3 > 0\}$$

which is the upper sheet of the hyperboloid.

We have $\mathrm{O}(2,1)$ the group of automorphisms preserving this bilinear form. Let $G = \mathrm{O}^+$ the subgroup preserving X, it acts transitively on X.

We can endow X with a Riemannian G-invariant metric. More precisely, the tangent space at $x \in X$ can be identified with

$$T_x X = \{u \in \mathbb{R}^3, [u, x] = 0\}$$

The restriction of $[\cdot,\cdot]$ to this tangent space is a scalar product.

The hyperbolic line between x and y is the intersection of the vector plane passing through x and y with X.

The Poincaré disk model is the projection of the hyperboloid model on the unit disk in $\mathbb{R}^2 \times \{0\}$ viewed from $(-1, 0, 0)$.

The group of isometries of this model is the group of linear transformation of $\mathbb{R}^3$ preserving the bilinear form, and preserving the upper half sheet of the hyperboloid. So one can write this group as $\mathrm{SO}_0(2,1)$.

C - Klein-Beltrami model

The Klein-Beltrami model : We take the projectivisation of the hyperboloid model in the projective space $\mathbb{R}P^2$. We obtain a subset Ω (that we can identify with the unit disk in $\mathbb{R}^2 \times \{1\}$) of projective space, and we endow Ω with the induced metric. In that case, the geodesic lines are projective lines.

The base space is still the unit disc but the metric is different. In that model, the lines are euclidean segments (chords). But this model is not conformal, and the angles are not the euclidean angles. The distance between two points in this model is half the logarithm of the cross-ratio of the four (aligned) points.

D - Others

There are other more exotic models, like the hemisphere model : the projection on the unit upper hemisphere in R^3 of the hyperboloid, with respect to the point $(0, 0, -1)$. The lines are arcs of circles orthogonal to the Equator. This model is conformal (but not planar).

Or also the projective model, where we see the hyperbolic plane as a subset of $\mathbb{R}P^2$.

IV Elements of hyperbolic geometry

A - Elementary hyperbolic geometry

The metric defined on $\mathbb{H}$ is conformal, which means that at each point $z \in \mathbb{H}$, the inner product is proportionnal to the usual euclidean inner product, and hence hyperbolic angles between vectors in $T_z\mathbb{H}$ are equal to the euclidean angles. The isometries of $\mathbb{H}$ are conformal so isometries preserve angles.

With this setting, we have a set of points, a set of lines, and angles between these lines. We also have a group acting on that space, so we can do elementary Geometry "à la Euclid".

Definition IV.1
Two lines are said to be parallel if they do not intersect in $\mathbb{H}$. Two lines are perpendicular if they intersect at an angle $\pi/2$.

Proposition IV.2
Hyperbolic geometry satisfies all the axioms of Euclidean geometry, at the exception of the parallel postulate.

Proof : For your culture, here is a possible list of axioms defining Euclidean geometry (see Hilbert's axioms) :

- Incidence axioms
  - (I.1) For any points $A \neq B$, there exists a unique line between A et B, denoted (AB).
  - (I.2) Any line has at least two points.
  - (I.3) There are three points not on the same line.
- Order axioms
  - (O.1) Each line is a totally ordered set without smallest or largest element.
  - (O.2) If d is a line, the relation $A \sim B$ defined by $(AB) \cap d = \emptyset$ is an equivalence relation with two classes (being on the same side).
- Congruence Axioms. Let G be the group acting on E.
  - (G.1) The action preserves the order.
  - (G.2) The action is free and transitive.
  - (G.3) For any pair of point, there is an isometry exchanging them.
  - (G.4) For any pair of geodesic rays based on the same point, there is an isometry exchanging them.
- Continuity axiom
  - (C) On any line d, if $d = T_1 \cup T_2$ so that $T_1 < T_2$ then there exists a unique point $A \in d$ such that $T_1 \leq A \leq T_2$.
- Parallel axiom
  - (P) Given a line l and a point $P \notin l$, There exists at most one line m passing through P not intersecting l.

For hyperbolic geometry, the last axiom is replaced by "There exists infinitely many lines passing through P not intersecting l".

It's a simple verification to check that hyperbolic geometry satisfies these axioms. □

What this means is that there are many propositions and theorem of Euclidean geometry that can be proven exactly in the same way in hyperbolic geometry. (In Euclid's Elements, you can get until Proposition 29 without using the parallel postulate). Let us illustrate a few of these propositions (the proof are elementary geometric proofs using "ruler and compass").

Proposition IV.3
If two triangles have congruent sides (with the same length) , then they are congruent (isometric).

Proof : Let ABC and DEF be the two triangles (with $AB = DE$ ...) . Choose an isometry such that $f(D) = A$ (Axiom G.(2)). Let $E' = f(E)$ and $F' = f(F)$. Now, there is an isometry g exchanging the lines AB and AE' (axiom $G.4$). Let $g(E') = E$" . We have $AE" = AE' = DE = AB$, and $E' \in [AB)$, so $E" = B$.

We have $AF"' = AF' = DF = AC$ and $BF'' = E'F' = EF = BC$. So the point F"' is at the intersection of the two circles centered on A and B passingt hthrough C. So either $F"' = C$ and we are done, or $F \neq C$ and we apply the symetry along the line AB. □

Proposition IV.4
Given two points, the set of points equidistant to A and B is the unique perpendicular line bisector.

Proof : Let A, B be two points. Construct the equilateral triangle ABC. Then construct the angle bisector of this triangle at C. It intersects (AB) at point D. The triangles ACD and BCD have a ASAS congruence so they are isometric. Hence, $BD = CD$, and the angle ACD is equal to BCD, which means that they are right angle. So (CD) is the line perpendicular to (AB). Now, for any point $E \in (CD)$, the triangles ACE and BCE have an SAS relation, and hence they are isometric.

Reciprocally, given a point F such that $AF = BF$. The triangle ADF and BDF have an SSS relation, and hence they are congruent. So the angle at D are equal and hence are right angles. So F is on the line (CD) □

Proposition IV.5
Given a line and a point not on this line, there exists a unique hyperbolic line perpendicular passing through the point.

***Remark* :** This allows us to easily compute the distance from a point to a given line, by finding the base point of the unique perpendicular through that point

B - New phenomenons of hyperbolic geometry

1) *Parallelism*

The notion of parallelism is very different in hyperbolic geometry. In particular, parallelism is no longer a transitive relation. We can split the notion of parallelism into two cases

Definition IV.6
Two parallel lines are said to be limiting parallel *if they share a common endpoint on $\partial\mathbb{H}$. Otherwise, they are said to be* ultraparallel.

Given a line l and a point P, there are exactly two limiting parallel to l passing through P.

Theorem IV.7
Given two ultraparallel lines, there exists a unique line perpendicular to both. Reciprocally, if two lines have a common perpendicular, then they are ultraparallel

Proof : It suffices to prove the theorem for the lines $(0, \infty)$ and $(1, a)$ for some $a > 0$. A circle perpendicular to the vertical line is necessarily centered at 0. The radius R of such a circle needs to satisfy $(\frac{1+a}{2})^2 = R^2 + (\frac{a-1}{2})^2$ for the two circles to be orthogonal. There is a unique positive solution to this equation, so the theorem is proved in that situation.

For any two ultraparallel hyperbolic lines with endpoint (x, y) and (z, w), one can find an isometry sending (x, y, z) to $(0, \infty, 1)$. As the lines are not intersecting, the w is sent on a point $a > 0$. □

So the distance between two ultraparallel lines is given by the length of the unique line segment perpendicular to both. This situation is ver different from the Euclidean setting because in Euclidean plane, if two lines are parallel, then any third line perpendicular to one of them will be perpendicular to both of them.

In Euclidean plane, the set of points at a given distance from a line is itself a line. In the hyperbolic plane, the set of points at a given distance from a line is not a line. This is called an hypercycl**Exercise 8:** In the upper half plane model, prove that an hypercircle will be represented by the arc of a circle intersecting $\mathbb{R}$ at two points, or by a straight non horizontal line.

2) Polygons

A polygon is usually defines as a finite number of straight line segments connected to form a closed polygonal chain. In that case, the points where two edges meet is called a vertex.

In hyperbolic geometry, the notion of asymptotic parallel allows us to define a new kind of triangle (and polygons in general). A point on $\partial\mathbb{H}$ will be called an ideal point. So we can consider polygons where two edges do not intersect in $\mathbb{H}$ but share a point in $\partial\mathbb{H}$ which is called an ideal vertex. (So the sides adjacent to an ideal vertex are limiting parallel) A polygon that have only ideal vertices will be called an ideal polygon.

