Classification of Seifert fibrations through orbifold Euler classes.

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1 Aim of these notes.

These notes grew out of my attempt to gain a clear understanding of the classical Seifert invariants of Seifert fibre spaces. I struggled with them for quite some time, mainly because of the following issues:

1. conventions vary significantly from one author to another;

2. the invariants are introduced through local models, although they clearly depend on global data;
3. the rules governing them often look like an ad hoc collection of recipes;
4. the classical description by surgeries is very well suited to construction and classification problems, but much less convenient for recognition problems;
5. there are many possible approaches (complex line bundles, plane bundles, classifying spaces, obstruction theory, Čech cohomology, group cohomology etc.), making it difficult to decide which framework is the most appropriate;
6. most references deal only with the orientable case, sometimes without stating this explicitly;
7. many references define the integral or rational Euler number only for "good" orbifolds. Although bad orbifolds are rather exceptional, this always felt somewhat unsatisfactory, as it obscures the geometric phenomenon these invariants are meant to capture.

My interest in these questions originally came from problems in birational geometry. In the situations I needed to study, Seifert fibrations naturally appeared in all possible forms: orientable and non-orientable, over both good and bad orbifolds. This made it difficult to rely on the existing literature, where one or more of these assumptions are often built into the exposition. I therefore looked for a framework that treats all these cases on an equal footing.

My intuition was that, just as circle bundles are classified by characteristic classes, the same should be true for Seifert fibrations. The key feature I was looking for was functoriality, again because of the applications I had in mind. From this point of view, the basic ingredients are certainly classical. Ordinary circle bundles are classified by their Stiefel–Whitney and Euler classes, with twisted coefficients when the fibre orientation is not globally defined. Likewise, principal $\mathbb{S}^1$-orbibundles are classified by their orbifold Euler class in

$$H^2_{orb}(\mathcal{O};\mathbb{Z});$$

this goes back, in a substantially more general equivariant setting, to work of Hattori and Yoshida [HY76], and has been used in several later treatments of orbifold circle bundles (see for instance [KL21]). On the other hand, the classical theory of Seifert fibrations describes the same objects by numerical invariants attached to the exceptional fibres together with a global Euler invariant; see for instance Seifert's original classification [Sei+80] and the account of Scott [Sco83].

What I have tried to do here is to put these two descriptions together in a form which also accommodates non-orientable fibrations. For a Seifert fibration over an orbifold $\mathcal{O}$ with underlying surface B, the orientation monodromy of the fibres gives

$$w_1 \in H^1(B;\mathbb{Z}_2).$$

After choosing a cocycle w representing w_1, the remaining gluing data define a twisted orbifold Euler class

$$e_w \in H^2_{orb}(\mathcal{O};\mathcal{Z}_w).$$

If no orientation of the fibres is retained, only its orbit

$$[e_w]_\pm \in H^2_{orb}(\mathcal{O};\mathcal{Z}_w)/\{\pm 1\}$$

is intrinsic after identifying the coefficient system with the chosen model $\mathcal{Z}_w$. The local manifold condition is expressed by requiring the restrictions of this class to the isotropy groups to be generators.

The resulting description treats, in the same language, orientable and non-orientable Seifert fibrations and does not require the base orbifold to be good. Moreover, for a closed 2-orbifold with only cone points, the relevant twisted orbifold cohomology group can be computed explicitly. The classical global invariant b and the local invariants β_i then arise as coordinates of a representative of the single class e_w. In particular, the usual transformations of Seifert invariants acquire a simple cohomological interpretation: some come from the relations in the cohomology group itself, while the various sign changes come from changing the generators used to identify the intrinsic local coefficient groups with copies of $\mathbb{Z}_{\alpha_i}$, or from the residual global reversal of the fibre orientation.

I do not know to what extent this precise formulation already appears in the literature. The classification of Seifert fibrations themselves, the use of Euler classes for principal circle orbibundles, and all the general machinery of orbifold and groupoid cohomology are classical. Thus, no novelty is claimed for these ingredients. What I have not found explicitly developed is their combination in the form used here: a classification by a twisted orbifold Euler class, together with an explicit computation of the relevant group and a direct interpretation of the classical Seifert equivalence relations in terms of that class, uniformly including the non-orientable and bad-orbifold cases. It is entirely possible that this viewpoint is implicit, or even explicit, in sources that I have missed.

The main purpose of these notes is therefore first expository: to give a coherent and reasonably concrete account of this viewpoint, beginning with ordinary circle bundles and then passing to Seifert fibrations. The emphasis is on understanding how the characteristic classes arise directly from transition functions and how the familiar numerical

invariants are recovered from them. When appropriate, the corresponding orientation-preserving classification is also discussed. Thus the classification considered here is purely topological—or equivalently smooth, since both $\mathrm{Homeo}(\mathbb{S}^1)$ and $\mathrm{Diff}(\mathbb{S}^1)$ have the homotopy type of $\mathrm{O}(2)$.

To me, the cleanest conceptual approach would probably be through classifying spaces and Postnikov invariants. However, this requires substantially more algebraic topology than is needed for the questions considered here. I have instead chosen a Čech cohomological approach. For Seifert fibrations, ordinary Čech cohomology is replaced by the cohomology of the nerve of an étale groupoid presenting the orbifold. This provides a useful balance between abstraction and concreteness: the characteristic classes remain intrinsic, while the dependence on local choices and the passage back to the classical Seifert invariants remain visible.

I decided to make these notes available because they may help readers who, like me, are looking for a coherent mental picture of Seifert fibrations and their invariants. Although I have tried to make the exposition as self-contained as reasonably possible, some standard facts about Lie groupoids, orbifold sheaves, and topological stacks are used without proof. My aim has not been to develop these theories, but to use the small part of them needed to organise the topology of circle bundles and Seifert fibrations. The notes should therefore be read as an attempt to clarify and connect classical constructions rather than as a systematic treatment of the general theory.

Finally, I should again emphasize that I am by no means an expert on orbifolds, topological stacks, or Seifert fibrations, so these notes should be read with both a critical and a benevolent eye.

2 Circle bundles: definitions and basic concepts.

We introduce here the standard concept of (smooth) circle bundles. The definitions are quite standard and the reader familiar with this notion can safely skip all this section, coming back here occasionally for precise definitions or assumptions.

2.1 What is it?

A (smooth) *circle bundle* is a smooth submersion $\pi : M \to B$ between two smooth manifolds M and B, satisfying the following *local triviality* property: for every point $b \in B$, there exists an open neighbourhood $U \subset B$ containing b and a diffeomorphism $\varphi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{S}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times \mathbb{S}^1 \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

where $\mathrm{pr}_1(x, u) = x$.

Such a pair (U, φ_U) is called a *local trivialisation*. A family of local trivialisations whose open sets cover B is called a *trivialising cover* of the bundle. The manifold M is called the *total space*, and B the *base space*. By definition, every fibre of π is diffeomorphic to $\mathbb{S}^1$. Throughout this text, we shall assume that B is connected. However, we make no orientability assumptions on either M or B. In particular, we do not restrict ourselves to principal $\mathbb{S}^1$-bundles.

If one assumes π to be only a continuous map between topological manifolds M and B satisfying the local triviality property with φ_U only a homeomorphism, then we say that π is a *continuous* circle bundle.

2.2 Bundle morphisms.

Suppose $\pi_1 : M_1 \to B_1$ and $\pi_2 : M_2 \to B_2$ are two circle bundles. A *bundle morphism* from π_1 to π_2 is a pair (η, f) of smooth functions such that

$$\begin{array}{ccc} M_1 & \xrightarrow{\eta} & M_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

commutes. If $B_1 = B_2$ and $f = \mathrm{id}$, we say that the bundle morphism *covers the identity*. Throughout this text, a *bundle isomorphism* will mean a bundle morphism covering the identity (η, id) where η is a diffeomorphism.

2.3 Transition functions.

Consider a circle bundle $\pi : M \to B$ and two local trivialisations (U_i, φ_i) and (U_j, φ_j). On the overlap, the change of trivialisation is given by the diffeomorphism $\varphi_{ij} : \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{S}^1 \to (U_i \cap U_j) \times \mathbb{S}^1$ which satisfies

$$\begin{array}{ccc} (U_i \cap U_j) \times \mathbb{S}^1 & \xrightarrow{\varphi_{ij}} & (U_i \cap U_j) \times \mathbb{S}^1 \ , \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U_i \cap U_j & \end{array}$$

that is, $\varphi_{ij}(x, u) = (x, g_{ij}(x)(u))$ where $g_{ij} : U_i \cap U_j \to \mathrm{Diff}(\mathbb{S}^1)$ is called the *transition function* between the local trivialisations. These transition functions satisfy the following:

Lemma 2.1. i. $g_{ii} = \mathrm{id}$;

ii. $g_{ji} = g_{ij}^{-1}$;

iii. $g_{ij} \circ g_{jk} \circ g_{ki} = \mathrm{id}$.

The third point is defined on the triple overlaps $U_i \cap U_j \cap U_k$ and is called the *cocycle condition*. The following lemma shows that the data of the transition functions is essentially equivalent to the data of the bundle itself.

Lemma 2.2. Let $\pi : M \to B$ be a circle bundle, and let $\{g_{ij}\}_{i,j}$ be its transition functions with respect to a trivialising open cover $\{U_i\}_{i \in \mathcal{I}}$ of B. Then, knowing only the transition functions $\{g_{ij}\}_{i,j}$, one can construct a circle bundle $\pi_N : N \to B$ which is isomorphic to M.

Proof. Consider the disjoint union

$$\bigsqcup_{i \in \mathcal{I}} (U_i \times \mathbb{S}^1).$$

We want to glue them together according to the transition functions. Define an equivalence relation $\sim$ by

$$(x, u, i) \sim (x, u', j) \quad \Longleftrightarrow \quad x \in U_i \cap U_j \text{ and } u' = g_{ji}(x)(u).$$

This relation is well-defined thanks to 2.1 and we then define

$$N := \left(\bigsqcup_{i \in \mathcal{I}} (U_i \times \mathbb{S}^1) \right) \Big/ \sim .$$

Let $\pi_N : N \to B$ be given by $\pi_N([x, u, i]) = x$. For each i, the map

$$\phi_i : \pi_N^{-1}(U_i) \to U_i \times \mathbb{S}^1, \qquad [x, u, j] \mapsto (x, g_{ij}(x)(u))$$

is a well-defined local trivialisation, and on overlaps the induced transition functions are precisely the given maps g_{ij}. Finally, since M has the same local trivialisations and the same transition functions, the local identifications

$$\psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{S}^1$$

glue together to define a global bundle isomorphism $M \xrightarrow{\sim} N$ over B. □

Let us now describe how transition functions change under a bundle isomorphism

$$\begin{array}{ccc} M & \xrightarrow{\eta} & N \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & B & \end{array}$$

First observe that if π is trivial over an open set $U_i \subset B$, so is π'. Choose local trivialisations (U_i, φ_i) for π and (U_i, φ_i') for π'. Then we must have

$$\varphi_i' \circ \eta \circ \varphi_i^{-1}(x, u) = (x, a_i(x)(u))$$

where $a_i : U_i \to \mathrm{Diff}(\mathbb{S}^1)$ so that, on $U_i \cap U_j$, the transition function g_{ij}' for π' satisfies $g_{ij}'(x) = a_i(x) \circ g_{ij}(x) \circ a_j(x)^{-1}$.

2.4 Monodromy.

The *monodromy* is a map that measures how the fibers are twisted along paths or loops. Without additional structure, the only canonical information one can extract from a circle bundle is whether transport along a loop preserves or reverses the orientation of the fibre. Throughout this text, the term monodromy will refer to this information.
Let $b_0, b_1 \in B$ be two points in the base and let $\gamma : [0,1] \to B$ be a path joining them. Any choice of local trivialisation covering the path yields a lift of the motion along γ and a diffeomorphism

$$T_\gamma : M_{b_0} \to M_{b_1}$$

where $M_b \approx \mathbb{S}^1$ denotes the fibre above b. This map is called a *parallel transport* along γ. Note that in general, it depends both on the path γ and on the choices of trivialisations.
If γ is a loop based at $b \in B$, $T_\gamma : M_b \to M_b$ becomes, after a choice of identification $M_b \approx \mathbb{S}^1$, an element of $\mathrm{Diff}(\mathbb{S}^1)$. The dependence of the parallel transport in choices of trivialisations leads to restrict attention on the weaker information of its connected component in $\mathrm{Diff}(\mathbb{S}^1)$. The topological group $\mathrm{Diff}(\mathbb{S}^1)$ has two connected components, the identity component $\mathrm{Diff}^+(\mathbb{S}^1)$ consisting of orientation-preserving diffeomorphisms, and the component of orientation-reversing diffeomorphisms. The two-point set $\pi_0(\mathrm{Diff}(\mathbb{S}^1))$ carries a group structure where the law is induced by composition:

$$[f] \cdot [g] = [f \circ g].$$

One can show that the class of T_γ in $\pi_0(\mathrm{Diff}(\mathbb{S}^1))$ does not depend on the choice of trivialisations and depends only on the homotopy class of γ. Therefore, parallel transport induces a homomorphism

$$\pi_1(B, b) \to \pi_0(\mathrm{Diff}(\mathbb{S}^1)) \simeq \mathbb{Z}_2$$

called the *monodromy representation*. In general, since B is path-connected, changing the base point b changes this representation by conjugation. In the case of circle bundles the target group $\pi_0(\mathrm{Diff}(\mathbb{S}^1))$ is abelian, the resulting homomorphism is independent of the base point and determines a cohomology class $w(\pi) \in H^1(B; \mathbb{Z}_2)$. Hence, throughout this text the *monodromy representation of* π will refer to this element $w(\pi) \in H^1(B; \mathbb{Z}_2)$. We shall see this monodromy representation coincides with the Stiefel–Whitney class.

3 Topological classification of circle bundles.

Before turning to Seifert fibrations, we begin with the simpler and classical case of circle bundles. Besides providing a model for the constructions to come, this case already contains the main idea of these notes: a bundle is locally trivial, so its topology is not contained in the local models themselves, but in the way these local models are glued together. The purpose of this section is to see explicitly how this gluing information reorganises itself into intrinsic characteristic classes.

Our goal is to recover the following classification theorem.

Theorem 3.1. Let B be a fixed connected smooth manifold and let $w_1 \in H^1(B;\mathbb{Z}_2)$. Choose a cocycle w representing w_1. Then isomorphism classes, over the identity of B, of circle bundles with first Stiefel–Whitney class w_1 are classified by the quotient

$$H^2(B;\underline{\mathbb{Z}}_w)\big/_{\{\pm 1\}},$$

where the non-trivial element of $\{\pm 1\}$ acts through the sheaf automorphism $-1 : \underline{\mathbb{Z}}_w \longrightarrow \underline{\mathbb{Z}}_w$.

If $w' = w + \delta a$ is another representative of w_1, the isomorphism of sheaves induced by a descends to a canonical bijection

$$H^2(B;\underline{\mathbb{Z}}_w)\big/_{\{\pm 1\}} \xrightarrow{\sim} H^2(B;\underline{\mathbb{Z}}_{w'})\big/_{\{\pm 1\}}.$$

The same statement holds for continuous circle bundles, with the obvious necessary changes. Notice that the passage to the quotient removes one last ambiguity which remains after the determinant cocycle has been fixed. Namely, a simultaneous reversal of all local fibre coordinates leaves w unchanged but replaces the Euler class by its opposite. We shall return to this point in Subsection 3.5.2.

Once the two invariants w_1 and e have been constructed appropriately, the classification will be essentially built into their definition. This is one of the advantages of the Čech cohomological approach: rather than introducing the characteristic classes afterwards as invariants of an already classified object, we shall obtain them directly by successively reorganising the transition data of the bundle. The twisted sheaf notation $\underline{\mathbb{Z}}_w$ appearing in the statement will be introduced in Subsection 3.4.1.

Let us describe this process more precisely. We consider a circle bundle through a collection of local trivialisations. Over each trivialising chart, there is nothing to understand: the bundle is simply a product with $\mathbb{S}^1$. All the global topology is therefore contained in the discrepancies between these local pictures on their overlaps, namely in the transition functions. The problem is to encode this information in a form which is both intrinsic and manageable.

The first step is almost tautological. Transition functions are precisely a $\mathscr{C}^\infty(-,\mathrm{Diff}(\mathbb{S}^1))$-valued Čech 1-cocycle, while changing the local trivialisations acts by the corresponding 0-cochain. Consequently, isomorphism classes of circle bundles are naturally identified with the non-abelian Čech cohomology set

$$\check{H}^1\big(\mathcal{U};\mathscr{C}^\infty(-,\mathrm{Diff}(\mathbb{S}^1))\big)$$

(Subsection 3.1).

This description is already a classification, but not yet a very enlightening one. The rest of the section consists essentially in unpacking this rather opaque cohomology set until its geometric content becomes apparent. The classical homotopy equivalence $\mathrm{Diff}(\mathbb{S}^1) \simeq \mathrm{O}(2)$ allows us to replace arbitrary diffeomorphisms of the circle by rotations and reflections. The semidirect-product decomposition $\mathrm{O}(2) \simeq \mathbb{S}^1 \rtimes \mathbb{Z}_2$ then separates the transition data into two parts (Subsection 3.2). The first records whether the fibre orientation is preserved or reversed along the base and gives the Stiefel–Whitney class $w_1 \in H^1(B;\mathbb{Z}_2)$ (Subsection 3.3). Once this orientation monodromy has been fixed, the remaining rotational part is no longer an ordinary $\mathbb{S}^1$-valued cocycle: it is naturally twisted by w, and defines a class in $H^1(B;\mathcal{S}^1_w)$ (Subsection 3.4.1). Finally, the twisted exponential sequence identifies this group with $H^2(B;\underline{\mathbb{Z}}_w)$, and the resulting class is precisely the Euler class of the circle bundle (Subsection 3.4.3).

Thus the passage

$$\mathrm{Diff}(\mathbb{S}^1) \rightsquigarrow \mathrm{O}(2) = \mathbb{S}^1 \rtimes \mathbb{Z}_2 \rightsquigarrow \big(w_1, e\big)$$

should not be regarded merely as a convenient computation of the classification set. It progressively extracts from the transition functions the two intrinsic pieces of information they contain: first the orientation monodromy of the fibres, and then their remaining twisting. This mechanism will serve as the model for the treatment of Seifert fibrations in the next part.

The final subsections are devoted to the explicit computation of the groups $H^2(B;\underline{\mathbb{Z}}_w)$ when B is a closed surface (Subsection 3.5.3). Since we have adopted the sheaf-theoretic viewpoint, we first relate these groups to the more classical singular cohomology groups with local coefficients before carrying out the computations (Subsection 3.5.1). This identification makes the question of the intrinsic nature of the Euler class more apparent, so we deal with it in details in Subsection 3.5.2.

3.1 Isomorphism classes of circle bundles and $\check{H}^1\big(\mathcal{U};\mathscr{C}^\infty(-,\mathrm{Diff}(\mathbb{S}^1))\big)$.

We will use just a bit of (non-abelian) Čech cohomology. Because our needs are very restricted, we do not give general definitions. These can be found very easily in the literature. Fix an open cover $\mathcal{U}=\{U_i\}_{i\in\mathcal{I}}$ of the base B consisting of local trivialisation charts. Refining $\mathcal{U}$ if necessary, we may assume it is a *good cover*, that is all non-empty intersections of finitely many sets are contractible. Čech cohomology with respect to a good cover computes the sheaf cohomology. Let G be a subgroup of $\mathrm{Diff}(\mathbb{S}^1)$ and $\mathcal{G}:=\mathscr{C}^\infty(-,G)$ the sheaf of smooth G-valued functions on B.

A 0-cochain is a collection of local sections $a_i\in\mathcal{G}(U_i)$ for each $U_i\in\mathcal{U}$. The set of 0-cochains is denoted $\check{C}^0(\mathcal{U};\mathcal{G})$ and forms a group under pointwise composition. Those collections $\{a_i\}_i$ that agree on overlaps, *i.e.* $a_i|_{U_i\cap U_j}=a_j|_{U_i\cap U_j}$, form a subset $\check{Z}^0(\mathcal{U};\mathcal{G})$ called the set of 0-cocycles.
A 1-cochain is a collection of sections $h_{ij}\in\mathcal{G}(U_i\cap U_j)$ on every non-empty intersections. The set of 1-cochains is denoted $\check{C}^1(\mathcal{U};\mathcal{G})$. Those satisfying the cocycle condition

$$h_{ij}\circ h_{jk}\circ h_{ki}=\mathrm{id}$$

on triple overlaps $U_i\cap U_j\cap U_k$ form a subset $\check{Z}^1(\mathcal{U};\mathcal{G})$ called the set of 1-cocycles. Note that because the $\mathcal{G}(U)$ are not abelian groups in general, $\check{Z}^1(\mathcal{U};\mathcal{G})$ is not a subgroup. There is a natural action of 0-cochains on 1-cocycles defined by:

$$\{a_i\}_i\cdot\{h_{ij}\}_{i,j}=\{h'_{ij}\}_{i,j}\qquad\text{with}\qquad h'_{ij}(x)=a_i(x)\circ h_{ij}(x)\circ a_j(x)^{-1}\quad\forall x\in U_i\cap U_j.$$

The space of orbits $\check{H}^1(\mathcal{U};\mathcal{G}):={}^{\check{Z}^1(\mathcal{U};\mathcal{G})}\!/_{\check{C}^0(\mathcal{U};\mathcal{G})}$ is the first Čech cohomology set of $\mathcal{G}$ with respect to the cover $\mathcal{U}$. Again, beware this is not a group in general.

The observation of the section 2.3 may be rephrased in these terms. Every circle bundle determines a 1-cocycle $\{g_{ij}\}_{i,j}\in\check{Z}^1(\mathcal{U};\mathcal{G})$ with $G=\mathrm{Diff}(\mathbb{S}^1)$ through its transition functions. Changing local trivialisations modifies this cocycle by the action of a 0-cochain, and two cocycles define isomorphic bundles if and only if they lie in the same orbit. Hence, according to Lemma 2.2, $\check{H}^1(\mathcal{U};\mathcal{G})$ is naturally identified with the set of isomorphism classes of circle bundles over B.

3.2 Splitting the cocycles in two.

The group $\mathrm{Diff}(\mathbb{S}^1)$ is rather complicated, in particular it is non-abelian. However, a classical result states that $\mathrm{Diff}(\mathbb{S}^1)$ is homotopy equivalent to the group of isometry of the circle $\mathrm{O}(2)$. Notice that more is true, since $\mathrm{Homeo}(\mathbb{S}^1)$ deformation retract to $\mathrm{O}(2)$. This is the fundamental reason why smooth and continuous circle bundles have the same classification.

