manuscripta math.

Francisco-Jesús Castro-Jiménez · Dorin Popescu · Guillaume Rond

Linear nested Artin approximation theorem for algebraic power series

Received: 1 July 2017 / Accepted: 29 March 2018

Abstract. We give an elementary proof of the nested Artin approximation theorem for linear equations with algebraic power series coefficients. Moreover, for any Noetherian local subring of the ring of formal power series, we clarify the relationship between this theorem and the problem of the commutation of two operations for ideals: the operation of replacing an ideal by its completion and the operation of replacing an ideal by one of its elimination ideals. In particular we prove that a Grothendieck conjecture about morphisms of analytic/formal algebras and Artin's question about linear nested approximation problem are equivalent.

1. Introduction

The aim of the paper is to investigate the nested Artin approximation problem for linear equations. Namely the nested Artin approximation problem is the following: if

$$F(x, y) = 0$$

is a system of algebraic or analytic equations which are linear in y, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, and if $y(x)$ is a formal power series solution

$$F(x, y(x)) = 0$$

Francisco-Jesús Castro-Jiménez was partially supported by Ministerio de Economía y Competitividad (MTM2013-40455-P and Feder). Dorin Popescu was partially supported by the Project ID-PCE-2011-1023, granted by the Romanian National Authority for Scientific Research, CNCS - UEFISCDI. Guillaume Rond was partially supported by ANR Projects STAAVF (ANR-2011 BS01 009) and SUSI (ANR-12-JS01-0002-01).

F.-J. Castro-Jiménez: Departamento de Álgebra and IMUS, Universidad de Sevilla, Sevilla, Spain.
e-mail: castro@us.es

D. Popescu: Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 5, University of Bucharest, P.O. Box 1-764, 014700 Bucharest, Romania.
e-mail: dorin.popescu@imar.ro

G. Rond (✉): UMR 7373, I2M, Centrale Marseille, CNRS, Aix-Marseille Université, 13453 Marseille, France. e-mail: guillaume.rond@univ-amu.fr

Mathematics Subject Classification: Primary: 13B40; Secondary: 13J05, 14B12

https://doi.org/10.1007/s00229-018-1025-0
Published online: 09 April 2018

with the property that

$$y_i(x) \text{ depends only on the variables } x_1, \ldots, x_{\sigma_i} \tag{1.1}$$

for some integers σ_i, is it possible to find algebraic or analytic solutions satisfying (1.1)?

Popescu [21] proved the nested Artin approximation theorem for any vector $F(x, y)$ of algebraic power series (not necessarily linear in y). In this paper we give an elementary proof of this theorem when $F(x, y)$ is linear in y. Moreover we provide a characterization for a certain class of germs of functions $F(x, y)$, linear in y, to satisfy the nested Artin approximation property. From an example of Gabrielov [12] we know that the answer to the nested Artin approximation problem is negative for the ring of convergent power series.

In order to explain the situation let us consider the following theorem (proven by M. Artin in characteristic zero and by M. André in positive characteristic):

Theorem 1.1. [2,3] *Let $\mathbb{k}$ be a complete valued field and let $F(x, y)$ be a vector of convergent power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ vanishing at* 0*,*

$$F(x, \widehat{y}(x)) = 0.$$

Then, for any $c \in \mathbb{N}$, there exists a convergent power series solution $\widetilde{y}(x)$,

$$F(x, \widetilde{y}(x)) = 0$$

which coincides with $\widehat{y}(x)$ up to degree c,

$$\widetilde{y}(x) \equiv \widehat{y}(x) \textit{ modulo } (x)^c.$$

Then M. Artin (see [5, p. 7]) asked, whether or not, given a formal solution $\widehat{y}(x) = (\widehat{y}_1(x), \ldots, \widehat{y}_m(x))$ satisfying

$$\widehat{y}_j(x) \in \mathbb{k}[\![x_1, \ldots, x_{\sigma_j}]\!] \quad \forall j$$

for some integers $\sigma_j \in \{1, \ldots, n\}$, there exists a convergent solution $\widetilde{y}(x)$ as in Theorem 1.1 such that

$$\widetilde{y}_j(x) \in \mathbb{k}\{x_1, \ldots, x_{\sigma_j}\} \quad \forall j.$$

Shortly after, Gabrielov [12] gave an example showing that the answer to Artin's question is negative in general.

On the other hand since Theorem 1.1 remains valid if we replace convergent power series by algebraic power series (cf. [4]) the question of M. Artin is also relevant in this context and in this case this question has a positive answer. Let us recall that a formal power series $f(x) \in \mathbb{k}[\![x_1, \ldots, x_n]\!]$ is called *algebraic* if it is algebraic over the ring of polynomials $\mathbb{k}[x_1, \ldots, x_n]$. The ring of algebraic power series is denoted by $\mathbb{k}\langle x_1, \ldots, x_n\rangle$. Indeed after A. Gabrielov gave a negative answer to Artin's question, D. Popescu showed that it has a positive answer in the case the ring of convergent power series is replaced by the ring of algebraic power series:

Theorem 1.2. [21] *Let $\mathbb{k}$ be a field and $F(x, y)$ be a vector of algebraic power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x) = (\widehat{y}_1(x), \ldots, \widehat{y}_m(x))$ vanishing at* 0*,*

$$F(x, \widehat{y}(x)) = 0.$$

Moreover let us assume that $\widehat{y}_j(x) \in \mathbb{k}[\![x_1, \ldots, x_{\sigma_j}]\!]$, $1 \leq j \leq m$, for some integers σ_j, $1 \leq \sigma_j \leq n$.

Then for any $c \in \mathbb{N}$ there exists an algebraic power series solution $\widetilde{y}(x)$ such that for all j, $\widetilde{y}_j(x) \in \mathbb{k}\langle x_1, \ldots, x_{\sigma_j}\rangle$ and $\widetilde{y}(x) - \widehat{y}(x) \in (x)^c$.

Let us remark that if $F(x, y)$ is a vector of polynomials in y with coefficients in $\mathbb{k}\langle x\rangle$ we may drop the condition that $\widehat{y}(x)$ vanishes at 0 by replacing $F(x, y)$ (resp. $\widehat{y}(x)$) by $F(x, y + \widehat{y}(0))$ (resp. $\widehat{y}(x) - \widehat{y}(0)$). This result has a large range of applications (see [11,18] or [25] for some recent examples). Its proof relies on an idea of Kurke from 1972 and the Artin approximation property of rings of type $\mathbb{k}[\![x]\!]\langle z\rangle$ based on the General Néron Desingularization Theorem which is quite involved (see [21] or [26]).

The first goal of this paper is to provide a new and elementary proof of Theorem 1.2 for equations $F(x, y) = 0$ which are linear in y (see Theorem 2.1). This shows that Theorem 1.2 is really easier in the case $F(x, y)$ is linear in y. Let us mention that in the case where there is only one nest (i.e. when there is a given $k \leq n$ such that $\sigma_i = k$ or n for every i) this has been already proven by E. Bierstone and P. Milman (see [7, Theorem 12.6]). In fact our proof is based on a reduction to this case.

In the second part of this paper we investigate the relationship between Artin's question and the following conjecture of A. Grothendieck (see [14, p. 13-08]):

If $\varphi : \mathbb{C}\{x\}/I \longrightarrow \mathbb{C}\{y\}/J$ is an injective morphism of analytic algebras then the corresponding morphism $\widehat{\varphi} : \mathbb{C}[\![x]\!]/I\mathbb{C}[\![x]\!] \longrightarrow \mathbb{C}[\![y]\!]/J\mathbb{C}[\![y]\!]$ is again injective.

In fact the counterexample of A. Gabrielov to Artin's question is built from a counterexample to the conjecture of A. Grothendieck he gave in [12]. Even if it is obvious that the counterexample of Gabrielov to Grothendieck's conjecture provides a negative answer to the question of M. Artin, the relationship between these two problems is not clear in general.