The angle at an ideal vertex of a polygon is defined as 0. This is consistent with the fact that such an ideal vertex is the limit of a sequence of vertices converging towards that endpoint in $\mathbb{H} \cup \partial\mathbb{H}$. The closer you get from the boundary, the smaller the angle. The length adjacent to an ideal vertex will be infinite. Again, this is consistent.

Proposition IV.8
All ideal triangles are isometric

Proof : This is an obvious consequence of the 3-transitivity of the $\mathrm{PGL}(2,\mathbb{R})$ action on $\partial\mathbb{H}$ □

Remark : Note that ideal quadrilateral are no longer all isometric. In fact two quadrilateral will be isometric if they share the same cross-ratio.

Proposition IV.9
The angle sum of a triangle is strictly less than π. The difference is called the angle defect

Proof : See later with area of a triangle □

In Euclidean geometry, three points are either aligned or cocyclic. This is no longer the case in hyperbolic geometry. A consequence is that certain triangles do not have a circuscribed (hyperbolic) circle.

Exercise 9: Prove that three points in $\mathbb{H}$ that are not aligned and not cocylic iff and only if they are on a unique horocyle or hypercycle.

Hyperbolic triangles are thin. Which means that there exists a universal constant δ such that for all triangle, a side is included in the δ neighborhood of the union of the two other sides. There exists an optimal constant delta that can be computed as $\ln(1+\sqrt{2})$ (again,use a well chosen ideal triangle). This property is known as δ-hyperbolicity and will be very important to define hyperbolic groups.

Any hyperbolic triangle has an inscribed circle, whose radius is bounded above by $\ln(\sqrt{(3)})$ (which is the value for an ideal triangle).

3) Trigonometry

Exercise 10: The angle between two lines (a,b) et (c,d) that interrsect satisfies $\cos(\theta) = 2[a,c,b,d]-1$. (up to sign and correct choice orientation of the lines)

Proposition IV.10
In a right-angle with sides a,b,c and angles α,β,γ, with $\gamma=\frac{\pi}{2}$ then $\cos(\alpha) = \dfrac{\tanh(b)}{\tanh(c)}$.

Proof : This is a rather intricate computation but in a well-chosen triangle, this can be done. This exercise is easier in the Poincare Disk model, because one can choose a triangle where two sides are represented by straight line segments, and hence one can use a more Euclidean geometrical approach. □

Right-angle triangles in hyperbolic geometry do not satisfy Pythagoras Theorem. But the following statement is somehow the equivalent of Pythagorean theorem :

Proposition IV.11
If Δ is a right-angle triangle and a, b, c the lengths of the sides, then

$$\cosh(c) = \cosh(a)\cosh(b)$$

A more surprising formula is the following

Proposition IV.12
With the same hypothesis, $\cosh(a) = \dfrac{\cos(\beta)}{\cos(\alpha)}$.

An important corollary of this is

Corollary IV.13
The angles of a triangle entirely determine the triangle up to isometry

This is a very different behavior compared to euclidean geometry. A consequence is that if two triangles are similar, then they are isometric. In fact there are no similarities in hyperbolic geometry, except isometries.

C - Area and measure

We can define an area form on $\mathbb{H}$ with the following formula :

$$\mu(A) = \int_A \frac{dxdy}{y^2} =$$

Proposition IV.14
This volume is invariant by $\mathrm{PSL}(2, \mathbb{R})$.

Proof : Consider that the Mobius map $T(z) = \dfrac{az+b}{cz+d}$. It is an holomorphic function on $\mathbb{H}$ and so using Cauchy-Riemann equations, we have that the Jacobian is

$$\mathrm{J}T = \left|\frac{dT}{dz}\right|^2 = \frac{1}{|cz+d|^4}$$

so $(\mathrm{Im}(T(z)))^2 = v^2 = y^2 \times \mathrm{Jac}(T(z))$. Now, one can compute doing a change of variable $z \to T(z)$

$$\mu(T(A)) = \int_{T(A)} \frac{dxdy}{y^2} = \int_A \frac{\mathrm{Jac}(T)(z)dudv}{v^2} = \mu(A)$$

□

Given this we can measure the area of various geometric figures such as triangles and discs.

Proposition IV.15
The area of a disc of radius R is $4\pi(\sinh^2(R/2))$.

Proof : We consider the circle centered on i of radius R. it is the Euclidean circle of center $\cosh(R)$ and radius $\sinh(R)$. The rest is an integration exercise. □

Theorem IV.16
An hyperbolic triangle with angles α, β and γ has an area equal to

$$A = \pi - \alpha - \beta - \gamma$$

Proof : We first prove that a triangle with one vertex on the boundary satisfies the formula. Up to isometry, we can assume that this vertex is ∞ and that the two other points are on th unit circle. One can then do a direct computation relating the angles with the coordinates of the two points.

Now assume that no vertex is on the boundary. Then we extend one of the sides up to infty to get a new triangle split in two. And the area of this new triangle is the sum of the two part, ending the proof. □

This proves that the sum of the angles of a triangle are always less than π.

Corollary IV.17
For an hyperbolic polygon, the area is $A = (n-2)\pi - (\theta_1 + \cdots + \theta_n)$.

Proof : Cut the polygon into triangles □

There is a reciprocal to this theorem

Proposition IV.18
There exists a polygon with prescribed angles if and only if $\theta_1 + \cdots + \theta_n < (n-2)\pi$.

V Isometries

A - The group of isometries of the hyperbolic plane

We first prove that there are no other isometries than the ones coming from Mobius maps and anti-Mobius maps.

Theorem V.1
We have $\mathrm{PGL}(2, \mathbb{R}) \simeq \mathrm{Isom}(\mathbb{H})$.

Proof : Recall that isometries send geodesics to geodesics.

Let f be an isometry. We can find a Möbius transformation ϕ such that $f \circ \phi = g$ is a map that fixes i and preserve the vertical line. Hence it fixes the entire line. Let $z \in \mathbb{H}$, we have $d(z, it) = d(g(z), it)$ for all $t \in \mathbb{R}$. Using formulas for the distance, we get that $g(z) = z$ or $g(z) = -\bar{z}$. By continuity of g, the choice is consistant for all z, and we get the result. □

In the rest of the course, we will mostly consider orientation preserving isometries, denoted $\mathrm{Isom}^+(\mathbb{H})$ which correspond to the group $\mathrm{PSL}(2,\mathbb{R})$. The group $\mathrm{PSL}(2,\mathbb{R})$ has some remarkable subgroups listed below :

- $R = \mathrm{Stab}_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \ , \ a > 0, b \in \mathbb{R} \right\}$.
- $A = \mathrm{Stab}_{0,\infty} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ , \ \lambda > 0 \right\}$
- $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \ , \ t \in \mathbb{R} \right\}$
- $K = \mathrm{Stab}_i = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \ , \ \theta \in \mathbb{R} \right\} = \mathrm{PSO}_2(\mathbb{R})$.

We have various properties relating these groups : $N \triangleleft R$, $N \simeq \mathbb{R} \simeq A$, $R = A \ltimes N$.

Remark : The R-action is transitive on $\mathbb{H}$, and the K-action is transitive on $\partial\mathbb{H}$

Proposition V.2

(Iwasawa decomposition or KAN decomposition) For any $g \in \mathrm{PSL}(2,\mathbb{R})$ there exists unique $(k,a,n) \in K \times A \times N$ such that $g = kan$.

Proof : Assume that $g^{-1}(i) = x + iy$. The let $n = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. So $n \circ g^{-1}(i) = iy$. Then choose $a = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & -\sqrt{y} \end{pmatrix}$, so that $ang^{-1}(i) = i$. Which means that there exists $k \in K$ such that $ang^{-1} = k^{-1}$.

For uniqueness, let $kan = k'a'n'$. We have that $kan(\infty) = k(\infty) = k'(\infty) = k'a'n'(\infty)$. Which is equal to $\frac{\cos(\theta)}{\sin(\theta)}$. So $k = k'$. Now $n^{-1}a^{-1}(0) = n^{-1}(0) = n'^{-1}(0)$, so that $n = n'$. Hence we can conclude that $k = k'$.