Thus, the above discussion remains valid with $G=\mathrm{O}(2)\simeq\mathbb{S}^1\rtimes\mathbb{Z}_2$ where $\mathbb{Z}_2$ is identified with a subgroup generated by a reflection. Thus, every transition function h_{ij} can be written uniquely as $h_{ij}=(\eta_{ij},w_{ij})$ with $\eta_{ij}\in\mathscr{C}^\infty(U_i\cap U_j,\mathbb{S}^1)$ and $w_{ij}\in\mathscr{C}^0(U_i\cap U_j,\mathbb{Z}_2)$. If $\mathrm{O}(2)$ were a direct product, the collections $\{\eta_{ij}\}$ and $\{w_{ij}\}$ would define independent 1-cocycle with values in the sheaves $\mathcal{S}^1:=\mathscr{C}^\infty(-,\mathbb{S}^1)$ and the constant sheaf $\underline{\mathbb{Z}_2}$ respectively. However, since it's only a semi-direct product, the $\mathbb{S}^1$-part *is twisted* by the action of $\mathbb{Z}_2$. Indeed, the 1-cocycle condition

$$h_{ij}h_{jk}h_{ki}=1$$

is equivalent to

$$\eta_{ij}\,(w_{ij}\cdot\eta_{jk})\,(w_{ij}w_{kj}\cdot\eta_{ki})=1\qquad\text{and}\qquad w_{ij}w_{jk}w_{ki}=1.$$

Remark. We have traded one problem with coefficients in the non-abelian group $\mathrm{Diff}(\mathbb{S}^1)$ for two almost independent problems with coefficients in the abelian groups $\mathbb{S}^1$ and $\mathbb{Z}_2$.

The $\mathbb{Z}_2$-part leads to a first invariant, the Stiefel–Whitney class $w_1(\pi)$. The $\mathbb{S}^1$-part is used to define a second invariant, the Euler class $e(\pi)$ which, built that way, *depends on the Stiefel–Whitney class* (see 3.5.2 for a full discussion of this point).

3.3 The Stiefel–Whitney class $w_1(\pi)$.

The collection $\{w_{ij}\}_{i,j\in\mathcal{I}}$ from above defines a genuine 1-cocycle in $\check{Z}^1(\mathcal{U};\underline{\mathbb{Z}_2})$. Since $\mathcal{U}$ is assumed to be a good cover, the Čech cohomology groups $\check{H}^\bullet(\mathcal{U};\underline{\mathbb{Z}_2})$ are isomorphic to the sheaf cohomology groups $H^\bullet(B;\underline{\mathbb{Z}_2})$ of the sheaf $\underline{\mathbb{Z}_2}$ on B. And because B is a CW-complex and $\mathbb{Z}_2$ is abelian, these in turn are isomorphic to the singular cohomology $H^\bullet(B;\mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$. The corresponding cohomology class $[w_{ij}]\in H^1(B;\mathbb{Z}_2)$ through

$$\check{C}^1(\mathcal{U};\underline{\mathbb{Z}_2})\to\check{H}^1(\mathcal{U},\underline{\mathbb{Z}_2})\xrightarrow{\sim}H^1(B;\underline{\mathbb{Z}_2})\xrightarrow{\sim}H^1(B;\mathbb{Z}_2)$$

is called the (first) Stiefel–Whitney class of the bundle π, and is denoted by $w_1(\pi)$.

Lemma 3.2. The following equivalent assertions hold:

(i) The Stiefel-Whitney class $w_1(\pi)$ coincides with the monodromy representation $w(\pi)$.

(ii) For each loop $c : \mathbb{S}^1 \to B$, one has
$$\langle w_1(\pi), [c] \rangle = 0$$
if and only if parallel transport along c induces an orientation-preserving diffeomorphism of the fibre.

(iii) For each loop $c : \mathbb{S}^1 \to B$, one has
$$\langle w_1(\pi), [c] \rangle = 0$$
if and only if the pullback circle bundle $c^*\pi$ over $\mathbb{S}^1$ is trivialisable.

Proof. By construction of the monodromy representation, (i) and (ii) are equivalent.
Anticipating a bit and using Theorem 3.1, since $H^1(\mathbb{S}^1; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and that for any $w \in H^1(\mathbb{S}^1; \mathbb{Z}_2)$ we have $H^2(\mathbb{S}^1; \underline{\mathbb{Z}}_w) = 0$, there are only two circle bundles over the circle. On the other hand, the two following are non-isomorphic:

- the trivial bundle, with total space a torus $\mathbb{S}^1 \times \mathbb{S}^1$ and a trivial monodromy representation;
- the circle bundle structure
$$\mathrm{Kl}^2 = {}^{\mathbb{S}^1 \times [0,1]}\!\diagup_{(u,0) \sim (\overline{u},1)} \to \mathbb{S}^1, \qquad [u,t] \mapsto [t]$$
on the Klein bottle with a non-trivial monodromy representation.

This shows that a circle bundle over $\mathbb{S}^1$ is trivialisable if and only if its monodromy representation is trivial. In addition, a simple computation shows w_1 is trivial for the torus and non-trivial for the Klein bottle. So (i) is true for $B = \mathbb{S}^1$.
Let $p : c^*M \to \mathbb{S}^1$ be the induced circle bundle over $\mathbb{S}^1$. Its transition functions are $\{(\eta_{ij} \circ c, w_{ij} \circ c)\}$ and thus $w_1(p) = c^*(w_1(\pi))$. Finally, by functoriality,
$$\langle w_1(\pi), [c] \rangle = \langle c^*(w_1(\pi)), [\mathbb{S}^1] \rangle = \langle w_1(p), [\mathbb{S}^1] \rangle$$
which concludes. □

3.4 The Euler class $e(\pi)$.

Let $\pi : M \to B$ be a circle bundle and $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ a good local trivialisation cover. Denote
$$\{(\eta_{ij}, w_{ij})\}_{i,j \in \mathcal{I}}, \quad \text{with} \quad \eta_{ij} \in \mathcal{S}^1(U_i \cap U_j) = \mathscr{C}^\infty(U_i \cap U_j, \mathbb{S}^1) \quad \text{and} \quad w_{ij} \in \underline{\mathbb{Z}_2}(U_i \cap U_j) = \mathscr{C}^0(U_i \cap U_j, \mathbb{Z}_2)$$
the transition functions of π so that $\{w_{ij}\}$ represents $w_1(\pi)$. We have shown in section 3.2 that the $\mathcal{S}^1$ part $\{\eta_{ij}\}$ of the transition functions of π satisfies:
$$\eta_{ij} \, (w_{ij} \cdot \eta_{jk}) \, (w_{ij} w_{kj} \cdot \eta_{ki}) = 1.$$
In general, it does not define a 1-cocycle for the sheaf $\mathcal{S}^1$. In the next section, we shall show how to twist a sheaf according to an element $w \in H^1(B; \mathbb{Z}_2)$.

To do so, It will be convenient to switch the group $\mathbb{Z}_2$ to the isomorphic one $\mu_2 = \{-1, 1\}$ with a multiplicative notation. By this, we mean that for this section, we consider $\mathrm{O}(2) = \mathbb{S}^1 \rtimes \mu_2$ with μ_2 identified with a subgroup generated by any reflection. The group $\mathbb{Z}_2$ has the advantage to give directly the more standard notation $H^1(B; \mathbb{Z}_2)$ while the group μ_2 gives the more direct expression
$$w \cdot \eta = \eta^w$$
for the group action $\mu_2 \curvearrowright \mathbb{S}^1$. This way, the above relation becomes
$$\eta_{ij} \eta_{jk}^{w_{ij}} \eta_{ki}^{w_{ij} w_{jk}} = 1.$$

3.4.1 Twisting of a sheaf via an element $w \in \check{Z}^1(B;\underline{\mu_2})$.

Consider, for each $i \in \mathcal{I}$, a sheaf $\mathcal{G}_i$ of abelian group on $U_i \subset B$. As explained in section 3.3, to each cohomology class $w_1 \in H^1(B;\mathbb{Z}_2)$, one can associate a 1-cocycle $\{w_{ij}\} \in \check{Z}^1(\mathcal{U};\underline{\mu_2})$ representing it. On the overlaps we define the isomorphisms of sheaves:

$$\phi_{ij} : \mathcal{G}_i|_{U_i\cap U_j} \longrightarrow \mathcal{G}_j|_{U_i\cap U_j}, \qquad \mathcal{G}_i|_{U_i\cap U_j}(U) \ni g \longmapsto g^{w_{ij}} \in \mathcal{G}_j|_{U_i\cap U_j}(U).$$

Then for each open set U contained in a triple overlap $U_i \cap U_j \cap U_k$ and $g \in \mathcal{G}_i|_{U_i\cap U_j\cap U_k}(U)$

$$\phi_{ik}(g) = g^{w_{ik}} = g^{w_{ij}} a^{w_{jk}}$$

because $w_{ij}w_{jk}w_{ki} = 1.$ and $w_{ik} = w_{ki}$. This means

$$\begin{array}{ccc} \mathcal{G}_i|_{U_i\cap U_j\cap U_k} & \xrightarrow{\quad\phi_{ik}\quad} & \mathcal{G}_k|_{U_i\cap U_j\cap U_k} \\ & {\scriptstyle\phi_{ij}}\searrow \quad \nearrow{\scriptstyle\phi_{jk}} & \\ & \mathcal{G}_j|_{U_i\cap U_j\cap U_k} & \end{array}$$

commutes, hence the $(\mathcal{G}_i, \phi_{ij})$ are glueing data. The sheaf gluing theorem yields a sheaf of abelian groups $\mathcal{G}_w$ on B together with isomorphisms $\phi_i : \mathcal{G}_w|_{U_i} \xrightarrow{\sim} \mathcal{G}_i$ for each $i \in \mathcal{I}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{G}_w|_{U_i\cap U_j} & \xrightarrow[\phi_i]{} & \mathcal{G}_i|_{U_i\cap U_j} \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\phi_{ij}} \\ \mathcal{G}_w|_{U_i\cap U_j} & \xrightarrow{\phi_j} & \mathcal{G}_j|_{U_i\cap U_j} \end{array}$$

are commutative.

The following point is quite subtle but plays an important role throughout these notes. For two different representative w and w' of the same cohomology class w_1, the resulting sheaves $\mathcal{G}_w$ and $\mathcal{G}_{w'}$ are not equal, only naturally isomorphic. Indeed, if $\{\varepsilon_i\} \in \check{C}^0(\mathcal{U};\underline{\mu_2})$ is a 0-cochain, let's consider the isomorphisms

$$\mathcal{G}_i \longrightarrow \mathcal{G}_i, \qquad g \longmapsto g^{\varepsilon_i}.$$

Then the gluing functions are changed into $\tilde{\phi}_{ij} : g \mapsto g^{\varepsilon_i w_{ij} \varepsilon_j^{-1}}$.

Remark. A section $g \in \mathcal{G}_w(U)$ is equivalent to the data of a collection of sections $g_i \in \mathcal{G}_i(U \cap U_i)$ satisfying

$$g_i|_{U\cap U_i\cap U_j} = g_j^{w_{ij}}|_{U\cap U_i\cap U_j}$$

on the overlaps $U \cap U_i \cap U_j$.

Notation. For a section g_i over the open set U_i, the corresponding collection will be denoted $\{g_{i,j}\}_{j\in\mathcal{I}}$. The same way, for a section g_{ij} over $U_i \cap U_j$, the corresponding collection of sections will be denoted $\{g_{ij,k}\}_{k\in\mathcal{I}}$.

3.4.2 The Euler class as an element of $\check{H}^1\left(\mathcal{U};\mathcal{S}^1_w\right)$.

Applying the above procedure to the sheaves $\mathcal{G}_i = \mathcal{S}^1_i := \mathscr{C}^\infty(-,\mathbb{S}^1)$ on U_i and w a fixed representative of the Stiefel–Whitney class $w_1(\pi)$, one gets the twisted sheaf $\mathcal{S}^1_w$. We want now to associate with each orbit $\check{C}^0(\mathcal{U};\mathcal{S}^1)\cdot\{\eta_{ij}\}$ of 1-cochains in $\check{C}^1(\mathcal{U};\mathcal{S}^1)$ an element of $\check{H}^1(\mathcal{U};\mathcal{S}^1_w)$. This is done by the following lemma.

Lemma 3.3. The maps

$$\zeta_0 : \left|\begin{array}{ccc} \check{C}^0(\mathcal{U};\mathcal{S}^1) & \to & \check{C}^0(\mathcal{U};\mathcal{S}^1_w) \\ \{a_i\}_{i\in\mathcal{I}} & \mapsto & \{\{a_{i,j} = a_i^{w_{ij}}|_{U_i\cap U_j}\}_{j\in\mathcal{I}}\}_{i\in\mathcal{I}} \end{array}\right. \quad \text{and} \quad \zeta_1 : \left|\begin{array}{ccc} \check{C}^1(\mathcal{U};\mathcal{S}^1) & \to & \check{C}^1(\mathcal{U};\mathcal{S}^1_w) \\ \{\eta_{ij}\}_{i,j\in\mathcal{I}} & \mapsto & \{\{\eta_{ij,k} = \eta_{ij}^{w_{ik}}|_{(U_i\cap U_j)\cap U_k}\}_{k\in\mathcal{I}}\}_{i,j\in\mathcal{I}} \end{array}\right.$$

are well-defined. The map ζ_0 induces an action $\check{C}^0(\mathcal{U};\mathcal{S}^1) \curvearrowright \check{C}^1(\mathcal{U},\mathcal{S}^1_w)$ via $\{a_i\}\cdot\{\eta_{ij,k}\} = \zeta_0(\{a_{ij}\})\cdot\{\eta_{ij,k}\}$. The map ζ_1 is $\check{C}^0(\mathcal{U};\mathcal{S}^1)$-equivariant and send $\{\eta_{ij}\}$ to $\check{Z}^1(\mathcal{U};\mathcal{S}^1_w)$.

Proof. We show only that ζ_1 is well-defined, the proof for ζ_0 being analogous. We just need to check that

$$\eta_{ij,k}|_{(U_i\cap U_j)\cap U_k\cap U_l} = \eta_{ij,l}|_{(U_i\cap U_j)\cap U_k\cap U_l}.$$

But it is enough to check this equality as sections of $\mathcal{S}^1_w|_{U_l} \simeq \mathcal{S}^1_l$ and there it becomes

$$\eta_{ij}^{w_{ik}\epsilon_{kl}}|_{(U_i\cap U_j)\cap U_k\cap U_l} = \eta_{ij}^{w_{il}}|_{(U_i\cap U_j)\cap U_k\cap U_l},$$

which holds.
Let us show that the image of $\{\eta_{ij}\}$ is a 1-cocycle. It is enough to show that on the triple overlap $U_i \cap U_j \cap U_k$:

$$\eta_{ij,i}\eta_{jk,i}\eta_{ki,i} = 1 \quad \text{in} \quad \mathcal{S}^1_w|_{U_i}(U_i \cap U_j \cap U_k) \simeq \mathcal{S}^1_i(U_i \cap U_j \cap U_k).$$

Since

$$\eta_{ij,i}\eta_{jk,i}\eta_{ki,i} = \eta_{ij}\eta_{jk}^{w_{ji}}\eta_{ki}^{w_{ik}} = \eta_{ij}\eta_{jk}^{w_{ij}}\eta_{ki}^{w_{ij}w_{jk}} = 1,$$

it is indeed satisfied. It remains to check that ζ_1 is equivariant. With obvious notations,

$$\begin{aligned}\zeta_1(\{a_i\}\cdot\{\eta_{ij}\}) &= \zeta_1(\{a_i\eta_{ij}a_j^{-1}\})\\ &= \{\{(a_i\eta_{ij}a_j^{-1})^{w_{ik}}\}_{j\in\mathcal{I}}\}\\ &= \{\{a_{i,k}\}_{k\in\mathcal{I}}\}\cdot\{\{\eta_{ij,k}\}_{k\in\mathcal{I}}\}\\ &= \zeta_0(\{a_i\})\cdot\zeta_1(\{\eta_{ij}\})\end{aligned}$$

which concludes the proof. □

Thus, the orbit of $\{\eta_{ij}\}$ determines a well-defined class $\overline{e}(\pi) \in \check{H}^1(\mathcal{U};\mathcal{S}^1_w)$. This may already be regarded as a *first form of the Euler class*, although it is not yet expressed in its classical form. Let us emphasise once again that its definition depends on the choice of a specific cocycle w representing $w_1(\pi)$, and not merely on its cohomology class. Changing this representative will change the coefficient sheaf $\mathcal{S}^1_w$, together with the corresponding representative of $\overline{e}(\pi)$, through the natural identifications described in the preceding section.

The geometric reason for this twisting is quite intuitive. Consider, for instance, a loop c in B, and suppose that it is covered successively by $U_0, U_1, \dots, U_{k-1}$, with $U_i \cap U_{i+1} \neq \emptyset$, where the indices are taken in $\mathbb{Z}_k$. As we travel along c, the transition functions compare the circle coordinates chosen over consecutive charts. Their rotational parts $\eta_{i,i+1}$ measure the successive rotations of the fibres, but these rotations cannot simply be multiplied together without keeping track of the orientations in which they are measured.

On each overlap $U_i \cap U_j$, the cocycle w tells us whether the two chosen fibre coordinates have the same orientation. If they do, the rotation η_{ij} is read in the same direction and contributes as it stands. If their orientations are opposite, one must first identify the two circle coordinates by the chosen reflection identified with $-1 \in \mu_2$. Conjugation by this reflection reverses the direction of rotations, so that the same rotational discrepancy is now represented by η_{ij}^{-1}.

This is precisely the role of the twisted sheaf $\mathcal{S}^1_w$: it keeps track of the orientation in which each local rotation has to be read. The map ζ_1 simply rewrites the rotational parts of the transition functions as sections of this single twisted coefficient sheaf. In this way, $\overline{e}(\pi)$ records the cumulative rotational twisting of the fibres in a manner which remains meaningful even when their orientations cannot be chosen globally.

3.4.3 Isomorphism $\check{H}^1(\mathcal{U};\mathcal{S}^1_w) \simeq H^2(B;\underline{\mathbb{Z}}_w)$ and Euler class $e(\pi)$.

Since $\mathcal{U}$ is assumed to be a good cover, it is easy to see that $\check{H}^\bullet(\mathcal{U};\mathcal{S}^1_w)$ computes the sheaf cohomology $H^\bullet(B;\mathcal{S}^1_w)$. For each $i \in \mathcal{I}$, we can consider the exponential exact sequence on U_i:

$$0 \longrightarrow \underline{\mathbb{Z}}_i \longrightarrow \mathscr{C}^\infty_{U_i}(-,\mathbb{R}) \xrightarrow{u\mapsto e^{2i\pi u}} \mathcal{S}^1_i \longrightarrow 0.$$

The group μ_2 acts on

- $\mathbb{S}^1$ by inversion $s \mapsto s^w$;
- $\mathbb{R}$ by sign $t \mapsto (-1)^w t$;
- $\mathbb{Z}$ by sign $n \mapsto (-1)^w n$.

Those actions are compatible with the exponential map because $e^{2i\pi(-t)} = (e^{2i\pi t})^{-1}$. Building the corresponding twisted sheaves as described in section 3.4.1, one gets a twisted exponential exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_w \longrightarrow \mathscr{C}^\infty_w(-,\mathbb{R}) \longrightarrow \mathcal{S}^1_w \longrightarrow 0$$

of sheaves of abelian groups on B and thus the following long exact sequence:

$$\cdots \longrightarrow H^1\left(B;\mathscr{C}^\infty_w(-,\mathbb{R})\right) \longrightarrow H^1\left(B;\mathcal{S}^1_w\right) \longrightarrow H^2\left(B;\underline{\mathbb{Z}}_w\right) \longrightarrow H^2\left(B;\mathscr{C}^\infty_w(-,\mathbb{R})\right) \longrightarrow \dots.$$

Note that $\mathscr{C}^\infty_w(-,\mathbb{R})$ is the sheaf of smooth sections of a real line bundle. Those are fine, and thus acyclic, which yields

$$H^1\left(B;\mathscr{C}^\infty_w(-,\mathbb{R})\right) = H^2\left(B;\mathscr{C}^\infty_w(-,\mathbb{R})\right) = 0$$

and then

$$H^1\left(B;\mathcal{S}^1_w\right) \xrightarrow{\sim} H^2\left(B;\underline{\mathbb{Z}}_w\right).$$

The image of $\bar{e}(\pi)$ defined above by this isomorphism is called the *Euler class* of the circle bundle $\pi : M \to B$, denoted by $e(\pi) \in H^2(B;\underline{\mathbb{Z}}_w)$.

Remark. The construction above associates with a choice of local fibre coordinates, whose determinant cocycle is the fixed representative w, a class $e_w(\pi) \in H^2(B;\underline{\mathbb{Z}}_w)$. There remains, however, one global ambiguity. Reversing simultaneously all the fibre coordinates does not modify the determinant cocycle w, since the corresponding $\mathbb{Z}_2$-valued 0-cochain is constant and therefore has zero coboundary. On the other hand, it induces the automorphism $-1 : \underline{\mathbb{Z}}_w \longrightarrow \underline{\mathbb{Z}}_w$ and replaces $e_w(\pi) \longmapsto -e_w(\pi)$. Consequently, if no fibre orientation is retained, the choice-free invariant expressed in the fixed coefficient system $\underline{\mathbb{Z}}_w$ is the orbit

$$[e_w(\pi)]_\pm \in {H^2(B;\underline{\mathbb{Z}}_w)}\big/{\{\pm 1\}}.$$

There is a second dependence on choices: the sheaf $_w$ itself depends on the chosen representative w of the Stiefel–Whitney class. If $w' = w + \delta a$, the 0-cochain a induces an isomorphism

$$\underline{\mathbb{Z}}_w \xrightarrow{\sim} \underline{\mathbb{Z}}_{w'}.$$

Different choices of a inducing the same change from w to w' differ by a $\mathbb{Z}_2$-valued 0-cocycle. Since B is connected, such a cocycle is constant, and the corresponding two isomorphisms differ by the global automorphism -1. They therefore induce the same map after passing to the quotients

$${H^2(B;\underline{\mathbb{Z}}_w)}\big/{\{\pm 1\}} \xrightarrow{\sim} {H^2(B;\underline{\mathbb{Z}}_{w'})}\big/{\{\pm 1\}}.$$

Thus the quotient depends only on the class w_1. For the genuinely intrinsic Euler class, before any identification of its coefficient system with a standard model $\underline{\mathbb{Z}}_w$, see Subsection 3.5.2.

3.5 About computations.

3.5.1 Cohomology of twisted sheaves and singular cohomology with local coefficients.

Let A be an abelian group, and let M be an abelian group endowed with an action

$$\rho : A \longrightarrow \mathrm{Aut}(M).$$

Let B be a connected manifold with base point b_0, and let $w_1 \in H^1(B;A)$. Under the canonical identification

$$H^1(B;A) \simeq \mathrm{Hom}(\pi_1(B,b_0),A),$$

the class w_1 determines a homomorphism

$$\pi_1(B,b_0) \longrightarrow A \xrightarrow{\rho} \mathrm{Aut}(M),$$

and hence an action of $\pi_1(B,b_0)$ on M. Since B is connected, this is equivalent to the data of a local coefficient system on B, which we denote by M_{w_1}; see Theorem VI.(1.12) in [Whi78].

There are two equivalent ways of associating cohomology groups with this local system. First, one may form the singular cohomology groups with local coefficients

$$H^\bullet_{\mathrm{sing}}(B;M_{w_1});$$

see for instance Section VI.2 of [Whi78].

On the other hand, choose a sufficiently fine open cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ and a Čech cocycle $w \in \check{Z}^1(\mathcal{U};\underline{A})$ representing w_1. Starting with the constant sheaf $\underline{M}$ on each U_i and twisting the gluing morphisms by $\rho(w_{ij})$, as described in Section 3.4.1, one obtains a locally constant sheaf which we denote by $\underline{M}_w$.