The second goal of this paper is to clarify the relationship between Grothendieck's conjecture and Artin's question. We show in a general framework (i.e. not only for the rings of convergent power series or algebraic power series but for more general families of rings—cf. Definition 3.1) that Grothendieck's conjecture is equivalent to the question of M. Artin in the case where $F(x, y)$ is linear in y (see Theorem 3.9). Let us mention that it is well known that Grothendieck's conjecture is equivalent to Artin's question for some very particular $F(x, y)$ which are linear in y (see [6,22]) but, to the best of our knowledge, it was not known that they are equivalent for all $F(x, y)$ linear in y.

We also prove (see Theorem 3.9) that these two problems are equivalent to the problem of the commutation of two operations: the operation of replacing an ideal by its completion and the operation of replacing an ideal by one of its elimination ideals (see 3.2).

Finally we mention that the question of Grothendieck has been widely studied in the case of convergent power series rings and it has been shown that the answer is positive for some particular cases (see for instance [1,10,13,15,17] or [28]). One of them, similar to our situation, is the case of a morphism $\varphi : \mathbb{k}\{x\}/I \longrightarrow \mathbb{k}\{y\}/J$ where the images of the x_i are algebraic power series and the ideals I and J are prime and generated by algebraic power series. For such morphisms it is shown that φ is injective if and only if $\widehat{\varphi}$ is injective (it has been proven in several steps in [6,17,23,27]).

2. Linear nested Artin approximation for algebraic series

We will prove the following linear version of Theorem 1.2:

Theorem 2.1. (Linear Nested Artin Approximation Theorem) *Let m, n, p be positive integers, T be a $p \times m$ matrix with entries in $\mathbb{k}\langle x\rangle := \mathbb{k}\langle x_1, \dots, x_n\rangle$, $b = (b_1, \dots, b_p) \in \mathbb{k}\langle x\rangle^p$ and $\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ be a map. Let $y = (y_1, \dots, y_m)$ be a vector of new variables. Then for any solution $\widehat{y}(x)$ in*

$$\mathbb{k}[\![x_1, \dots, x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1, \dots, x_{\sigma(m)}]\!]$$

of the following system of linear equations

$$Ty = b \tag{2.1}$$

and for any integer c there exists a solution $y(x)$ in

$$\mathbb{k}\langle x_1, \dots, x_{\sigma(1)}\rangle \times \cdots \times \mathbb{k}\langle x_1, \dots, x_{\sigma(m)}\rangle$$

such that $y(x) - \widehat{y}(x) \in (x)^c \mathbb{k}[\![x]\!]^m$.

We begin by giving some intermediate results:

Lemma 2.2. *Let $(A, \mathfrak{m})$ be a complete normal local domain, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$. Let $B = A[\![x]\!]\langle y\rangle$ be the algebraic closure of $A[\![x]\!][y]$ in $A[\![x, y]\!]$ and $f \in B$. Then there exist g in the algebraic closure $A\langle y, z\rangle$ of $A[y, z]$ in $A[\![y, z]\!]$, with $z = (z_1, \dots, z_s)$ for some $s \in \mathbb{N}$, and $\widehat{z} \in A[\![x]\!]^s$ such that $f = g(y, \widehat{z})$.*

Proof. By replacing f by $f - f(0, y)$ we may assume that $f \in (x)B$. Note that B is the Henselization of $C = A[\![x]\!][y]_{(\mathfrak{m},x,y)}$ by [19, 44.1] and so there exists some étale neighborhood of C containing f. Using for example [26, Theorem 2.5] there exists a monic polynomial F in u over $A[\![x]\!][y]$ and $h \in (\mathfrak{m}, x, y)A[\![x]\!]\langle y\rangle$ such that $F(h) = 0$, $(\partial F/\partial u)(h) \notin (\mathfrak{m}, x, y)$ and $f \in A[\![x]\!][y, h]_{(\mathfrak{m},x,y)\cap A[\![x]\!][y,h]}$, let us say $f = P(y, h)/Q(y, h)$ for some $P(y, u), Q(y, u) \in A[\![x]\!][y, u]$, $Q(y, h) \notin (\mathfrak{m}, x, y) \cap A[\![x]\!][y, h]$.

Let us write

$$Q(y, h) = \sum_{\alpha,i} (q_{\alpha i} + \widehat{w}_{\alpha i}) y^\alpha h^i$$

where $q_{\alpha i} \in A$ and $\widehat{w}_{\alpha i} \in (\mathfrak{m} + (x))A[\![x]\!]$ for every α, i. We set

$$\tilde{Q} = \sum_{\alpha,i} (q_{\alpha i} + w_{\alpha i}) y^\alpha u^i$$

for new indeterminates $w_{\alpha i}$. Since $Q(y, h) \notin (\mathfrak{m}, x, y)$, $\widehat{w}_{\alpha i} \in (\mathfrak{m} + (x))A[\![x]\!]$ and $h \in (\mathfrak{m}, x, y)A[\![x]\!]\langle y \rangle$, we have that q_{00} is a unit of A. So $\tilde{Q}$ is invertible in $A\langle y, u, w_{\alpha,i} \rangle$. Moreover

$$\tilde{Q}^{-1}(y, h, \widehat{w}_{\alpha i}) = Q(y, h)^{-1}$$

by uniqueness of the inverse.

Thus by adding the new $w_{\alpha i}$ and the coefficients of P from $A[\![x]\!]$ as new $\widehat{w}$, we see that our lemma works for f as soon as it works for h. So we can replace f by h and assume $f \in (\mathfrak{m}, x, y)A[\![x]\!]\langle y \rangle$, $F(f) = 0$ and $F'(f) := (\partial F/\partial u)(f) \notin (\mathfrak{m}, x, y)$. Let us write $F = \sum_{\alpha, j} F_{\alpha j} y^\alpha u^j$ for some $F_{\alpha j} \in A[\![x]\!]$.

Set $\widehat{z}_{\alpha i} = F_{\alpha i} - F_{\alpha i}(0) \in (x)A[\![x]\!]$, $\widehat{z} = (\widehat{z}_{\alpha i})$ and $G := G(y, u, z) = \sum_{\alpha i} (F_{\alpha i}(0) + z_{\alpha i}) y^\alpha u^i$ for some new variables $z = (z_{\alpha i})$. We have $G(y, u, \widehat{z}) = F$. Set $G' = \partial G/\partial u$. As

$$G(y, f, 0) \equiv G(y, f, \widehat{z}) \equiv F(f) \equiv 0 \text{ modulo } (\mathfrak{m}, x, y, u),$$
$$G'(y, f, 0) \equiv G'(y, f, \widehat{z}) \equiv F'(f) \not\equiv 0 \text{ modulo } (\mathfrak{m}, x, y, u)$$

we get $G(y, 0, z) \equiv 0$, $G'(y, 0, z) \not\equiv 0$ modulo $(\mathfrak{m}, y, z)A\langle y, z \rangle$. By the Implicit Function Theorem there exists $g \in (\mathfrak{m}, y, z)A\langle y, z \rangle$ such that $G(y, g, z) = 0$. It follows that $G(y, g(y, \widehat{z}), z) = 0$. But $F = G(y, u, \widehat{z}) = 0$ has just one solution $u = f$ in $(\mathfrak{m}, x, y)B$ by the Implicit Function Theorem and so $f = g(y, \widehat{z})$. □

The following result can be rephrased as a particular case of Theorem 12.6 [7], and the proof we give here, for the sake of completeness, follows essentially the same principle as the proof given in [7].

Proposition 2.3. *We set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ and let M be a submodule of $\mathbb{k}\langle x, y \rangle^p$. Then*

$$\mathbb{k}[\![x]\!](M \cap \mathbb{k}\langle x \rangle^p) = \widehat{M} \cap \mathbb{k}[\![x]\!]^p$$

where $\widehat{M} = \mathbb{k}[\![x, y]\!]M$ denotes the (x, y)-adic completion of M. Moreover, if $c \in \mathbb{N}$ and $\widehat{u} = \sum_{i=1}^r \widehat{v}_i \omega_i \in \widehat{M} \cap \mathbb{k}[\![x]\!]^p$ for some $\omega_i \in M$, $\widehat{v}_i \in \mathbb{k}[\![x, y]\!]$ then there exist $v_{ic} \in \mathbb{k}\langle x, y \rangle$ such that $v_{ic} \equiv \widehat{v}_i$ modulo $(x, y)^c \mathbb{k}[\![x, y]\!]$, $u_c = \sum_{i=1}^r v_{ic} \omega_i \in M \cap \mathbb{k}\langle x \rangle^p$ and $\widehat{u}$ is the limit of $(u_c)_c$ in the (x)-adic topology.

