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Approximation of Holomorphic Solutions of a System of Real Analytic Equations

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Abstract. We prove the existence of an approximation function for holomorphic solutions of a system of real analytic equations. For this we use ultraproducts and Weierstrass systems introduced by J. Denef and L. Lipshitz. We also prove a version of the Płoski smoothing theorem in this case.

1 Introduction

Let $\mathbb{k}$ be a (nontrivially) valued field of characteristic zero and let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_p)$ be indeterminates. For any convergent power series

$$f_1(x, y), \dots, f_q(x, y) \in \mathbb{k}\{x, y\}$$

with constant terms equal to 0, the famous Artin approximation theorems [1,2] (see [13] for a complete presentation of them) assert the following.

(A1) *Density of analytic solutions among formal solutions* [1,2]. The set of analytic solutions $y(x) \in \mathbb{k}\{x\}^p$, $y(0) = 0$, of the following system of equations (S) $f_1(x, y(x)) = \cdots f_q(x, y(x)) = 0$ is dense with respect to the Krull topology (or (x)-adic topology) in the set of formal solutions, *i.e.*, $\forall \hat{y}(x) \in \mathbb{k}[[x]]^p$, $\hat{y}(0) = 0$, such that $f_j(x, \hat{y}(x)) = 0$, $j = 1, \dots, q$, $\forall i \in \mathbb{N}$, $\exists y^i \in \mathbb{k}\{x\}^p$ such that $y^i(x) - \hat{y}(x) \in (x)^{i+1}\mathbb{k}[[x]]^p$ and $f_j(x, y^i(x)) = 0$, $j = 1, \dots, q$.

(A2) *Smoothing* [12] (see [13] and its references). For any formal solution $\hat{y}(x) \in \mathbb{k}[[x]]^p$, $\hat{y}(0) = 0$, of the system (S) $f_j(x, \hat{y}(x)) = 0$, $j = 1, \dots, q$, $\exists h(\lambda, x) \in \mathbb{k}\{\lambda, x\}^p$, $h(0,0) = 0$, $\lambda = (\lambda_1, \dots, \lambda_m)$, $\exists \hat{\lambda}(x) \in \mathbb{k}[[x]]^m$ such that $f_j(x, h(\lambda, x)) = 0$, $j = 1, \dots, q$ and $\hat{y}(x) = h(\hat{\lambda}(x), x)$.

(A3) *Strong approximation, existence of an approximation function* [3,8,11,14]. There exists $\beta\colon \mathbb{N} \to \mathbb{N}$ satisfying the following property: $\forall \hat{y}(x) \in \mathbb{k}[[x]]^p$, $\hat{y}(0) = 0$ such that $f_j(x, \hat{y}(x)) \in (x)^{\beta(i)+1}$, $j = 1 \dots, q$, $\exists y(x) \in \mathbb{k}\{x\}^p$ such that $y(x) - \hat{y}(x) \in (x)^{i+1}$ and $f_j(x, y(x)) = 0$.

For any of the three former situations, if a formal solution $\hat{y}(x)$ does not depend of one of the variables, say x_i, it is not possible for a general system of analytic equations to find analytic solutions approximating the original solution and not depending on x_i (see [9] for a counterexample). It is natural to investigate the properties of a formal solution that can be preserved by approximating convergent solutions. In this direction P. Milman [10], motivated by questions coming from pluri-complex

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analysis [4], showed that any formally holomorphic solution of a real analytic system of equations in $\mathbb{C}^p \times \mathbb{C}^q$ can be approximated to any order by holomorphic solutions, *i.e.*, the analogue of (A1).

More precisely, let $z = (z_1, \dots, z_p)$ (resp. $w = (w_1, \dots, w_q)$) be coordinates at the origin of $\mathbb{C}^p$ (resp. $\mathbb{C}^q$), $z_i = x_i + iy_i$ (resp. $w_i = u_i + iv_i$), and $f_1(x, y, u, v) = \rho_1(z, \overline{z}, w, \overline{w}), \dots, f_l(x, y, u, v) = \rho_l(z, \overline{z}, w, \overline{w})$ be germs at 0 of real analytic power series with no constant term. Then for any formal solution $(u(x, y), v(x, y))$ of $f_i(x, y, u(x, y), v(x, y)) = 0$ satisfying the Cauchy–Riemann equations, *i.e.*, $w(z) = u(x, y) + iv(x, y) \in \mathbb{C}[[z]]^q$ and $\rho_i(z, \overline{z}, w(z), \overline{w(z)}) = 0$, for all $c \in \mathbb{N}$, there exists $w^c \in \mathbb{C}\{z\}^q$ such that $\rho_i(z, \overline{z}, w^c(z), \overline{w^c(z)}) = 0$ and $w(z) - w^c(z) \in (z)^{c+1}\mathbb{C}^q\{z\}$.