□

Remark : One could choose any geodesic (z,w) with $z, w \in \partial\mathbb{H}$ and any $x \in (z,w)$, and obtain a similar decomposition using the subgroups $K_x = \mathrm{Stab}(x)$, $A_{z,w} = \mathrm{Stab}(z,w)$ and N_z conjugated to K, A, N by the same element.

B - Classification of Isometries

Theorem V.3

Let $g \in \mathrm{PSL}(2,\mathbb{R})$ with $g \neq \mathrm{Id}$. Then g is conjugated to a matrix in one of the three subgroup A, N or K. More precisely we have the following classification :

1. $|\mathrm{tr}(g)| < 2$ iff *g is conjugated to an element of K* iff *g fixes a single point in $\mathbb{H}$. In that case we say that g is elliptic.*
2. $|\mathrm{tr}(g)| = 2$ iff *g is conjugated to an element of N* iff *g fixes a single point in $\partial\mathbb{H}$. In that case we say that g is parabolic.*
3. $|\mathrm{tr}(g)| > 2$, iff *g is conjugated to an element of A* iff *g fixes two points in $\partial\mathbb{H}$. We say that g is hyperbolic.*

Proof : Let $g \neq \mathrm{Id}$, with $g = \frac{az+b}{cz+d}$ and $ad - bc = 1$. We are looking at the fixed points in $\mathbb{H} \cup \partial\mathbb{H}$. We solve the equation $\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0$. If $c \neq 0$, then this equation has discriminant $\Delta = (d-a)^2 + 4(bc) = a^2 + d^2 - 2ad + 4bc = (a+d)^2 - 4(ad - bc) = (a+d)^2 - 4$.

We get that if $\mathrm{tr}(g) = |a+d| < 2$, then the equation has two distinct complex solutions which are complex conjugates. Only one is in the upper half plane, so g fixes a unique point in $\mathbb{H}$.

If $\mathrm{tr}(g) = |a+d| = 2$, then the equation has a unique real solution, so g fixes a unique point in $\partial\mathbb{H}$.

If $\mathrm{tr}(g) = |a+d| > 2$, then the equation has two distinct real solutions, so g fixes exactly two points in $\partial\mathbb{H}$.

For the other part of the theorm, if g fixes a point p in $\mathbb{H}$. Then there exists an isometry ϕ such that $\phi(i) = p$. Then the map $\phi^{-1} \circ g \circ \phi$ is an isometry and it fixes the points i. So $\phi^{-1} \circ g \circ \phi \in K$. Finally, if g is conjugated to an element in K, then its trace is $2\cos\theta \in [-2, 2]$ with equality only if $g = \mathrm{Id}$.

The other cases are done similarly (Exercise) □

We can give geometric interpretation of these isometries. As we will see elliptic elements correspond to "rotations", and hyperbolic elements will behave a little bit like translations. Parabolic elements are new features of hyperbolic geometry that do not exist in Euclidean geo,etry.

Definition V.4
The translation length of an element g is defined by $l(g) = \inf\{d(z, g(z))\ ,\ z \in \mathbb{H}$.

Here is a more detailed description of each type of element

Proposition V.5
Let g be a non-identity orientation preserving isometry

1. *If g is elliptic, then :*
   - *We have $l(g) = 0$ and the infimum is reached at the unique fixed point.*
   - *g preserves all the circles centered on that fixed point.*
   - *All the orbits of g are bounded.*
   - *g acts as a rotation. A line passing through the center is sent to another line that forms an angle of θ with the first.*
2. *If g is parabolic, then :*
   - *$l(g) = 0$ but is not reached.*
   - *g preserve all the horospheres passing through p.*
   - *No orbit is bounded in $\mathbb{H}$.*
   - *For any $x \in \mathbb{H} \cup \partial\mathbb{H}$, we have $g^n(x) \to p$ and $g^{-n}(x) = p$.*
3. *If g is hyperbolic, then :*
   - *$l(g) > 0$ and the infimum is reached on the unique geodesic through the fixed points, denoted g^- and g^+.*
   - *On this geodesic, g acts as a translation of length $l(g)$, from g^- towards g^+.*
   - *g preserves the hypercycles based on g^- and g^+.*
   - *No orbit is bounded in $\mathbb{H}$. For all $x \in \mathbb{H}$, we have $g^n(x) \to g^+$ and $g^{-n}(x) \to g^-$.*
   - *For any point open neighborhood U^+ and U^- of g^+ and g^-, there exists $n_0 > 0$ such that $g^n(\partial\mathbb{H} \setminus U^-) \subset U^+$. (This behavior is called North-South Dynamic).*

Proof : All these properties are easy to show on particular examples of elliptic, parabolic or hyperbolic elements. □

Let's finish with another way of writing elements of $\mathrm{PSL}(2, \mathbb{R})$.

Proposition V.6
(Cartan Decomposition - KAK) Let $M \in \mathrm{PSL}(2,\mathbb{R})$*, then there exists* $P, Q \in K$ *and* $\Delta \in A$ *such that* $M = P\Delta Q$*. If* $M \notin K$ *this decomposition is unique if we choose* $\Delta_{1,1} \geq 1$*.*

Proof : We can sent i on $M \cdot i$ by first doing a stretching Δ along the imaginary axis (so that $d(i, \Delta i) = d(i, Mi)$ and then applying a rotation P around i. So we have $M(i) = P(\Delta(i))$. Which means that $\Delta^{-1}P^{-1}M(i) = i$ And hence $\Delta^{-1}P^{-1}M \in K$.

For uniqueness, we can see that $d(i, P\Delta Q(i)) = d(i, \Delta(i)) = l(\Delta)$ and hence Δ is uniquely determined. The rest is a simple computation.

□

VI Some additional Exercises

Exercise 11: (Fellow travelling) Let $(x_n) \in \mathbb{H}^{\mathbb{N}}$ such that $x_n \to x_\infty \in \partial\mathbb{H}$. (Recall that this is the usual convergence in $\hat{\mathbb{C}}$). Let $(y_n) \in \mathbb{H}^{\mathbb{N}}$ such that $d_{\mathbb{H}}(x_n, y_n) \leq R$. Then $y_n \to x_\infty$.

Exercise 12: Let l an hyperbolic line in $\mathbb{H}$.

1. Prove that for any $z \in \mathbb{H}$ there is a unique point $S(z)$ such that $d(z, S(z)) = 2d(z, l) = 2d(S(z), l)$. The map S is called the reflexion about line l.
2. Show that if $z \neq S(z)$ then the line l is the perpendicular bisector of the segment $[zS(z)]$.
3. Prove that reflexions are isometries (but not orientation-preserving).
4. Let $M \in \mathrm{PGL}(2,\mathbb{R})$ be the matrix representative of a reflexion. Prove that $\mathrm{tr}(M) = 0$.

Exercise 13: Isometries as product of reflexions.

1. Prove that any element of $\mathrm{Isom}^+(\mathbb{H})$ can be written as the product of two reflexions about hyperbolic lines. (It suffices to prove that you can do this for any element in K, A and N.)
2. Prove that if ϕ is the product of S_1 and S_2 the reflexions about lines l_1 and l_2, then we have that ϕ is elliptic (resp. hyperbolic, parabolic) if and only if l_1 and l_2 intersect (resp. are ultraparallel, are limiting parallels).
3. Assume that g, h are two elliptic isometries without a common fixed point. Using the previous question, give a geometric proof that the commutator $ghg^{-1}h^{-1}$ is an hyperbolic isometry.

Exercise 14: Let g, h be two hyperbolic elements without common fixed points.

1. Prove that for all large $n, m > 0$, the element $g^n h^m$ is hyperbolic.
2. Show that the fixed points of $g^n h^m$ are disjoint with those of g, h.
3. Same questions with g parabolic and h hyperbolic.

Hyperbolic Surfaces

I Fuchsian Groups

Definition I.1
In a topological group G, a subgroup Γ is called discrete*, if the identity is an isolated element, i.e for any sequence $T_n \to \mathrm{Id}$ in Γ, we must have $T_n = \mathrm{Id}$ for all large n.*

We can endow $\mathrm{Isom}^+(\mathbb{H})$ with the topology coming from the natural topology on $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{M}(2,\mathbb{R})$.

Definition I.2
A Fuchsian group Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$,

Exemple 1 : Any finite group is a discrete group.

Exemple 2 : The group $\mathrm{PSL}(2,\mathbb{Z})$ of Möbius transformations with integer coefficients is a Fuchsian group. All its subgroup are discrete groups.