Proof. Of course we always have $\mathbb{k}[\![x]\!](M \cap \mathbb{k}\langle x \rangle^p) \subset \widehat{M} \cap \mathbb{k}[\![x]\!]^p$. So we only have to prove the opposite inclusion.

Let $\omega_1, \ldots, \omega_r$ be generators of M and $\widehat{u}(x)$ be an element of $\widehat{M} \cap \mathbb{k}[\![x]\!]^p$. Such an element $\widehat{u}(x)$ has the form

$$\widehat{u}(x) = \sum_{\ell=1}^r \widehat{v}_\ell(x, y) \omega_\ell \tag{2.2}$$

for some formal power series $\widehat{v}_\ell(x, y)$. The components of Eq. (2.2) provide a system of p linear equations as follows:

$$T\widehat{v}(x, y) = \widehat{u}(x) \tag{2.3}$$

where T is a $p \times r$ matrix with entries in $\mathbb{k}\langle x, y\rangle$ and $\widehat{v}(x, y)$ is the vector of entries $\widehat{v}_\ell(x, y)$.

The morphism $\mathbb{k}[\![x]\!]\langle y\rangle \longrightarrow \mathbb{k}[\![x, y]\!]$ being faithfully flat, for any integer c there exists a solution $\widetilde{v}(x, y) \in \mathbb{k}[\![x]\!]\langle y\rangle^r$ of (2.3) such that

$$\widetilde{v}(x, y) - \widehat{v}(x, y) \in (x, y)^c \mathbb{k}[\![x, y]\!]^r.$$

Indeed, choose $v'(x, y) \in \mathbb{k}[\![x]\!]\langle y\rangle^r$ such that $v'(x, y) - \widehat{v}(x, y) \in (x, y)^c \mathbb{k}[\![x, y]\!]$. By faithfully flatness the linear system

$$\widehat{u}(x) = \sum_{\ell=1}^{r} v_\ell \omega_\ell, \;\; v'(x, y) - v = \sum_{|\alpha|+|\beta|=c} x^\alpha y^\beta w_{\alpha,\beta}$$

has a solution $\widetilde{v}(x, y)$, $\widetilde{w}(x, y)$ in $\mathbb{k}[\![x]\!]\langle y\rangle$ since it has one in $\mathbb{k}[\![x, y]\!]$.

Thus from now on we may assume that $\widehat{v}(x, y) \in \mathbb{k}[\![x]\!]\langle y\rangle^r$. By Lemma 2.2 there exist a new set of variables $z = (z_1, \ldots, z_s)$, algebraic power series $g_\ell(y, z) \in \mathbb{k}\langle y, z\rangle$ for $1 \leq \ell \leq r$ and formal power series $\widehat{z}_1(x), \ldots, \widehat{z}_s(x) \in (x)\mathbb{k}[\![x]\!]$ such that

$$\widehat{v}_\ell(x, y) = g_\ell(y, \widehat{z}_1(x), \ldots, \widehat{z}_s(x)).$$

Then, by replacing v_ℓ by $g_\ell(y, z)$ for $\ell = 1, \ldots, r$ in the linear system of equations $T \cdot v = \widehat{u}(x)$ we obtain a new system of (non linear) equations

$$f(x, y, \widehat{u}(x), \widehat{z}(x)) = 0$$

where $f(x, y, u, z)$ is a vector of algebraic power series.

Let $\mathcal{I}$ denote the ideal of $\mathbb{k}\langle x, u, z\rangle$ generated by all the coefficients of the monomials in y in the expansion of the components of f as power series in $(y_1, \ldots, y_m)$. Let $h_1, \ldots, h_t$ be a system of generators of $\mathcal{I}$. By assumption $(\widehat{u}(x), \widehat{z}(x))$ is a formal power series solution of the system

$$h_1(x, u, z) = \cdots = h_t(x, u, z) = 0. \tag{2.4}$$

Thus by Artin approximation theorem for algebraic power series [4], for any integer $c \geq 0$ there exists $(\widetilde{u}(x), \widetilde{z}(x)) \in \mathbb{k}\langle x\rangle^{p+s}$ solution of the system (2.4) with

$$\widetilde{u}_\kappa(x) - \widehat{u}_\kappa(x) \in (x)^c \mathbb{k}[\![x]\!], \widetilde{z}_k(x) - \widehat{z}_k(x) \in (x)^c \mathbb{k}[\![x]\!] \quad \forall \kappa, k.$$

Thus $(\widetilde{u}(x), \widetilde{v}(x, y))$ is a solution of the system (2.3) where

$$\widetilde{v}_\ell(x, y) = g_\ell(y, \widetilde{z}_1(x), \ldots, \widetilde{z}_s(x)) \quad \forall \ell.$$

In particular $\widetilde{u}(x) \in M \cap \mathbb{k}\langle x\rangle^p$ and $M \cap \mathbb{k}\langle x\rangle^p$ is dense in $\widehat{M} \cap \mathbb{k}\langle x\rangle$. Moreover by Taylor's formula we have that

$$\widetilde{v}_\ell(x, y) - \widehat{v}_\ell(x, y) \in (x, y)^c \mathbb{k}[\![x, y]\!] \quad \text{for } 1 \leq \ell \leq r.$$

□

The next theorem is a key result to reduce the proof of Theorem 2.1 to the case of only one nest (i.e. when there is $k \leq n$ such that $\sigma_i = k$ or n for every i). This one nest case is Proposition 2.3. As we have previously said, the linear one nest case was proven by E. Bierstone and P. Milman (see [7, Theorem 12.6]).

Theorem 2.4. *Let $M \subset \mathbb{k}\langle x\rangle^p$ be a finitely generated $\mathbb{k}\langle x\rangle$-submodule and $\sigma : \{1,\dots,p\} \longrightarrow \{1,\dots,n\}$ be a weakly increasing function. Then*

$$\mathcal{N} = M \cap (\mathbb{k}\langle x_1,\dots,x_{\sigma(1)}\rangle \times \cdots \times \mathbb{k}\langle x_1,\dots,x_{\sigma(p)}\rangle)$$

is dense in

$$\mathcal{N}' = (\mathbb{k}[\![x]\!]M) \cap (\mathbb{k}[\![x_1,\dots,x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1,\dots,x_{\sigma(p)}]\!]).$$

Moreover, if $c \in \mathbb{N}$ and $\widehat{u} = \sum_{i=1}^{t} \widehat{v}_i\omega_i \in \mathcal{N}'$ for some $\omega_i \in M$, $\widehat{v}_i \in \mathbb{k}[\![x]\!]$ then there exist $v_{ic} \in \mathbb{k}\langle x\rangle$ such that $v_{ic} \equiv \widehat{v}_i$ modulo $(x)^c\mathbb{k}[\![x]\!]$, $u_c = \sum_{i=1}^{t} v_{ic}\omega_i \in \mathcal{N}$ and $\widehat{u}$ is the limit of $(u_c)_c$ in the (x)-adic topology.