We prove here the analogues of (A2) and (A3) for formally holomorphic solutions of a system of real analytic equations. The method used (using ultraproducts) is the one introduced in [3, 8] in order to deduce the strong approximation from the approximation. Like the general case, this method needs to obtain the approximation (A1) for more general fields than $\mathbb{R}$ and for families of rings (called systems of Weierstrass according to [8]). These rings may seem to be exotic but their use allows us to handle algebraically the approximated solutions.

Let $\mathbf{R}$ be a real closed field and let $\mathbf{R}\llbracket t \rrbracket$ be a Weierstrass system over $\mathbf{R}$ (see [2] for precise definitions). Let $\mathbf{C} := \mathbf{R} + \sqrt{-1}\,\mathbf{R} = \mathbf{R}[X]/(X^2 + 1)$. Let $x := (x_1, \dots, x_p)$, $y := (y_1, \dots, y_p)$, $u := (u_1, \dots, u_q)$, and $v := (v_1, \dots, v_q)$, $z_i = x_i + \sqrt{-1}\,y_i$, $w_j = u_i + \sqrt{-1}\,v_j$. We have the following results.

Theorem 1.1 ([10]) *Let $f_1(x, y, u, v), \dots, f_l(x, y, u, v) \in \mathbf{R}\llbracket x, y, u, v \rrbracket$. Then the following system of equations satisfies the Artin approximation property.*

$$
\text{(S)} \qquad \begin{cases} f_i(x, y, u, v) = 0, \quad 1 \le i \le l, \\ \dfrac{\partial u_k}{\partial x_j} - \dfrac{\partial v_k}{\partial y_j} = 0, \quad 1 \le j \le p,\ 1 \le k \le q, \\ \dfrac{\partial v_k}{\partial x_j} + \dfrac{\partial u_k}{\partial y_j} = 0, \quad 1 \le j \le p,\ 1 \le k \le q. \end{cases}
$$

i.e. for any $\hat{u}_k(x, y), \hat{v}_k(x, y) \in \mathbf{R}[[x, y]]$, $1 \le k \le q$, solutions of the system (S) *and for any $c \in \mathbb{N}$, there exists $u_k(x, y), v_k(x, y) \in \mathbf{R}\llbracket x, y \rrbracket$, $1 \le k \le q$, solutions of the system* (S) *such that $\hat{u}_k - u_k, \hat{v}_k - v_k \in (x, y)^c$, $1 \le k \le q$.*

Theorem 1.2 *Let $\hat{u}_k(x, y), \hat{v}_k(x, y) \in \mathbf{R}[[x, y]]$, $1 \le k \le q$, solutions of the system* (S). *(In particular $\hat{u} + \sqrt{-1}\,\hat{v} \in \mathbf{C}[[z]]^q$). Then there exist $u_k(x, y, \alpha, \beta), v_k(x, y, \alpha, \beta) \in \mathbf{R}\llbracket x, y, \alpha, \beta \rrbracket$, $1 \le k \le q$ (with $\alpha := (\alpha_1, \dots, \alpha_s)$, $\beta := (\beta_1, \dots, \beta_s)$, $\lambda := (\lambda_1, \dots, \lambda_s)$, $\lambda_i := \alpha_i + \sqrt{-1}\,\beta_i$), solutions of the system* (S) *and satisfying the Cauchy–Riemann equations with respect to λ (hence $u + \sqrt{-1}\,v \in \mathbf{C}\llbracket z, \lambda \rrbracket^q$), and $\hat{\lambda}(z) \in \mathbf{C}[[z]]^s$ such that $(\hat{u}_k + \sqrt{-1}\,\hat{v}_k)(x, y) = (u_k + \sqrt{-1}\,v_k)(z, \hat{\lambda}(z))$, $1 \le k \le q$.*

Theorem 1.3 *There is an approximation function for* (S), i.e., *there exists a function $\beta \colon \mathbb{N} \to \mathbb{N}$ such that for all $c \in \mathbb{N}$ $\forall (u(x, y), v(x, y)) \in \mathbf{R}[[x, y]]^{2q}$ such that $f_i(x, y, u(x, y), v(x, y)) \in (x, y)^{\beta(c)+1}$, $1 \le i \le l$, and*

$$
\frac{\partial u_k}{\partial x_j}(x, y) - \frac{\partial v_k}{\partial y_j}(x, y) = \frac{\partial v_k}{\partial x_j}(x, y) + \frac{\partial u_k}{\partial y_j}(x, y) = 0,\ 1 \le j \le p,\ 1 \le k \le q,
$$

$\exists(\widetilde{u}(x,y),\widetilde{v}(x,y)) \in \mathbf{R}[\![x,y]\!]^{2q}$ *such that* $f(x,y,\widetilde{u}(x,y),\widetilde{v}(x,y))=0,$

$$\frac{\partial \widetilde{u}_k}{\partial x_j}(x,y)-\frac{\partial \widetilde{v}_k}{\partial y_j}(x,y)=\frac{\partial \widetilde{v}_k}{\partial x_j}(x,y)+\frac{\partial \widetilde{u}_k}{\partial y_j}(x,y)=0,\ 1\leq j\leq p,\ 1\leq k\leq q$$

and $\widetilde{u}_k-u_k,\ \widetilde{v}_k-v_k\in(x,y)^{c+1}$.