A - Action on $\mathbb{H}$

Definition I.3
Let G be a group acting on a topological set X. We say that the action is properly discontinuous on X if for all compact subsets $K \subset \mathbb{H}$, the set $\{g \in G, gK \cap K \neq \emptyset\}$ is finite.

For groups of isometries acting on $\mathbb{H}$ we have the following characterisation of proper discontinuous actions.

Proposition I.4
Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. The following are equivalent :

- *Γ does not act properly discontinuously on $\mathbb{H}$*
- *Some G-orbit in $\mathbb{H}$ has accumulation points in $\mathbb{H}$*
- *All G orbits in $\mathbb{H}$ have accumulation points in $\mathbb{H}$, with the possible exception of a single orbit which consists of a single fixed point.*

Recall that an accumulation point p in $\mathbb{H}$ is an accumulation point of a set X, if there exists a sequence of distinct elements $p_n \in X$ such that $p_n \to p$.
Proof :

- Assume (i). There exists a compact K with $g_n K \cap K \neq \emptyset$ for infinitely many distinct $g_n \in \Gamma$. So there are poins $z_n \in K$ such that $g_n z_n \in K$. As K is compact, we can take a subsequence such that $z_n \to w$ for some $w \in K$. As K is bounded, it is included in $B_R(w)$ for some $R > 0$.

Now take the orbit $\Gamma \cdot w$. We have $d(g_n w, w) \leq d(g_n w, g_n z_n) + d(g_n z_n, w) = d(w, z_n) + d(g_n z_n, w) < 1 + R$ for n large enough. Which means that $g_n w$ is at bounded distance from w, and hence has an accumulation point.

- Assume (ii). Let z_0 such that $\Gamma \cdot z_0$ has an accumulation point w_0 such that $g_n z_0 \to w_0$. Let $z \in \mathbb{H}$. We have

$$d(g_n, z) \leq d(g_n z, g_n z_0) + d(g_n z_0, w_0) + d(w_0, z) = d(z, z_0) + d(g_n z_0, w_0) + d(w_0, z)$$

So this distance is bounded independently of z and hence there is a subsequence $g_n z \to z_\infty$.
If the $g_n z$ coincide for infintely many distinct n, then we have $h_{n,m}(z) = g^n g^{-m}(z) = z$ for infinitely many distinct elements $h_{n,m}$. So these $h_{n,m}$ are distinct elliptic elements with a common fixed point. Any point that is not the fixed point will have accumulation points.

- Assume (iii). Pick an orbit $\Gamma \cdot z$ with an accumulation point w so that $g_n z \to w$. Let $R > 0$ such that $g_n z \in \overline{B_R(w)}$. So $\overline{B_R(w)}$ contains infinitely many points $g_n z$ which proves that $g_n \overline{B_R(w)} \cap \overline{B_R(w)} \neq \emptyset$ for infinitely many g_n.

□

The two notions are related by the following theorem.

Theorem I.5
A group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is Fuchsian if and only if it acts properly discontinuously on $\mathbb{H}$

Proof : If Γ is not discrete, then let $g_n \in \Gamma$ a sequence of distinct elements such that $g_n \to \mathrm{Id}$. So we have that $g_n z \to z$ for all $z \in \mathbb{H}$. Let $z_0 \in \mathbb{H}$ which is not a fixed point of any element of the form $g_m^{-1} g_n, n, m \in \mathbb{N}$ (which exists because there are only countably many such fixed points). Then the points $g_n z_0$ are all distinct and $g_n z_0 \to z_0$. So z_0 is an accumulation point of the Γ-orbit $\Gamma \cdot z_0$. Which means that Γ does not act properly discontinuously.

If Γ is discrete. Let K be a compact set in $\mathbb{H}$. We have to show that $g(K) \cap K = \emptyset$ for all but finitely many $g \in \Gamma$. Without loss of generality, we can assume that $K = \overline{B_R(i)}$, that is the closed hyperbolic disk with center i and radius R. (We take R large enough so that $K \subset B_R(i)$). Note that $B_R(i) \cap g B_R(i) \neq \emptyset \Rightarrow g(i) \in B_{2R}(i)$.

Consider the set $E = \{g \in \mathrm{PSL}(2, \mathbb{R}), g(i) \in \overline{B_{2R}(i)}\}$. This is a compact set. E is closed because the map $g \mapsto g(i)$ is continuous. E is bounded because $g(i) \in \overline{B_{2R}(i)}$ implies that $|g(i)| < M$ and $\mathrm{Im}(g(i)) > m$. This shows that the coefficients of g are bounded (if one chooses $ad - bc = 1$. As any infinite subset of a compact set has accumulation points, we see that $E \cap \Gamma$ is finite. Which means that $g(\overline{B_R(i)}) \cap \overline{B_R(i)} = \emptyset$ for all but finitely many $g \in \Gamma$. So the action is properly discontinuous. □

Corollary I.6

1. *If Γ is Fuchsian, then no Γ-orbit has accumulation points in $\mathbb{H}$.*
2. *If Γ is Fuchsian and infinite, then every orbit of z_0 has accumulation points on $\partial\mathbb{H}$. This is called the limit set of Γ.*

Proof :

1. Obvious
2. The set $\mathbb{H} \cap \partial\mathbb{H}$ is compact.

□

Remark : A group Γ acts freely, if and only if it does not contain any elliptic elements

Corollary I.7
If a discrete group Γ acts freely, then for all $z \in \mathbb{H}$, there exists a neighborhood U of z such that $gU \cap U = \emptyset$ for all $g \in \Gamma$.

Proof : We let $\epsilon = \inf\{d(z, g \cdot z), g \in \Gamma \setminus \{Id\}\}$ which is non-zero because z is not an accumulation point of $\Gamma \cdot z$. Then $B_\epsilon(z) = U$ is such a neighborhood. □

B - Elementary groups

Definition I.8
A group is said to be elementary if there exists a finite orbit in $\mathbb{H} \cup \partial\mathbb{H}$.

Proposition I.9
A cyclic subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is Fuchsian if it is generated by an hyperbolic or parabolic element (in whic h case, it is infinite), or by an elliptic element of finite order (in which case it is finite).

Proof : Powers of hyperbolic elements (resp. parabolic, elliptic) are hyperbolic elements (resp parabolic, elliptic). The only possibility to have a cyclic group that is not discrete is to have an elliptic element with an angl which is an irrational multiple of π. (See rotation on a circle) □

Proposition I.10
An elementary Fuchsian group is either cyclic, or conjugated to the group $\left\langle z \to \lambda z, z \to \frac{-1}{z} \right\rangle$.

Proof : We distinguish four cases depending on the cardinal of a finite orbit.

- If there is a global fixed point in $\mathbb{H}$. Then all elements are elliptic and centered on that point. As the group is Fuchsian, Γ is a cyclic finite group.
- If there is a global fixed point in $\partial\mathbb{H}$. Then all elements are parabolic and hyperbolic with this fixed point. There cannot be both types of elements. INdeed, if $g(z) = z + k$ and $h(z) = \lambda z$ are two elements fixing ∞, we have $k_n = h^n \circ g \circ h^n(z) = z + \lambda^{-n} k$. The sequence $k_n \to \mathrm{Id}$ which contradicts the fact that Γ is discrete.

  If the elements are all parabolic, then the group is infinite cyclic.

  If the elements are all hyperbolic, one can show that all elements have the same fixed points. Otherwise a commutator $[g, h]$ would be parabolic.
- If there is an orbit with two elements in $\partial\mathbb{H}$. Each $g \in \Gamma$ either fixes both point (hyperbolic) or permutes both elements (elliptic of order 2). So there can be no parabolic. And using the previous case, we are in the situation of a group generated by an hyperbolic and an elliptic of order 2.
- If there is an orbit with 3 elements. Then there can be no parabolic or hyperbolic. So there are only elliptic elements, and they have the same fixed point. The group is finite cyclic.

□

Proposition I.11
If a group is non-elementary, then it contains an hyperbolic element (and hence infinitely many).