Proof. Apply induction on p, the case $p = 1$ being done in Proposition 2.3. Assume that $p > 1$. We may reduce to the case when $\sigma(p) = n$ replacing M by $M \cap \mathbb{k}\langle x_1,\dots,x_{\sigma(p)}\rangle^p$ if $\sigma(p) < n$. Let

$$q : \mathbb{k}[\![x]\!]^p \to \mathbb{k}[\![x]\!]^{p-1}$$

be the projection on the first $p-1$ components and

$$q' : \mathbb{k}[\![x]\!]^p \to \mathbb{k}[\![x]\!]$$

be the projection on the last component. Let $\widehat{u} = (\widehat{u}_1,\dots,\widehat{u}_p) \in \mathcal{N}'$, and $M_1 = q(M)$. Assume that $\widehat{u} = \sum_{i=1}^{t} \widehat{v}_i\omega_i$ for some $\widehat{v}_i \in \mathbb{k}[\![x]\!]$, $\omega_i \in M$. By the induction hypothesis applied to M_1 and $q(\widehat{u})$, for every $c \in \mathbb{N}$ there exists $v_{ic} \in \mathbb{k}\langle x\rangle$ with $v_{ic} \equiv \widehat{v}_i$ modulo $(x)^c\mathbb{k}[\![x]\!]$ such that

$$\begin{aligned} u'_c = \sum_{i=1}^{t} v_{ic}q(\omega_i) &\in q(\mathcal{N}) \\ &= M_1 \cap (\mathbb{k}\langle x_1,\dots,x_{\sigma(1)}\rangle \times \cdots \times \mathbb{k}\langle x_1,\dots,x_{\sigma(p-1)}\rangle) \end{aligned}$$

and $q(\widehat{u})$ is the limit of $(u'_c)_c$ in the (x)-adic topology.

Now, let $u''_c = \sum_{i=1}^{t} v_{ic}q'(\omega_i) \in \mathbb{k}\langle x_1,\dots,x_n\rangle$. We have $u''_c \equiv q'(\widehat{u})$ modulo $(x)^c\mathbb{k}[\![x]\!]$. Then $u_c = (u'_c, u''_c) = \sum_{i=1}^{t} v_{ic}\omega_i \in \mathcal{N}$ since $\sigma(p) = n$, $u_c \equiv \widehat{u}$ modulo $(x)^c\mathbb{k}[\![x]\!]^p$ and $\widehat{u}$ is the limit of $(u_c)_c$ in the (x)-adic topology. □

Proof of Theorem 2.1. First of all we may assume that σ is weakly increasing after permuting the y_i.

If $b = 0$ then it is enough to apply Theorem 2.4 for the module M of the solutions of $Ty = 0$ in $A = \mathbb{k}\langle x\rangle$. Suppose that $b \neq 0$. Replace the system $Ty = b$ by the homogeneous system of linear polynomials

$$T'y' := Ty - by_0 = 0$$

from $A[y_0, y]^p$ where $y' = (y_0, y)$. A nested formal solution $\widehat{y}$ of $Ty = b$ in $\Bbbk[\![x]\!]^m$ with $\widehat{y}_i \in \Bbbk[\![x_1, \ldots, x_{\sigma(i)}]\!]$, $1 \leq i \leq m$ induces a nested formal solution $(\widehat{y}_0, \widehat{y})$, $\widehat{y}_0 = 1$ of $T'y' = 0$ with $\sigma(0) = \sigma(1)$. As above, for all $c \in \mathbb{N}$ we get a nested algebraic solution $(y_0(x), y(x))$ of $T'y' = 0$ with $y_i(x) \in \Bbbk\langle x_1, \ldots, x_{\sigma(i)}\rangle$ and $y_i(x) \equiv \widehat{y}_i$ modulo $(x)^c\Bbbk[\![x]\!]$ for all $0 \leq i \leq m$. In particular $y_0(0) = 1 \neq 0$ and $y_0(x)$ is a unit. Thus

$$\left(y_0(x)^{-1}y_1(x), \ldots, y_0(x)^{-1}y_m(x)\right)$$

is an algebraic nested solution of $Ty - b = 0$. Moreover, for all $j \geq 1$, we have:

$$y_0(x)^{-1}y_j(x) - \widehat{y}_j(x) = \left(y_0(x)^{-1} - 1\right)y_j(x) + (y_j(x) - \widehat{y}_j(x)) \in (x)^c.$$

□

3. Linear nested approximation property

In the second part of this paper we generalize the method used to prove Theorem 2.1 in order to show that the question of A. Grothendieck, for local subrings of the ring of formal power series, is equivalent to the nested Artin approximation property for linear equations. We begin by giving several definitions.

Definition 3.1. Let $\Bbbk$ be a field. An *admissible family of rings* is an increasing sequence of rings $\mathcal{F} = (R_n)_{n\in\mathbb{N}}$ satisfying the following properties:

(1) For every integer $n \geq 0$ the ring R_n is a $\Bbbk$-subalgebra of $\Bbbk[\![x_1, \ldots, x_n]\!]$ (in particular $R_0 = \Bbbk$).
(2) For every integer $n \geq 0$, $\Bbbk[x_1, \ldots, x_n] \subset R_n$.
(3) For every integer $n > 0$ the ring R_n is a Noetherian local ring whose maximal ideal is generated by $x_1, \ldots, x_n$.
(4) For every integer n the completion of R_n is $\Bbbk[\![x_1, \ldots, x_n]\!]$.
(5) For every integers m, n with $0 \leq m \leq n$ we have

$$R_n \cap \Bbbk[\![x_1, \ldots, x_m]\!] = R_m.$$

When an admissible family of rings is given, any element of a member of this family is called an *admissible power series*.

Sometimes we will emphasize the dependency of R_n on the variables $(x_1, \ldots, x_n)$ by writing $R_n = \Bbbk\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ for $n \in \mathbb{N}$.

Example 3.2. The following families of rings are admissible:

- The rings of convergent power series over a valued field $\Bbbk$.
- The rings of algebraic power series over a field $\Bbbk$.
- The rings of formal power series.
- The rings of germs of rational functions at $0 \in \Bbbk^n$, $\Bbbk[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$.

3.1. Krull topology

Let $(A, \mathfrak{m})$ be a Noetherian local ring. The *Krull topology* of A is the topology in which the ideals $\mathfrak{m}^c$ constitute a basis of neighborhoods of the zero of A. For a A-module M the Krull topology of M is the one in which the submodules $\mathfrak{m}^c M$ constitute a basis of neighborhoods of the zero of M. The completion of A (resp. M) for the Krull topology is denoted by $\widehat{A}$ (resp. $\widehat{M}$). We have the following lemma asserting that the topological closure of a finite module and its completion coincide:

Lemma 3.3. ([29, Corollary 2, p. 257]) *If N is a A-submodule of a finite A-module M then the closure of N in $\widehat{M}$ is $\widehat{N} = \widehat{A}N$.*

Definition 3.4. If M is a A-module where $(A, \mathfrak{m})$ is a Noetherian local ring and E is a subset of M, we say that an element $f \in M$ may *be approximated by elements of E* if f is in the closure (for the Krull topology) of E in M, i.e. if for every integer c there exists $f_c \in E$ such that $f - f_c \in \mathfrak{m}^c M$.

3.2. Strong elimination property

One says that an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ has the *strong elimination property for ideals* if for every two sets of variables x and y and every ideal I of $\Bbbk\langle\langle x, y\rangle\rangle$ we have

$$(I \cap \Bbbk\langle\langle x\rangle\rangle)\Bbbk[\![x]\!] = \widehat{I} \cap \Bbbk[\![x]\!] \tag{3.1}$$

where $\widehat{I}$ denotes the ideal of $\Bbbk[\![x, y]\!]$ generated by I.

One says that the admissible family $\mathcal{F}$ has the *strong elimination property for modules* if for every two sets of variables x and y, every positive integer p and every $\Bbbk\langle\langle x, y\rangle\rangle$-submodule M of $\Bbbk\langle\langle x, y\rangle\rangle^p$ we have

$$\Bbbk[\![x]\!](M \cap \Bbbk\langle\langle x\rangle\rangle^p) = \widehat{M} \cap \Bbbk[\![x]\!]^p \tag{3.2}$$

where $\widehat{M}$ denotes the $\Bbbk[\![x, y]\!]$-submodule of $\Bbbk[\![x, y]\!]^p$ generated by M.

Remark 3.5. Since $I \cap \Bbbk\langle\langle x\rangle\rangle \subset \widehat{I} \cap \Bbbk[\![x]\!]$ (resp. $M \cap \Bbbk\langle\langle x\rangle\rangle^p \subset \widehat{M} \cap \Bbbk[\![x]\!]^p$, Lemma 3.3 shows that (3.1) (resp. (3.2)) is equivalent to say that the elements of $\widehat{I} \cap \Bbbk[\![x]\!]$ (resp. $\widehat{M} \cap \Bbbk[\![x]\!]^p$) may be approximated by elements of $I \cap \Bbbk\langle\langle x\rangle\rangle$ (resp. $M \cap \Bbbk\langle\langle x\rangle\rangle^p$).