Since B is a manifold, the sheaf cohomology of this locally constant sheaf agrees with singular cohomology with the corresponding local coefficients:

$$H^\bullet(B;\underline{M}_w) \simeq H^\bullet_{\mathrm{sing}}(B;M_{w_1}).$$

In particular, although the left-hand side is written using a chosen representative w, its isomorphism class depends only on w_1. Keeping track of the chosen representative remains important, however, when one wants to identify

specific cohomology classes under a change $w \mapsto w + \delta a$, see the following subsection for more details.

This identification provides useful alternative methods for computations. In particular, singular cohomology with local coefficients may be computed on the universal cover by means of equivariant cochains. We do not introduce the general theory here, but restrict ourselves to the case needed below. For more details, see Section VI.3 of [Whi78].

Denote by $\widetilde{B}$ the universal cover of B and choose a lift $\widetilde{b}_0$ of b_0. The group $\pi_1(B, b_0)$ acts naturally on $\widetilde{B}$ by deck transformations and hence on its singular chain complex $C_\bullet(\widetilde{B})$ with integral coefficients. Together with the action of $\pi_1(B, b_0)$ on M determined by w_1, this allows us to consider the complex of equivariant homomorphisms

$$\mathrm{Hom}^{\pi_1(B,b_0)}\big(C_\bullet(\widetilde{B}), M\big).$$

Its cohomology groups are denoted by $E^\bullet(\widetilde{B}; M)$. Since B is connected, we shall usually omit the base points from the notation.

Theorem 3.4 (Eilenberg, VI.3.(3.4*) in [Whi78])**.** There are natural isomorphisms

$$H^\bullet_{\mathrm{sing}}(B; M_{w_1}) \simeq E^\bullet(\widetilde{B}; M).$$

For computational purposes, especially in the next section, it is useful to consider the cellular version of this result. For details, see Section VI.4 of [Whi78]. Let $\Gamma_\bullet(\widetilde{B})$ denote the cellular chain complex of $\widetilde{B}$ with integral coefficients. Again, $\pi_1(B, b_0)$ acts on $\Gamma_\bullet(\widetilde{B})$ through deck transformations, and one considers the complex

$$\mathrm{Hom}^{\pi_1(B,b_0)}\big(\Gamma_\bullet(\widetilde{B}), M\big).$$

Its cohomology groups are the equivariant cellular cohomology groups, denoted by $E^\bullet(\Gamma(\widetilde{B}); M)$.

Theorem 3.5 (Theorem VI.(4.9) in [Whi78])**.** The cellular cohomology groups with local coefficients are naturally isomorphic to the equivariant cellular cohomology groups:

$$H^\bullet_{\mathrm{cell}}(B; M_{w_1}) \simeq E^\bullet(\Gamma(\widetilde{B}); M).$$

Finally, the usual comparison between singular and cellular cohomology remains valid with local coefficients.

Theorem 3.6 (Theorem VI.(4.4*) in [Whi78])**.** There are natural isomorphisms

$$H^\bullet_{\mathrm{sing}}(B; M_{w_1}) \simeq H^\bullet_{\mathrm{cell}}(B; M_{w_1}).$$

3.5.2 On the intrinsic nature of the Euler class.

There is a small but important distinction between the intrinsic Euler class of a circle bundle and the different concrete representatives of it that appear in the preceding constructions. We make this distinction explicit, as it will become particularly useful for Seifert fibrations.

Let $\pi : M \longrightarrow B$ be a circle bundle. Associated with π is an intrinsic rank-one integral local system $\mathcal{Z}_\pi$ on B, whose stalk at $b \in B$ is

$$(\mathcal{Z}_\pi)_b := H_1(M_b; \mathbb{Z}) \simeq \mathbb{Z}.$$

Transport along a path in B induces an isomorphism between the homology groups of the corresponding fibres, and therefore defines the monodromy of $\mathcal{Z}_\pi$. No orientation of the fibres is involved in this definition. The associated homomorphism

$$\pi_1(B, b_0) \longrightarrow \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$$

is precisely the orientation monodromy of the circle bundle, and its cohomology class is the first Stiefel–Whitney class $w_1(\pi) \in H^1(B; \mathbb{Z}_2)$.

At this intrinsic level, the Euler class is *canonically* defined as

$$e(\pi) \in H^2(B; \mathcal{Z}_\pi).$$

Thus neither the coefficient system nor the Euler class requires a choice of orientation of the fibres, a representative of $w_1(\pi)$, or a system of local trivialisations. This is the genuinely choice-free form of the Euler class.

One may pass from this intrinsic local system to the more familiar notation involving $w_1(\pi)$ by making a first choice. Fix a base point $b_0 \in B$ and choose a generator of

$$(\mathcal{Z}_\pi)_{b_0} = H_1(M_{b_0}; \mathbb{Z}),$$

or equivalently an orientation of the fibre M_{b_0}. Parallel transport then identifies $\mathcal{Z}_\pi$ with the standard local coefficient system associated with the representation

$$\pi_1(B, b_0) \longrightarrow \{\pm 1\}, \qquad \gamma \longmapsto (-1)^{\langle w_1(\pi), \gamma\rangle}.$$

Denote this standard local coefficient system by $\mathbb{Z}_{w_1}$. The chosen generator therefore gives an isomorphism

$$\mathcal{Z}_\pi \xrightarrow{\sim} \mathbb{Z}_{w_1}$$

and transports the intrinsic Euler class to an element

$$e_{w_1}(\pi) \in H^2_{\text{sing}}(B; \mathbb{Z}_{w_1}).$$

This second description is no longer completely canonical. There are two possible choices of generator of $H_1(M_{b_0}; \mathbb{Z})$, and replacing one by its opposite changes the above identification by the automorphism $-1 : \mathbb{Z}_{w_1} \longrightarrow \mathbb{Z}_{w_1}$. Accordingly, the corresponding numerical description of the Euler class is changed by $e_{w_1}(\pi) \longmapsto -e_{w_1}(\pi)$.

Thus the notation $e_{w_1}(\pi) \in H^2_{\text{sing}}(B; \mathbb{Z}_{w_1})$ implicitly contains a choice of identification of the intrinsic fibre-orientation local system with the standard model $\mathbb{Z}_{w_1}$. The class w_1 determines the isomorphism class of the coefficient system, but not such an identification.

There is finally a third, more concrete, level corresponding to the Čech description used above. Choose a trivialising cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ and local orientations of the circle coordinates. Comparing these orientations on the overlaps produces a cocycle $w \in \check{Z}^1(\mathcal{U}; \mathbb{Z}_2)$ representing $w_1(\pi)$. These choices give a concrete presentation of the intrinsic local system as the twisted sheaf

$$\mathcal{Z}_\pi \simeq \underline{\mathbb{Z}}_w.$$

The intrinsic Euler class is then represented by $e_w(\pi) \in H^2(B; \underline{\mathbb{Z}}_w)$.

At this level the dependence on the choices is more visible. If the local orientations are changed by a 0-cochain a, the cocycle w is replaced by $w' = w + \delta a$. The sheaves $\underline{\mathbb{Z}}_w$ and $\underline{\mathbb{Z}}_{w'}$ are not literally the same sheaf, but a determines an isomorphism

$$\varphi_a : \underline{\mathbb{Z}}_w \xrightarrow{\sim} \underline{\mathbb{Z}}_{w'}.$$

The corresponding isomorphism in cohomology transports the Euler class:

$$\varphi_a\big(e_w(\pi)\big) = e_{w'}(\pi).$$

Consequently, the apparent dependence of the Euler class on the particular cocycle w is simply the dependence of a concrete presentation on the chosen local generators. All the classes $e_w(\pi)$ obtained in this way are presentations of the same intrinsic class $e(\pi) \in H^2(B; \mathcal{Z}_\pi)$.

It is useful to summarise these three levels as follows:

$$\boxed{e(\pi) \in H^2(B; \mathcal{Z}_\pi)} \qquad \text{intrinsic and canonical,}$$

where $\mathcal{Z}_\pi$ itself is canonically associated with the circle bundle; then, after choosing a generator of one fibre,

$$\boxed{e_{w_1}(\pi) \in H^2_{\text{sing}}(B; \mathbb{Z}_{w_1})} \qquad \text{defined up to the global automorphism } -1;$$

and finally, after choosing local fibre coordinates,

$$\boxed{e_w(\pi) \in H^2(B; \underline{\mathbb{Z}}_w)} \qquad \text{depending on a cocycle } w \text{ representing } w_1,$$

with different choices related by the canonical isomorphisms induced by the corresponding 0-cochains.

3.5.3 Computation of $H^2_{\text{sing}}(B; \mathbb{Z}_{w_1})$ for closed connected surface B.

This section is devoted to the proof of the following proposition:

Proposition 3.7. For a connected closed surface B and $w_1 \in H^1(B; \mathbb{Z}_2)$,

$$H^2_{\text{sing}}(B; \mathbb{Z}_{w_1}) \simeq \begin{cases} \mathbb{Z} & \text{if } w_1 = w_1(B); \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

where $w_1(B) \in H^1(B; \mathbb{Z}_2)$ is the orientation class of B.

Proof. As explained in the previous section,

$$H^2_{\text{sing}}(B;\mathbb{Z}_w) \simeq E^2_{cell}(\widetilde{B};\mathbb{Z}_{w_1}) = H^2\left(\text{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_\bullet(\widetilde{B}),\mathbb{Z}_{w_1})\right),$$

where $\Gamma_\bullet(\widetilde{B})$ denotes the cellular chain complex with integral coefficients of the universal cover. The coboundary operator is

$$d^n : \text{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_n(\widetilde{B}),\mathbb{Z}_{w_1}) \longrightarrow \text{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_{n+1}(\widetilde{B}),\mathbb{Z}_w), \qquad d^n(\varphi) = \varphi \circ \partial_{n+1}$$

with $\partial_{n+1} : \Gamma_{n+1}(\widetilde{B}) \to \Gamma_n(\widetilde{B})$ the usual cellular boundary operator.

Consider a CW-decomposition of B with one 0-cell, some 1-cells $e_1,\dots,e_r$ and one 2-cell E (see for instance Section 2.2. in [FM97]). The universal cover $\widetilde{B}$ has a CW-decomposition where all cells are lifts of cells of B. Choose lifts $\widetilde{e_1},\dots,\widetilde{e_r}$ to $\widetilde{B}$ with the same initial point and a lift $\widetilde{E}$ of E. Different lifts are related by an element of $\pi_1(B)$, thus the cellular chains are free $\mathbb{Z}[\pi_1(B)]$-modules:

$$\Gamma_2(\widetilde{B}) \simeq \mathbb{Z}[\pi_1(B)]\widetilde{E}, \qquad \Gamma_1(\widetilde{B}) \simeq \bigoplus_{i=1}^{r} \mathbb{Z}[\pi_1(B)]\widetilde{e}_i.$$

Therefore,

$$\text{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_2(\widetilde{B}),\mathbb{Z}_{w_1}) \simeq \mathbb{Z}, \qquad \text{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_1(\widetilde{B}),\mathbb{Z}_{w_1}) \simeq \mathbb{Z}^r.$$

Case of an orientable surface.
Assume B is an orientable closed surface of genus g. Then

$$\pi_1(B) \simeq \langle a_1,b_1,\dots,a_g,b_g \;\mid\; [a_1,b_1]\cdots[a_g,b_g]\rangle$$

and one can choose a CW-decomposition of B with one 0 cell, $2g$ 1-cells $e_{a_1},e_{b_1},\dots,e_{a_g},e_{b_g}$ representing the generators a_i,b_i, and one 2-cell attached, up to cyclic order, to the loop

$$R = [a_1,b_1]\cdots[a_g,b_g].$$

Choose lifts $\widetilde{e}_{a_1},\widetilde{e}_{b_1},\dots,\widetilde{e}_{a_g},\widetilde{e}_{b_g}$ of the 1-cells with common initial point. A suitable lift of the 2-cell is then attached along a lift of the word R. Let us compute the contribution of a single commutator

$$[a_i,b_i] = a_ib_ia_i^{-1}b_i^{-1}.$$

Starting from the chosen lift of the 0-cell, the corresponding lifted path is

$$\widetilde{e}_{a_i}\cdot(a_i\cdot\widetilde{e}_{b_i})\cdot\left(-(a_ib_ia_i^{-1})\cdot\widetilde{e}_{a_i}\right)\cdot\left(-[a_i,b_i]\cdot\widetilde{e}_{b_i}\right).$$

Indeed, $(a_ib_ia_i^{-1})\cdot\widetilde{e}_{a_i}$ runs from $a_ib_ia_i^{-1}$ to a_ib_i, and must therefore be traversed in the opposite direction; similarly, $[a_i,b_i]\cdot\widetilde{e}_{b_i}$ runs from $[a_i,b_i]$ to $a_ib_ia_i^{-1}$ and must also be traversed backwards. Hence the contribution of the commutator $[a_i,b_i]$ to the cellular boundary is

$$\left(1-a_ib_ia_i^{-1}\right)\cdot\widetilde{e}_{a_i} + \left(a_i-[a_i,b_i]\right)\cdot\widetilde{e}_{b_i}.$$

Let $\mu \in \text{Hom}_{\mathbb{Z}[\pi_1(B)]}\left(\Gamma_1(\widetilde{B}),\mathbb{Z}_{w_1}\right)$ correspond to $(u_{a_1},u_{b_1},\dots,u_{a_g},u_{b_g}) \in \mathbb{Z}^{2g}$. Since the action of $\pi_1(B)$ on $\mathbb{Z}_{w_1}$ is given by

$$\gamma\cdot n = (-1)^{w_1(\gamma)}n,$$

the contribution of the commutator $[a_i,b_i]$ to $d^1(\mu)$ is

$$\left(1-(-1)^{w_1(a_ib_ia_i^{-1})}\right)u_{a_i} + \left((-1)^{w_1(a_i)}-(-1)^{w_1([a_i,b_i])}\right)u_{b_i} = \left(1-(-1)^{w_1(b_i)}\right)u_{a_i} + \left((-1)^{w_1(a_i)}-1\right)u_{b_i}.$$

Therefore the image of d^1 is generated by

$$\left\{1-(-1)^{w_1(b_i)},\ (-1)^{w_1(a_i)}-1 \;\middle|\; i=1,\dots,g\right\}.$$

If $w_1=0$, all these generators vanish and $\text{Im}(d^1)=0$. If $w_1\neq 0$, then $w_1(a_i)=1$ or $w_1(b_i)=1$ for some i, so one of the above generators is equal to ± 2. Hence $\text{Im}(d^1)=2\mathbb{Z}$.

Case of a non-orientable surface.
Assume B is a non-orientable closed surface of genus q, that is

$$B \simeq \#^q\mathbb{RP}^2.$$

Then

$$\pi_1(B) \simeq \langle a_1, \ldots, a_q \mid a_1^2 \cdots a_q^2 \rangle.$$

One can choose a CW-decomposition with one 0-cell, q 1-cells $e_{a_1}, \ldots, e_{a_q}$ representing the generators a_i and one 2-cell attached along the loop

$$R = e_{a_1}^2 \cdots e_{a_q}^2.$$

Choose lifts $\widetilde{e}_{a_1}, \ldots, \widetilde{e}_{a_q}$ of these 1-cells with common initial point. A suitable lift of the 2-cell $\widetilde{E}$ is attached along

$$\widetilde{R} = \widetilde{e}_{a_1}.(a_1 \cdot \widetilde{e}_{a_1}).\widetilde{e}_{a_2}.(a_2 \cdot \widetilde{e}_{a_2}).\cdots.(a_q \cdot \widetilde{e}_{a_q}).$$

Hence, applying $\mu \in \mathrm{Hom}_{\mathbb{Z}[\pi_1(B)]}(\Gamma_1(\widetilde{B}), \mathbb{Z}_{w_1})$ corresponding to $(u_{a_1}, \ldots, u_{b_q}) \in \mathbb{Z}^q$, gives the contribution

$$\big(1 + (-1)^{w_1(a_i)}\big) u_{a_i}$$

for each 1-cell $\widetilde{e}_i$. Therefore the image of d^1 is generated by

$$\{1 + (-1)^{w_1(a_i)} \mid i = 1, \ldots, q\}.$$

and is 0 if $w_1 = w_1(B)$, that is $w_1(a_i) = \bar{1} \in \mathbb{Z}_2$ for each i, and $2\mathbb{Z}$ otherwise. □

Remark. Another way to compute this group is by using twisted Poincaré duality:

Theorem 3.8. For a closed and connected n-manifold M, any $\mathbb{Z}[\pi_1(M)]$-module L, and $\mathbb{Z}_{w_1(M)}$ the orientation local system:

$$H^k(M; L) \simeq H_{n-k}(M; L \otimes \mathbb{Z}_{w_1(M)}).$$

In the case of a connected closed surface B and $L = \mathbb{Z}_w$ for some $w \in H^1(B; \mathbb{Z}_2)$, this yields

$$H^2(B; \mathbb{Z}_w) \simeq H_0(B; \mathbb{Z}_w \otimes \mathbb{Z}_{w_1(B)}) = H_0(B; \mathbb{Z}_{w+w_1(B)}).$$

The latter is the group of coinvariants of the $\mathbb{Z}[\pi(B)]$-module $\mathbb{Z}_{w+w_1(B)}$. If the local system is non-trivial, then some loop acts by -1 so that $m \sim -m$ for any $m \in \mathbb{Z}$. Hence $H_0(B, \mathbb{Z}_{w+w_1(B)}) \simeq \mathbb{Z}_2$. Otherwise $H_0(B, \mathbb{Z}_{w+w_1(B)}) \simeq \mathbb{Z}$. This recovers the result above.

4 Seifert fibre spaces and their invariants.

The theory of Seifert manifolds naturally extends that of circle bundles over surfaces. Besides the standard fibred solid torus

$$\mathrm{pr}_1 : \mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow \mathbb{D}^2,$$

one also allows local models of the form

$$\overline{\mathrm{pr}_1} : T(\alpha,\beta) = {}^{\mathbb{D}^2 \times \mathbb{S}^1}\!/_{\mathbb{Z}_\alpha} \approx \mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_\alpha} \approx \mathbb{D}^2,$$

where $\alpha > 1$ and $\mathbb{Z}_\alpha$ acts on $\mathbb{D}^2 \times \mathbb{S}^1$ by

$$\bar{1} \cdot (z,u) = \left(e^{\frac{2\pi i\beta}{\alpha}} z, e^{\frac{2\pi i}{\alpha}} u\right),$$

with $\gcd(\alpha,\beta) = 1$. Such neighbourhoods contain an *exceptional fibre* (or *critical fibre*), and exceptional fibres are isolated. Away from them, the projection is an ordinary circle bundle. The quotient space is therefore naturally a 2–dimensional orbifold whose cone points correspond precisely to the exceptional fibres.

Seifert classified Seifert fibrations in his seminal paper [Sei33]; an English translation by W.Heil may be found in [Sei+80]. His classification is expressed in terms of a collection of numerical invariants associated with the isomorphism class of the fibration. This point of view has become standard in low-dimensional topology, although the precise conventions for these *Seifert invariants* vary from one author to another. It is particularly well suited to classification and construction problems.

Our perspective is different. Rather than working with numerical invariants, we seek a cohomological description extending the familiar theory of circle bundles. The intrinsic data associated with a Seifert bundle will consist of its base orbifold with only conic points $\mathcal{O} = B(b_1,\ldots,b_m)$, its first Stiefel–Whitney class $w_1 \in H^1(B;\mathbb{Z}_2)$, and an Euler class

$$e \in H^2(\mathcal{O};\mathbb{Z}_w).$$

This parallels exactly what we had already for circle bundles The main difficulty lies in defining this Euler class and the "orbifold cohomology group" in which it takes values.

For an ordinary circle bundle over a surface B, the Euler class records the obstruction to patching together local trivialisations. More concretely, if

$$U_i \times \mathbb{S}^1 \qquad \text{and} \qquad U_j \times \mathbb{S}^1$$

are two local trivializations, then over an intersection $U_i \cap U_j$ the discrepancy between the two parametrisations of the fibres is given by a continuous map

$$U_i \cap U_j \longrightarrow \mathrm{Diff}(\mathbb{S}^1) \overset{hom.\,equ.}{\cong} \mathrm{O}(2) \simeq \mathbb{S}^1 \rtimes \mathbb{Z}_2.$$

The $\mathbb{S}^1$-part of these transition functions form a Čech cocycle whose cohomology class is the Euler class. One would like to proceed similarly for Seifert bundles. However, a new phenomenon appears near the exceptional fibres.

Let $\pi : M \to \mathcal{O}$ be a Seifert bundle, and let $b \in U_i \cap U_j$, where U_i contains a cone point of order $\alpha > 1$, whereas U_j is an ordinary manifold chart. Over U_i, the bundle is identified with

$$T(\alpha,\beta) \longrightarrow {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_\alpha} \approx U_i,$$

while over U_j it is identified with the trivial bundle $\mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow \mathbb{D}^2$. To compare these two local models, one must compare the corresponding parametrisations of the fibre $M_b := \pi^{-1}(b)$. As in the case of ordinary circle bundles, one may reduce to the case where the discrepancy between the two parametrisations is given by a rotation of the circle and therefore by an element $\theta_b \in \mathbb{S}^1$.

The difficulty is that the parametrisation coming from the exceptional chart is not uniquely defined. Indeed, a point of M_b corresponds to an orbit in $T(\alpha,\beta)$ rather than to a distinguished point of $\mathbb{D}^2 \times \mathbb{S}^1$. Choosing another representative of the same orbit changes the parametrisation by the action of $\mathbb{Z}_\alpha$ on $\mathbb{S}^1$. Moreover, when one travels once around the cone point, these choices cannot be made consistently: the monodromy around the singular point precisely records the isotropy action. Consequently, there is no globally defined transition function on the ordinary overlap $U_i \cap U_j$.

The remedy is to refine the notion of transition function. Instead of assigning transition functions only to overlaps of orbifold charts $U_i \cap U_j$, one must also assign them to the isotropy arrows $V_i \to V_i$ of the orbifold atlas. This naturally leads to the étale groupoid associated with the orbifold and to its simplicial nerve, whose cohomology replaces ordinary Čech cohomology. The resulting Euler class lies in the orbifold cohomology group

$$H^2_{orb}(\mathcal{O};\mathbb{Z}_w),$$

and recovers, in a functorial and intrinsic way, the information encoded by the classical Seifert invariants.

It should be emphasised that our discussion applies to a fairly general class of Seifert bundles, although not to the most general notion considered in the literature. More precisely, we do not allow local models obtained by taking quotient of the standard fibred solid torus by an involution acting by a reflection on the $\mathbb{D}^2$-factor (see [Sco83]). Subject to this restriction, our framework includes Seifert bundles with non-orientable total space, as well as Seifert bundles whose base orbifold is bad.

These cases, and especially the latter, are precisely those for which the classical description by Seifert invariants becomes less satisfactory. In particular, the usual notion of Euler number is no longer straightforward to define, whereas the orbifold Euler class introduced below remains a natural and functorial invariant.