Proof : Assume that there is no hyperbolic elements. If the group contains only elliptic elements, it is elementary. So there exists a parabolic element conjugated to $t : z \to z + 1$. For any $g \in \Gamma$, we compute $\mathrm{tr}(t^n \circ g) = \mathrm{tr}(g) + nc$. But this element cannot be hyperbolic, so $|\mathrm{tr}(t^n \circ g)| \leq 2$ for all $n \in \mathbb{N}$, which means that $c = 0$ for all $g \in \Gamma$. So ∞ is a fixed point of all elements in Γ. Contradiction. □

II Geometry of Fuchsian groups

A - Fundamental domains

Definition II.1
If Γ acts on $\mathbb{H}$, then a closed subset $R \subset \mathbb{H}$ is a fundamental domain for Γ if :

1. *For all $z \in \mathbb{H}$, there exists $g \in G$ such that $gz \in R$.*
2. *For all $g \in \mathrm{PSL}(2, \mathbb{R})$, $\mathring{R} \cap g\mathring{R} = \emptyset$.*

Lemma II.2
If Γ admits a fundamental domain, then G is a Fuchsian group.

Proof : If Γ is not Fuchsian, then there exists o in the interior of R, such that $\Gamma \cdot o$ has an accumulation point p in $\mathbb{H}$, ie $g_n o \to p$. Let $\epsilon > 0$ such that $B_\epsilon(o)$ is contained in the interior of R. As the fundamental domain covers $\mathbb{H}$ we know that $p \in gR$ for some $g \in \Gamma$. If $g \neq \mathrm{id}$ then $d(g_n o, p) \geq d(g_n o, gR) = d(o, g_n^{-1} gR) \geq \epsilon$ for infinitely many n. Contradiction. □

Theorem II.3
Let Γ be a Fuchsian group. Then there exists fundamental domains R for Γ. Moreover, this fundamental domain can be chosen as a locally finite convex geodesic polygon.

We will see one particular type of fundamental domain that satisfy the conclusions of the theorem.

1) Dirichlet domains

Let $z_0 \in \mathbb{H}$, which is not fixed by any element of $\Gamma \setminus \{\mathrm{Id}\}$. Let $g \in \Gamma$. We define the set

$$H_g(z_0) = \{z \in \mathbb{H}, d(z, z_0) \leq d(z, g \cdot z_0)\}$$

This is the half-space containing z_0 cut out by the perpendicular bisector of z_0 and $g \cdot z_0$.

Definition II.4
The Dirichlet domain for Γ with center z_0 is

$$R_{z_0} = \bigcap_{g \in \Gamma} H_g(z_0) = \{z \in \mathbb{H}, d(z, z_0) \leq d(z, g \cdot z_0) \forall g \in \Gamma \setminus \{id\}\}$$

Theorem II.5
If z_0 is not fixed by any element of Γ, then the Dirichlet domain R_{z_0} is a fundamental domain for the Γ action on $\mathbb{H}$. The boundary of the Dirichlet domain is included in the set of bisectors.

Proof : Let $z \in \mathbb{H}$. As the group is discrete, the set $\{d(z_0, g \cdot z), g \in \Gamma\}$ has a smallest element. (The group acts properly discontinuously). Let p be a point where this minimum is obtained. Then we have $d(z_0, p) \leq d(z_0, g^{-1} p) = d(g z_0, p)$ for all $g \in \Gamma \setminus \{Id\}$, and hence $p \in R$. So R contains a point of each orbit, which proves the first property.

For (ii), we note that $g(R_z) = R_{gz}$ and also that $\mathring{R} = \{z \in \mathbb{H}, d(z, z_0) < d(z, g \cdot z_0) \forall g \in \Gamma \setminus \{id\}\}$.

Let $g \in \Gamma$ and $z \in \mathring{R} \cap g\mathring{R}$. Then $d(z, z_0) < d(z, gz_0)$ because $z \in R_{z_0}$. And we also have $d(z, gz_0) < d(z, g^{-1}gz_0) = d(z, z_0)$, because $z \in gR_{z_0} = R_{gz_0}$. Which gives a contradiction. □

Exemple 3 : Dirichlet domains for elementary groups

Exemple 4 : The modular group $\mathrm{PSL}(2, \mathbb{Z})$ and the Dirichelt domain centered in $2i$.

We take the generatores $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We draw the first three bisectors corresponding to the three elements g, h, h^{-1}. This gives a region

$$\Omega = \{z \in \mathbb{H}, |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}\}$$

We can prove that Ω is the Dirichlet domain. We obviously have $R_{2i} \subset \Omega$. To prove equality it suffices to show that $g\mathring{\Omega} \cap \mathring{\Omega} = \emptyset$. Let $z \in \mathring{\Omega}$ and $g \in \Gamma$ such that $gz \in \Omega$. We note $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As $z \in \Omega$ and g has integer value we get

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz + d|^2} < \mathrm{Im}(z) \frac{1}{(|c| - |d|)2 + |cd|} \leq \mathrm{Im}(z)$$

Applying the same reasoning to gz and $g^{-1}gz$ we get the opposite inequality, which is absurd.

2) *Geometry of the Dirichlet domain*

Proposition II.6

Let Γ be a Fuchsian group and R_{z_0} be a Dirichlet fundamental domain. Then the orbit $\Gamma \cdot R$ is locally finite, which means that for any K compact, $\{g \in \Gamma, gR \cap R \neq \emptyset\}$ is finite.

Proof : Without loss of generality, we can take K to be the closed disk $B_r(z_0)$ of radius r and center z_0. (As any compact is included in such a disk). Assume that $gR_{z_0} \cap K \neq \emptyset$, so there exists $w \in R_{z_0}$ such that such that $d(gw, z_0) < r$. So

$$d(gz_0, z_0) \leq d(gz_0, gw) + d(gw, z_0) \leq d(z_0, w) + r \leq 2r$$

Because, as $w \in R_{z_0}$, we have that $d(w, z_0) \leq d(gw, z_0) \leq r$ (definition of the Dirichlet domain).

So $d(z_0, gz_0) \leq 2R$. So there can be only finitely many such g. □

Proposition II.7

R is a convex geodesic polygon, ie $\partial R \cap \mathbb{H}$ is a countable union of geodesic segments of positive length, only finitely many of which intersect any compact set

Proof : Let $\mathcal{B}$ be the set of all bisectors of $[z_0, gz_0]$. If K is a compact set, then K meets only finitely many elements in $\mathcal{B}$. Indeed, if $w \in K \subset B_r(z_0)$ is on the bisector of $[z_0, gz_0]$ then $d(z_0, gz_0) \leq d(z_0, w) + d(w, gz_0) \leq 2d(z_0, w) \leq 2r$.

As a consequence, each side s must contain at least two points, so has positive length. □

3) *side pairings*

For now, a *side* is just a maximal convex subset of the boundary ∂R. W (We do not include points in $\partial\mathbb{H}$ in the set of sides). Two sides intersect at a vertex.

Lemma II.8
For any side $s \in \partial R$, there exists a unique $g \in \Gamma \setminus \{Id\}$ such that $s = R \cap gR$.

Proof : Any side of R belongs to exactly two tiles, R and gR for some $g \in \Gamma$. ☐

We can have $gs = s$ in the previous lemma. In that case, g exchange the two endpoints of the side s, and hence preserves the midpoint of that side. In that case g is an elliptic element of order 2 whose center is in the middle of that side. To simplify notations, we add these midpoints to the vertices of the polygon R, and hence the sides containing such a point will be cut in half.

Definition II.9
The side-pairings of polygon R are the elements g_s for all sides s

Given a side s and $g \in \Gamma$ such that $s = R \cap gR$, then $g^{-1}s$ is also a side of R. So any side pairing associates a side s with another side $g^{-1}s$.

We can consider the set of all side pairings associated to the sides of R.

Proposition II.10
The side pairings generate Γ

Proof : Let $g \in \Gamma$ and γ be the geodesic path in $\mathbb{H}$ between $z \in R$ and gz avoiding any orbit of vertices of R (for a suitable choice of z). This path crosses finitely many images of the fundamental domain R, denoted $R = R_0, R_1, \ldots, R_n = gR$. We let $R_i = h_i R$. If we apply h_i^{-1} to the adjacent regions $h_i R, h_{i+1}(R)$, we get the regions R and $h_i^{-1}h_{i+1}R$ which need to be adjacent. So $h_i^{-1}h_{i+1}$ is a side pairing. As $g = h_n = (h_0^{-1}h_1) \cdot (h_1^{-1}h_2) \cdots (h_{n-1}^{-1}h_n)$, the proof is complete. ☐

As a corollary, if a Fuchsian group is finitely generated, then it is geometrically finite.