3.3. Linear nested approximation property

We say that an admissible family of rings $\mathcal{F} = (\Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle)_n$ has the *linear nested approximation property* if the following property holds:

For every positive integers m, n, p, every $p \times m$ matrix T with entries in $\Bbbk\langle\langle x\rangle\rangle := \Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle$, every $b = (b_1, \ldots, b_p) \in \Bbbk\langle\langle x\rangle\rangle^p$ and every map $\sigma : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ we have the following: let $y = (y_1, \ldots, y_m)$ be a vector of new variables. Then the set of solutions $y(x)$ in

$$\Bbbk\langle\langle x_1, \ldots, x_{\sigma(1)}\rangle\rangle \times \cdots \times \Bbbk\langle\langle x_1, \ldots, x_{\sigma(m)}\rangle\rangle$$

of the following system of linear equations

$$Ty = b \tag{3.3}$$

is dense in the set of formal solutions in

$$\mathbb{k}[\![x_1, \ldots, x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1, \ldots, x_{\sigma(m)}]\!].$$

3.4. Strongly injective morphisms

Definition 3.6. Let $\varphi : A \longrightarrow B$ be a morphism of local rings. We denote by $\widehat{\varphi}$ the induced morphism $\widehat{A} \longrightarrow \widehat{B}$. One says that φ is *strongly injective* if $\widehat{\varphi}$ is injective.

Definition 3.7. We say that an admissible family of rings $\mathcal{F} = (\mathbb{k}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle)_n$ has the *strong injectivity property* if for every integers n and m and every ideals I of $\mathbb{k}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ and J of $\mathbb{k}\langle\!\langle y_1, \ldots, y_m\rangle\!\rangle$, every injective morphism of local rings

$$\frac{\mathbb{k}\langle\!\langle x\rangle\!\rangle}{I} \longrightarrow \frac{\mathbb{k}\langle\!\langle y\rangle\!\rangle}{J}$$

is strongly injective.

Remark 3.8. Definition 3.6 is not the classical one. In [1] a morphism $\varphi : A \longrightarrow B$ is called strongly injective if $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$. This definition, which is the classical one, is stronger than the one we use in this paper. Nevertheless we will prove that if an admissible family of rings $(\mathbb{k}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle)_n$ has the strong injectivity property then for any morphism of local rings $\varphi : A = \frac{\mathbb{k}\langle\!\langle x\rangle\!\rangle}{I} \longrightarrow B = \frac{\mathbb{k}\langle\!\langle y\rangle\!\rangle}{J}$ we have $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$ (see Corollary 3.10).

The main result of this part is the following:

Theorem 3.9. *For an admissible family of rings $\mathcal{F} = (\mathbb{k}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle)_n$ the following properties are equivalent:*

(i) *$\mathcal{F}$ has the strong elimination property for ideals.*
(ii) *$\mathcal{F}$ has the strong elimination property for modules.*
(iii) *$\mathcal{F}$ has the linear nested approximation property.*
(iv) *$\mathcal{F}$ has the strong injectivity property.*

Corollary 3.10. *Let $\mathcal{F} = (\mathbb{k}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle)_n$ be an admissible family having the strong injectivity property. Then for any morphism of local rings*

$$\varphi : A = \frac{\mathbb{k}\langle\!\langle x\rangle\!\rangle}{I} \longrightarrow B = \frac{\mathbb{k}\langle\!\langle y\rangle\!\rangle}{J}$$

we have

$$\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A).$$

In particular if $\widehat{\varphi}$ is surjective then φ is surjective too.

Proof. Clearly $\varphi(A) \subset \widehat{\varphi}(\widehat{A}) \cap B$. Let us prove the reverse inclusion.

We can replace φ by $\varphi \circ \pi$ where $\pi : \mathbb{k}\langle\langle x\rangle\rangle \longrightarrow \frac{\mathbb{k}\langle\langle x\rangle\rangle}{I}$ is the natural quotient morphism. This allows us to assume that $I = 0$ and $A = \mathbb{k}\langle\langle x\rangle\rangle$. Let $\widehat{f} \in \mathbb{k}[\![x]\!]$ such that $\widehat{\varphi}(\widehat{f}) = b \in B$. Let us denote by $\varphi_i(y)$ an admissible power series of $\mathbb{k}\langle\langle y\rangle\rangle$ which is the image of x_i by φ modulo J, for $i = 1, \ldots, n$. Let $q_1(y), \ldots, q_s(y)$ be generators of J. Thus, by assumption, there exist formal power series $\widehat{l}_j, \widehat{k}_i$, for $1 \leq j \leq s$ and $1 \leq i \leq n$, such that

$$\widehat{f}(x) = b(y) + \sum_{j=1}^{s} q_j(y)\widehat{l}_j(x, y) + \sum_{i=1}^{n}(x_i - \varphi_i(y))\widehat{k}_i(x, y).$$

By the previous theorem the family of rings $\mathcal{F}$ has the linear nested approximation property, thus there exist admissible power series

$$f(x), l_j(x, y), k_i(x, y)$$

such that

$$f(x) = b(y) + \sum_{j=1}^{s} q_j(y)l_j(x, y) + \sum_{i=1}^{n}(x_i - \varphi_i(y))k_i(x, y).$$

In particular, by replacing x_i by $\varphi_i(y)$ for all i we see that $\varphi(f) = b$. Thus $b \in \varphi(\mathbb{k}\langle\langle x\rangle\rangle)$. □

Remark 3.11. Let $\mathcal{F} = (R_n)_n$ be an admissible family. Let $f \in R_n$ such that $f(0) = 0$ and $\frac{\partial f}{\partial x_n}(0) \neq 0$. By the Implicit Function Theorem for formal power series there exists a unique formal power series $h(x')$ with $x' = (x_1, \ldots, x_{n-1})$ such that

$$f(x', h(x')) = 0 \text{ and } h(0) = 0.$$

Thus, by Taylor's formula, there exists a formal power series $g(x)$ such that

$$f(x) + (x_n - h(x'))g(x) = 0.$$

Since $\frac{\partial f}{\partial x_n}(0) \neq 0$ and $h(0) = 0$ we have $g(0) \neq 0$, i.e. $g(x)$ is a unit. Hence we have, where $u(x)$ denotes the inverse of $g(x)$:

$$f(x)u(x) + x_n - h(x') = 0.$$

Moreover, since $h(x')$ is unique, $u(x)$ is also unique and the linear equation

$$f(x)y_2 + x_n - y_1 = 0$$

has a unique nested formal solution $(h(x'), u(x))$ whose first component vanishes at 0. Thus if the family $\mathcal{F}$ satisfies the equivalent properties of Theorem 3.9 then this family has to satisfy the Implicit Function Theorem (which is equivalent to say that the rings R_n are Henselian local rings).

In particular the family of germs of rational functions at the origin of $\mathbb{k}^n$ does not satisfy the properties of Theorem 3.9.

Since the ring of algebraic power series in n variables is the Henselization of the ring of germs of rational functions at the origin of $\mathbb{k}^n$, this also shows that the family of algebraic power series is the smallest admissible family containing the family of germs of rational functions at the origin of $\mathbb{k}^n$ and satisfying the properties of Theorem 3.9 (by Theorem 2.1).

Remark 3.12. Let $\mathcal{F} = (R_n)_n$ be an admissible family and f, g two elements of R_n. Let us assume that f is x_n-regular of order d, i.e. $f(0, x_n) = x_n^d u(x_n)$ for some unit $u(x_n)$. By the Weierstrass division Theorem for formal power series there exists a unique vector

$$(q(x), a_0(x'), \ldots, a_{d-1}(x')) \in \mathbb{k}[\![x]\!] \times \mathbb{k}[\![x']\!]^d$$

with $x' = (x_1, \ldots, x_{n-1})$ such that

$$g(x) = f(x)q(x) + \sum_{\kappa=0}^{d-1} a_\kappa(x')x_n^\kappa.$$

By the uniqueness of $(q(x), a_0(x'), \ldots, a_{d-1}(x'))$ if the family $\mathcal{F}$ has the linear nested approximation property then

$$(q(x), a_0(x'), \ldots, a_{d-1}(x')) \in R_n \times R_{n-1}^d.$$

Thus $\mathcal{F}$ satisfies the Weierstrass division Theorem if it satisfies the equivalent properties of Theorem 3.9.