4.1 Seifert bundles.

Let $\alpha > 0$ and β be integers. The cyclic group $\mathbb{Z}_\alpha$ acts on $\mathbb{D}^2 \times \mathbb{S}^1$ by

$$\bar{1} \cdot (z, u) = \left(e^{2\pi \mathrm{i}\beta/\alpha} z, e^{2\pi \mathrm{i}/\alpha} u \right).$$

This action is free if and only if $\gcd(\alpha, \beta) = 1$. Thus coprimality condition is exactly the condition that the total space of the local model be a manifold. We then suppose $\gcd(\alpha, \beta) = 1$, and the quotient

$$T(\alpha, \beta) := {}^{\mathbb{D}^2 \times \mathbb{S}^1}\!/_{\mathbb{Z}_\alpha}$$

is again diffeomorphic to a solid torus. This Seifert structure may be obtained from the trivial structure on a solid torus by cutting it open along $\mathbb{D}^2 \times \{u\}$, rotating one end of the obtained cylinder by β/α of a full twist, and gluing back together. When $\alpha = 1$, the action is trivial and $T(1, 0) \cong \mathbb{D}^2 \times \mathbb{S}^1$ is the usual trivial circle bundle over a disc. Notice that the models $T(\alpha, \beta)$ are exactly the one covered by $\mathbb{D}^2 \times \mathbb{S}^1$ with its trivial fibration through a fibre-preserving covering of finite degree α. The image of the core circle $\{0\} \times \mathbb{S}^1$ is called the *central fibre*.

Definition 4.1. A *Seifert fibration* on a closed 3-manifold M is a decomposition of M into disjoint circles, called *fibres*, such that every fibre admits a saturated neighbourhood fibre-preserving diffeomorphic to one of the standard models

$$T(\alpha, \beta) \longrightarrow {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_\alpha},$$

where the projection is induced by $\mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{D}^2$. A fibre with $\alpha > 1$ is called an *exceptional fibre*; otherwise it is called *regular*. The pair (α, β) is called the *local Seifert invariant*.

Ordinary circle bundles correspond to the special case in which all local invariants are $(1, 0)$. It follows directly from the construction that two fibred solid tori $T(\alpha, \beta)$ and $T(\alpha', \beta')$ are isomorphic if and only if $\alpha = \alpha'$ and $\beta' = \pm\beta + k\alpha$ for some integer k. Hence the local invariant is not uniquely determined. When $\alpha > 1$, a unique representative is obtained by imposing $0 < \beta \leqslant \alpha/2$. Furthermore, they are *orientation-preservingly* isomorphic if and only if $\alpha = \alpha'$ and $\beta' = \beta + k\alpha$ for some integer k. Then for *oriented* fibred solid tori, it is enough to normalise with $0 < \beta < \alpha$.

We describe here our orientations convention, which is the one chosen in [Sco83]. For every oriented manifold V, its boundary ∂V is oriented so that adding the inward normal direction gives back the orientation of V. We endow $\mathbb{D}^2$, $\mathbb{S}^1$ and consequently $\mathbb{D}^2 \times \mathbb{S}^1$ with their standard orientations. With this convention, the local invariant (α, β) is represented by an anticlockwise twist through an angle $2\pi\beta/\alpha$ on the $\mathbb{D}^2$ factor.

The quotient of M by the equivalence relation identifying points lying on the same fibre is naturally a closed 2-dimensional orbifold $\pi : M \to \mathcal{O}$, called the *base orbifold*. Its underlying topological space is a closed surface B, and the images of the exceptional fibres become cone points. More precisely, if an exceptional fibre has local invariant (α, β), then its image is a cone point of order α. Consequently,

$$\mathcal{O} = B(\alpha_1, \ldots, \alpha_m).$$

Let b_i denote the image of the exceptional fibre with multiplicity α_i. Over the complement of the cone points,

$$\pi^{-1}\big(B \setminus \{b_1, \ldots, b_m\}\big) \longrightarrow B \setminus \{b_1, \ldots, b_m\}$$

is an ordinary circle bundle. The local invariants do not determine the Seifert bundle. As for ordinary circle bundles, there are also global topological invariants measuring the twisting of the fibration. In the orientable case, this information is classically encoded by an integer b, the *integral Euler class*, or equivalently a rational e called the *rational Euler class*, together with the local invariants $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$. For general Seifert bundles, one must also specify the first Stiefel–Whitney class of the underlying circle bundle over the regular part of the base. The α_i are already encoded in the base orbifold. Our aim is to reinterpret the integer b (as well as e) and the local invariants β_i as a single orbifold Euler class.

4.2 The groupoid associated with an orbifold atlas.

Let $\mathcal{O}$ be a 2-dimensional orbifold whose singularities are cone points, and let $B = |\mathcal{O}|$ be its underlying topological space. Choose a locally finite *orbifold atlas*

$$\mathcal{U} = \{(V_i, \Gamma_i, \varphi_i, U_i)\}_{i\in I}\,.$$

Here V_i is a connected open subset of $\mathbb{R}^2$, the finite group Γ_i acts effectively and diffeomorphically on V_i, and

$$\varphi_i \colon V_i \longrightarrow U_i \subset B$$

induces a homeomorphism $V_i\!\!\diagup_{\Gamma_i} \xrightarrow{\sim} U_i$. The U_i define an open cover of B and these charts must be locally compatible: if $x \in U_i \cap U_j$ there exists a chart $(V_k, \Gamma_k, \varphi_k, U_k) \in \mathcal{U}$ with $x \in U_k \subset U_i \cap U_j$ and two smooth embeddings $\lambda_{ki} : V_k \hookrightarrow V_i$ and $\lambda_{kj} : V_k \hookrightarrow V_j$ satisfying

$$\varphi_i \circ \lambda_{ki} = \varphi_k = \varphi_j \circ \lambda_{kj}.$$

Since the only singularities of $\mathcal{O}$ are cone points, the groups Γ_i may be taken to be cyclic. More precisely, if U_i does not contain a cone point, then Γ_i is trivial. If U_i contains a cone point of order α_i, then, after identifying V_i with a disc in $\mathbb{C}$, one may take $\Gamma_i \simeq \mathbb{Z}_{\alpha_i}$ acting by rotations:

$$\bar{1} \cdot z = e^{\frac{2\pi \mathrm{i}}{\alpha_i}} z.$$

In the following, we will always assume that the chosen orbifold atlas satisfies the following conditions.

1. Each cone point is contained in a unique singular chart.
2. The singular charts are pairwise disjoint.
3. The images U_i form a good cover of the underlying surface B, that is, every nonempty finite intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is contractible.

We now recall how the atlas $\mathcal{U}$ determines an *étale groupoid*. Set

$$G_0 := \coprod_{i\in I} V_i.$$

The points of G_0 are the objects of the groupoid. Thus, an object is simply a point in one of the local uniformizing charts. It projects to a point of the orbifold $\mathcal{O}$, but distinct objects may project to the same point. This occurs either because the point belongs to several orbifold charts, or because it has several lifts in a singular chart. Note that G_0 has a natural topology and smooth structure.

The arrows encode changes of orbifold charts, as well as change of representatives inside a singular chart. Suppose that

$$x \in V_i, \qquad y \in V_j, \qquad \varphi_i(x) = \varphi_j(y).$$

A local change of charts from a neighbourhood of x to a neighbourhood of y is a germ of a diffeomorphism $\lambda \colon W \longrightarrow W'$, where $W \subset V_i$ and $W' \subset V_j$ are open neighbourhoods of x and y, such that

$$\lambda(x) = y \qquad \text{and} \qquad \varphi_j \circ \lambda = \varphi_i$$

on W. The set of arrows is therefore

$$G_1 = \{[\lambda]_x \mid \lambda \text{ is a local change of orbifold charts defined near } x\}\,.$$

For an arrow $[\lambda]_x$, its source and target are

$$s([\lambda]_x) = x, \qquad t([\lambda]_x) = \lambda(x).$$

The identity arrow at x is the germ of the identity map $1_x = [\mathrm{id}]_x$, the inverse of $[\lambda]_x$ is $[\lambda]_x^{-1} = [\lambda^{-1}]_{\lambda(x)}$, and the composition is induced by composition of germs: $[\mu]_{\lambda(x)} \circ [\lambda]_x = [\mu \circ \lambda]_x$. In this way, one obtains a groupoid

$$G_1 \rightrightarrows G_0,$$

called the *orbifold groupoid associated with the atlas* $\mathcal{U}$. The notation above indicates the two distinguished source and target maps $s, t : G_1 \to G_0$.

The *isotropy group* at an object $x \in G_0$ is

$$G_x = \{g \in G_1 \mid s(g) = t(g) = x\}\,.$$

If $x \in V_i$, this isotropy group identifies with the stabiliser $(\Gamma_i)_x$. In particular, the isotropy group is trivial over a regular point of $\mathcal{O}$, while over a lift of a cone point of order α_i it is isomorphic to $\mathbb{Z}_{\alpha_i}$.

The topology and smooth structure on G_1 are defined using the local changes of charts themselves. If $\lambda\colon W \longrightarrow W'$ is a change of charts, then the map

$$W \longrightarrow G_1, \qquad x \longmapsto [\lambda]_x,$$

defines a local chart on G_1. With this structure, both the source and target maps $s, t\colon G_1 \longrightarrow G_0$ are local diffeomorphisms. The groupoid is therefore a *étale* Lie groupoid. Moreover, one can show that

$$(s,t)\colon G_1 \longrightarrow G_0 \times G_0$$

is proper. Thus, $G_1 \rightrightarrows G_0$ is a *proper* étale Lie groupoid. The orbit space of this groupoid is naturally identified with the underlying topological space of the orbifold:

$$G_0 \big/ G_1 \simeq B.$$

4.3 The nerve of the orbifold groupoid.

We have constructed the Euler class of a circle bundle using Čech cohomology. Let us briefly recall the underlying idea, as it suggests the construction that will be needed for Seifert bundles.

Choose a good open cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ of the base space B. It gives rise to the Čech cover

$$\coprod_{i\in\mathcal{I}} U_i \longrightarrow \coprod_{i,j\in\mathcal{I}} U_i \cap U_j \longrightarrow \coprod_{i,j,k\in\mathcal{I}} U_i \cap U_j \cap U_k \longrightarrow \cdots,$$

where the arrows are the restriction maps. Each multiple intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ admits $p+1$ natural restriction maps to higher-order intersections. A circle bundle is then described by transition functions

$$g_{ij} \in \mathscr{C}^\infty\big(U_i \cap U_j, \mathrm{O}(2)\big),$$

whose compatibility with these restriction maps is precisely the cocycle condition. This leads to the usual Čech complex. Trying to apply the same construction directly to an orbifold immediately runs into a difficulty. The Čech cover records only the topology of the overlaps: each intersection is canonically embedded in the corresponding open sets. For an orbifold, however, one must also keep track of the different ways in which local uniformizing charts (the V_i) are identified. Around a cone point, these identifications are no longer unique because of the local isotropy group. The difficulty is therefore the fact that each intersection carries several possible identifications between the corresponding local uniformizing charts. Ideally, one would like a complex whose objects record these choices explicitly. In other words, the analogue of the Čech cover should consist of spaces equipped with natural maps (in the opposite direction)

$$\coprod_{i_0,\dots,i_p\in\mathcal{I}} U_{i_0} \cap \cdots \cap U_{i_p} \longrightarrow \coprod_{i_0,\dots i_{p-1}\in\mathcal{I}} U_{i_0} \cap \cdots \cap U_{i_{p-1}}$$

where each map remembers not only the underlying intersection but also the chosen embedding into a lower-order intersection. This is achieved by replacing points of multiple intersections with chains of compatible changes of orbifold charts. Thus, the basic objects are no longer points but transition maps between local charts, and the natural maps between them are obtained by composing or forgetting these transitions.

Accordingly, for each integer $p \geqslant 2$, let G_p denote the space of composable p-tuples of arrows:

$$G_p = \underbrace{G_1 \times_{G_0} \cdots \times_{G_0} G_1}_{p \text{ factors}}.$$

Thus,

$$G_2 = G_1 \times_{s,G_0,t} G_1$$

is the space of pairs (g_2, g_1) satisfying $s(g_2) = t(g_1)$. Equivalently, such a pair represents a diagram $x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2$. More generally, an element of G_p is a chain

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_p} x_p$$

of p composable arrows. Just as the restriction maps between multiple intersections that allow to define the Čech differential, there are natural maps

$$d_j : G_p \longrightarrow G_{p-1}, \qquad 0 \leq j \leq p,$$

obtained by composing two consecutive arrows or deleting one of the two endpoints of the chain. Explicitly,

$$d_0(g_p, \dots, g_1) = (g_p, \dots, g_2), \qquad d_p(g_p, \dots, g_1) = (g_{p-1}, \dots, g_1),$$

and, for $0 < j < p$,

$$d_j(g_p, \dots, g_1) = (g_p, \dots, g_{j+1}g_j, \dots, g_1).$$

In particular, on G_1, $d_0 = t$ and $d_1 = s$.

Remark. The collection of manifolds $(G_p)_{p\geq 0}$, together with the maps d_j above and the maps obtained by inserting identity arrows, forms a *simplicial manifold*, called the *nerve* of the groupoid. The maps d_j are the *face maps*, while those inserting identity arrows are the *degeneracy maps*. Although this terminology is standard, only the face maps will be used in the sequel.

4.4 Cohomology on the nerve and sheaves on orbifolds.

The analogy with the Čech complex extends naturally to the groupoid setting. The spaces G_p play the role of the multiple intersections of a good cover, while the face maps play the role of the restriction maps. Accordingly, in the smooth setting considered here, cochains are smooth maps on the manifolds G_p. Just as for the classical Čech theory, the cohomology of the corresponding complex computes, under some suitable hypotheses, the cohomology of a sheaf. This section will be divided in three, corresponding to three levels of generality. The definition of an orbifold sheaf, which is equivalent to an equivariant sheaf on the groupoid, will be postponed to the beginning of the third part. In the rest of the text, what will be needed is the definition of the Čech complex on the nerve for the twisted sheaves $\mathcal{S}^1_w$, $\mathcal{R}_w$ and $\mathcal{Z}_w$. These fit into the second level of generality. The third part is devoted to the full generality definition because, at this point, it comes almost for free. It may also help some readers who need more global pictures.

The first stage concerns smooth sections in a Lie group A and we discuss here only the nerve complex, which is the analogue of the standard Čech complex for orbifolds.

4.4.1 Nerve cohomology for smooth A-valued functions.

Let A be a Lie group.

Definition 4.2. A 0-*cochain* with values in A is a smooth map

$$a : G_0 \longrightarrow A,$$

while a 1-*cochain* is a smooth map

$$h : G_1 \longrightarrow A.$$

The set of 0-cochains (resp. 1-cochains) is denoted by $\check{C}^0(G_\bullet;\mathcal{A})$ (resp. $\check{C}^1(G_\bullet;\mathcal{A})$). A 1-cochain h is called a 1-*cocycle* if it is compatible with the composition of arrows, namely if

$$h(g_2 g_1) = h(g_2)h(g_1)$$

for every composable pair $(g_2, g_1) \in G_2$. The set of 1-cocycles is $\check{Z}^1(G_\bullet;\mathcal{A})$.

The notation $\mathcal{A}$ refers to the orbifold sheaf of smooth A-valued functions (see the third part of this section). Exactly as in Čech theory, every 0-cochain a acts on the set of 1-cochains by

$$(a \cdot h)(g) = a(t(g))\, h(g)\, a(s(g))^{-1}.$$

Since

$$(a \cdot h)(g_2 g_1) = (a \cdot h)(g_2)\, (a \cdot h)(g_1),$$

this action preserves the subset of 1-cocycles.

Definition 4.3. The first cohomology set of the groupoid $G_1 \rightrightarrows G_0$ is then the pointed set of orbits

$$\check{H}^1(G_\bullet;\mathcal{A}) = {\check{Z}^1(G_\bullet;\mathcal{A})}\big/{\check{C}^0(G_\bullet;\mathcal{A})}.$$

Assume now A is an *abelian* Lie group. In this case, the construction extends to cochains of arbitrary degree and one obtains higher cohomology groups. The nerve gives a cochain complex

$$\check{C}^p(G_\bullet;\mathcal{A}) = \mathscr{C}^\infty(G_p, A),$$

whose differential $\delta^p : \check{C}^p(G_\bullet;\mathcal{A}) \to \check{C}^{p+1}(G_\bullet;\mathcal{A})$ is the alternating sum of pullbacks by the face maps:

$$\delta^p = \sum_{j=0}^{p+1} (-1)^j d_j^*.$$

In degree zero, a cochain is a smooth function $a\colon G_0 \longrightarrow A$. Its coboundary is the function on G_1 given, in additive notation, by

$$\forall x \xrightarrow{g} y \in G_1, \qquad (\delta^0 a)(g) = a(t(g)) - a(s(g)) = a(y) - a(x).$$

A degree-one cochain is a smooth function $h\colon G_1 \longrightarrow A$. Its coboundary is the function on G_2 given, in additive notation, by

$$\forall x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2 \in G_2, \qquad (\delta^1 h)(g_2, g_1) = h(g_2) - h(g_2 g_1) + h(g_1).$$

Denote by $\check{Z}^p(G_\bullet; \mathcal{A})$ the p-cocycles, that is the kernel of δ^p. Since A is abelian, the coboundary operator is a group homomorphism, so its image $\check{B}^{p+1}(G_\bullet; \mathcal{A}) := \operatorname{Im} \delta^p$ is a subgroup of $\check{C}^{p+1}(G_\bullet; \mathcal{A})$.

Definition 4.4. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid and let A be an abelian Lie group. The *nerve cohomology groups with values in* A are then

$$\check{H}^p(G_\bullet; \mathcal{A}) = {}^{\check{Z}^p(G_\bullet; \mathcal{A})}\!\!\diagup\!\!_{\check{B}^p(G_\bullet; \mathcal{A})}.$$

4.4.2 Nerve cohomology for smooth A-valued functions with locally constant twisting.

To define the Euler class, we shall need some twisting that perfectly parallels the one for circle bundles. Because it goes though an action of $\mathbb{Z}_2$, we explain the construction for this group. Also, in the following A will be $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{Z}$. But this construction applies in more general settings.

Let $w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$ and let A be an abelian Lie group endowed with an action of $\mathbb{Z}_2$. The cocycle w associates with every arrow $g \in G_1$ an automorphism

$$\rho_g : A \longrightarrow A, \qquad \rho_g(z) = w(g) \cdot z.$$

Since w is a cocycle, these automorphisms are compatible with the composition of arrows:

$$\rho_{g_2 g_1} = \rho_{g_2} \circ \rho_{g_1}.$$

The definition of the twisted coboundary operator is motivated by the same principle as for cohomology with local coefficients. Indeed, a cochain assigns values to chains of composable arrows, and these values are naturally regarded as living over the terminal object of the chain. When taking the alternating sum over the faces of a simplex, all faces except the last have the same terminal object as the original simplex, so their values already belong to the same copy of A. The last face, however, has terminal object one step earlier and must therefore be transported along the omitted arrow before it can be compared with the others.

Accordingly, for $c \in \check{C}^p(G_\bullet; \mathcal{A})$, define

$$(\delta^p_w c)(g_{p+1}, \dots, g_1) = \sum_{j=0}^{p} (-1)^j c\left(d_j(g_{p+1}, \dots, g_1)\right) + (-1)^{p+1} \rho_{g_{p+1}}\left(c\left(d_{p+1}(g_{p+1}, \dots, g_1)\right)\right).$$

In degree 0, this gives

$$(\delta^0_w a)(g) = a(t(g)) - w(g) \cdot a(s(g)),$$

while in degree 1,

$$(\delta^1_w h)(g_2, g_1) = h(g_2) - h(g_2 g_1) + w(g_2) \cdot h(g_1)$$

Proposition 4.1. The twisted coboundary operator satisfies

$$\delta^{p+1}_w \circ \delta^p_w = 0.$$

Hence $\big(\check{C}^\bullet(G_\bullet; \mathcal{A}), \delta_w\big)$ is a cochain complex.

Example. To see how it works, let us do the computation in degree 0. Let $a : G_0 \to A$ be a 0-cochain, and consider a composable pair $x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2$. Then

$$\begin{aligned}\big(\delta^1_w \delta^0_w a\big)(g_2, g_1) &= (\delta^0_w a)(g_2) - (\delta^0_w a)(g_2 g_1) + \rho_{g_2}\big((\delta^0_w a)(g_1)\big). \\ &= \big(a(x_2) - \rho_{g_2} a(x_1)\big) - \big(a(x_2) - \rho_{g_2 g_1} a(x_0)\big) + \rho_{g_2}\big(a(x_1) - \rho_{g_1} a(x_0)\big).\end{aligned}$$

After cancellation, this reduces to

$$\rho_{g_2 g_1} a(x_0) - \rho_{g_2} \rho_{g_1} a(x_0),$$

which vanishes since $\rho_{g_2 g_1} = \rho_{g_2} \circ \rho_{g_1}$.

Proof. Let $c \in \check{C}^p(G_\bullet; \mathcal{A})$. As in the untwisted case, the terms appearing in $\delta^{p+1}_w \delta^p_w c$ are indexed by pairs of successive face maps. The simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j$$

pair these terms with opposite signs. For pairs not involving the last face map, the cancellation is exactly the same as for the ordinary coboundary operator. If precisely one of the two face maps is the last face map, the corresponding terms are acted upon by the same automorphism $\rho_{g_{p+2}}$ and therefore still cancel, since $\rho_{g_{p+2}}$ is a homomorphism of A.

The only exceptional pair is the one for which the two successive face operations remove the last two vertices. In one order, the coefficient is transported successively along g_{p+1} and g_{p+2}, giving $\rho_{g_{p+2}} \circ \rho_{g_{p+1}}$. In the other order, the arrows g_{p+1} and g_{p+2} are first composed, and the resulting transport is $\rho_{g_{p+2}g_{p+1}}$. Since w is a 1-cocycle,

$$w(g_{p+2}g_{p+1}) = w(g_{p+2}) + w(g_{p+1}), \qquad \text{and hence} \qquad \rho_{g_{p+2}g_{p+1}} = \rho_{g_{p+2}} \circ \rho_{g_{p+1}}.$$

These two terms therefore cancel as well. □

We set

$$\check{Z}^p(G_\bullet; \mathcal{A}_w) = \operatorname{Ker} \delta_w^p, \qquad \check{B}^p(G_\bullet; \mathcal{A}_w) = \operatorname{Im} \delta_w^{p-1},$$

and define

$$H^p_{orb}(\mathcal{O}; \mathcal{A}_w) := \check{H}^p(G_\bullet; \mathcal{A}_w) = {\check{Z}^p(G_\bullet; \mathcal{A}_w)}\big/{\check{B}^p(G_\bullet; \mathcal{A}_w)}$$

(see the next subsection for a discussion on the above notations).

4.4.3 Sheaves on orbifolds and their nerve cohomology groups.

The nerve cohomology groups introduced above admit a natural interpretation in terms of sheaf cohomology on the orbifold. Let $\mathcal{O}$ be an orbifold with underlying topological space B, and let $\mathcal{U} = (V_i, \Gamma_i, \varphi_i, U_i)_{i\in\mathcal{I}}$ be an orbifold atlas satisfying the assumptions 4.2. Denote by $G = G_1 \rightrightarrows G_0$ the associated étale groupoid, whose object space is $G_0 = \coprod_{i\in\mathcal{I}} V_i$ and whose arrows are given by germs of chart embeddings.