B - Vertex Cycles

We distinguish three types of vertices of the polygon R

1. The vertex v is inside the hyperbolic plane $\mathbb{H}$. This vertex is necessarily adjacent to two sides.
2. The vertex v is in the boundary $\partial\mathbb{H}$ and is the endpoint of two asymptotic sides of R.
3. The vertex v is in $\partial\mathbb{H}$ and bounds a non-empty interval in $\partial\mathbb{H}$.

1) First case

Let v be a vertex of R in $\mathbb{H}$. Consider a small circular path around v in $\mathbb{H}$. This path crosses finitely many tiles $g_i R$ with $i = 0, \ldots, n-1$, that all share the vertex v, with $g_0 = Id$, and such that $g_i R$ and $g_{i+1}R$ are adjacent, and share a side. We denote by $v_i = g_i^{-1}(v)$, which is a vertex in R for all i.

As $g_i R$ and $g_{i+1}R$ are adjacent, the tiles R and $g_{i+1}^{-1}g_i(R)$ they are also adjacent and share a side denoted s_i. We infer naturally that the side pairing associated to s_i is $h_{i+1} = g_{i+1}^{-1}g_i$. We see that in that case, $v_{i+1} = h_{i+1}(v_i)$.

As there are only finitely many vertices, there exists a smallest $r > 0$ such that $v_r = v$, and we can write the (reduced) vertex cycle :

$$v = v_0 \xrightarrow{h_1} v_1 \xrightarrow{h_2} v_2 \xrightarrow{h_3} \ldots \xrightarrow{h_{r-1}} vr - 1 \xrightarrow{h_r} v_r = v$$

We now have two possibilities :

1. If $g_r R = R$ (in that case, we made a full turn around vertex v) then the cycle is full and $h_1^{-1} \cdots h_r^{-1} = \mathrm{Id}$.
2. If $g_r R \neq R$. Then g_r is an elliptic element fixing v. We can continue following the path around v, until $g_n R = R$ for a certain n. (Such a n exists because of the local compactness of the fundamental domain).

In that last case, we get that $g_{i+r} = g_r \circ g_i$ for all i. Which means that $h_{i+r} = h_i$ and hence the (full) vertex cycle around v :

$$v = v_0 \xrightarrow{h_1} v_1 \xrightarrow{h_2} v_2 \xrightarrow{h_3} \dots \xrightarrow{h_{n-1}} vn - 1 \xrightarrow{h_n} v_n = v$$

is perdiodic. So there exists $m \geq 2$ such that $(h_1^{-1} \cdots h_r^{-1})^m = \mathrm{Id}$.

This implies the following proposition :

Proposition II.11

Let R be the Dirichlet domain, and $v \in \mathbb{H}$ a vertex of R. If $\theta_1, \dots, \theta_n$ are the internal angles of the polygon R at the vertices $v_1, \dots, v_r$ of the (reduced) vertex cycle associated to v, then there exists $m_v \geq 1$, such that $\theta_1 + \cdots + \theta_n = \frac{2\pi}{m_v}$. This number m is the order of the stabilizer in Γ of v (and hence of any of the vertices in the same cycle).

This m_v is called a *period* of the Fuchsian group associated to vertex v.

2) Parabolic vertices

The same principle can be made with an horocyclic path around an ideal vertex. As there are only fintely many vertices, the (reduced) vertex cycle can be written as

$$v = v_0 \xrightarrow{h_1} v_1 \xrightarrow{h_2} v_2 \xrightarrow{h_3} \dots \xrightarrow{h_{r-1}} v_{r-1} \xrightarrow{h_r} v_r = v$$

(Here, we never go back to the initial tile)

We see that in that case $h_r \cdots h_1$ is a parabolic or hyperbolic element that fixes v.

Proposition II.12

The element $h_r \cdots h_1$ appearing in the vertex cycle is a parabolic element.

Proof : Suppose it's hyperbolic. Then it fixes another point w. Wlog, assume that $v = \infty$ and $w = 0$.

Let L be a vertical line passing through the interior of R. Choose a sequence $b_n \in L \cap R$ such that $b_n \to \infty$. Let $a_n \in i\mathbb{R}$ such that $Im(a_n) = Im(b_n)$. Then we have $d(a_n, b_n) \to 0$.

Now choose a compact fundamental segment $I \subset i\mathbb{R}$ of length $l(h)$. Then for all n there exists m_n such that $h^m(a_n) \in I$. So the sequence $h^{m_n} b_n$ will accumulate on I, which contradicts the local finiteness. □

***Remark* :** For parabolic vertices, one can consider ∞ as the period of that vertices.

3) hyperbolic vertices

The vertex bound an interval $I \subset \partial\mathbb{H}$. This interval does not contain any fixed point of hyp or par elements in Γ. Again consider the reduced vertex cycle

$$v = v_0 \xrightarrow{h_1} v_1 \xrightarrow{h_2} v_2 \xrightarrow{h_3} \dots \xrightarrow{h_{r-1}} v_{r-1} \xrightarrow{h_r} v_r = v$$

Proposition II.13

The element $h = h_r \cdots h_1$ is hyperbolic, or Γ is elementary

Proof : We consider the set $K = \bigcup_{i\in\mathbb{Z}} I_i$ where $I_i + 1 = h_i I_i$ and $I_0 = I$ (we extend take the full vertex cycle which is infinite). Three cases are possible :

1. $K = \partial\mathbb{H}$. Then there are no fixed point of elements of Γ on $\partial\mathbb{H}$. The group is elementary generated by an elliptic.
2. $K = \partial\mathbb{H} \setminus \{p\}$. A single fixed point. The groupe is elementary generated by a parabolic element.
3. $K = \partial\mathbb{H} \setminus J$ where J is an interval. Then the points h^n accumulate on the endpoints of this interval, and hence h is hyperbolic.

□

For now on, we will assume now that there are no hyperbolic vertices.

C - Topology of the quotient

The quotient $\mathbb{H}/\Gamma$ is isometric to the quotient R/Γ^*, where Γ^* is the set of side pairings of the polygon R. This quotient is a topological two-manifold, possibly non compact, which is obtained by identification of the sides of a polygon. The classification of (compact) surfaces states that the 2-manifold R/Γ^* is a genus g surface with m points removed.

To understand the topology, one can look at the polygon after identification as a CW complex. Where ideal vertices are added to the polygon.

If we assume that R has $2n$ sides, and r different vertex cycles. Then the CW complex has n edges , r vertices and 1 face. Which means that the Euler characteristic $\chi(R/\Gamma) = r-n+1 = 2-2g$.

The number g is called the genus of the surface, and corresponds to the number of "holes". (A torus has genus 1)

Definition II.14
The collection $(g; m_1, \ldots, m_r)$ is called the signature of the Fuchsian group Γ.

One can define an area form on $\mathbb{H}/\Gamma$ just using the one induced from the hyperbolic area on $\mathbb{H}$.

We can see that the total area of the surface $\mathbb{H}/\Gamma$ is equal to the area of the polygon R. If there are finitely many sides, and there are no hyperbolic vertices, then R has a finite area. In that case, we say that Γ is a lattice in $\mathrm{PSL}(2,\mathbb{R})$.

Theorem II.15
If Γ is a lattice and has signature $(g; m_1, \ldots, m_r)$, then

$$\mathrm{Area}(\mathbb{H}/\Gamma) = 2\pi\left[(2g-2) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right]$$

Proof : The area of the polygon R is $(2n-2)\pi - \sum_{j=1} n\theta_i$. But by definition $\sum_j \theta_j = \sum_i \frac{2\pi}{m_i}$. Hence

$$\mu(\mathbb{H}/\Gamma) = 2\pi\left((n-1-r) + r + \sum_{i=1} r\frac{1}{m_i}\right) = 2\pi\left((2g-2) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right)$$

□

Exemples 5 :

- The signature of the group $\mathrm{PSL}(2,\mathbb{Z})$ is $(0; 2, 3, \infty)$. This is called the modular surface. The area of the quotient is

$$\mu = 2\pi(-2 + 1 + \frac{1}{2} + \frac{2}{3}) = \frac{\pi}{3}$$

- If Γ has a fundamental domain which is a regular octogon with angle $\pi/4$ at each vertex, and each side is paired with the opposite side. Then there is only one vertex cycle with total angle 2π. So $\mathbb{H}/\Gamma$ is a closed surface of genus 2 without any cusp or conical point.

Proposition II.16

Let Γ be a Fuchsian group. Then $\mu(\mathbb{H}/\Gamma) \geq \dfrac{\pi}{21}$.