Let us mention that an admissible family of rings that satisfies the Weierstrass division Theorem is necessarily a family of Henselian local rings (see for instance [9]). But it is still unknown if an admissible family of Henselian local rings satisfies the Weierstrass division Theorem (see for instance [24, Remark 5.20]). See also [16] for a partial result in this direction.

Remark 3.13. The example of Gabrielov [12] shows that the family of convergent power series over a characteristic zero valued field does not satisfy the properties of Theorem 3.9 (but this family satisfies the implicit function Theorem, it even satisfies the Weierstrass division Theorem). This example is the following one:

Let

$$\varphi : \mathbb{C}\{x_1, x_2, x_3\} \longrightarrow \mathbb{C}\{y_1, y_2\}$$

be the morphism of analytic $\mathbb{C}$-algebras defined by

$$\varphi(x_1) = y_1, \quad \varphi(x_2) = y_1 y_2, \quad \varphi(x_3) = y_1 e^{y_2}.$$

It is not very difficult to show that φ and $\widehat{\varphi}$ are both injective (see [20]). Then A. Gabrielov remarked that there exists a formal but not convergent power series $\widehat{g}(x)$ whose image $h(y)$ by $\widehat{\varphi}$ is convergent (see [12]). This shows that Corollary 3.10 is not satisfied for convergent power series rings. Thus the properties of Theorem 3.9 are not satisfied in the case of convergent power series rings.

4. Proof of Theorem 3.9

We will prove the following implications:

$$(\text{i}) \Longrightarrow (\text{ii}) \Longrightarrow (\text{iii}) \Longrightarrow (\text{i}v) \Longrightarrow (\text{i})$$

4.1. Proof of (i) $\Longrightarrow$ (ii)

In fact we will prove a stronger result that we will also use in the proof of (ii) $\Longrightarrow$ (iii). The proof of (i) $\Longrightarrow$ (ii) follows from the following lemma with $p = 0$:

Lemma 4.1. *Let $(\mathbb{k}\langle\langle x_1, \dots, x_n\rangle\rangle)_n$ be an admissible family satisfying the strong elimination property for ideals and let M be a $\mathbb{k}\langle\langle x, y\rangle\rangle$-submodule of $\mathbb{k}\langle\langle x, y\rangle\rangle^{p+t}$. Then $M \cap (\{0\}^p \times \mathbb{k}\langle\langle x\rangle\rangle^t)$ is dense in $\widehat{M} \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t)$.*

Proof. Let S be the ring $\mathbb{k}\langle\langle x, y, z, w\rangle\rangle/(z, w)^2$ where $z = (z_1, \dots, z_p)$ and $w = (w_1, \dots, w_t)$ are new variables. Then the morphism of $\mathbb{k}\langle\langle x, y\rangle\rangle$-modules

$$\begin{aligned} \varphi : \mathbb{k}\langle\langle x, y\rangle\rangle \times \mathbb{k}\langle\langle x, y\rangle\rangle^{p+t} &\longrightarrow S \\ (a, b_1, \dots, b_p, c_1, \dots, c_t) &\longmapsto a + \sum_{i=1}^{p} b_i z_i + \sum_{j=1}^{t} c_j w_j \end{aligned}$$

is a $\mathbb{k}\langle\langle x, y\rangle\rangle$-isomorphism. We denote by $\widehat{\varphi}$ the isomorphism from $\mathbb{k}[\![x, y]\!]^{p+t+1}$ to $\widehat{S} = \frac{\mathbb{k}[\![x,y,z,w]\!]}{(z,w)^2}$ defined in the same way:

$$\widehat{\varphi}(a, b_1, \dots, b_p, c_1, \dots, c_t) = a + \sum_{i=1}^{p} b_i z_i + \sum_{j=1}^{t} c_j w_j.$$

The image of $\{0\} \times M$ under φ is an ideal of S denoted by I and the image of $\{0\} \times \widehat{M}$ under $\widehat{\varphi}$ is $\widehat{I}$. This is the idealization principle of Nagata.

Moreover the image of $\{0\} \times (M \cap (\{0\}^p \times \mathbb{k}\langle\langle x\rangle\rangle^t))$ under φ is the ideal $I \cap \dfrac{\mathbb{k}\langle\langle x, w\rangle\rangle}{(w)^2}$ and the image of $\{0\} \times (\widehat{M} \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t))$ is the ideal $\widehat{I} \cap \dfrac{\mathbb{k}[\![x, w]\!]}{(w)^2}$.

By the strong elimination property for ideals $I \cap \dfrac{\mathbb{k}\langle\langle x, w\rangle\rangle}{(w)^2}$ is dense in $\widehat{I} \cap \dfrac{\mathbb{k}[\![x, w]\!]}{(w)^2}$ hence $M \cap (\{0\}^p \times \mathbb{k}\langle\langle x\rangle\rangle^t)$ is dense in $\widehat{M} \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t)$. □

4.2. Proof of (ii) $\Longrightarrow$ (iii)

We assume that $\mathcal{F}$ has the strong elimination property for modules and we fix a system of linear equations as (3.3). After a permutation of the y_i we may assume that σ is weakling increasing.

We call an *admissible nested solution* (resp. *formal nested solution*) of such a system (3.3) a solution in

$$\mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(1)}\rangle\rangle \times \cdots \times \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(m)}\rangle\rangle$$
$$(\text{resp. } \mathbb{k}[\![x_1, \ldots, x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1, \ldots, x_{\sigma(m)}]\!]).$$

We will show that the set of admissible nested solutions is dense, for the $\mathfrak{m}$-adic topology, in the set of formal nested solutions.

- First we claim that we can assume that $b = 0$, i.e. the system (3.3) of linear equations is homogeneous. Indeed let us assume that the set of admissible nested solutions of any linear homogeneous system is dense in the set of formal nested solutions and let us fix a linear (non-homogenous) system as (3.3). Let $y(x) \in \mathbb{k}[\![x]\!]^m$ be a formal nested solution of the system (3.3): $Ty = b$.

Let us write $a_{i,j}$ the entries of the $p \times m$ matrix T and denote by T' the matrix

$$T' = \left[-b \,|\, T \right]$$

and set $y' = (y_0, y_1, \ldots, y_m)$.

Let us extend the previous function σ to $\{0, \ldots, m\}$ by $\sigma(0) = \sigma(1)$. Since $y(x)$ is a formal nested solution of (3.3), $y'(x) = (1, y(x))$ is a formal nested solution of the following linear homogeneous system:

$$T'y' = 0 \tag{4.1}$$

By assumption, for any given integer $c \geq 1$, there exists an admissible nested solution $y'_c(x) = (y_{0,c}(x), y_{1,c}(x), \ldots, y_{m,c}(x))$ of (4.1) such that

$$y_{0,c}(x) - 1 \in (x)^c \text{ and } y_{j,c}(x) - y_j(x) \in (x)^c \quad \forall j \geq 1.$$

In particular $y_{0,c}(0) = 1 \neq 0$ and $y_{0,c}(x)$ is a unit. Thus

$$\left(y_{0,c}(x)^{-1} y_{1,c}(x), \ldots, y_{0,c}(x)^{-1} y_{m,c}(x) \right)$$

is an admissible nested solution of (3.3). Moreover, for all $j \geq 1$, we have:

$$y_{0,c}(x)^{-1} y_{j,c}(x) - y_j(x) = \left(y_{0,c}(x)^{-1} - 1 \right) y_{j,c}(x) + (y_{j,c}(x) - y_j(x)) \in (x)^c.$$

Thus the set of admissible nested solutions of (3.3) is dense in the set of formal nested solutions of (3.3) and the claim is proven.