Let us remark that the last two conditions of the system (S) are equivalent to $u+\sqrt{-1}\,v\in\mathbf{C}[\![z]\!]^q$. Of course, the family of rings of convergent power series with real coefficients forms a Weierstrass system. Thus we obtain the analogues of (A2) and (A3) that complete the result of [10]. The proof of Theorem 1.1 is very similar to the one given in [10]. Theorem 1.2 can be deduced easily from [12] and the proof of Theorem 1.1. Finally the proof of Theorem 1.3 is given in Section 3.

2 Preliminaries

We give here a brief survey of the facts we use in the sequel. The interested reader may refer to [3, 5, 6, 8].

2.1 Real Closed Fields

Definition 2.1 An ordered field $(\mathbf{R},\leq)$ is a field $\mathbf{R}$ along with a total order on it satisfying:

- $x\leq y\Rightarrow\forall z\in\mathbf{R},\ x+z\leq y+z$;
- $0\leq x$ and $0\leq y\Rightarrow 0\leq xy$.

Thus 1, like any square, is positive and the characteristic of $\mathbf{R}$ is zero. A field $\mathbf{R}$ can be ordered if and only if it is real, *i.e.*, $\forall a_1,\ldots,a_p\in\mathbf{R}$, $a_1^2+a_2^2+\cdots+a_p^2=0\Rightarrow a_i=0$, $\forall i$. A real field $\mathbf{R}$ is closed if and only if it satisfies one of the following equivalent properties:

- $\mathbf{R}$ has no nontrivial algebraic extension that is a real field.
- $\mathbf{R}$ has an ordered field structure such that any positive element is a square and any polynomial of odd degree with coefficient in $\mathbf{R}$ has at least one root.
- $\mathbf{C}=\mathbf{R}[X]/(X^2+1)$ is algebraically closed.

In this case we denote by $\sqrt{-1}$ the image of X in the field $\mathbf{C}:=\mathbf{R}[X]/(X^2+1)$ and we have $\mathbf{C}=\mathbf{R}+\sqrt{-1}\,.\,\mathbf{R}$. If $z=a+\sqrt{-1}\,.b\in\mathbf{C}$, we denote by $\overline{z}=a-\sqrt{-1}\,.\,b$ its conjugate.

The field of real numbers $\mathbb{R}$ and the field $\mathbf{P}$ of Puiseux power series (formal or convergent) with real coefficients are real closed fields. The field $\mathbf{P}$ is equipped with an absolute value defined by $|s|=e^{-\mathrm{ord}(s)}$. Hence it allows us to speak about convergent power series with coefficients in $\mathbf{P}$.

2.2 Ultraproducts

A *filter* D (over $\mathbb{N}$) is a non empty subset of $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$ that satisfies the following properties:

$$\varnothing \notin D, \quad \mathcal{E}, \mathcal{F} \in D \Longrightarrow \mathcal{E} \cap \mathcal{F} \in D, \quad \mathcal{E} \in D, \mathcal{E} \subset \mathcal{F} \Longrightarrow \mathcal{F} \in D.$$

A filter D is *principal* if there exists $\mathcal{E} \in D$ such that $D = \{\mathcal{F} \mid \mathcal{E} \subset \mathcal{F}\}$. An *ultrafilter* is a filter which is maximal under inclusion. It is easy to check that a filter D is an ultrafilter if and only if for any $\mathcal{E} \subset \mathbb{N}$, $\mathcal{E} \in D$ or $\mathbb{N} - \mathcal{E} \in D$. In the same way an ultrafilter is non-principal if and only if it contains the filter $E := \{\mathcal{E} \subset \mathbb{N} \mid \mathbb{N} - \mathcal{E} \text{ is finite}\}$.

Let $(A_i)_{i\in\mathbb{N}}$ be a family of noetherian commutative rings. Let D be a non-principal ultrafilter. We define the *ultraproduct* $\mathrm{Ul}((A_i)_i)$ in the following way:

$$\mathrm{Ul}((A_i)_i) := \frac{\{(a_i)_{i\in\mathbb{N}} \in \prod_i A_i\}}{((a_i) \sim (b_i) \text{ if and only if } \{i \mid a_i = b_i\} \in D)}.$$

We have the following fundamental result.

Theorem 2.2 ([6]) *Let L be a first order language, let A_i, $i \in \mathbb{N}$ be structures for L, and let D be an ultrafilter over $\mathbb{N}$. Then for any $(a_i)_{i\in\mathbb{N}} \in \mathrm{Ul}((A_i))$ and for any first order formula $\varphi(x)$, $\varphi((a_i))$ is true in $\mathrm{Ul}((A_i))$ if and only if $\{i \in \mathbb{N} \mid \varphi(a_i)$ is true in $A\} \in D$.*

In particular we can deduce the following properties. The ultraproduct $\mathrm{Ul}((A_i))$ is equipped with a commutative ring structure. If every A_i is a field, then $\mathrm{Ul}((A_i))$ is a field. If every A_i is a local ring with maximal ideal $\mathfrak{m}_i$, then $\mathrm{Ul}((A_i))$ is a local ring with maximal ideal $\mathrm{Ul}((\mathfrak{m}_i))$ defined by $(a_i)_i \in \mathrm{Ul}((\mathfrak{m}_i))$ if and only if $\{i \mid a_i \in \mathfrak{m}_i\} \in D$. Elementary proofs of these results are given in [3]. In the same way, if every A_i is a real closed field, then $\mathrm{Ul}((A_i))$ is a real closed field.