Proof : The minimum is obtained for $(0; 2, 3, 7)$ □

The remarkable thing is that there is a reciprocal to the previous discussion. If P is an hyperbolic polygon, with side pairings (so, isometries sending each side on another side in a unique way), then the condition on the angles of the vertex cycles and the genus is sufficient to ensure that the group generated by the side pairings is Fuchsian.

We don't state the full generalisation of the theorem, but restrict ourselves to the case where the polygon has no ideal vertices.

Theorem II.17

(Poincaré Polygon) Let P an hyperbolic polygon with no ideal vertices, together with a set S of side pairings given by isometries. If for each vertex cycle, there exists $m_i \in \mathbb{N}$ such that $\sum_{j=1}^{r} \theta_j = \frac{2\pi}{m_i}$. And if the genus g of P/S satisfies

$$(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0$$

Then the group generated by S is a Fuchsian group (of signature $(g; m_1, \ldots, m_c)$)

As a consequence, there exists Fuchsian groups for any given signature satisfying the condition, because one can always construct a polygon with the appropriate angles and side-pairings.

III Hyperbolic surfaces

Definition III.1

An hyperbolic surface is a Surface (S, g) with a metric, such that S is locally isometric to $\mathbb{H}$.

We restrict ourselves to complete surface, which are the surfaces where we can extend geodesics infinitely. (The Hopf-Rinow theorem states that this definition of complete coincides with the usual definition of completeness using Cauchy sequences)

For simplicity, we do not consider orbifold.

Theorem III.2

Let X be a complete hyperbolic (orientable) surface. Then X is isometric to $\mathbb{H}/\Gamma$ where Γ is a torsion-free Fuchsian group.

(Recall that Torsion-free means that there are no element of finite order in Γ. In this setting, it means that Γ does not have any elliptic element)

Proof : We won't get into any details of the proof. The key steps are the following.

1. Construct the developping map $\widetilde{X} \to \mathbb{H}$ from the universal cover $\widetilde{X}$ using the local isometries.
2. Show that this map is an isometry, where the metric on $\widetilde{X}$ is naturally induced by the one on X.
3. Construct the holonomy map, which is a group morphism hol : $\pi_1(X) \to \mathrm{Isom}(\mathbb{H})$ from the fundamental group into the group of isometries.
4. The group $\Gamma = \mathrm{hol}(\pi_1(X))$ is the desired Fuchsian group.

□

Geodesics on an hyperbolic surface, are just the projection of geodesics on $\mathbb{H}$ using $\mathbb{H} \to \mathbb{H}/\Gamma$.

1) Teichmuller Space

Let Σ be a topological surface. A *marked* hyperbolic structure on Σ is a pair (X, ϕ) where X is an hyperbolic surface and $\phi : \Sigma \to X$ is an homeomorphism.

Two hyperbolic structure (X, ϕ) and (X', ϕ') on Σ are said to be equivalent, if there exists an isometry $i : X \to X'$ such that $i \circ f$ is homotope to g.

Homotope means that we can continuously deform $i \circ f$ into g.

Teichmuller space can be defined as $\mathcal{T}(\Sigma)$ the set of equivalence class of marked hyperbolic structures on Σ.

The previous theorem states that Teichmuller space can also be defined as the space of conjugacy classes of representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}(\mathbb{H})$ where ρ is an injective morphism with discrete image.

IV Dynamics

We end this lecture with some dynamical properties of geodesic flows on hyperbolic surface. Geodesics flows on manifolds with negative curvature are one of the basic example of ergodic behavior. We are going to leave out certain technical details in the various definitions and proofs, and simplify exposition as much as possible, to be able to prove the main theorem of this section which is the ergodicity of the geodesic flow on the unit tangent bundle.

A - Unit tangent bundle

Recall that for a smooth manifold M and a point $x \in M$ the tangent space at x, denoted $T_x M$ is the vector space of tangent vectors to curve in S passing through x. If M is an n-dimensional manifold, then the tangent space $T_x M$ is naturally isomorphic to $\mathbb{R}^n$.

The tangent bundle TM is formally the disjoint union of all tangent space. It is a fiber bundle with base space M, and fiber $\mathbb{R}^n$. So elements are pairs $(x, \vec{v})$ where $x \in M$ and $\vec{v}$ is some tangent vector at x.

The unit tangent bundle, is the fiber bundle where we only consider unit tangent vectors. It is denoted UTM (or sometimes T^1M). There is a natural projection $\pi : UTM \to M$.

For the hyperbolic plane, we have a simple description of the unit tangent bundle.

Lemma IV.1
For any $(x, \vec{v}) \in UT\mathbb{H}$ there exists a unique oriented geodesic L such that γ passes through x and is tangent to v.

Proof : Simple exercise in Euclidean geometry. There is a unique circle (or vertical line) that is tangent to $\vec{v}$ at point x and whose center is on $\mathbb{R}$. □

Given this proposition, there are several equivalent way to describe uniquely a point in $UT\mathbb{H}$:

- Pairs (x, L) where L is an oriented geodesic passing throught x.
- Pairs $(x, L^+) \in \mathbb{H} \times \partial\mathbb{H}$. Here L^+ is the positive end point L^+ of the oriented geodesic L.

- Triples $(L^-, \xi, L^+) \in \partial\mathbb{H}^{(3)+}$ of positively ordered points on $\partial\mathbb{H}$. The point ξ is the endpoint point on $\partial\mathbb{H}$ of the perpendicular ray to L based at x, and on the right side of L.

The group $\mathrm{PSL}(2,\mathbb{R})$ acts naturally on $UT\mathbb{H}$ by isometries. As geodesics are sent to geodesics, and angles are preserved, the action can be understood in any of the previous parametrisations in a simple way. The last identification proves that :

Proposition IV.2
The action of $\mathrm{PSL}(2,\mathbb{R})$ *is simply transitive on* $UT\mathbb{H}$.

Given this, we can identify $\mathrm{PSL}(2,\mathbb{R})$ with $UT\mathbb{H}$, with the map $\Phi : \mathrm{PSL}(2,\mathbb{R}) \to UT\mathbb{H}$ such that $\Phi(g) = g(0,1,\infty)$.

Note that the point $(0,1,\infty) \in \partial\mathbb{H}^(3)$ is the point $(i,\vec{v})$ where $\vec{v}$ is a positive vertical vector.

Using this identification, we see that the action of $\mathrm{PSL}(2,\mathbb{R})$ on $UT\mathbb{H}$ is equivalent to the multiplication on the left in the group $\mathrm{PSL}(2,\mathbb{R})$.

B - Geodesic and Horocyclic flows

The natural space to look at flows is the unit tangent bundle.

Definition IV.3
The geodesic flow is the flow $(g_t)_{t\in\mathbb{R}} : UT\mathbb{H} \to UT\mathbb{H}$ *defined by* $g_t(x,L) = (y,L)$ *where the* $d(x,y) = |t|$ *and* y *is in the positive direction on* L *when* $t \geq 0$*, and negative otherwise.*

It's the action of moving along geodesics with unit speed. The geodesics on $\mathbb{H}$ are the projections of the orbits of that flow.

Definition IV.4
The positive (negative) horocylic flow $(h_s^\pm) : UT\mathbb{H} \to UT\mathbb{H}$ *is defined by* $h_s^\pm(x,L^\pm) = (y,L^\pm)$*, where* x *and* y *are on the same horocycle based on* $L^\pm$*, and the distance between* x *and* y*, along the horocyle is* s*, where the horocyle is oriented positively.*

The positive horocyclic flow is hence defined by moving distance s along the horocycle perpendicular to v at x with v pointing inward. The negative one is the same thing with v pointing outwards.

One can describe these two flows in a very simple way using the identification of $\mathrm{PSL}(2,\mathbb{R})$ with $UT\mathbb{H}$. We define the following matrices

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} , \; n_t^+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} , \; n_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

Theorem IV.5
Under the identification $UT\mathbb{H} \simeq \mathrm{PSL}(2,\mathbb{R})$*, the geodesic flow and horocyclic flows on* $UT\mathbb{H}$ *are given by the one-parameter family of subgroups given by* $a_t, n_s^\pm$ *acting on the right on* $\mathrm{PSL}(2,\mathbb{R})$.