Definition 4.5. A *sheaf of groups on the orbifold* $\mathcal{O}$ is a sheaf of groups on the étale groupoid G, that is, the data of

- a sheaf of groups $\mathcal{F}$ on G_0;
- an isomorphism
$$\alpha : s^{-1}\mathcal{F} \xrightarrow{\sim} t^{-1}\mathcal{F}$$
on G_1, whose pullback along the identity arrows is the identity and whose pullback along a composite arrow is the composition of the corresponding pullbacks.

This definition admits the following concrete description. Since $G_0 = \coprod_{i\in\mathcal{I}} V_i$, a sheaf on G_0 is simply a collection of sheaves $\{\mathcal{F}_i\}_{i\in\mathcal{I}}$, where $\mathcal{F}_i$ is a sheaf on V_i. The isotropy group $\Gamma_i \subset G_1$ acts on $\mathcal{F}_i$ by sheaf automorphisms, while every embedding of orbifold charts $\lambda_{ij} : V_i \longrightarrow V_j$ induces an isomorphism

$$\theta_{ij} : \lambda_{ij}^{-1}\mathcal{F}_j \xrightarrow{\sim} \mathcal{F}_i.$$

These isomorphisms are compatible with the group actions and satisfy the natural cocycle condition: whenever $V_i \xrightarrow{\lambda_{ij}} V_j \xrightarrow{\lambda_{jk}} V_k$, the transition isomorphism associated with the composite embedding agrees with the composition of the two transition isomorphisms.

The category of orbifold sheaves is an abelian category with enough injectives, and one may therefore define sheaf cohomology by derived functors. For the proper étale groupoid associated with the chosen orbifold atlas, the Čech complex of the nerve computes these derived-functor cohomology groups for the sheaves of interest (see the remark below). Accordingly, throughout this article we write

$$H^p_{orb}(\mathcal{O}; \mathcal{F}) := \check{H}^p(G_\bullet; \mathcal{F}),$$

which agrees with the usual sheaf cohomology of the orbifold in many cases.

The only sheaf of non-abelian groups we will encounter is the one of smooth O_2-valued (or $\mathrm{Diff}(\mathbb{S}^1)$-valued) functions (with trivial equivariance) and this one fit into the first settings of this section.

We now describe the corresponding complex for an arbitrary sheaf of abelian groups on the groupoid. Let $\mathcal{F}$ be a sheaf of abelian groups on the étale groupoid $G_1 \rightrightarrows G_0$. Thus $\mathcal{F}$ is a sheaf on G_0, together with an equivariance isomorphism

$$\alpha : s^{-1}\mathcal{F} \xrightarrow{\sim} t^{-1}\mathcal{F}$$

on G_1, compatible with identities and composition.

Recall that an element of G_p is a composable sequence

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_p} x_p.$$

For every $p \geqslant 0$, let

$$v_p : G_p \longrightarrow G_0, \qquad (g_p, \ldots, g_1) \longmapsto x_p$$

denote the map which remembers the terminal object. We define $\mathcal{F}_p := v_p^{-1}\mathcal{F}$, which is a sheaf on G_p. and

$$\check{C}^p(G_\bullet;\mathcal{F}) := \Gamma(G_p;\mathcal{F}_p).$$

Thus a p-cochain assigns to every composable sequence $x_0 \xrightarrow{g_1} \cdots \xrightarrow{g_p} x_p$ a germ of a section of $\mathcal{F}$ at x_p, in a manner locally described by sections of the pulled-back sheaf $\mathcal{F}_p$.

The face maps

$$d_j : G_{p+1} \longrightarrow G_p$$

induce morphisms between these groups of sections. For $j \leqslant p$, deleting or composing arrows does not change the terminal object, so that

$$v_p \circ d_j = v_{p+1}.$$

Consequently, ordinary pullback gives a map

$$d_j^* : \Gamma(G_p;\mathcal{F}_p) \longrightarrow \Gamma(G_{p+1};\mathcal{F}_{p+1}).$$

The face d_{p+1} is different: it forgets the last arrow $x_{p-1} \xrightarrow{g_p} x_p$ and therefore changes the terminal object from x_p to x_{p-1}. The ordinary pullback by d_{p+1} consequently takes values in the sheaf whose stalk is $\mathcal{F}_{x_{p-1}}$ rather than $\mathcal{F}_{x_p}$. The equivariant structure

$$\alpha : s^{-1}\mathcal{F} \xrightarrow{\sim} t^{-1}\mathcal{F}$$

is precisely what allows us to transport this value back along g_p and regard it as an element of $\mathcal{F}_{x_p}$. Combining pullback by d_p with this transport gives the required map, which we again denote by

$$d_{p+1}^* : \check{C}^p(G_\bullet;\mathcal{F}) \longrightarrow \check{C}^{p+1}(G_\bullet;\mathcal{F}).$$

The equivariance and composition conditions on α imply the usual cosimplicial identities $d_i d_j = d_{j-1} d_i$ for $i < j$ between these operators. Hence the alternating sum

$$\delta^p = \sum_{j=0}^{p+1} (-1)^j d_j^* : \check{C}^p(G_\bullet;\mathcal{F}) \longrightarrow \check{C}^{p+1}(G_\bullet;\mathcal{F})$$

whose expression is

$$(\delta^p c)(g_{p+1},\dots,g_1) = \sum_{j=0}^{p} (-1)^j c(d_j(g_{p+1},\dots,g_1)) \;+\; (-1)^{p+1} \alpha_{g_{p+1}}\big(c(d_{p+1}(g_{p+1},\dots,g_1))\big)$$

satisfies

$$\delta^{p+1} \circ \delta^p = 0.$$

We therefore obtain a cochain complex

$$\check{C}^\bullet(G_\bullet;\mathcal{F}) = \big(\Gamma(G_p; v_p^{-1}\mathcal{F}), \delta^p\big)_{p \geqslant 0},$$

whose cohomology groups are denoted by

$$\check{H}^p(G_\bullet;\mathcal{F}).$$

In low degrees, this construction is particularly transparent. A 0-cochain is simply a section

$$a \in \Gamma(G_0;\mathcal{F}).$$

For an arrow $g : x \to y$, its coboundary is

$$(\delta^0 a)(g) = a(y) - \alpha_g\big(a(x)\big) \in \mathcal{F}_y,$$

where $\alpha_g : \mathcal{F}_x \longrightarrow \mathcal{F}_y$ denotes the transport induced by the equivariant structure.

Similarly, a 1-cochain assigns to every arrow $g : x \to y$ an element $h(g) \in \mathcal{F}_y$. For a composable pair $x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2$, its coboundary is

$$(\delta^1 h)(g_2, g_1) = h(g_2) - h(g_2 g_1) + \alpha_{g_2}\big(h(g_1)\big) \in \mathcal{F}_{x_2}.$$

We now connect this to the previous settings.

When $\mathcal{F}$ is the sheaf $\mathcal{A} = \mathscr{C}^\infty(-, A)$ of smooth functions with values in an abelian Lie group A, endowed with the trivial equivariant structure α, the pullbacks above reduce to the ordinary pullbacks by the face maps. We recover exactly the complex introduced in the first subsection:

$$\check{C}^p(G_\bullet;\mathcal{A}) = \mathscr{C}^\infty(G_p, A).$$

The second subsection corresponds to the case where the equivariant structure is

$$\alpha_g = \rho_g, \qquad \text{namely} \qquad \alpha_g(u) = (-1)^{w(g)}(u).$$

Remark. The proper étale groupoid associated with an orbifold depends on the chosen orbifold atlas. Nevertheless, two atlases defining the same orbifold admit a common refinement, and their associated groupoids are *Morita equivalent*. Morita equivalence induces an equivalence between the corresponding categories of groupoid sheaves and, consequently, canonical isomorphisms between their groupoid sheaf cohomology groups. Some care is required, however, when groupoid cohomology is computed using the elementary complex of smooth cochains introduced above. For an arbitrary presentation, this complex need not compute the derived sheaf cohomology of the orbifold. It does so whenever the presentation is Leray for the coefficient sheaf, namely when the restriction of the coefficient sheaf to every connected component of every G_p is acyclic. In the present setting, the assumptions made on the orbifold atlas make it Leray for the three sheaves of interest $\mathcal{Z}_w$, $\mathcal{R}_w$ and $\mathcal{S}^1_w$. The complexes introduced above compute the Morita-invariant cohomology of $\mathcal{O}$ for them.

4.5 Seifert fibrations over $\mathcal{O}$ and $H^1_{orb}\big(\mathcal{O};\mathscr{C}^\infty(-,\mathrm{O}(2))\big)$.

Let $\pi : M \to \mathcal{O}$ be a Seifert fibration. We choose a good open cover $(U_i)_{i\in\mathcal{I}}$ of the underlying surface $B = |\mathcal{O}|$, adapted to the cone points, together with fixed identifications

$$h_i : U_i \longrightarrow {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_{\alpha_i}}.$$

A local trivialization of the Seifert fibration over U_i is an isomorphism

$$\Phi_i : \pi^{-1}(U_i) \longrightarrow T(\alpha_i,\beta_i)$$

with $\alpha_i \geqslant 1$ and β_i coprime to α_i such that the square

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\Phi_i]{\sim} & T(\alpha_i,\beta_i) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle p_{\beta_i}} \\ U_i & \xrightarrow[h_i]{\sim} & {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_{\alpha_i}} \end{array}$$

commutes. Let us emphasise that the identification of the base is fixed. For an ordinary circle bundle, local triviality provides an identification

$$\pi^{-1}(U_i) \simeq U_i \times \mathbb{S}^1$$

under which the bundle projection becomes the first projection $U_i \times \mathbb{S}^1 \to U_i$. In the present setting, the standard local model is naturally defined over the quotient disc rather than directly over U_i. The fixed identification h_i is therefore part of the local trivialisation data. Recall that we use the convention

$$T(\alpha,\beta) = {}^{\mathbb{D}^2\times\mathbb{S}^1}\!/_{\mathbb{Z}_\alpha} \xrightarrow{p_\beta} {}^{\mathbb{D}^2}\!/_{\mathbb{Z}_\alpha}$$

where

$$\bar{1}\cdot(z,u) = \Big(e^{\frac{2\pi\mathrm{i}\beta}{\alpha}}z, e^{\frac{2\pi\mathrm{i}}{\alpha}}u\Big) \qquad \text{and} \qquad p_\beta : [z,u]_\beta \longmapsto [z].$$

Then we may choose an orbifold atlas $\mathcal{U} = (V_i,\Gamma_i,\varphi_i,U_i)_{i\in\mathcal{I}}$ compatible with the open cover and satisfying the hypotheses 4.2. Consider the pullback of the fibration to V_i:

$$P_i = V_i \times_{U_i} \pi^{-1}(U_i) \longrightarrow V_i.$$

The chosen local model induces a trivialisation $\tau_i : P_i \xrightarrow{\sim} V_i \times \mathbb{S}^1$.

Let $G_1 \rightrightarrows G_0$ be the étale groupoid associated with the atlas $\mathcal{U}$. An arrow $g \in G_1$, with

$$s(g) = x \in V_i, \qquad t(g) = y \in V_j,$$

identifies a neighbourhood of x with a neighbourhood of y. It also induces an identification between the corresponding fibres of P_i and P_j. In the trivialisations τ_i and τ_j, this fibre identification is represented by an element

$$h(g) \in \mathrm{Diff}(\mathbb{S}^1).$$

Note that $h : g \mapsto h(g)$ must be smooth on G_1 and then define a 1-cochain in $\check{C}^1\big(G_\bullet;\mathscr{C}^\infty(-,\mathrm{Diff}(\mathbb{S}^1))\big)$. The compatibility of changes of charts with composition gives

$$h(g_2g_1) = h(g_2)h(g_1)$$

whenever $g_2 g_1$ is defined. Thus every choice of local trivialisations determines a 1-cocycle $h \in \check{Z}^1\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{Diff}(\mathbb{S}^1))\big)$.

We now examine how this cocycle changes when the local trivialisations are replaced. If the local model $T(\alpha_i, \beta_i)$ remains fixed, two trivialisations of P_i differ by a smooth map

$$a_i : V_i \longrightarrow \mathrm{Diff}(\mathbb{S}^1).$$

This yields 0-cochain $a \in \check{C}^0\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{Diff}(\mathbb{S}^1))\big)$. The corresponding cocycles are related by the usual action of this 0-cochain:

$$h'(g) = a_j(t(g))\, h(g)\, a_i(s(g))^{-1}.$$

There is however an additional subtlety. The local models $T(\alpha, \beta)$ and $T(\alpha, -\beta)$ are isomorphic over the identity of the quotient disc. Indeed, the map

$$F_{\alpha,\beta} : T(\alpha, \beta) \longrightarrow T(\alpha, -\beta), \qquad F_{\alpha,\beta}([z, u]_\beta) = [z, \overline{u}]_{-\beta},$$

is well defined and satisfies

$$p_{-\beta} \circ F_{\alpha,\beta} = p_\beta.$$

The effect of this replacement becomes visible after pulling the fibration back to the uniformizing chart V_i. It is important here to keep the fixed generator γ_i of Γ_i. In the model $T(\alpha_i, \beta_i)$, the generator of the action used in the definition of the local model rotates the base by $e^{\frac{2\pi \mathrm{i}\beta_i}{\alpha_i}}$. Consequently, the fixed chart generator

$$\gamma_i : z \longmapsto e^{\frac{2\pi \mathrm{i}\beta_i}{\alpha_i}} z$$

corresponds to the element

$$\overline{\beta_i}^{-1} \in \mathbb{Z}_{\alpha_i}$$

of the group defining $T(\alpha_i, \beta_i)$. Its action on the fibre of the pulled-back bundle is therefore

$$u \longmapsto e^{\frac{2\pi \mathrm{i}\beta_i^{-1}}{\alpha_i}} u.$$

If the local model is replaced by $T(\alpha_i, -\beta_i)$, the same fixed chart generator acts on the fibre by

$$u \longmapsto e^{-\frac{2\pi \mathrm{i}\beta_i^{-1}}{\alpha_i}} u.$$

Let

$$r : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \qquad r(u) = \overline{u},$$

be a reflection. Since

$$r \circ \left(u \longmapsto e^{\frac{2\pi \mathrm{i}\beta_i^{-1}}{\alpha_i}} u \right) \circ r^{-1} = \left(u \longmapsto e^{-\frac{2\pi \mathrm{i}\beta_i^{-1}}{\alpha_i}} u \right),$$

the two equivariant trivialisations are related by the vertical fibre-coordinate change $(z, u) \longmapsto (z, \overline{u})$.

Thus the replacement $\beta_i \longmapsto -\beta_i$ is represented, on the trivial pullback bundle over V_i, by composing the chosen trivialisation with a reflection of the fibre. It is therefore encoded by the same 0-cochain action as an ordinary change of fibre trivialisation. Hence two choices of local trivialisations give cohomologous cocycles.

It is nevertheless useful to distinguish the two possible origins of an orientation-reversing term a_i. Such a term may arise either from an orientation-reversing change of fibre coordinates within a fixed local model, or from replacing the local invariant β_i by $-\beta_i$. Although these operations induce the same transformation of the cocycle upstairs, their geometric interpretations are different. We shall return to this distinction in Section 4.11.

Conversely, a $\mathrm{Diff}(\mathbb{S}^1)$-valued 1-cocycle h defines a Seifert bundle, provided it satisfies the two following conditions:

1. its restriction to each local isotropy group is faithful (cocycle-theoretic form of the coprimality condition);
2. every isotropy arrow is mapped to an orientation-preserving diffeomorphism of $\mathbb{S}^1$.

Indeed, it is obtained simply by taking the quotient of $G_0 \times \mathbb{S}^1$ by the identifications

$$(x, u) \sim \big(t(g), h(g)u\big), \qquad s(g) = x.$$

Since h satisfies the cocycle condition, the gluings are compatible on composable changes of charts, and therefore define a Seifert bundle over $\mathcal{O}$. Moreover, two such cocycles define isomorphic Seifert bundles over $\mathcal{O}$ if and only if they differ by the action of a $\mathrm{Diff}(\mathbb{S}^1)$-valued 0-cochain as above.

As $\mathrm{O}(2) \hookrightarrow \mathrm{Diff}(\mathbb{S}^1)$ is a homotopy equivalence, every $\mathrm{Diff}(\mathbb{S}^1)$-bundle admits a reduction of its structure group to $\mathrm{O}(2)$, unique up to isomorphism. We obtain the following.

Theorem 4.2. The assignment sending a Seifert fibration over $\mathcal{O}$ to the cohomology class of its transition cocycle induces a natural bijection between isomorphism classes of Seifert fibrations over $\mathcal{O}$ and the subset of

$$H^1_{orb}\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{O}(2))\big).$$

consisting of elements represented by 1-cocycles whose restriction to each local isotropy group is orientation-preserving and faithful.

Remark. Here two Seifert fibrations are said to be isomorphic if there exists a fibre-preserving diffeomorphism between their total spaces covering the identity of the orbifold base. Although the replacement $T(\alpha_i, \beta_i) \mapsto T(\alpha_i, -\beta_i)$ is represented upstairs by an orientation-reversing change of fibre coordinates, it does not alter the first Stiefel–Whitney class: it changes the determinant cocycle only by a coboundary. Questions involving orientations of the total space and orientation-preserving isomorphisms will be considered separately in Section 4.11.

4.6 Splitting cocycles in two.

Recall that the orthogonal group decomposes as a semidirect product $\mathrm{O}(2) \simeq \mathbb{S}^1 \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ is identified with a subgroup generated by any reflection. Thus the multiplication is given by

$$(u, \varepsilon)(v, \delta) = \big(u(\varepsilon \cdot v),\, \varepsilon + \delta\big), \qquad \text{where} \qquad \bar{1} \cdot v = v^{-1}.$$

Consequently, every $\mathrm{O}(2)$-valued 1-cochain $h : G_1 \longrightarrow \mathrm{O}(2)$ may be written uniquely as

$$h(g) = \big(\eta(g), w(g)\big),$$

where

$$\eta : G_1 \longrightarrow \mathbb{S}^1, \qquad w : G_1 \longrightarrow \mathbb{Z}_2$$

are smooth functions. Suppose now that h is a 1-cocycle. Then the relation

$$h(g_2 g_1) = h(g_2) h(g_1)$$

for every composable pair $(g_2, g_1) \in G_2$ becomes

$$\eta(g_2 g_1) = \eta(g_2)\,\big(w(g_2) \cdot \eta(g_1)\big), \qquad \text{and} \qquad w(g_2 g_1) = w(g_2) + w(g_1).$$

Hence the $\mathbb{Z}_2$-component is an ordinary $\mathbb{Z}_2$-valued 1-cocycle, whereas the $\mathbb{S}^1$-component satisfies a twisted cocycle relation determined by w as introduced above.

Proposition 4.3. There is a one-to-one correspondence between $\mathrm{O}(2)$-valued 1-cocycles $h \in \check{Z}^1\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{O}(2))\big)$ and pairs (w, η) consisting of cocycles

$$w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2) \qquad \text{and} \qquad \eta \in \check{Z}^1(G_\bullet; \mathcal{S}^1_w).$$

The $\mathbb{Z}_2$-valued part is naturally obtained via the determinant $\det : \mathrm{O}(2) \to \{\pm 1\} \simeq \mathbb{Z}_2$. It descends to a map in cohomology

$$\det_* : H^1_{orb}\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{O}(2))\big) \longrightarrow H^1_{orb}(\mathcal{O}; \mathcal{Z}_2).$$

Indeed, if two $\mathrm{O}(2)$-valued 1-cocycles differ by the action of an $\mathrm{O}(2)$-valued 0-cochain, then their $\mathbb{Z}_2$-components differ by the action of the corresponding $\mathbb{Z}_2$-valued 0-cochain. Hence they define the same cohomology class in $H^1_{orb}(\mathcal{O}; \mathcal{Z}_2)$.

Fix now a cocycle $w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$. We examine more carefully the equivalence relation induced by $\mathrm{O}(2)$-valued 0-cochains on the $\mathrm{O}(2)$-valued cocycles whose determinant represents $[w]$.

Write an $\mathrm{O}(2)$-valued 0-cochain as $a = (\lambda, \varepsilon)$, with

$$\lambda : G_0 \longrightarrow \mathbb{S}^1, \qquad \varepsilon : G_0 \longrightarrow \mathbb{Z}_2.$$

If an $\mathrm{O}(2)$-valued cocycle (η, w) is transformed by a, its determinant becomes $w' = w + \delta\varepsilon$. Thus every cocycle whose determinant is cohomologous to w may first be transformed into one whose determinant is exactly w.

Suppose now that the determinant is already equal to w. A 0-cochain (λ, ε) preserves this equality precisely when $\delta\varepsilon = 0$. Since the orbifold $\mathcal{O}$ is connected, $\check{H}^0(G_\bullet; \mathcal{Z}_2) \simeq \mathbb{Z}_2$, so ε is globally constant.

If $\varepsilon = 0$, the transformation is given by an $\mathbb{S}^1$-valued 0-cochain, and its action on η is precisely the twisted coboundary action defining $H^1_{orb}(\mathcal{O}; \mathcal{S}^1_w)$. If $\varepsilon = 1$, the transformation contains a simultaneous reflection of all the fibre coordinates. Modulo the preceding $\mathbb{S}^1$-valued transformations, this acts on twisted cohomology by

$$[\eta] \longmapsto -[\eta].$$

Consequently, the residual action after fixing w is the natural action of $H^0_{orb}(\mathcal{O};\mathcal{Z}_2)\simeq\pm1$ on $H^1_{orb}(\mathcal{O};\mathcal{S}^1_w)$.

Let now $w,w'\in\check{Z}^1(G_\bullet;\mathcal{Z}_2)$ be cohomologous cocycles, say $w'=w+\delta\varepsilon$ for some 0-cochain $\varepsilon:G_0\to\mathbb{Z}_2$. As above, ε induces an isomorphism of cochain complexes

$$\Phi_\varepsilon:\big(\check{C}^\bullet(G_\bullet;\mathcal{S}^1),\delta_w\big)\xrightarrow{\sim}\big(\check{C}^\bullet(G_\bullet;\mathcal{S}^1),\delta_{w'}\big),$$

given by

$$(\Phi_\varepsilon c)(g_p,\dots,g_1)=(-1)^{\varepsilon(t(g_p))}c(g_p,\dots,g_1).$$

If ε' is another 0-cochain satisfying $w'=w+\delta\varepsilon'$, then $\varepsilon-\varepsilon'$ is a 0-cocycle, hence is constant. Therefore Φ_ε and $\Phi_{\varepsilon'}$ differ at most by the global automorphism -1. In particular, they induce the same bijection after quotienting by ±1.

Proposition 4.4. The determinant induces a well-defined map

$$\det_*:H^1_{orb}\big(G_\bullet;\mathscr{C}^\infty(-,\mathrm{O}(2))\big)\longrightarrow H^1_{orb}(\mathcal{O};\mathcal{Z}_2).$$

Let $w\in\check{Z}^1(G_\bullet;\mathcal{Z}_2)$ represent a class $[w]$ in its image. Then there is a natural bijection

$$\overset{-1}{\det_*}([w])\simeq H^1_{orb}(\mathcal{O};\mathcal{S}^1_w)\big/\{\pm1\},$$

where the non-trivial element acts by $[\eta]\longmapsto-[\eta]$. Moreover, if w' is another representative of the same class, the induced bijection

$$H^1_{orb}(\mathcal{O};\mathcal{S}^1_w)\big/\{\pm1\}\xrightarrow{\sim}H^1_{orb}(\mathcal{O};\mathcal{S}^1_{w'})\big/\{\pm1\}$$

is independent of the choice of 0-cochain relating w and w'.