2.3 Weierstrass Systems

In the following $\mathbb{k}$ denotes a field of characteristic zero.

Definition 2.3 ([8]) A *Weierstrass system of* $\mathbb{k}$*-algebras* or a *W-system* over $\mathbb{k}$ is a family of $\mathbb{k}$-algebras $\mathbb{k}\llbracket x_1, \dots, x_n \rrbracket$, $n \in \mathbb{N}$ such that:

- For $n = 0$, the $\mathbb{k}$-algebra is $\mathbb{k}$. For $n \geq 1$, $\mathbb{k}[x_1, \dots, x_n]_{(x_1,\dots,x_n)} \subset \mathbb{k}\llbracket x_1, \dots, x_n \rrbracket \subset \mathbb{k}[[x_1, \dots, x_n]]$ and $\mathbb{k}\llbracket x_1, \dots, x_{n+m} \rrbracket \cap \mathbb{k}[[x_1, \dots, x_n]] = \mathbb{k}\llbracket x_1, \dots, x_n \rrbracket$ for $m \in \mathbb{N}$. For any permutation σ of $\{1, \dots, n\}$, $\mathbb{k}\llbracket x_{\sigma(1)}, \dots, x_{\sigma(n)} \rrbracket = \mathbb{k}\llbracket x_1, \dots, x_n \rrbracket$.
- Any element of $\mathbb{k}\llbracket x \rrbracket$, $x = (x_1, \dots, x_n)$, invertible in $\mathbb{k}[[x]]$ is invertible in $\mathbb{k}\llbracket x \rrbracket$.
- Let $f \in (x)\mathbb{k}\llbracket x \rrbracket$ such that $f(0, \dots, 0, x_n) \neq 0$. Let $d := \mathrm{ord}_{x_n} f(0, \dots, 0, x_n)$. Then for any $g \in \mathbb{k}\llbracket x \rrbracket$ there exist a unique $q \in \mathbb{k}\llbracket x \rrbracket$ and a unique $r \in \mathbb{k}\llbracket x_1, \dots, x_{n-1} \rrbracket[x_n]$ with $\deg_{x_n} r < d$ such that $g = qf + r$.

The reader may refer to [8] for properties of these rings. In particular they satisfy the Artin approximation property (see [8, Theorem 1.1]).

Example 2.4 The family of rings of formal power series with coefficients in $\Bbbk$ that are algebraic over the rings of polynomials forms a Weierstrass system over $\Bbbk$. If $\Bbbk$ is a valued field, the family of rings of convergent power series with coefficients in $\Bbbk$ forms a Weierstrass system over $\Bbbk$.

Remark 2.5 If $\Bbbk[\![x]\!]$ is a Weierstrass system, then it is stable under derivation, *i.e.,* if $f \in \Bbbk[\![x_1, \ldots, x_n]\!]$, then $\partial f / \partial x_i \in \Bbbk[\![x_1, \ldots, x_n]\!]$ (see [8, Remark 1.3.3]).

3 Proofs of the Results

3.1 Proof of Theorem 1.1

The proof is very similar to the one given in [10]. Let us write $f_i(x, y, u, v)$ as

$$\rho_i(z, \bar{z}, w, \overline{w}) \in \mathbf{C}[\![z, \bar{z}, w, \overline{w}]\!]$$

with $z = x + \sqrt{-1}\, y$ and $w = u + \sqrt{-1}\, v$. Let $\hat{\zeta} = (\hat{u}(x, y), \hat{v}(x, y))$ be a formally holomorphic solution of (S). Let $\hat{w} = \hat{u} + \sqrt{-1}\, \hat{v}$. Then $\hat{\zeta}$ is a formally holomorphic solution of (S) if and only if $\hat{w} \in \mathbf{C}[[z]]^q$ (and not only $\hat{w} \in \mathbf{C}[[z, \bar{z}]]^q$) and $\rho_i(z, \bar{z}, \hat{w}(z), \overline{\hat{w}(z)}) = 0, i = 1, \ldots, l$. We define the morphisms

$$\hat{\zeta}^* \colon \mathbf{C}[\![z, \bar{z}, w, \overline{w}]\!] \longrightarrow \mathbf{C}[[z, \bar{z}]], \quad \hat{w}^* \colon \mathbf{C}[\![z, w]\!] \longrightarrow \mathbf{C}[[z]],$$

respectively by

$$\hat{\zeta}^*(h(z, \bar{z}, w, \overline{w})) = h(z, \bar{z}, \hat{w}(z), \overline{\hat{w}(z)}), \quad \hat{w}^*(s(z, w)) = s(z, \hat{w}(z)).$$

We denote by $\mathrm{Ker}(\hat{\zeta}^*)$ and $\mathrm{Ker}(\hat{w}^*)$ the respective kernels. As in [10], in order to obtain Theorem 1.1 it is enough to show the following claim.