- Let us consider a homogeneous linear system (3.3) where $b = 0$. The set of (non-nested) admissible solutions of such a system is a $\mathbb{k}\langle\langle x\rangle\rangle$-submodule of $\mathbb{k}\langle\langle x\rangle\rangle^m$ denoted by M. By Noetherianity this module is finitely generated. The set of (non-nested) formal solutions is the completion of M denoted by $\widehat{M}$ (by flatness of $\mathbb{k}\langle\langle x\rangle\rangle \longrightarrow \mathbb{k}[\![x]\!]$ since $\mathbb{k}\langle\langle x\rangle\rangle$ is a Noetherian local ring). Thus the following lemma shows that the nested admissible solutions are dense in the set of nested formal solutions and $\mathcal{F}$ has the linear nested approximation property:

Lemma 4.2. *Let us assume that $\mathcal{F}$ has the strong elimination property for modules and M be a finite submodule of $\mathbb{k}\langle\langle x\rangle\rangle^m$. Then*

$$M \cap \left(\mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(1)}\rangle\rangle \times \cdots \times \mathbb{k}\langle\langle x_1, \ldots, x_{\sigma(m)}\rangle\rangle\right)$$

is dense in

$$\widehat{M} \cap \left(\mathbb{k}[\![x_1, \ldots, x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1, \ldots, x_{\sigma(m)}]\!]\right).$$

To prove this lemma we proceed in a similar way as for Theorem 2.4. But before we need to state a preliminary result since the strong elimination property is apparently weaker than the condition used in Proposition 2.3. This statement is the following lemma which is an analogue of Chevalley's Lemma for summands of modules (classical Chevalley's Lemma concerns decreasing sequences of ideals in complete local rings—see [8, Lemma 7]):

Lemma 4.3. (Chevalley's Lemma) *Let M be a $\mathbb{k}[\![x, y]\!]$-submodule of $\mathbb{k}[\![x, y]\!]^{p+t}$. Then there exists a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that*

$$M \cap ((x)^{\beta(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)) \subset M \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t) + (x)^c\mathbb{k}[\![x]\!]^{p+t} \quad \forall c \in \mathbb{N}.$$

Proof. For simplicity let us set $N := M \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t)$. Let us assume that there is an integer $c_0 \in \mathbb{N}$ such that

$$M \cap ((x)^{\beta}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)) \not\subset N + (x)^{c_0}\mathbb{k}[\![x]\!]^{p+t} \quad \forall \beta \in \mathbb{N}.$$

So we have that

$$M \cap ((x)^{\beta}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) \not\subset N + (x)^{c}\mathbb{k}[\![x]\!]^{p+t} \quad \forall \beta \in \mathbb{N}, \ \forall c \geq c_0.$$

By replacing M by $M \cap \mathbb{k}[\![x]\!]^{p+t}$ we may assume that M is a submodule of $\mathbb{k}[\![x]\!]^{p+t}$. The module $M/(M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t})$ is an Artinian module thus there is an integer $a(c)$ such that for all $\beta \geq a(c)$:

$$\begin{aligned} &M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) + M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t} \\ &\quad = M \cap ((x)^{\beta}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)) + M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t}. \end{aligned}$$

We may assume that $a(c) < a(c+1)$ for every c. Since

$$\begin{aligned} &M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) \subset M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) + M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t} \\ &\quad = M \cap ((x)^{a(c+1)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) + M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t} \end{aligned}$$

for a given $u_c \in M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)$ there exists an element

$$u_{c+1} \in M \cap ((x)^{a(c+1)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)$$

such that

$$u_c - u_{c+1} \in M \cap (x)^c\mathbb{k}[\![x]\!]^{p+t}.$$

Thus by choosing $u_{c_0} \in M \cap ((x)^{a(c_0)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) \setminus (N + M \cap (x)^{c_0})$ we may construct a sequence $(u_c)_c$ as above. Hence this sequence is a Cauchy sequence and has a limit $u \in M$ since M is a complete module. But $u_{c'} \in M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)$ for every $c' \geq c$ and $M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t)$ is a complete module thus

$$u \in \bigcap_{c \geq c_0} M \cap ((x)^{a(c)}\mathbb{k}[\![x]\!]^p \times \mathbb{k}[\![x]\!]^t) = M \cap (\{0\}^p \times \mathbb{k}[\![x]\!]^t) = N$$

by Nakayama's Lemma.

On the other hand we have

$$u - u_{c_0} \in M \cap (x)^{c_0}\mathbb{k}[\![x]\!]^{p+t}$$

so $u_{c_0} \in N + M \cap (x)^{c_0}\mathbb{k}[\![x]\!]^{p+t}$ which contradicts the assumption on u_{c_0}. □

Proof of Lemma 4.2. We prove the lemma by induction on m, the case $m = 1$ being equivalent to the strong elimination property for modules. Assume that $m > 1$. By the strong elimination property for modules we may reduce to the case when $\sigma(m) = n$ by replacing M by $M \cap \mathbb{k}\langle\!\langle x_1, \ldots, x_{\sigma(m)} \rangle\!\rangle^m$ if $\sigma(m) < n$. Let

$$q : \mathbb{k}[\![x]\!]^m \to \mathbb{k}[\![x]\!]^{m-1}$$

be the projection on the first $m - 1$ components. Let

$$\widehat{u} \in \widehat{M} \cap \left(\mathbb{k}[\![x_1, \ldots, x_{\sigma(1)}]\!] \times \cdots \times \mathbb{k}[\![x_1, \ldots, x_{\sigma(m)}]\!]\right)$$

and set $M_1 = q(M)$. By induction hypothesis applied to M_1 and $q(\widehat{u})$, for every $c \in \mathbb{N}$ there exists $u'_c \in M \cap \left(\mathbb{k}\langle\!\langle x_1, \ldots, x_{\sigma(1)} \rangle\!\rangle \times \cdots \times \mathbb{k}\langle\!\langle x_1, \ldots, x_{\sigma(m)} \rangle\!\rangle\right)$ such that $q(\widehat{u}) - q(u'_c) \in (x)^c$.

Now $\widehat{u} - u'_c \in \left((x)^c\mathbb{k}[\![x]\!]^{m-1} \times \mathbb{k}[\![x]\!]\right) \cap \widehat{M}$. Thus by Lemmas 4.3 and 4.1 there exists a function β such that

$$\widehat{u} - u'_{\beta(c)} \in (x)^c\mathbb{k}[\![x]\!]^m \cap \widehat{M} + \overline{(\{0\}^{m-1} \times \mathbb{k}\langle\!\langle x \rangle\!\rangle) \cap M}$$

where $\overline{(\{0\}^{m-1} \times \mathbb{k}\langle\!\langle x \rangle\!\rangle) \cap M}$ denotes the closure of $(\{0\}^{m-1} \times \mathbb{k}\langle\!\langle x \rangle\!\rangle) \cap M$. Thus there exists $u''_c \in (\{0\}^{m-1} \times \mathbb{k}\langle\!\langle x \rangle\!\rangle) \cap M$ such that

$$\widehat{u} - \left(u'_{\beta(c)} + u''_c\right) \in (x)^c\mathbb{k}[\![x]\!]^m$$

and

$$u'_{\beta(c)} + u''_c \in M \cap \left(\mathbb{k}\langle\!\langle x_1, \ldots, x_{\sigma(1)} \rangle\!\rangle \times \cdots \times \mathbb{k}\langle\!\langle x_1, \ldots, x_{\sigma(m)} \rangle\!\rangle\right)$$

since $\sigma(m) = n$. □

4.3. Proof of (iii) $\Longrightarrow$ (iv)

Let

$$\varphi : \frac{\Bbbk\langle\langle x\rangle\rangle}{I} \longrightarrow \frac{\Bbbk\langle\langle y\rangle\rangle}{J}$$

be an injective morphism of local rings and let $\widehat{f} \in \mathrm{Ker}(\widehat{\varphi})$. The morphism φ is defined by admissible power series $\varphi_1(y), \ldots, \varphi_n(y)$ such that

$$g(\varphi_1(y), \ldots, \varphi_n(y)) \in J \quad \forall g \in I$$

and, for any power series g, the image of g modulo I is equal to

$$g(\varphi_1(y), \ldots, \varphi_n(y)) \text{ modulo } J.$$

We still denote by $\widehat{f}$ a lifting of $\widehat{f}$ in $\Bbbk[\![x]\!]$. Thus

$$\widehat{f}(\varphi_1(y), \ldots, \varphi_n(y)) \in \widehat{J},$$

i.e. there exist formal power series $\widehat{h}_1(y), \ldots, \widehat{h}_s(y)$ such that

$$\widehat{f}(\varphi_1(y), \ldots, \varphi_n(y)) = \sum_{j=1}^{s} q_j(y)\widehat{h}_j(y)$$

where the $q_j(y)$ are generators of the ideal J. By Taylor's formula there exist formal power series $\widehat{k}_i(x, y)$ such that

$$\widehat{f}(x) - \sum_{j=1}^{s} q_j(y)\widehat{h}_j(y) = \sum_{i=1}^{n}(x_i - \varphi_i(y))\widehat{k}_i(x, y). \tag{4.2}$$