4.7 The (first) Stiefel–Whitney class of a Seifert fibre space.

We now identify the $\mathbb{Z}_2$-component of an O(2)-valued cocycle with an ordinary cohomology class of the underlying surface. Let $\mathcal{J}\subset\mathcal{I}$ be the set of indices corresponding to charts of $\mathcal{O}$ containing a cone point, and set

$$B^\circ=B\setminus\bigcup_{j\in\mathcal{J}}U_j.$$

After shrinking the cone charts if necessary, we may assume that they are pairwise disjoint topological discs. Thus B° has the homotopy type of the surface obtained from B by deleting pairwise disjoint open discs around the cone points. For $i\in\mathcal{I}\setminus\mathcal{J}$, set

$$U_i^\circ=U_i\cap B^\circ.$$

Since U_i contains no cone point, its isotropy group is trivial, and the uniformising map $\varphi_i:V_i\to U_i$ is a homeomorphism. Hence the family

$$\mathcal{U}^\circ=\{U_i^\circ\}_{i\in\mathcal{I}\setminus\mathcal{J}}$$

is an ordinary open cover of B°. Let $G_1^\circ\rightrightarrows G_0^\circ$ be the corresponding groupoid, which is a subgroupoid of $G_1\rightrightarrows G_0$.

Lemma 4.5. The groups $H^1_{orb}(B^\circ;\mathcal{Z}_2)=\check{H}^1(G_\bullet^\circ;\mathcal{Z}_2)$ and $\check{H}^1\big(B^\circ;\mathcal{U}^\circ;\underline{\mathbb{Z}_2}\big)$ identifies.

Proof. Let $w\in\check{Z}^1(G_\bullet^\circ;\mathcal{Z}_2)$. For every nonempty overlap $U_i^\circ\cap U_j^\circ$ and every $x\in U_i^\circ\cap U_j^\circ$, the points

$$\varphi_j^{-1}(x)\in V_j\qquad\text{and}\qquad\varphi_i^{-1}(x)\in V_i$$

are joined by a unique arrow of the restricted groupoid, which we denote by

$$g_{ij}(x):\varphi_j^{-1}(x)\longrightarrow\varphi_i^{-1}(x)\in G_1^\circ.$$

Define $w_{ij}(x)=w\big(g_{ij}(x)\big)$. Since the overlaps are connected and $\mathbb{Z}_2$ is discrete, each w_{ij} is constant on $U_i^\circ\cap U_j^\circ$. On a triple intersection, the corresponding arrows satisfy $g_{ik}(x)=g_{ij}(x)g_{jk}(x)$. The groupoid cocycle identity therefore gives $w_{ik}=w_{ij}+w_{jk}$. Thus $\{w_{ij}\}_{i,j}\in\check{Z}^1(B^\circ;\mathcal{U}^\circ;\underline{\mathbb{Z}_2})$. We consequently obtain a homomorphism

$$r:\check{Z}^1(G_\bullet^\circ;\mathbb{Z}_2)\longrightarrow\check{Z}^1(B^\circ;\mathcal{U}^\circ;\mathbb{Z}_2),\qquad w\longmapsto\{w_{ij}\}_{i,j\in\mathcal{I}\setminus\mathcal{J}}.$$

Moreover, if $w=\delta a$ for some $a\in\check{C}^0(G_\bullet^\circ;\mathbb{Z}_2)$, then

$$w_{ij}=a\big(\varphi_i^{-1}(x)\big)-a\big(\varphi_j^{-1}(x)\big),$$

so $\{w_{ij}\}$ is a Čech coboundary. Hence r induces a homomorphism

$$\bar{r}:H^1_{orb}(B^\circ;\mathcal{Z}_2)\longrightarrow\check{H}^1(B^\circ;\mathcal{U}^\circ;\underline{\mathbb{Z}_2}).$$

Conversely, given a Čech cocycle $\{w_{ij}\}$, define $w(g)=w_{ij}$ whenever $g:\varphi_j^{-1}(x)\to\varphi_i^{-1}(x)$. The Čech cocycle identity implies that w is a groupoid cocycle. This construction is inverse to r. □

Now, there is a natural restriction $\check{Z}^1(G_\bullet;\mathcal{Z}_2) \to \check{Z}^1(G_\bullet^\circ;\mathcal{Z}_2)$ and this descends trivially to cohomology groups. Since $\mathcal{U}^\circ$ is a good cover of B°, its Čech cohomology is naturally identified with the singular cohomology of B°:

$$\check{H}^1(B^\circ;\mathcal{U}^\circ;\underline{\mathbb{Z}_2}) \simeq H^1(B^\circ;\mathbb{Z}_2).$$

Let $\pi : M \longrightarrow \mathcal{O}$ be a Seifert bundle, and let $[w] \in H^1_{orb}(\mathcal{O};\mathcal{Z}_2)$ be the image of its isomorphism class under the determinant map. Since B° contains no orbifold point, the restriction $\pi^{-1}(B^\circ) \longrightarrow B^\circ$ is an ordinary circle bundle. Its Stiefel–Whitney class is exactly the class in $H^1(B^\circ;\mathbb{Z}_2)$ corresponding to $[w] \in \check{H}^1_{orb}(\mathcal{O};\mathcal{Z}_2)$.

The image of the above homomorphism admits a simple geometric description. Recall that the bundle around a critical fibre is modelled on

$$T(\alpha,\beta) = {}^{\mathbb{D}^2 \times \mathbb{S}^1}\!/_{\mathbb{Z}_\alpha},$$

where the generators of $\mathbb{Z}_\alpha$ acts by

$$(z,u) \mapsto \big(e^{\frac{2\pi i\beta}{\alpha}} z,\, e^{\frac{2\pi i}{\alpha}} u\big).$$

In particular, the induced action on the fibre is always a rotation. Hence every isotropy element acts by an orientation-preserving homeomorphism of $\mathbb{S}^1$, and its image under $\det : \mathrm{O}(2) \to \mathbb{Z}_2$ is trivial. Therefore the restriction of the Stiefel–Whitney class to each boundary component $\partial\overline{U_j}$, $j \in \mathcal{J}$ is trivial so lies in the kernel of the restriction map

$$H^1(B^\circ;\mathbb{Z}_2) \longrightarrow \bigoplus_{i=1}^{m} H^1(\partial\overline{U_j};\mathbb{Z}_2).$$

On the other hand, a simple computation using the Mayer-Vietoris sequence shows that this kernel is naturally identified with $H^1(B;\mathbb{Z}_2)$. We have proved the following.

Proposition 4.6. The determinant map induces an identification between the set of determinant classes of Seifert bundles and $H^1(B;\mathbb{Z}_2)$.

Definition 4.6. The image of the determinant class $[w] \in H^1_{orb}(\mathcal{O};\mathcal{Z}_2)$ of π through the above identification is called the *the (first) Stiefel–Whitney class* of the Seifert bundle, denoted by $w_1(\pi) \in H^1(B;\mathbb{Z}_2)$.

4.8 The Euler class of a Seifert bundle.

Let $w \in \check{Z}^1(G_\bullet;\mathcal{Z}_2)$ be a cocycle representing the first Stiefel–Whitney class of π. The action of $\mathbb{Z}_2$ on $\mathbb{S}^1$ by inversion is induced by the action on $\mathbb{R}$ and $\mathbb{Z}$ by multiplication by -1. Hence the short exact sequence

$$0 \to \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{t\mapsto e^{2\pi i t}} \mathbb{S}^1 \to 0,$$

is $\mathbb{Z}_2$-equivariant. Since by our choice of orbifold atlas every connected component of every G_p is contractible, every $\mathbb{S}^1$-valued map admits a smooth lift to $\mathbb{R}$. Therefore the sequence

$$0 \to \mathscr{C}^\infty(G_p,\mathbb{Z}) \longrightarrow \mathscr{C}^\infty(G_p,\mathbb{R}) \longrightarrow \mathscr{C}^\infty(G_p,\mathbb{S}^1) \to 0$$

is exact for every $p \geqslant 0$, giving a short exact sequence of cochain complexes. Twisting by the cocycle w therefore yields a short exact sequence of cochain complexes

$$0 \to \check{C}^\bullet(G_\bullet;\mathcal{Z}_w) \longrightarrow \check{C}^\bullet(G_\bullet;\mathcal{R}_w) \longrightarrow \check{C}^\bullet(G_\bullet;\mathcal{S}^1_w) \to 0.$$

Passing to cohomology gives the associated long exact sequence

$$\cdots \longrightarrow H^1_{orb}(\mathcal{O};\mathcal{R}_w) \longrightarrow H^1_{orb}(\mathcal{O};\mathcal{S}^1_w) \xrightarrow{\delta} H^2_{orb}(\mathcal{O};\mathcal{Z}_w) \longrightarrow H^2_{orb}(\mathcal{O};\mathcal{R}_w) \longrightarrow \cdots.$$

Proposition 4.7. The connecting homomorphism associated with the twisted exponential sequence induces a natural isomorphism

$$\delta : H^1_{orb}(\mathcal{O};\mathcal{S}^1_w) \xrightarrow{\sim} H^2_{orb}(\mathcal{O};\mathcal{Z}_w).$$

Proof. For our chosen orbifold atlas every connected component of every G_p is contractible. Furthermore, one can show using equivariant partitions of unity that the orbifold sheaf of smooth real-valued functions is fine, hence acyclic, and therefore

$$H^q_{orb}(\mathcal{O};\mathcal{R}_w) = 0, \qquad q > 0.$$

Consequently, the connecting homomorphism δ is an isomorphism. □

Definition 4.7. Let $\pi : M \longrightarrow \mathcal{O}$ be a Seifert fibre space, and let (η, w) be a pair of cocycles associated with a choice of local trivialisations, with $w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$ representing its Stiefel–Whitney class. The image of

$$[\eta] \in H^1_{orb}(\mathcal{O}; \mathcal{S}^1_w)$$

under the connecting isomorphism

$$H^1_{orb}(\mathcal{O}; \mathcal{S}^1_w) \xrightarrow{\sim} H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$$

will be denoted $e_w(\pi)$ and called an Euler class associated with the chosen presentation.

If no orientation of the fibres is retained, the Euler invariant of the Seifert fibration is the orbit

$$[e_w(\pi)]_\pm \in {}^{H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)}\!/_{\{\pm 1\}}.$$

Remark. A simultaneous reversal of all fibre coordinates leaves the determinant cocycle w unchanged but induces the coefficient-system automorphism $-1 : \mathcal{Z}_w \longrightarrow \mathcal{Z}_w$. It therefore replaces $e_w(\pi) \longmapsto -e_w(\pi)$. This explains the quotient by $\{\pm 1\}$ in the preceding definition.

If $w' = w + \delta a$ is another representative of the Stiefel–Whitney class, the 0-cochain a identifies the two coefficient systems and transports $e_w(\pi)$ to an Euler class in $H^2_{orb}(\mathcal{O}; \mathcal{Z}_{w'})$. Different choices of a differ by a global sign and hence induce the same map on the corresponding quotients. Thus the orbit $[e_w(\pi)]_\pm$ depends only on the Seifert fibration and its Stiefel–Whitney class. For a genuinely intrinsic formulation using the fibre-orientation local system, compare Subsection 3.5.2.

4.9 Computation of $H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$.

Theorem 4.8. Let $\mathcal{O}$ be a compact connected 2-dimensional orbifold whose underlying space is the closed surface B, and whose cone points $b_1, \dots b_m$ have orders $\alpha_1, \dots, \alpha_m$. Let $w_1 \in H^1(B; \mathbb{Z}_2)$ and denote by $w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$ an isotropy-trivial representing cocycle. Then we have a short exact sequence

$$0 \longrightarrow H^2(B; \underline{\mathbb{Z}}_w) \longrightarrow H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \xrightarrow{r} \bigoplus_{i=1}^{m} H^2\left(\mathbb{Z}_{\alpha_i}; (\mathcal{Z}_w)_{\widetilde{b_i}}\right) \longrightarrow 0$$

where the $\widetilde{b_i} \in V_i$ are the lifts of the cone points. Furthermore, one has the following presentations:

$$H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \simeq \langle c, q_1, \dots, q_m \mid \alpha_i q_i = c,\ 1 \leqslant i \leqslant m \rangle$$

when $w_1 = w_1(B)$, whereas

$$H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \simeq \langle c, q_1, \dots, q_m \mid 2c = 0,\ \alpha_i q_i = c,\ 1 \leqslant i \leqslant m \rangle$$

when $w_1 \neq w_1(B)$.

In the remaining, we will use identifications similar to 4.5 for orbifold sheaves and cohomology groups on actually regular manifolds.

Lemma 4.9. Let $\mathcal{D}_\alpha = \mathbb{D}^2(\alpha)$ be an orbifold disc whose origin is a cone point of order α, and let $\mathcal{Z}$ be an orbifold sheaf of rank-one integral groups on $\mathcal{D}_\alpha$ whose isotropy action at the cone point is trivial. Denote by $(\mathcal{Z})_0$ its stalk at the origin. Then there are natural isomorphisms

$$H^q_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) \simeq H^q\left(\mathbb{Z}_\alpha; (\mathcal{Z})_0\right).$$

In particular,

$$H^1_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) = 0, \qquad \text{and} \qquad H^2_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) \simeq {}^{(\mathcal{Z})_0}\!/_{\alpha(\mathcal{Z})_0}.$$

Since $(\mathcal{Z})_0$ is an infinite cyclic group, the last quotient is abstractly cyclic of order α, but no identification with a distinguished copy of $\mathbb{Z}_\alpha$ is made here. Such an identification requires the choice of a generator of the stalk $(\mathcal{Z})_0$.

Proof. Let $G_\alpha = G_{\alpha,1} \rightrightarrows G_{\alpha,0}$ be the groupoid associated with $\mathcal{D}_\alpha$. Since $\mathcal{D}_\alpha = \left[\mathbb{D}^2/\mathbb{Z}_\alpha\right]$, the space of p-simplices of its nerve is

$$G_{\alpha,p} \simeq \mathbb{Z}_\alpha^p \times \mathbb{D}^2.$$

Indeed, a chain of p composable arrows is uniquely determined by its initial point in $\mathbb{D}^2$ together with p elements of the isotropy group $\mathbb{Z}_\alpha$.

The coefficient sheaf on $G_{\alpha,p}$ is obtained by pulling back $\mathcal{Z}$ from the relevant vertex of the simplex. Since $\mathbb{D}^2$ is connected and the stalks of $\mathcal{Z}$ are discrete rank-one groups, every smooth section over a component $\{\gamma\} \times \mathbb{D}^2$ is constant. Restriction to the fixed point $0 \in \mathbb{D}^2$ therefore gives a natural identification

$$\check{C}^p(G_{\alpha,\bullet}; \mathcal{Z}) \simeq \mathrm{Map}\left(\mathbb{Z}_\alpha^p, (\mathcal{Z})_0\right).$$

Under this identification, the differential induced by the face maps of the nerve is precisely the usual inhomogeneous group-cohomology differential for the $\mathbb{Z}_\alpha$-module $(\mathcal{Z})_0$. Hence

$$H^q_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) \simeq H^q\left(\mathbb{Z}_\alpha; (\mathcal{Z})_0\right).$$

In our situation the isotropy acts trivially on $(\mathcal{Z})_0$. The standard computation of the cohomology of a finite cyclic group with trivial coefficients gives the last statement. Let us give also an elementary proof for this last point involving no group cohomology. The objects defined here will play a role in the proof of the theorem.

For clarity, fix an isomorphism $(\mathcal{Z})_0 \simeq \mathbb{Z}$. A 1-cocycle is a map $a : \mathbb{Z}_\alpha \longrightarrow \mathbb{Z}$ satisfying $a(i+j) = a(i) + a(j)$. It is therefore a group homomorphism from the finite group $\mathbb{Z}_\alpha$ to $\mathbb{Z}$, and hence vanishes. This proves that $H^1_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) = 0$.

Let now $c : \mathbb{Z}_\alpha \times \mathbb{Z}_\alpha \longrightarrow \mathbb{Z}$ be a 2-cocycle. Its cocycle identity is

$$c(j,k) - c(i+j,k) + c(i,j+k) - c(i,j) = 0.$$

Associate with c the integer

$$N(c) = \sum_{r=0}^{\alpha-1} c(r,1).$$

If $c = \delta a$ is a coboundary, then

$$(\delta a)(r,1) = a(r) + a(1) - a(r+1),$$

and therefore $N(\delta a) = \alpha a(1)$, the other terms cancelling telescopically. Thus the residue class $[N(c)] \in \mathbb{Z}_\alpha$ depends only on the cohomology class of c.

Consider the cocycle

$$\kappa(i,j) = \begin{cases} 0, & i+j < \alpha, \\ 1, & i+j \geqslant \alpha, \end{cases} \qquad 0 \leqslant i,j \leqslant \alpha - 1.$$

It records whether the addition of i and j produces a carry modulo α. A direct verification shows that κ satisfies the cocycle identity, and $N(\kappa) = 1$. We claim that every 2-cocycle c is cohomologous to $N(c)\kappa$. Define a 1-cochain a by

$$a(0) = 0, \qquad a(k) = \sum_{r=0}^{k-1} c(r,1), \qquad 1 \leqslant k \leqslant \alpha - 1.$$

Using the cocycle identity successively with the third argument equal to 1, one obtains

$$c(i,j) = a(i+j) - a(i) - a(j)$$

when $i + j < \alpha$, and

$$c(i,j) = a(i+j-\alpha) - a(i) - a(j) + N(c)$$

when $i + j \geqslant \alpha$. Equivalently, $c = N(c)\kappa - \delta a$.

Finally, the class of $m\kappa$ is trivial if and only if $m \equiv 0 \pmod{\alpha}$. Indeed, the preceding invariant sends $m\kappa$ to $m \pmod{\alpha}$. Consequently, $H^2_{orb}(\mathcal{D}_\alpha; \mathcal{Z}) \simeq \mathbb{Z}_\alpha$, with generator represented by the carry cocycle κ. □

Proof of the theorem. Choose pairwise disjoint orbifold discs $\mathcal{D}_1, \ldots, \mathcal{D}_m$ centred at the cone points $b_1, \ldots, b_m$, with respective orders $\alpha_1, \ldots, \alpha_m$. Let B° be the complement of smaller discs around the cone points, so that

$$\mathcal{O} = B^\circ \cup \bigcup_{i=1}^{m} \mathcal{D}_i,$$

and each intersection $B^\circ \cap \mathcal{D}_i$ is an annulus homotopy equivalent to a circle $C_i := \partial\mathcal{D}_i$. The sheaf $\mathcal{Z}_w$ is a sheaf of rank-one integral groups whose isotropy actions at the cone points are trivial. Its restriction the regular locus identifies with $\underline{\mathbb{Z}}_w$ where w also denotes the corresponding usual Čech 1-cocycle (see 4.5). Therefore by the lemma above

$$H^1_{orb}(\mathcal{D}_i; \mathcal{Z}_w) = 0, \qquad H^2_{orb}(\mathcal{D}_i; \mathcal{Z}_w) \simeq H^2\left(\mathbb{Z}_{\alpha_i}; (\mathcal{Z}_w)_{\widetilde{b_i}}\right),$$

and

$$H^1_{orb}(C_i; \mathcal{Z}_w) \simeq H^1(C_i; \underline{\mathbb{Z}}_w) \simeq \mathbb{Z}, \qquad H^2_{orb}(C_i; \mathcal{Z}_w) = 0, \qquad H^2_{orb}(B^\circ; \mathcal{Z}_w) \simeq H^2(B^\circ; \underline{\mathbb{Z}}_w) = 0,$$

since B° is a compact surface with nonempty boundary. The Mayer–Vietoris sequence for the above orbifold decomposition therefore contains

$$H^1(B^\circ;\underline{\mathbb{Z}}_w) \xrightarrow{\rho} \bigoplus_{i=1}^m H^1(C_i;\underline{\mathbb{Z}}_w) \xrightarrow{\delta_{\mathcal{O}}} H^2_{orb}(\mathcal{O};\mathcal{Z}_w) \xrightarrow{r} \bigoplus_{i=1}^m H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w) \longrightarrow 0.$$

Here ρ is the direct sum of the restriction maps (the terms $H^1(\mathcal{D}_i;\mathcal{Z}_w)$ do not appear because they vanish). We now apply the ordinary Mayer–Vietoris sequence to the decomposition of the underlying surface

$$B = B^\circ \cup \bigcup_{i=1}^m D_i,$$

where D_i is the underlying topological disc of $\mathcal{D}_i$. The relevant part of this sequence is

$$H^1(B^\circ;\underline{\mathbb{Z}}_w) \xrightarrow{\rho} \bigoplus_{i=1}^m H^1(C_i;\underline{\mathbb{Z}}_w) \xrightarrow{\delta_B} H^2(B;\underline{\mathbb{Z}}_w) \longrightarrow 0.$$

Notice that the first map is the same map ρ so these sequences fit into the following diagram.

$$\begin{array}{ccccccccc} & & & & & & H^2(B;\underline{\mathbb{Z}}_w) & \longrightarrow & 0 \\ & & & & & \overset{\delta_B}{\nearrow} & & & \\ H^1(B^\circ;\underline{\mathbb{Z}}_w) & \xrightarrow{\rho} & \bigoplus_i H^1(C_i;\underline{\mathbb{Z}}_w) & \xrightarrow{\delta_{\mathcal{O}}} & H^2_{orb}(\mathcal{O};\mathcal{Z}_w) & \xrightarrow{r} & \bigoplus_i H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w) & \longrightarrow & 0 \end{array}$$

The exactness of the two paths yields

$$H^2(B;\underline{\mathbb{Z}}_w) \simeq \operatorname{coker}(\rho) \simeq \operatorname{Im}(\delta_{\mathcal{O}}) = \ker(r).$$

This defines an isomorphism yielding the short exact sequence

$$0 \longrightarrow H^2(B;\underline{\mathbb{Z}}_w) \longrightarrow H^2_{orb}(\mathcal{O};\mathcal{Z}_w) \xrightarrow{r} \bigoplus_{i=1}^m H^2\left(\mathbb{Z}_{\alpha_i};(\mathcal{Z}_w)_{\widetilde{b_i}}\right) \longrightarrow 0.$$