Claim 1

$$\mathrm{Ker}(\hat{\zeta}^*) = \mathrm{Ker}(\hat{w}^*) \,.\, \mathbf{C}[\![z, \bar{z}, w, \overline{w}]\!] + \overline{\mathrm{Ker}(\hat{w}^*)} \,.\, \mathbf{C}[\![z, \bar{z}, w, \overline{w}]\!].$$

Indeed $\mathrm{Ker}(\hat{w}^*)$ (as any ideal of $\mathbf{C}[\![z, w]\!]$) satisfies the Artin approximation theorem (the analogue of (A1)) [3,8]. We deduce from this that for all $c \in \mathbb{N}$, there exists $w^c \in \mathbf{C}[\![z]\!]^q$ such that $\hat{w} - w^c \in (z)^{c+1}\mathbf{C}[[z]]^q$ and $s(z, w^c(z)) = 0, \forall s \in \mathrm{Ker}(\hat{w}^*)$. According to Claim 1, we have $\rho(z, \bar{z}, w^c(z), \overline{w^c(z)}) = 0$, for all $\rho \in \mathrm{Ker}(\hat{\zeta}^*)$. Hence we get the conclusion since the ρ_j's are in this last kernel.

Proof of Claim 1 First let us assume that $\hat{w}^*$ is injective and let us show that $\hat{\zeta}^*$ is also injective. Even if we add new variables w_i and we write $\hat{w}_i(z) = z_i$ for them, we may consider an element $F \in \mathbf{C}[\![w, \overline{w}]\!]$ such that $F(\hat{w}(z), \overline{\hat{w}(z)}) = 0$ and show that F is also zero. In order to do this, let us write $F(w, \overline{w}) = \Sigma_{\beta \in \mathbb{N}^n} F_\beta(w).\overline{w}^\beta$ in $\mathbf{C}[[w, \overline{w}]]$. Since $F_\beta(w) = 1/\beta! \frac{\partial F}{\partial \overline{w}^\beta}(w, 0)$ and $\mathbf{C}[\![w, \overline{w}]\!]$ is closed under derivation (Remark 2.5), we have $F_\beta(w) \in \mathbf{C}[\![w]\!]$. Let us denote $F(w, \overline{\hat{w}(z)}) = \Sigma_{\alpha \in \mathbb{N}^n} a_\alpha(w) \,.\, \bar{z}^\alpha$ in $\mathbf{C}[\![w, \bar{z}]\!]$. Now it is easy to check that $a_\alpha(w)$ is a finite linear combination of some $F_\beta(w)$. Thus every a_α is in $\mathbf{C}[\![w]\!]$. Hence, since $F(\hat{w}(z), \overline{\hat{w}(z)}) = 0$, we have $\forall \alpha \in \mathbb{N}^n$,

$a_\alpha(\hat{w}(z)) = 0$ and thus $a_\alpha \in \mathrm{Ker}(\hat{w}^*) = (0)$. Hence, $F(w, \overline{\hat{w}(z)}) = 0$. Let us now write $F(w, \overline{w}) = \Sigma_{\beta \in \mathbb{N}^n} G_\beta(\overline{w})\,.\,w^\beta$ in $\mathbf{C}[[w, \overline{w}]]$. Still, because $\mathbf{C}\llbracket w, \overline{w}\rrbracket$ is stable under derivation, we have $G_\beta(\overline{w}) \in \mathbf{C}\llbracket \overline{w}\rrbracket$. Since $F(w, \overline{\hat{w}(z)}) = 0$, we have $G_\beta(\overline{\hat{w}(z)}) = 0$. Thus $\overline{G_\beta}(w) \in \mathrm{Ker}(\hat{w}^*) = (0)$. Hence $\overline{G_\beta}(w) = 0$ and then $G_\beta(\overline{w}) = 0$ by conjugation. Finally we have $F = 0$ and $\mathrm{Ker}(\hat{\zeta}^*) = (0)$.

In order to prove the equality in Claim 1in the general case, we shall need the following lemma (it is on this point that our proof is different from the one given in [10]).