In other words, if $u = (i,\infty)$ *and* $v = \phi(u)$ *with* $\phi \in \mathrm{PSL}(2,\mathbb{R})$*, then*

$$g_t(v) = (\phi a_t)(u) \, , \; h_t^+(v) = (\phi n_t^+)(u) \, , \; h_t^-(v) = (\phi n_t^-)(u)$$

Proof : The action of the flows g_t and h_s on $UT\mathbb{H}$ naturally commute with the action of isometries of $\mathbb{H}$.

If $(x, L^+) = k(i, \infty)$, with $k \in \mathrm{Isom}(\mathbb{H})$, then $g_t(x, L)$ is the element (y, L^+) where $d(x, y) = t$. Hence, we see that $k^{-1}(y, L^+) = (y', \infty)$ where $y' \in i\mathbb{R}$ and $d(i, y') = t$, because the action is by isometries (and hence preserves the geodesics and distances). Which means that $(y', \infty) = g_t(i, \infty)$. So $k^{-1} g_t k = g_t$ and the two actions commute.

So it suffices to check that the action of each 1-parameter subgroup above is correct at the basepoint (i, ∞), corresponding to the identity element in the identification $G = UT\mathbb{H}$. This is now a simple check.

For the geodesic flow, we have that $a_t(i, \infty) = (\frac{e^{t/2}}{e^{-t/2} i}, \infty) = (e^t i, \infty)$. For the horocyclic flow, we have and $n_s^+(i, \infty) = (i + s, \infty)$.

For the negative horocyclic flow, we simply reverse the orientation of the vector as $n_s^+(x, \vec{v}) = n_{-s}^-(x, -\vec{v})$. □

Proposition IV.6
Let $s, t \in \mathbb{R}$. We have $g_t h_s^+ = h_{se^{-t}} g_t$, and $g_t h_s^- = h_{se^t}^- g_t$.

Proof : Using the previous identification, we just have to check the following matrix identity

$$n_s^+ a_t = a_t n_{se^{-t}}^+ \ , \ n_s^- a_t = a_t n_{se^t}^-$$

(Note that the action is on the right) □

This is one of the fundamental property of the geodesic flow that allows us to prove ergodicity.

C - Measure on $UT\mathbb{H}$

We can define a natural measure on $UT\mathbb{H}$ using the area form on $\mathbb{H}$ and the Lebesgue measure on $UT_x\mathbb{H} \simeq \mathbb{S}^1$. This measure is called the Liouville measure, denoted $\mathcal{L}$.

As $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathbb{H}$ by isometries, and preserve angle, it's relatively clear that Liouville measure is invariant by the (left) action of $\mathrm{PSL}(2, \mathbb{R})$.

Under the identification $\mathrm{PSL}(2, \mathbb{R}) \to UT\mathbb{H}$, we can infer that the pullback measure on $\mathrm{PSL}(2, \mathbb{R})$ is invariant by (left) multiplication. By regularity of the Liouville measure and of the identification, it means that this measure is the unique (left) Haar measure on $\mathrm{PSL}(2, \mathbb{R})$, up to a multiplicative constant. By unimodularity of $\mathrm{PSL}(2, \mathbb{R})$, this measure is also invariant by the right action.

From this we deduce :

Proposition IV.7
The Liouville measure $\mathcal{L}$ is invariant by the geodesic and horocyclic flows.

D - Flows on the quotient $\mathbb{H}/\Gamma$

Let S be an hyperbolic surface. For clarity, we will assume that S is compact (and without boundary). So S is isometric to $\mathbb{H}/\Gamma$ for some torsion free Fuchsian group, whose fundamental domain is a compact polygon with only neutral vertices.

The unit tangent bundle UTS is identified with $UT\mathbb{H}/\Gamma$. The geodesic flow, and horocyclic flow on UTS are defined as the projection of any lift of the geodesic and horocyclic flow on $\mathbb{H}$.

In other words, if $\pi : UT\mathbb{H} \to UT\mathbb{H}/\Gamma \simeq UTS$. If $u \in UTS$, and $\widetilde{u} \in UT\mathbb{H}$ a lift of u in $UT\mathbb{H}$. Then we define $g_t(u) = \pi(g_t(\widetilde{u}))$.

(One can check that this does not depend on any of the choices that we made)

The Liouville measure $\mathcal{L}$ induces a unique locally finite measure on UTS, that we also denote by $\mathcal{L}$. By compactness of the surface, the total measure is finite. We normalize the measure so that (UTS, μ) is a probability space.

E - Ergodicity

Definition IV.8
A measure preserving flow $\phi_t : X \to X$ is ergodic, if every invariant measurable set E (such that $\phi_t(E) = E$ for all t) satisfies $\mu(E) = 0$ or $\mu(E^c) = 0$.

Theorem IV.9
The geodesic flow (g_t) is ergodic on the unit tangent bundle UTS, for the Liouville measure

Proof : We begin by proving that we can move through $UT\mathbb{H}$ using only geodesic and horocyclic flows.

Lemma IV.10
For almost any $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, there exists $s_1, s_2, t \in \mathbb{R}$ such that $\gamma = n^-_{s_1} a_t n^+_{s_2}$.

Proof : If $a \neq 0$ we simply write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix}$$

and notice that the matrices such that $a = 0$ have measure 0 in $\mathrm{PSL}(2,\mathbb{R})$.

(This decomposition is called the Bruhat decomposition) □

We will use without proof the following general result from Ergodic Theory, (which is a consequence of Birkhoff Ergodic Theorem).

Proposition IV.11
Let (X,μ) compact probability space (with μ a Radon measure), and ϕ_t a measure preserving flow. Let f be a continuous function on X. Then there exists an invariant function F such that for μ-almost every $x \in X$ we have

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\phi_t x)dt = F(x)$$

Moreover, if for every such f, the function F is almost everywhere constant, then μ is an ergodic measure for (ϕ_t).

We now take f a continuous function on UTS, and F defined as in the previous proposition.

Lemma IV.12
Let $u \in UTS$. For any $t_0 \in \mathbb{R}$, if $v = g_{t_0}u$, then $F(v) = F(u)$.

Proof : We have

$$\int_0^T f(g_t v)dt = \int_0^T f(g_t(g_{t_0}u)dt = \int_0^T f(g_{t_0+t}u)dt = \int_{t_0}^{T+t_0} f(g_t u)dt = \int_0^{T+t_0} f(g_t u)dt - \int_0^{t_0} f(g_t u)dt$$

When doin the average and going to the limit we get

$$\lim_{T\to\infty}\frac{1}{T}(\int_0^{T+t_0} f(g_t u)dt - \int_0^{t_0} f(g_t u)dt) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(g_t u)dt$$

□

So the value of the function F at u only depends on the geodesic defined by u. We now prove that it does only depend on the positive endpoint.

Lemma IV.13
Let $u \in UTS$. For any $s_0 \in \mathbb{R}$, if $v = h^+_{s_0} u$, then $F(v) = F(u)$.

Proof : Let $v = h^+_s x$ for a certain s.

Let $\epsilon > 0$, by uniform continuity of f, there exists δ such that for any $u, v \in UTS$, if $d(u,v) < \delta$ then $|f(u) - f(v)| < \epsilon$.

From the definition of the horocyclic flow, we see that $d(h_s u, u) \leq s$. So let t_0 such that $se^{-t_0} < \delta$. Now we have

$$F(v) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t v)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t h_s u)dt$$
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(h_{se^{-t}} g_t u)dt$$
$$\lim_{T\to\infty} \frac{1}{T} (\int_0^{t_0} f(h_{se^{-t}} g_t u)dt + \int_{t_0}^T f(h_{se^{-t}} g_t u)dt)$$

For all $t \geq t_0$ we have $|f(h_{se^{-t}} g_t u) - f(g_t u)| \leq \varepsilon$. Hence for T large enough, we have that

$$\frac{1}{T}(\int_0^{t_0} f(h_{se^{-t}} g_t u)dt + \int_{t_0}^T f(h_{se^{-t}} g_t u)dt) - \frac{1}{T} \int_0^T f(g_t u)dt < 2\epsilon$$

Which proves that $F(u) = F(v)$. □

Lemma IV.14
Let $v = h^-_{s_1} u$. Then $F(v) = F(u)$.

Proof : The proof is the same, but using the geodesic flow in the negative direction. □

Now for almost every $v \in UTS$ one can write $v = h^+_{s_1} g_t h^-_{s_2} u$. Then, just apply the three previous lemma to prove that $F(v) = F(u)$. Hence F is almost everywhere constant, which proves ergodicity of the geodesic flow. □