We now determine the extension appearing in the above short exact sequence. For each i, choose a generator $\varepsilon_i \in H^1(C_i;\underline{\mathbb{Z}}_w) \simeq \mathbb{Z}$. We first show that $\delta_B(\varepsilon_i) = c$ for every i, where c denotes a fixed generator of $H^2(B;\underline{\mathbb{Z}}_w)$. Let $B_i^\circ = B \setminus \operatorname{Int}(D_i)$. The inclusion of pairs $(D_i, C_i) \to (B, B_i^\circ)$ induces, by naturality of the long exact sequence of a pair, the commutative diagram

$$\begin{array}{ccccccccc} H^1(B;\underline{\mathbb{Z}}_w) & \longrightarrow & H^1(B_i^\circ;\underline{\mathbb{Z}}_w) & \longrightarrow & H^2(B,B_i^\circ;\underline{\mathbb{Z}}_w) & \longrightarrow & H^2(B;\underline{\mathbb{Z}}_w) & \longrightarrow & H^2(B_i^\circ;\underline{\mathbb{Z}}_w) \\ \downarrow & & \downarrow & & \downarrow \iota & & \downarrow & & \downarrow \\ H^1(D_i;\underline{\mathbb{Z}}_w) & \longrightarrow & H^1(C_i;\underline{\mathbb{Z}}_w) & \xrightarrow{\partial_i} & H^2(D_i,C_i;\underline{\mathbb{Z}}_w) & \longrightarrow & H^2(D_i;\underline{\mathbb{Z}}_w) & \longrightarrow & H^2(C_i;\underline{\mathbb{Z}}_w) \end{array}$$

where the vertical arrows are induced by inclusions. But $H^1(D_i;\underline{\mathbb{Z}}_w) = H^2(D_i;\underline{\mathbb{Z}}_w) = 0$ so the second row yields an isomorphism

$$\partial_i : H^1(C_i;\underline{\mathbb{Z}}_w) \xrightarrow{\simeq} H^2(D_i,C_i;\underline{\mathbb{Z}}_w).$$

Moreover, excision implies that the vertical arrow

$$\iota : H^2(B,B_i^\circ;\underline{\mathbb{Z}}_w) \xrightarrow{\simeq} H^2(D_i,C_i;\underline{\mathbb{Z}}_w)$$

is also an isomorphism. Thus, we get the composite homomorphism

$$H^1(C_i;\underline{\mathbb{Z}}_w) \xrightarrow[\partial_i]{\sim} H^2(D_i,C_i;\underline{\mathbb{Z}}_w) \xrightarrow[\iota^{-1}]{\sim} H^2(B,B_i^\circ;\underline{\mathbb{Z}}_w) \longrightarrow H^2(B;\underline{\mathbb{Z}}_w)$$

which is surjective as $H^2(B_i^\circ;\underline{\mathbb{Z}}_w) = 0$. This composite coincides, up to the sign determined by the convention for the Mayer–Vietoris differential, with the connecting homomorphism associated with the decomposition $B = B_i^\circ \cup D_i$. Indeed, the Mayer–Vietoris sequence gives

$$H^1(B_i^\circ;\underline{\mathbb{Z}}_w) \oplus H^1(D_i;\underline{\mathbb{Z}}_w) \longrightarrow H^1(C_i;\underline{\mathbb{Z}}_w) \longrightarrow H^2(B;\underline{\mathbb{Z}}_w) \longrightarrow 0$$

where the first arrow comes from the difference of the restrictions at the level of cocycles. Since $H^1(D_i;\underline{\mathbb{Z}}_w)=0$, this is just a restriction and thus coincides with the first vertical arrow of the above diagram. Thus, this identifies

$$H^2(B;\underline{\mathbb{Z}}_w)\simeq\frac{H^1(C_i;\underline{\mathbb{Z}}_w)}{\operatorname{Im}\left(H^1(B_i^\circ;\underline{\mathbb{Z}}_w)\longrightarrow H^1(C_i;\underline{\mathbb{Z}}_w)\right)}.$$

and under this identification, the quotient map is precisely the composite above. By compatibility of the Mayer–Vietoris connecting homomorphisms with the refinement from the decomposition $B=B_i^\circ\cup D_i$ to $B=B^\circ\cup\bigcup_{j=1}^m D_j$, this map is also, up to sign, the restriction to the i-th summand of

$$\delta_B:\bigoplus_{j=1}^m H^1(C_j;\underline{\mathbb{Z}}_w)\longrightarrow H^2(B;\underline{\mathbb{Z}}_w).$$

The groups $H^1(C_i;\underline{\mathbb{Z}}_w)$ and $H^2(B;\underline{\mathbb{Z}}_w)$ being cyclic, such a surjective homomorphism sends a generator to a generator. Changing the sign of ε_i when necessary, we may therefore arrange that $\delta_B(\varepsilon_i)=c$ for every i, where c denotes a fixed generator of $H^2(B;\underline{\mathbb{Z}}_w)$.

We now consider the relative orbifold cohomology of the pair $(\mathcal{D}_i,C_i)$. Its long exact sequence contains

$$H^1_{orb}(\mathcal{D}_i;\mathcal{Z}_w)\longrightarrow H^1(C_i;\underline{\mathbb{Z}}_w)\xrightarrow{\partial_i^{\mathrm{orb}}}H^2_{orb}(\mathcal{D}_i,C_i;\mathcal{Z}_w)\longrightarrow H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)\longrightarrow H^2(C_i;\underline{\mathbb{Z}}_w).$$

Since, by the lemma 4.9

$$H^1_{orb}(\mathcal{D}_i;\mathcal{Z}_w)=0,\qquad H^2(C_i;\underline{\mathbb{Z}}_w)=0,\qquad H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)\simeq\mathbb{Z}_{\alpha_i},$$

this gives a short exact sequence

$$0\longrightarrow H^1(C_i;\underline{\mathbb{Z}}_w)\xrightarrow{\partial_i^{\mathrm{orb}}}H^2_{orb}(\mathcal{D}_i,C_i;\mathcal{Z}_w)\longrightarrow H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)\longrightarrow 0.$$

Fix identifications $H^1(C_i;\underline{\mathbb{Z}}_w)\simeq\mathbb{Z}$ and $H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)\simeq\mathbb{Z}_\alpha$. We claim that $H^2_{orb}(\mathcal{D}_i,C_i;\mathcal{Z}_w)\simeq\mathbb{Z}$ and that the (up to change of the identifications) above sequence is

$$0\longrightarrow\mathbb{Z}\xrightarrow{\times\alpha_i}\mathbb{Z}\longrightarrow\mathbb{Z}_{\alpha_i}\longrightarrow 0.$$

To see this, write $\alpha=\alpha_i$. The generator of $H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)$ is represented by the carry cocycle

$$\kappa(s,t)=\begin{cases}0, & s+t<\alpha,\\ 1, & s+t\geqslant\alpha\end{cases}\qquad 0\leqslant s,t\leqslant\alpha-1.$$

The 1-cochain on $\mathcal{D}_i$:

$$a:\mathbb{Z}_\alpha\longrightarrow\mathbb{Z},\qquad a(s)=s,\quad 0\leqslant s\leqslant\alpha-1,$$

satisfies

$$\delta a=\alpha\kappa.$$

On the boundary circle, the primitive a induces a cocycle evaluating to 1 on a positively oriented generator of $\pi_1(C_i)$: a loop going once around C_i lifts to a path joining a point z to γz, where γ is a generator of $\mathbb{Z}_\alpha$, and the variation of a along this path is $a(\gamma)=1$. It follows that, with a suitable choice of sign, the restriction of a to C_i represents the generator $\varepsilon_i\in H^1(C_i;\underline{\mathbb{Z}}_w)$. Consequently, if u_i is a relative class in $\check{Z}^2(\mathcal{D}_i,C_i;\mathcal{Z}_w)$ mapping to $[\kappa]$, then

$$\alpha_i u_i=\partial_i^{\mathrm{orb}}(\varepsilon_i).$$

By excision for relative orbifold cohomology, there is a natural map

$$\check{H}^2(\mathcal{D}_i,C_i;\mathcal{Z}_w)\longrightarrow H^2_{orb}(\mathcal{O};\mathcal{Z}_w).$$

Let q_i be the image of u_i. Its restriction to $\mathcal{D}_i$ is the standard generator $[\kappa]$ of $H^2_{orb}(\mathcal{D}_i;\mathcal{Z}_w)$, and its restrictions to the other orbifold discs are zero. Moreover, by naturality of the connecting homomorphisms,

$$\alpha_i q_i=\delta_{\mathcal{O}}(\varepsilon_i).$$

Under the identification $H^2(B;\underline{\mathbb{Z}}_w)\simeq\ker(r)$, the class $\delta_{\mathcal{O}}(\varepsilon_i)$ corresponds to $\delta_B(\varepsilon_i)=c$. Hence

$$\alpha_i q_i=c.$$

The presentation given in the statement follows easily. □

4.10 Alternative viewpoint on the Euler class.

The preceding construction admits a direct interpretation parallel to the classical theory of circle bundles. Recall that a circle bundle over a topological space B, with first Stiefel–Whitney class $w_1 \in H^1(B;\mathbb{Z}_2)$ admitting a representative $w \in \check{Z}^1(\mathcal{B};U;\underline{\mathbb{Z}_2})$, has an Euler class $e \in H^2(B;\underline{\mathbb{Z}}_w)$. It arises from the twisted exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_w \longrightarrow \mathcal{R}_w \longrightarrow \mathcal{S}^1_w \longrightarrow 0.$$

Since the sheaf $\mathcal{R}_w$ is fine, the connecting homomorphism gives an isomorphism

$$H^1(B;\mathcal{S}^1_w) \xrightarrow{\sim} H^2(B;\underline{\mathbb{Z}}_w).$$

Thus the Euler class is the obstruction to lifting the transition functions of the circle bundle from $\mathcal{S}^1_w$ to $\mathcal{R}_w$. It records the essential full turns of a section above a small disc on the base space.

The same interpretation applies verbatim to Seifert bundles. The only difference is that the coefficient sheaves are now considered on the orbifold rather than on its underlying topological space. If $f : \mathcal{O} \to B$ denotes the natural projection to the underlying surface, the transition functions of a Seifert bundle define a class

$$[\eta] \in H^1_{orb}(\mathcal{O};\mathcal{S}^1_w),$$

as described in Section 4.3 and the twisted exponential sequence on $\mathcal{O}$ gives its Euler class $e \in H^2_{orb}(\mathcal{O};\mathcal{Z}_w)$. In the groupoid model used above, this group is

$$H^2_{orb}(\mathcal{O};\mathcal{Z}_w) := \check{H}^2(G_\bullet;\mathcal{Z}_w).$$

In the case of a circle bundle over a closed surface B, the obstruction group $H^2(B;\underline{\mathbb{Z}}_w) \simeq H^2(B;\mathbb{Z}_{w_1})$ admits the explicit description of Proposition 3.7. Theorem 4.8 gives the analogue description for Seifert bundles. Our proof proceeds by comparing, in a completely elementary way, the topology of the punctured surface $B \setminus \{b_1,\dots,b_m\}$ with that of the orbifold obtained by reinstating the cone points, first ignoring and then incorporating their isotropy groups. In this process, the classical obstruction group $H^2(B;\underline{\mathbb{Z}}_w)$ naturally reappears, while the contribution of each isotropy group $\mathbb{Z}_{\alpha_i}$ is isolated.

One may make the relation with the underlying surface more explicit by considering the derived direct image $Rf_*\mathcal{Z}_w$. By definition, the sheaf cohomology of the orbifold is computed from this complex of sheaves on B. Its cohomology sheaves are particularly simple. Over the regular part of B, the map f carries no additional information, and $Rf_*\mathcal{Z}_w$ identifies with the ordinary locally constant sheaf $\underline{\mathbb{Z}}_w$. At a cone point b_i of order α_i, however, its higher stalks record the cohomology of the isotropy group:

$$(R^q f_*\mathcal{Z}_w)_{b_i} \simeq H^q_{orb}(\mathcal{D}_i;\mathcal{Z}_w) \simeq H^q\left(\mathbb{Z}_{\alpha_i};(\mathcal{Z}_w)_{\widetilde{b_i}}\right).$$

The assumption that w is trivial on the isotropy groups ensures that the action on $(\mathcal{Z}_w)_{\widetilde{b_i}}$ is trivial. Hence

$$R^0 f_*\mathcal{Z}_w \simeq \underline{\mathbb{Z}}_w, \qquad R^1 f_*\mathcal{Z}_w = 0,$$

whereas

$$R^2 f_*\mathcal{Z}_w \simeq \bigoplus_{i=1}^m (\iota_i)_* H^2\left(\mathbb{Z}_{\alpha_i};(\mathcal{Z}_w)_{\widetilde{b_i}}\right) \simeq \bigoplus_{i=1}^m (\iota_i)_* \frac{(\mathcal{Z}_w)_{\widetilde{b_i}}}{\alpha_i(\mathcal{Z}_w)_{\widetilde{b_i}}},$$

with $\iota_i : \{b_i\} \hookrightarrow B$. The Leray spectral sequence

$$H^p\big(B;R^q f_*\mathcal{Z}_w\big) \Longrightarrow H^{p+q}(\mathcal{O};\mathcal{Z}_w)$$

therefore gives, in total degree two, the short exact sequence

$$0 \longrightarrow H^2(B;\underline{\mathbb{Z}}_w) \longrightarrow H^2_{orb}(\mathcal{O};\mathcal{Z}_w) \longrightarrow \bigoplus_{i=1}^m H^2\left(\mathbb{Z}_{\alpha_i};(\mathcal{Z}_w)_{\widetilde{b_i}}\right) \longrightarrow 0.$$

This is the same exact sequence obtained above from the Mayer–Vietoris sequence of the orbifold groupoid.

The Leray spectral sequence, however, does not immediately yield the presentation of $H^2_{orb}(\mathcal{O};\mathcal{Z}_w)$. It determines the subgroup $H^2(B;\underline{\mathbb{Z}}_w)$ and the quotient but not the way they are glued together. The missing information is precisely encoded by the relations $\alpha_i q_i = c$. These relations depend on the extension class of the short exact sequence, or equivalently on the additional information contained in the derived object $Rf_*\mathcal{Z}_w$, beyond its individual cohomology sheaves $R^q f_*\mathcal{Z}_w$. The groupoid Čech description makes this extension data explicit. On an orbifold disc of order α_i, the local generator is represented by the carry cocycle κ_i, and the identity $\delta a_i = \alpha_i \kappa_i$ gives an explicit primitive for its α_i-fold multiple. This primitive a_i induces a generator on the boundary circle, which is then sent by the Mayer–Vietoris connecting homomorphism to the global class c. This gives the relation $\alpha_i q_i = c$.

4.11 Classification of Seifert fibrations.

A summary of the preceding constructions is given by the following classification result.

Theorem 4.10. Let $\mathcal{O}$ be a fixed connected 2-dimensional orbifold with underlying surface B and cone points of orders $\alpha_1, \dots, \alpha_m$. Let $w_1 \in H^1(B; \mathbb{Z}_2)$ and choose an isotropy-trivial cocycle $w \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$ representing it. Then isomorphism classes, over the identity of $\mathcal{O}$, of Seifert fibrations with first Stiefel–Whitney class w_1 are classified by the admissible sign-orbits

$$\frac{\left\{e \in H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \mid e \text{ is admissible}\right\}}{\{\pm 1\}}.$$

Here admissibility means that the restriction of the corresponding $\mathcal{S}^1_w$-valued cocycle to every isotropy group is faithful.

If $w' = w + \delta a$ is another representative of w_1, the isomorphism induced by a descends to a canonical bijection between the corresponding sets of admissible sign-orbits.

Several subtleties are hidden in this statement, especially those involving orientations. A choice of local trivialisations determines local orientations of the $\mathbb{S}^1$-coordinates over the uniformizing charts V_i. The determinant cocycle

$$w = \det h \in \check{Z}^1(G_\bullet; \mathcal{Z}_2)$$

records how these local orientations are compared along the arrows of the orbifold groupoid. Once w has been fixed, the remaining freedom consists of the twisted $\mathbb{S}^1$-valued coboundaries together with one global reflection of all fibre coordinates. The former leave the Euler class unchanged, whereas the latter sends it to its opposite. This is precisely the origin of the quotient by $\{\pm 1\}$ in Theorem 4.10.

Theorem 4.8 describes a class in $H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$ through the restriction map r. The restriction of an Euler class to the i-th cone point is intrinsically an element

$$r_i(e) \in H^2\left(\Gamma_i; (\mathcal{Z}_w)_{\widetilde{b}_i}\right),$$

where $\widetilde{b}_i \in V_i$ is the lift of the cone point. Since the isotropy acts orientation-preservingly on the circle fibre, it acts trivially on the coefficient stalk $(\mathcal{Z}_w)_{\widetilde{b}_i}$, and the above group is abstractly cyclic of order α_i. An identification

$$H^2\left(\Gamma_i; (\mathcal{Z}_w)_{\widetilde{b}_i}\right) \simeq \mathbb{Z}_{\alpha_i}$$

requires both a generator of Γ_i and a generator of the rank-one free module $(\mathcal{Z}_w)_{\widetilde{b}_i}$. Reversing the orientation of V_i replaces the first generator by its inverse, while reversing the local fibre coordinate replaces the second generator by its negative. Either change reverses the sign of the resulting numerical coordinate.

This explains an apparent difficulty. Isomorphisms of Seifert fibrations allow the replacement

$$T(\alpha_i, \beta_i) \longmapsto T(\alpha_i, -\beta_i).$$

At the level of the pulled-back trivialisation, this replacement involves a reflection of the fibre coordinate. It therefore changes the determinant cocycle from w to $w' = w + \delta a$ for a suitable 0-cochain a, and simultaneously induces an identification

$$H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \to H^2_{orb}(\mathcal{O}; \mathcal{Z}_{w+\delta_a}).$$

For a local model $T(\alpha_i, \beta_i)$, and after choosing generators as above, the numerical restriction coordinate is $\mu_i = \beta_i^{-1} \mod \alpha_i$. After replacing the model by $T(\alpha_i, -\beta_i)$, the corresponding generator of the coefficient stalk is reversed. Relative to the original numerical identification, the restriction coordinate therefore becomes

$$\mu_i' = -\beta_i^{-1} \pmod{\alpha_i}.$$

Thus the change of sign does not contradict the intrinsic definition of the restriction map: it reflects the change in the local identifications used to represent its value as an element of $\mathbb{Z}_{\alpha_i}$. Changing it to $T(\alpha_i, -\beta_i)$, while preserving the same generator for Γ_i, will send the previous generator of $\mathbb{Z}_\alpha$ to its opposite, so the restriction coordinate will become $\mu_i' = -\beta_i^{-1} \mod \alpha$.

Suppose now that M is oriented. The orientation of the total space identifies the fibre-orientation local system with the orientation local system of the base, and therefore

$$w_1 = [w] = w_1(B).$$

Once the orientation of M is retained, the global fibre reflection is no longer an admissible change of trivialisation for the orientation-preserving classification: it reverses the orientation of the total space. Consequently, in the oriented

setting the quotient by $\{\pm 1\}$ disappears. After fixing the identification determined by the orientation, the admissible classes

$$\mathscr{E}_w = \left\{ e \in H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \;\middle|\; r_i(e) \text{ generates } H^2\left(\Gamma_i; (\mathcal{Z}_w)_{\widetilde{b}_i}\right) \text{ for every } i \right\}$$

classify oriented Seifert fibrations up to orientation-preserving, fibre-preserving diffeomorphisms covering the identity of the base.

4.12 Connection with classical Seifert invariants.

Let us now relate the preceding cohomological description to the classical Seifert invariants. Their precise definition and the conventions governing the signs vary from one author to another. Nevertheless, they always encode the underlying surface B, together with the behaviour of the fibre orientations, namely the Stiefel–Whitney class w_1 when the total space is non-orientable. Once these are fixed, the remaining data are often written as

$$\big(b; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\big).$$

All of them are integers, except when the total space is non-orientable, in which case the Euler number b becomes a binary information $b \in \mathbb{Z}_2$. Here the $|\alpha_i| > 1$ are the multiplicities of the exceptional fibres, the β_i describe their local models, and b is a global Euler invariant. The local manifold condition is $\gcd(\alpha_i, \beta_i) = 1$ for every i. A Seifert fibration with these invariants may be constructed by starting with a circle bundle over B having the prescribed Stiefel–Whitney class and Euler invariant and then performing Dehn surgeries of slopes β_i/α_i along regular fibres to create the exceptional fibres.

In the remainder of this subsection, the base orbifold and the Stiefel–Whitney class will be regarded as fixed. Thus the Seifert invariants reduce to the collection of integers above.

With these conventions, two oriented Seifert fibrations are orientation-preservingly isomorphic if and only if their Seifert invariants differ by a finite sequence of the following operations:

1. replacing (α_i, β_i) by $(-\alpha_i, -\beta_i)$;
2. replacing $(b; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$ by

$$(b + k; (\alpha_1, \beta_1 - k_1\alpha_1), \dots, (\alpha_m, \beta_m - k_m\alpha_m) \qquad \text{with} \qquad \sum_{i=1}^{k} k_i = k.$$

3. inserting or deleting pairs $(1, 0)$.

Notice that the orientability of the total space is controlled by the equality $w_1 = w_1(B)$ and that these two classes are preserved by any fibre-preserving diffeomorphism above the identity on the base. So it is relevant to focus on the orientation-preserving classification in this case. The literature generally focuses on orientation-preserving isomorphisms. The cohomological way has the advantage to make the classification clearer when dropping the orientation-preserving condition. We shall indeed see that if orientations are ignored, one must further quotient by the operations:

4. replacing (α_i, β_i) by $(\alpha_i, -\beta_i)$.
5. replacing b by $-b$ (irrelevant if $b \in \mathbb{Z}_2$).

The rules above make it clear that we may normalise the invariants so that

- $b \in \mathbb{Z}$ and $0 < \beta_i < \alpha_i$ when $w_1 = w_1(B)$ and we care about orientations;
- $b \in \mathbb{Z}_2$ and $0 < \beta_i \leqslant \alpha_i/2$ otherwise.

Finally, in the orientable case, the rational number

$$e_{\text{rat}} = b + \sum_{i=1}^{m} \frac{\beta_i}{\alpha_i}$$

is independent of the chosen normalised representative. It is called the *rational Euler number*. It is multiplicative under finite coverings and one can show it vanishes precisely when the Seifert fibration is finitely covered by a trivial circle bundle (Lemma 3.7 in [Sco83]). By convention, one sets $e_{\text{rat}} = 0$ when the total space is non-orientable.

Let us now explain how the classical equivalence relations arise naturally from the cohomological description. We start by extracting set of Seifert invariants. This requires some care. First, remark the $|\alpha_i| > 1$ are the multiplicities of the critical fibres and therefore are encoded in the orbifold $\mathcal{O}$. So we set the α_i as the multiplicities of the cone points. This way, they are naturally in their normalised positive form. The Euler class $e \in H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$ contains

exactly the data $(b; \beta_1, \dots, \beta_m)$.

With our convention for the local models, the numerical restriction of the Euler class to the i-th isotropy group is not $\overline{\beta_i}$, but its multiplicative inverse. More precisely, after choosing an orientation of the uniformising chart V_i, the cyclic isotropy group Γ_i acquires a preferred generator. Likewise, the chosen orientation of the fibre coordinate determines a generator of the stalk $(\mathcal{Z}_w)_{\widetilde{b}_i}$. These choices identify

$$H^2\left(\Gamma_i; (\mathcal{Z}_w)_{\widetilde{b}_i}\right) \simeq \mathbb{Z}_{\alpha_i},$$

and, with this convention,

$$r_i(e) = \overline{\mu_i} = \overline{\beta_i}^{-1} \in \mathbb{Z}_{\alpha_i}^{\times}.$$

Thus the integer entering additively in the cohomological description is a lift $\mu_i \in \mathbb{Z}$ of $\overline{\beta_i}^{-1}$ rather than β_i itself. In particular, $\gcd(\alpha_i, \mu_i) = 1$.