Lemma 3.1 *Let $\mathbb{k}$ be a field and $t = (t_1, \ldots, t_n)$, $s = (s_1, \ldots s_m)$ be indeterminates over $\mathbb{k}$,* i.e., *$t_1, \ldots, t_n, s_1, \ldots, s_m$ are $\mathbb{k}$-algebraically independent.*

(i) *Let I be an ideal of $\mathbb{k}\llbracket t\rrbracket$. Then I is prime if and only if $I\,.\,\mathbb{k}[[t]]$ is prime.*

(ii) *Let I (resp. J) be a prime ideal of $\mathbb{k}\llbracket t\rrbracket$ (resp. of $\mathbb{k}\llbracket s\rrbracket$) and assume that $\mathbb{k}$ is an algebraically closed field of characteristic zero. Then the ideal $I\,.\,\mathbb{k}\llbracket t, s\rrbracket + J\,.\,\mathbb{k}\llbracket t, s\rrbracket$ is a prime ideal of $\mathbb{k}\llbracket t, s\rrbracket$.*

Proof We remark that $\mathbb{k}\llbracket t\rrbracket$ is a noetherian local ring whose completion is $\mathbb{k}[[t]]$ [8].

(i) The implication $I\,.\,\mathbb{k}[[x]]$ prime $\Rightarrow I$ prime follows from the previous assertion by faithful flatness. The reverse implication follows from the approximation property (the analogue of (A1) between $\mathbb{k}\llbracket t\rrbracket$ and $\mathbb{k}[[t]]$.

(ii) Let I and J be as in (ii). Then the ideals $I\,.\,\mathbb{k}[[t]]$ and $J\,.\,\mathbb{k}[[s]]$ are prime by (i). From this we deduce that $I\,.\,\mathbb{k}[[s, t]] + J\,.\,\mathbb{k}[[s, t]]$ is prime (according to a classical property of formal power series rings; see [7, Proposition 12a]), because the variables s, t are $\mathbb{k}$-algebraically independent. Now $I\,.\,\mathbb{k}\llbracket t, s\rrbracket + J\,.\,\mathbb{k}\llbracket t, s\rrbracket$ is prime according to (i). ■

Let us come back to the proof of Claim 1. The ideal $\mathrm{Ker}(\hat{w}^*)$ of $\mathbf{C}\llbracket w\rrbracket$ is prime. Since the conjugation sends $\mathbf{C}\llbracket w\rrbracket$ isomorphically onto $\mathbf{C}\llbracket \overline{w}\rrbracket$, $\overline{\mathrm{Ker}(\hat{w}^*)}$ is a prime ideal of $\mathbf{C}\llbracket \overline{w}\rrbracket$. Thus, according to the previous lemma,

$$\mathrm{Ker}(\hat{w}^*)\,.\,\mathbf{C}\llbracket w, \overline{w}\rrbracket + \overline{\mathrm{Ker}(\hat{w}^*)}\,.\,\mathbf{C}\llbracket w, \overline{w}\rrbracket$$

is prime. Obviously, $\mathrm{Ker}(\hat{\zeta}^*)$ is prime and

$$\mathrm{Ker}(\hat{\zeta}^*) \supseteq \mathrm{Ker}(\hat{w}^*)\,.\,\mathbf{C}\llbracket z, \overline{z}, w, \overline{w}\rrbracket + \overline{\mathrm{Ker}(\hat{w}^*)}\,.\,\mathbf{C}\llbracket z, \overline{z}, w, \overline{w}\rrbracket.$$

In order to show the equality, it is enough to show the equality of the heights of both ideals.

Let $q = \mathrm{Dim}(\mathbf{C}\llbracket w\rrbracket / \mathrm{Ker}(\hat{w}^*))$. The Noether normalisation theorem and the preparation and division theorems apply to the Weierstrass systems [8]. Thus, after a linear change of variables, we may assume that the morphism

$$\Pi_1^* \colon \mathbf{C}\llbracket w_1, \ldots, w_q\rrbracket \to \mathbf{C}\llbracket w\rrbracket / \mathrm{Ker}(\hat{w}^*)$$

(constructed from the canonical injection $\mathbf{C}\llbracket w_1, \ldots, w_q\rrbracket \longrightarrow \mathbf{C}\llbracket w_1, \ldots, w_n\rrbracket$) is injective and finite. Moreover, there exist distinguished polynomials,

$$P_1(w_n), \ldots, P_{n-q}(w_{q+1}),$$

with $P_i(w_{n-i+1}) \in \mathbf{C}[\![w_1,\dots,w_{n-i}]\!][w_{n-i+1}]$, $1 \leq i \leq n-q$, such that $P_i \in \operatorname{Ker}(\hat{w}^*)$. A repeated use of the division theorem with the polynomials $P_1,\dots,P_{n-q}$, $\overline{P}_1,\dots,\overline{P}_{n-q}$ assures that the morphism

$$\Pi_2^*\colon \mathbf{C}[\![w_1,\dots,w_q,\overline{w}_1,\dots,\overline{w}_q]\!] \longrightarrow \mathbf{C}[\![w,\overline{w}]\!]/(\operatorname{Ker}(\hat{w}^*),\overline{\operatorname{Ker}(\hat{w}^*)})$$