By the linear nested approximation property, for any integer c, there exists a vector of admissible power series

$$(f_c(x), h_{1,c}(x, y), \ldots, h_{s,c}(x, y), k_{1,c}(x, y), \ldots, k_{n,c}(x, y))$$

such that

$$f_c(x) - \sum_{j=1}^{s} q_j(y)h_{j,c}(x, y) = \sum_{i=1}^{n}(x_i - \varphi_i(y))k_{i,c}(x, y)$$

and

$$f_c(x) - \widehat{f}(x) \in (x)^c,\ h_{j,c}(x, y) - \widehat{h}_j(y) \in (x, y)^c,\ k_{i,c}(x, y) - \widehat{k}_i(x, y) \in (x, y)^c$$

for all j and i. By replacing x_i by $\varphi_i(y)$ for $i = 1, \ldots, n$, we see that $\varphi(f_c(x)) = 0$, thus $f_c(x) = 0$ since φ is injective. Thus $\widehat{f}(x) \in (x)^c$ for all $c \geq 0$ thus $\widehat{f}(x) = 0$ by Nakayama's Lemma. This shows that φ is strongly injective.

4.4. Proof of (iv) $\Longrightarrow$ (i)

Let I be an ideal of $\mathbb{k}\langle\langle x, y\rangle\rangle$. Let φ be the following injective morphism induced by the inclusion $\mathbb{k}\langle\langle x\rangle\rangle \longrightarrow \mathbb{k}\langle\langle x, y\rangle\rangle$:

$$\frac{\mathbb{k}\langle\langle x\rangle\rangle}{I \cap \mathbb{k}\langle\langle x\rangle\rangle} \longrightarrow \frac{\mathbb{k}\langle\langle x, y\rangle\rangle}{I}.$$

Then $(I \cap \mathbb{k}\langle\langle x\rangle\rangle)\mathbb{k}[\![x]\!] = \widehat{I} \cap \mathbb{k}[\![x]\!]$ if and only if φ is strongly injective since

$$\text{Ker}(\widehat{\varphi}) = \frac{\widehat{I} \cap \mathbb{k}[\![x]\!]}{(I \cap \mathbb{k}\langle\langle x\rangle\rangle)\mathbb{k}[\![x]\!]}.$$

Acknowledgements This research was started in the frame of the Jean Morlet Chair *Artin Approximation in Singularity Theory*, held at CIRM (Marseille, France) from January until June 2015. The Chair was held by Prof. Herwig Hauser and the Local Project Leader was the third author. We are grateful for the hospitality and support of CIRM during the main stage of this work. We would like to thank Profs. H. Hauser and M.E. Alonso for their very useful comments and suggestions. We also would like to thank E. Bierstone for having indicated to us the reference [7] where is proven a result similar to our Proposition 2.3.

References

[1] Abhyankar, S.S., van der Put, M.: Homomorphisms of analytic local rings. J. Reine Angew. Math. **242**, 26–60 (1970)
[2] André, M.: Artin theorem on the solution of analytic equations in positive characteristic. Manuscr. Math. **15**, 314–348 (1975)
[3] Artin, M.: On the solutions of analytic equations. Invent. Math. **5**, 277–291 (1968)
[4] Artin, M.: Algebraic approximation of structures over complete local rings. Publ. Math. IHES **36**, 23–58 (1969)
[5] Artin, M.: Algebraic Spaces, Yale Mathematical Monographs, vol. 3. Yale University Press, New Haven (1971)
[6] Becker, J.: Exposé on a conjecture of Tougeron. Ann. Inst. Fourier **27**(4), 9–27 (1977)
[7] Bierstone, E., Milman, P.: Relations among analytic functions II. Ann. Inst. Fourier **37**(2), 49–77 (1987)
[8] Chevalley, C.: On the theory of local rings. Ann. Math. **44**, 690–708 (1943)
[9] Denef, J., Lipshitz, L.: Ultraproducts and approximation in local rings II. Math. Ann. **253**, 1–28 (1980)
[10] Eakin, P.M., Harris, G.A.: When $\Phi(f)$ convergent implies f convergent. Math. Ann. **229**, 201–210 (1977)
[11] Fernández de Bobadilla, J.: Nash Problem for surface singularities is a topological problem. Adv. Math. **230**, 131–176 (2012)
[12] Gabrielov, A.M.: The formal relations between analytic functions. Funkc. Anal. Prilovzen **5**, 64–65 (1971)
[13] Gabrielov, A.M.: Formal relations among analytic functions. Izv. Akad. Naut. SSSR **37**, 1056–1088 (1973)
[14] Grothendieck, A.: Techniques de construction en géométrie analytique VI. Sémin. Henri Cartan **13**(1), 1–13 (1960–1961)

[15] Izumi, S.: The rank condition and convergence of formal functions. Duke Math. J. **59**, 241–264 (1989)
[16] Lafon, J.-P.: Anneaux henséliens et théorème de préparation. C. R. Acad. Sci. Paris Sér. A B **264**, A1161–A1162 (1967)
[17] Milman, P.: Analytic and polynomial homomorphisms of analytic rings. Math. Ann. **232**(3), 247–253 (1978)
[18] Mir, N.: Algebraic approximation in CR geometry. J. Math. Pures Appl. (9) **98**(1), 72–88 (2012)
[19] Nagata, M.: Local Rings, Interscience Tracts in Pure and Applied Mathematics, 13th edn. Interscience, New York (1962)
[20] Osgood, W.F.: On functions of several complex variables. Trans. Am. Math. Soc. **17**, 1–8 (1916)
[21] Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. **104**, 85–115 (1986)
[22] Rond, G.: Approximation de Artin cylindrique et morphismes d'algèbres analytiques. In: Proceedings of the Lêfest—Singularities I: Algebraic and Analytic Aspects, Contemporary Mathematics, vol. 474, pp. 299–307 (2008)
[23] Rond, G.: Homomorphisms of local algebras in positive characteristic. J. Algebra **322**(12), 4382–4407 (2009)
[24] Rond, G.: Artin approximation. arXiv:1506.04717
[25] Shiota, M.: Analytic and Nash equivalence relations of Nash maps. Bull. Lond. Math. Soc. **42**(6), 1055–1064 (2010)
[26] Swan, R.: Néron–Popescu desingularization, Algebra and geometry (Taipei, 1995), Lectures in Algebra and Geometry, vol. 2, pp. 135–192. International Press, Cambridge (1998)
[27] Tougeron, J.-C.: Courbes analytiques sur un germe d'espace analytique et applications. Ann. Inst. Fourier **26**(2), 117–131 (1976)
[28] Tougeron, J.-C.: Sur les racines d'un polynôme à coefficients séries formelles. Real analytic and algebraic geometry (Trento 1988). Lectures Notes in Mathematics, vol. 1420, pp. 325–363 (1990)
[29] Zariski, O., Samuel, P.: Commutative Algebra II. D. Van Nostrand, Princeton (1960)