Recall the presentation obtained in Theorem 4.8:

$$H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \simeq \langle c, q_1, \dots, q_m \mid \alpha_i q_i = c,\ 1 \leqslant i \leqslant m \rangle$$

when $w = w_1(B)$, and

$$H^2_{\mathrm{orb}}(\mathcal{O}; \mathcal{Z}_w) \simeq \langle c, q_1, \dots, q_m \mid 2c = 0,\ \alpha_i q_i = c,\ 1 \leqslant i \leqslant m \rangle$$

when $w \neq w_1(B)$. Here c generates the image of $H^2(B; \underline{\mathbb{Z}}_w)$, while q_i maps to the chosen generator of the local group at the i-th cone point.

Choose integer lifts μ_i of the residues. Then

$$r\left(e - \sum_{i=1}^{m} \mu_i q_i\right) = 0,$$

so the difference belongs to the kernel of r, which is generated by c. Consequently,

$$e = bc + \sum_{i=1}^{m} \mu_i q_i$$

where b depends on the lifts μ_i. When $w = w_1(B)$, the coefficient b is an integer. When $w \neq w_1(B)$, the relation $2c = 0$ means that only the class $b \in \mathbb{Z}_2$ is determined. To recover the classical Seifert invariants, one must now choose integer representatives β_i of the residue classes $\overline{\mu_i}^{-1}$. Unlike the residues themselves, these representatives are not canonically related to the coefficient b, since changing a lift μ_i changes the decomposition of the Euler class. One therefore has to choose a convention relating these two choices. For instance, one may require that the representatives $0 < \mu_i < \alpha_i$ correspond to the representatives $0 < \beta_i < \alpha_i$, and define b accordingly.

We now explain in detail how the classical equivalence relations arise from the preceding cohomological description. The various transformations of the classical Seifert invariants have different origins. Some come directly from the relations in the presentation of the cohomology group, while others come from changing the local identifications used to extract numerical coordinates from the intrinsic Euler class.

Rule 2: Transfer of complete twists. Let $k_1, \dots, k_m \in \mathbb{Z}$, and replace the chosen lifts μ_i by $\mu_i' = \mu_i - k_i\alpha_i$. Using the relation $\alpha_i q_i = c$, we obtain

$$\begin{aligned} e &= bc + \sum_{i=1}^{m} \mu_i q_i \\ &= \left(b + \sum_{i=1}^{m} k_i\right) c + \sum_{i=1}^{m} (\mu_i - k_i\alpha_i) q_i. \end{aligned}$$

Thus the same Euler class is represented by

$$b' = b + \sum_{i=1}^{m} k_i, \qquad \mu_i' = \mu_i - k_i\alpha_i.$$

To reproduce the usual classical transformation, one chooses the lifts μ_i and β_i according to the same convention. With this convention, the preceding identity becomes

$$\big(b; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\big) \rightsquigarrow \left(b + \sum_{i=1}^{m} k_i; (\alpha_1, \beta_1 - k_1\alpha_1), \dots, (\alpha_m, \beta_m - k_m\alpha_m)\right).$$

Geometrically, this operation transfers complete turns between the global Euler invariant and the local gluing data. Cohomologically, it is exactly the relation $\alpha_i q_i = c$.

Rule 3: Insertion or deletion of a pair of multiplicity one. The multiplicities $\alpha_i > 1$ are part of the orbifold and correspond to its genuine cone points. Classical Seifert symbols nevertheless allow one to insert a formal pair of multiplicity one. Suppose that one adjoins a symbol $(1, 0)$. At the level of the presentation, this amounts to adding a generator q_{m+1} satisfying $q_{m+1} = c$ which can be absorbed into the global coefficient b. Conversely, a multiple of c may be represented by inserting a multiplicity-one term.

Rule 1: Simultaneous change of signs $(\alpha_i, \beta_i) \rightsquigarrow (-\alpha_i, -\beta_i)$. In the orbifold description, the positive integer α_i is the order of the isotropy group and therefore cannot literally become negative. However, there is an interpretation for it in our construction. Looking at the definition of the local models, we see that $\alpha \rightsquigarrow -\alpha_i$ amounts to replace the preferred generator of the cyclic group Γ_i by its inverse $\gamma_i \rightsquigarrow \gamma_i^{-1}$. This reverses the orientation convention on the uniformizing disc V_i, but it does not alter the fibre coordinate and therefore does not change the determinant cocycle w. It changes however the numerical identification

$$H^2\left(\Gamma_i; (\mathcal{Z}_w)_{\widetilde{b}_i}\right) \simeq \mathbb{Z}_{\alpha_i}$$

and sends the numerical coordinate $\overline{\mu_i}$ to $-\overline{\mu_i}$ and therefore $\overline{\beta_i}$ to $-\overline{\beta_i}$.

Rule 4: Replacement of the local model $T(\alpha_i, \beta_i)$ by $T(\alpha_i, -\beta_i)$. This operation is fundamentally different from the preceding one. Here the chosen generator of the isotropy group is kept fixed, while the local Seifert model is genuinely replaced. For the fixed generator of the chart isotropy group, the action on the fibre is changed from the rotation $u \longmapsto e^{2\pi \mathrm{i} \beta_i^{-1}/\alpha_i} u$ to its inverse $u \longmapsto e^{-2\pi \mathrm{i} \beta_i^{-1}/\alpha_i} u$. Let $a \in \check{C}^0(G_\bullet; \mathcal{Z}_2)$ record this local reversal of the fibre coordinate. It may be chosen to be supported on the cone chart under consideration. The determinant cocycle changes according to $w' = w + \delta a$. Hence the old and the new Euler classes do not initially belong to the same cochain complex. The 0-cochain a identifies the coefficient systems $\mathcal{Z}_w$ and $\mathcal{Z}_{w'}$, and induces an isomorphism

$$\Phi_a : H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \xrightarrow{\sim} H^2_{orb}(\mathcal{O}; \mathcal{Z}_{w'}).$$

On the i-th cone chart, this identification reverses the chosen generator of the coefficient stalk $(\mathcal{Z}_w)_{\widetilde{b}_i}$. The isotropy generator, by contrast, is unchanged. Therefore the numerical local coordinate changes sign: $\overline{\mu_i} \rightsquigarrow -\overline{\mu_i}$. Consequently $\overline{\beta_i} \rightsquigarrow -\overline{\beta_i}$. Thus the induced transformation of the classical local invariant is $(\alpha_i, \beta_i) \rightsquigarrow (\alpha_i, -\beta_i)$.

Because the 0-cochain a may be supported on a single cone chart, the same operation can be performed independently at the different exceptional fibres. This is the source of the independent equivalences $\beta_i \rightsquigarrow -\beta_i \pmod{\alpha_i}$ when the orientation of the local Seifert models is not retained.

Although the coordinate μ_i has changed sign, the intrinsic Seifert fibration has not changed: its two Euler classes correspond under the canonical identification Φ_a induced by the change of local trivialization.

Rule 5: Global reversal of the fibre coordinates. There remains one transformation which is invisible if one works only with $\mathbb{S}^1$-valued 0-cochains. Consider the constant $\mathbb{Z}_2$-valued 0-cochain $a \equiv 1$, corresponding to reversing the fibre coordinate simultaneously over every uniformizing chart. Since $\delta a = 0$, the determinant cocycle is unchanged: $w' = w$. However, the induced automorphism of the coefficient system is multiplication by -1, $-1 : \mathcal{Z}_w \longrightarrow \mathcal{Z}_w$, and consequently the Euler class is replaced by its opposite: $e \rightsquigarrow -e$.

Write, with the choices of generators made above,

$$e = bc + \sum_{i=1}^{m} \mu_i q_i.$$

Relative to these fixed generators, the transformation $e \mapsto -e$ gives

$$(b, \mu_1, \ldots, \mu_m) \rightsquigarrow (-b, -\mu_1, \ldots, -\mu_m),$$

and hence

$$(b, \beta_1, \ldots, \beta_m) \rightsquigarrow (-b, -\beta_1, \ldots, -\beta_m).$$

The local signs may then be restored independently by applying Rule 4 at every exceptional fibre. Combining the global fibre reflection with these local replacements gives

$$(b, \beta_1, \ldots, \beta_m) \rightsquigarrow (-b, \beta_1, \ldots, \beta_m).$$

This is the origin of the classical additional equivalence $b \rightsquigarrow -b$ when orientations are forgotten. If $b \in \mathbb{Z}_2$, this operation is of course trivial.

If the total space M is oriented, the global fibre reflection reverses the orientation of the fibre while leaving the base coordinates unchanged, and hence reverses the induced orientation of M. It is therefore excluded when oriented Seifert fibrations are classified up to orientation-preserving fibre-preserving diffeomorphisms.

The preceding analysis shows that the classical equivalence relations have two distinct cohomological origins. The transfer of complete twists and the insertion of a pair $(1, 0)$ arise directly from the relations $\alpha_i q_i = c$ in the presentation of the Euler-class group. By contrast, the sign operations arise from changing the local generators used to identify the intrinsic restriction groups with the numerical groups $\mathbb{Z}_{\alpha_i}$, or from replacing the cocycle w by a cohomologous representative and transporting the Euler class through the corresponding identification of coefficient systems.

Finally, suppose that $w_1 = w_1(B)$. After tensoring the presentation of $H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$ with $\mathbb{Q}$, the relations become $q_i = \frac{1}{\alpha_i} c$. Therefore

$$H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \otimes \mathbb{Q} \simeq \mathbb{Q}.\langle c \rangle$$

and

$$e = \left(b + \sum_{i=1}^{m} \frac{\mu_i}{\alpha_i} \right) c \ \in H^2_{orb}(\mathcal{O}; \mathcal{Z}_w) \otimes \mathbb{Q}.$$

We see that the coordinate of e in this augmented group is the above rational Euler number. When $w_1 \neq w_1(B)$, the global class c has order two and disappears after tensoring with Q. Accordingly, there is no nonzero rational Euler class of this form; by convention, the rational Euler number is then taken to be zero.

4.13 The Euler class and the fundamental group.

As explained for instance in [Sco83], a Seifert fibration $\pi : M \to \mathcal{O}$ imposes a strong algebraic structure on $\pi_1(M)$. The first step is obtained by considering the action of $\pi_1(M)$ on the universal covering $\widetilde{M}$.

Indeed, the foliation of M by Seifert fibres lifts to a foliation of $\widetilde{M}$ by circles or lines. The deck transformations preserve this foliation, and therefore act on the corresponding leaf space $\widetilde{\mathcal{O}}$, which is an orbifold covering of $\mathcal{O}$. By the correspondence between orbifold coverings and subgroups of the orbifold fundamental group, one obtains a natural homomorphism

$$\pi_1(M) \longrightarrow \pi_1^{orb}(\mathcal{O}).$$

Its kernel consists of the deck transformations acting trivially on the leaf space, or equivalently of those preserving every fibre of the lifted fibration. Such a transformation acts freely on each fibre, which is either $\mathbb{S}^1$ or $\mathbb{R}$, and therefore generates a cyclic group. Since the regular fibres form a connected subset of M, this cyclic subgroup is generated by the homotopy class of any regular fibre, independently of the chosen base point.

One thus obtains the following result (Lemma 3.2 of [Sco83]; see also Proposition 2.12 of [BMP03]).

Lemma 4.11. Let $\pi : M \longrightarrow \mathcal{O}$ be a Seifert fibration. Then there is a short exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(\mathcal{O}) \longrightarrow 1,$$

where K is the cyclic subgroup generated by any regular fibre. Moreover, K is infinite unless M is covered by $\mathbb{S}^3$.

Scott then proceeds in two complementary directions. In the case of genuine circle bundles, he shows that, under suitable hypotheses, the Euler class is precisely the class of the above central extension. For general Seifert fibrations, he explains how to compute the fundamental group explicitly from the classical Seifert invariants.

The cohomological description developed in the present paper naturally bridges these two points of view. Indeed, the Euler class is defined intrinsically as an obstruction class in $H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$, exactly as for ordinary circle bundles. This makes it possible to recover the above short exact sequence as only a fragment of a longer exact sequence, whose connecting morphism is governed by the Euler class. In turn, this immediately recovers the classical presentations of $\pi_1(M)$ from the cohomological description of the Euler class.

The appropriate framework for this statement is the homotopy theory of orbifolds, or more generally of topological stacks. The resulting long exact sequence is the orbifold analogue of the homotopy exact sequence of an ordinary fibration (see [Noo14]). Developing this machinery would take us too far afield, so we simply state the consequence needed here, with some heuristic explanations.

Following Haefliger and Moerdijk, the homotopy groups of an orbifold are defined through the classifying space of a presenting étale groupoid. Let $G_\bullet = (G_p)_{p \geqslant 0}$ be the nerve of the groupoid associated with a chosen orbifold atlas. Its geometric realisation (as simplicial space)

$$BG := |G_\bullet|$$

is called the *classifying space* of the orbifold. It is obtained by gluing together the products $G_p \times \Delta^p$, where Δ^p denotes the standard p-simplex, according to the face and degeneracy maps of the nerve. Different choices of orbifold atlases

yield weakly equivalent classifying spaces. For more details, see [Moe02; Seg74; Noo12]. The *orbifold homotopy groups* are then defined by

$$\pi_k^{orb}(\mathcal{O}) := \pi_k(BG).$$

Now let

$$h \in \check{Z}^1\big(G_\bullet; \mathscr{C}^\infty(-, \mathrm{Diff}(\mathbb{S}^1))\big)$$

be the cocycle associated with a Seifert fibration $\pi : M \to \mathcal{O}$. It defines an action of the groupoid G on the trivial bundle

$$G_0 \times \mathbb{S}^1 \longrightarrow G_0$$

by

$$g \cdot (x, u) = \big(t(g), h(g)(u)\big),$$

where $s(g) = x$. This action gives rise to the action groupoid

$$G \ltimes \mathbb{S}^1 \rightrightarrows G_0 \times \mathbb{S}^1.$$

It has object space $(G \ltimes \mathbb{S}^1)_0 = G_0 \times \mathbb{S}^1$ and arrow space $(G \ltimes \mathbb{S}^1)_1 = G_1 \times \mathbb{S}^1$. The source and target maps are

$$s(g, u) = (s(g), u), \qquad t(g, u) = (t(g), h(g)u)$$

and composition is

$$(g_2, h(g_1)u) \circ (g_1, u) = (g_2 g_1, u).$$

The action groupoid presents the manifold M. Consequently, its classifying space is naturally weakly equivalent to M, *i.e.* $B(G \ltimes \mathbb{S}^1) \simeq M$. This is the classifying-space version of the fact that the equivariant trivial bundles $V_i \times \mathbb{S}^1$ glue to the total space M. The projection

$$G \ltimes \mathbb{S}^1 \longrightarrow G$$

induces a continuous map

$$B(G \ltimes \mathbb{S}^1) \longrightarrow BG,$$

which is the analogue of the projection of an ordinary circle bundle. A fundamental result in the homotopy theory of topological stacks asserts that this map is a Serre fibration up to homotopy [Noo14]. Consequently, one obtains the long exact sequence

$$\cdots \longrightarrow \pi_2^{orb}(\mathcal{O}) \xrightarrow{\partial} \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(\mathcal{O}) \longrightarrow \pi_0(\mathbb{S}^1) \longrightarrow \pi_0(M) \longrightarrow \pi_0^{orb}(\mathcal{O}) \longrightarrow 1.$$

And because, $\pi_0(\mathbb{S}^1) = \pi_2(\mathbb{S}^1) = \{1\}$, we get the following exact sequence

$$0 \longrightarrow \pi_2(M) \longrightarrow \pi_2^{orb}(\mathcal{O}) \xrightarrow{\partial} \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(\mathcal{O}) \longrightarrow 1.$$

Denoting $K = \mathrm{Im}\,\big(\pi_1(M) \to \pi_1(\mathbb{S}^1)\big) \simeq {}^{\pi_1(\mathbb{S}^1)}\!/_{\mathrm{Im}\,\partial}$, the rightmost part yields the short exact sequence deribed above. The additional information is the connecting homomorphism

$$\partial : \pi_2^{orb}(\mathcal{O}) \longrightarrow \pi_1(\mathbb{S}^1),$$

whose description is completely analogous to the classical case of circle bundles. Choosing an orientation of a regular fibre identifies $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$. Reversing this orientation multiplies the identification by -1. The following description is made with respect to such a choice.

Indeed, let

$$[f] \in \pi_2^{orb}(\mathcal{O})$$

be represented by a continuous map $f : \mathbb{S}^2 \longrightarrow BG$. Pulling back the Seifert fibration along f yields an ordinary circle bundle $f^*M \longrightarrow \mathbb{S}^2$. Since $\mathbb{S}^2$ is simply connected, the pull-back sheaf $f^*\mathcal{Z}_w$ is canonically isomorphic te the constant one $\underline{\mathbb{Z}}$ up to sign. Choosing an orientation of the fibre identifies decides this sign and the pull-back Euler class becomes $f^*e \in H^2(\mathbb{S}^2; \mathbb{Z})$, where $e \in H^2_{orb}(\mathcal{O}; \mathcal{Z}_w)$ denotes the Euler class of the original Seifert fibration. The connecting homomorphism is then given by

$$\partial([f]) = \left\langle f^*e, [\mathbb{S}^2] \right\rangle,$$

up to the global sign determined by the orientation convention on the fibre.

4.14 The Seifert fibration of $\mathbb{S}^3$ over the spindle orbifold $\mathbb{S}^2(p,q)$.

Let $p, q \geqslant 1$ be coprime integers. Consider the circle action on

$$\mathbb{S}^3 = \big\{(z_1, z_2) \in \mathbb{C}^2 \;\big|\; |z_1|^2 + |z_2|^2 = 1\big\}$$

defined by

$$\lambda \cdot (z_1, z_2) = (\lambda^p z_1, \lambda^q z_2)\,, \qquad \lambda \in \mathbb{S}^1.$$

Since $\gcd(p,q) = 1$, the action is effective and has no finite stabiliser away from the two coordinate circles

$$K_p = \{(z_1, 0) \in \mathbb{S}^3\}, \qquad K_q = \{(0, z_2) \in \mathbb{S}^3\}.$$

The stabilisers of K_p and K_q have respective orders p and q. The orbit space is therefore the spindle orbifold $\mathcal{O} = \mathbb{S}^2(p,q)$. The action preserves the orientation of the circle fibres. We denote the corresponding Seifert fibration by $\pi_{p,q} : \mathbb{S}^3 \longrightarrow \mathbb{S}^2(p,q)$.

Its Stiefel–Whitney class is obviously trivial: $w_1(\pi_{p,q}) = 0$. We may consequently describe the Euler class as belonging to $H^2_{orb}\big(\mathbb{S}^2(p,q); \mathcal{Z}\big)$. We first determine the local models. Near K_p, the stabiliser is

$$\mu_p = \{\lambda \in \mathbb{S}^1 \mid \lambda^p = 1\}.$$

An element $\lambda \in \mu_p$ acts on a normal coordinate to K_p by $z_2 \longmapsto \lambda^q z_2$, and on the fibre coordinate by multiplication by λ. Thus, with our convention, a neighbourhood of K_p is the fibred solid torus

$$T(p,q) \longrightarrow \mathbb{D}^2\!\diagup\!\mathbb{Z}_p.$$

Similarly, a neighbourhood of K_q is modelled on

$$T(q,p) \longrightarrow \mathbb{D}^2\!\diagup\!\mathbb{Z}_q.$$

These statements depend on the choices of positive generators induced by the orientations of the two uniformizing discs. Reversing one of these choices changes the corresponding second entry by a sign. We fix the choices for which the local models are $T(p,q)$ and $T(q,p)$.

Let

$$r_p : H^2_{orb}\big(\mathbb{S}^2(p,q); \mathcal{Z}\big) \longrightarrow \mathbb{Z}_p, \qquad r_q : H^2_{orb}\big(\mathbb{S}^2(p,q); \mathcal{Z}\big) \longrightarrow \mathbb{Z}_q$$

be the resulting numerical restriction maps. Since our convention records the rotation on the base in the second entry of $T(\alpha,\beta)$, the cohomological restriction is its multiplicative inverse. Therefore

$$r_p\big(e(\pi_{p,q})\big) = \overline{q}^{\,-1} \in \mathbb{Z}_p, \qquad r_q\big(e(\pi_{p,q})\big) = \overline{p}^{\,-1} \in \mathbb{Z}_q.$$

We now compute the global group. By Theorem 4.8, it has the presentation

$$H^2_{orb}\big(\mathbb{S}^2(p,q); \mathcal{Z}\big) \simeq \langle c, q_p, q_q \mid pq_p = c, \quad qq_q = c\rangle\,.$$

Here c is the image of the positive generator of $H^2(\mathbb{S}^2;\mathbb{Z}) \simeq \mathbb{Z}$, while q_p and q_q restrict to the standard generators of $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Since p and q are coprime, this group is infinite cyclic. More precisely, there is a generator u such that

$$c = pqu, \qquad q_p = qu, \qquad q_q = pu.$$

The restrictions of u are then

$$r_p(u) = \overline{q}^{\,-1}, \qquad r_q(u) = \overline{p}^{\,-1}.$$

It follows that the Euler class of $\pi_{p,q}$ has the form $e(\pi_{p,q}) = nu$ for some integer n satisfying

$$n \equiv 1 \pmod{p}, \qquad n \equiv 1 \pmod{q}.$$

Since p and q are coprime, this is equivalent to $n \equiv 1 \pmod{pq}$. Thus the local restrictions determine the Euler class modulo the surface class $c = pq\,u$.

The remaining information is global and can be recovered from the topology of the total space. Since the total space is $\mathbb{S}^3$, the homotopy exact sequence of the preceding subsection reduces to

$$0 \longrightarrow \pi_2^{orb}\big(\mathbb{S}^2(p,q)\big) \xrightarrow{\partial} \pi_1(\mathbb{S}^1) \longrightarrow 0.$$

Hence ∂ is an isomorphism. By the description of the connecting homomorphism in terms of the Euler class, this implies that $e(\pi_{p,q})$ is primitive in

$$H^2_{orb}\big(\mathbb{S}^2(p,q); \mathbb{Z}\big) \simeq \mathbb{Z}.$$

Consequently, $|n| = 1$.

Combining this with the congruence $n \equiv 1 \pmod{pq}$, we already obtain $n = 1$ whenever $pq > 2$. Only the cases $pq = 1$ or $pq = 2$ are not distinguished by the local congruences alone. The remaining sign is fixed by the global convention used in the identification of the connecting homomorphism with evaluation of the Euler class. We calibrate this convention on the Hopf fibration, corresponding to $p = q = 1$. With the usual outward-normal orientation of $\mathbb{S}^3 = \partial B^4 \subset \mathbb{C}^2$, the fibre orientation induced by the circle action and the complex orientation of $\mathbb{CP}^1$, the Hopf fibration is the unit-circle bundle of the tautological line bundle $\mathscr{O}_{\mathbb{CP}^1}(-1)$, and hence has Euler class -1.

Our boundary convention uses instead the inward normal. This reverses the orientation of $\mathbb{S}^3$ and, while keeping the fibre orientation fixed, reverses the induced orientation of the base. Relative to the positive generator determined by this convention, the Hopf Euler class is therefore $+1$. This fixes the global sign in the description of the connecting homomorphism, and the same convention is used throughout. Hence in all cases

$$\boxed{e(\pi_{p,q}) = u.}$$

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