is finite. Thus we have $\operatorname{Dim}(\mathbf{C}[\![w,\overline{w}]\!]/(\operatorname{Ker}(\hat{w}^*),\overline{\operatorname{Ker}(\hat{w}^*)})) \leq 2q$. Let us denote $\Pi_3^*\colon \mathbf{C}[\![w_1,\dots,w_q,\overline{w}_1,\dots,\overline{w}_q]\!] \to \mathbf{C}[\![w,\overline{w}]\!]/\operatorname{Ker}(\hat{\zeta}^*)$. We have $\Pi_3^* = \Pi^* \circ \Pi_2^*$, where Π^* is the canonical surjection of $\mathbf{C}[\![w,\overline{w}]\!]/(\operatorname{Ker}(\hat{w}^*),\overline{\operatorname{Ker}(\hat{w}^*)})$ onto $\mathbf{C}[\![w,\overline{w}]\!]/\operatorname{Ker}(\hat{\zeta}^*)$. The morphism Π_3^* is finite since Π_2^* is finite. Moreover Π_3^* is injective because, since $\operatorname{Ker}(\hat{w}^*) \cap \mathbf{C}[\![w_1,\dots,w_q]\!] = (0)$, we have $\operatorname{Ker}(\hat{\zeta}^*) \cap \mathbf{C}[\![w_1,\dots,w_q,\overline{w}_1,\dots,\overline{w}_q]\!] = (0)$, according to the first case we proved. Hence,

$$\operatorname{Dim}(\mathbf{C}[\![w,\overline{w}]\!]/\operatorname{Ker}(\hat{\zeta}^*)) = 2q = \operatorname{Dim}(\mathbf{C}[\![w,\overline{w}]\!]/(\operatorname{Ker}(\hat{w}^*),\overline{\operatorname{Ker}(\hat{w}^*)})). \qquad \blacksquare$$

3.2 Proof of Theorem 1.2

According to Claim 1 of the proof of Theorem 1.1, in order to prove Theorem 1.2, we only need to prove the analogue of (A2) for Weierstrass systems. The proof is really the same as the one given in [12], replacing [1] by the approximation theorem of [8]. We do not give the proof here in order to be concise.

3.3 Proof of Theorem 1.3

We will deduce Theorem 1.3 from Theorem 1.1 fully used in the situation of general Weierstrass systems. Let us remark that the existence of an approximation function for $f_1,\dots,f_l$ is equivalent to the existence of an approximation function for $f_1^2+\cdots+f_l^2$. Thus we may assume that $l=1$ and we denote f_1 by f.

Let us assume that the result is false: there exists $c \in \mathbb{N}$ such that forall $j \in \mathbb{N}$ there exists $(\hat{u}^{(j)},\ \hat{v}^{(j)}) \in \mathbf{R}[\![x,y]\!]^{2q}$ such that

$$\begin{cases} f(x,y,\hat{u}^{(j)},\hat{v}^{(j)}) \in (x,y)^{j+1}, \\ (\hat{u}^{(j)} + \sqrt{-1}\,\hat{v}^{(j)}) \in \mathbf{C}[[z]]^q, \end{cases}$$

and there does not exist $(u,v) \in \mathbf{R}[[x,y]]^{2q}$ such that

$$\begin{cases} f(x,y,u,v) = 0, \\ (u + \sqrt{-1}\,v) \in \mathbf{C}[[z]]^q, \\ \hat{u}_k^{(j)} - u_k,\ \hat{v}_k^{(j)} - v_k \in (x,y)^c. \end{cases}$$

Notation Let D be a non-principal ultrafilter over $\mathbb{N}$. Let us denote by $\operatorname{Ul}(A)$ the ultraproduct $(\prod_{n\in\mathbb{N}} A)/D$ for any local ring A. Let us denote by $\operatorname{Ul}(\mathfrak{m})$ the maximal ideal of $\operatorname{Ul}(A)$ and $\operatorname{Ul}(A)_{\text{sep}}$ the quotient of $\operatorname{Ul}(A)$ by the intersection of the powers of its maximal ideal.

By assumption, if we denote by $\hat{u}$ and $\hat{v}$ the images of $(\hat{u}^{(j)})_{j\in\mathbb{N}}$ and $(\hat{v}^{(j)})_{j\in\mathbb{N}}$ in $\mathrm{Ul}(\mathbf{R}[[x,y]])$, we have $f(x,y,\hat{u},\hat{v})\in \mathrm{Ul}(\mathfrak{m})^j$ for all $j\in\mathbb{N}$. Let $\pi\colon \mathrm{Ul}(\mathbf{R}[[x,y]])\to \mathrm{Ul}(\mathbf{R}[[x,y]])_{\mathrm{sep}}$ be the quotient morphism. The morphism π restricted to $\mathrm{Ul}(\mathbf{R})[x,y]$ is injective and $\mathrm{Ul}(\mathbf{R}[[x,y]])_{\mathrm{sep}}$ is isomorphic to $\mathrm{Ul}(\mathbf{R})[[x,y]]$ as $\mathrm{Ul}(\mathbf{R})[x,y]$-algebra (see [3, Lemma 3.4]). We have $\pi(f(x,y,\hat{u},\hat{v}))=0$, thus $f(x,y,\pi(\hat{u}),\pi(\hat{v}))=0$ because π is continuous with respect to the Krull topology.

Moreover, we also denote by π the quotient morphism

$$\pi\colon\ \mathrm{Ul}(\mathbf{C}[[z,\overline{z}]])\to \mathrm{Ul}(\mathbf{C}[[z,\overline{z}]])_{\mathrm{sep}}\simeq \mathrm{Ul}(\mathbf{C})[[z,\overline{z}]].$$

Then we have $\pi(\hat{u})+\sqrt{-1}\,\pi(\hat{v})\in \mathrm{Ul}(\mathbf{C})[[z]]$ according to the assumption on $(\hat{u}^{(j)},\hat{v}^{(j)})_{j\in\mathbb{N}}$. For any field $\mathbb{K}$, let us denote

$$\mathrm{Ul}(\mathbb{K})[\![X_1,\ldots,X_n]\!] := \Big\{ g(X_1,\ldots,X_n)\in \mathrm{Ul}(\mathbb{K})[[X]]\ \Big|\ \exists s\in\mathbb{N},$$
$$t=(t_1,\ldots,t_s), g(t,X)\in\mathbb{K}[t][[X]] \text{ and } \widetilde{t}\in \mathrm{Ul}(\mathbb{K})^s \text{ such that } g(x)=g(\widetilde{t},x)\Big\}.$$

According to [8, Lemma 7.2], the family $\mathrm{Ul}(\mathbb{K})[\![X_1,\ldots,X_n]\!]$ forms a Weierstrass system over $\mathrm{Ul}(\mathbb{K})$. Hence, according to Theorem 1.1, there exists $(\widetilde{u},\widetilde{v})\in \mathrm{Ul}(\mathbf{R})[\![x,y]\!]^{2q}$ such that

$$\begin{cases} f(x,y,\widetilde{u},\widetilde{v})=0,\\ \widetilde{u}+\sqrt{-1}\,\widetilde{v}\in \mathrm{Ul}(\mathbf{C})[\![z]\!]\end{cases}$$

We may write

$$\widetilde{u}_k=\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} a_{k,\alpha,\beta}(\widetilde{t})x^\alpha y^\beta \ \text{ and }\ \widetilde{v}_k=\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} b_{k,\alpha,\beta}(\widetilde{t})x^\alpha y^\beta,\ \ 1\le k\le q$$

with $a_{k,\alpha,\beta}(t), b_{k,\alpha,\beta}(t)\in\mathbf{R}[t_1,\ldots,t_s]$ and $\widetilde{t}\in \mathrm{Ul}(\mathbf{R})^s$. Then we have

$$f\Big(x,y,\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} a_{k,\alpha,\beta}(t)x^\alpha y^\beta,\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} b_{k,\alpha,\beta}(t)x^\alpha y^\beta\Big)=\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} c_{\alpha,\beta}(t)x^\alpha y^\beta,$$

where $c_{\alpha,\beta}(t)\in\mathbf{R}[t]$ for any $(\alpha,\beta)\in\mathbb{N}^{2p}$. By noetherianity, the ideal of $\mathbf{R}[t]$ generated by $c_{\alpha,\beta}(t)$, $(\alpha,\beta)\in\mathbb{N}^{2p}$, is generated by a finite number of them. Let us assume that it is generated by $c_{\alpha,\beta}(t)$ for $(\alpha,\beta)\in I$, where I is a finite subset of $\mathbb{N}^{2q}$. By assumption the system of equations $c_{\alpha,\beta}(t)=0$, $(\alpha,\beta)\in I$, has a solution $\widetilde{t}\in \mathrm{Ul}(\mathbf{R})^s$. We may write $\widetilde{t}=(\widetilde{t}^{(j)})_{j\in\mathbb{N}}$, $\widetilde{t}^{(j)}\in\mathbf{R}^s$ for any $j\in\mathbb{N}$. Thus there exists $\mathcal{E}_1\in D$ such that $c_{\alpha,\beta}(\widetilde{t}^{(j)})=0$, for every $(\alpha,\beta)\in I$, $j\in\mathcal{E}_1$. By definition of I, $c_{\alpha,\beta}(\widetilde{t}^{(j)})=0$, for every $(\alpha,\beta)\in\mathbb{N}^{2p}$, $j\in\mathcal{E}_1$

On the other hand,

$$\widetilde{u}+\sqrt{-1}\,\widetilde{v}=\sum_{(\alpha,\beta)\in\mathbb{N}^{2p}}(a_{\alpha,\beta}(\widetilde{t})+\sqrt{-1}\,b_{\alpha,\beta}(\widetilde{t}))x^\alpha y^\beta\in \mathrm{Ul}(\mathbf{C})[\![z]\!].$$

Since $\mathrm{Ul}(\mathbf{C})[\![z]\!] = \mathrm{Ul}(\mathbf{R})[\![z]\!] + \sqrt{-1}\ \mathrm{Ul}(\mathbf{R})[\![z]\!]$, we may write

$$\widetilde{u} + \sqrt{-1}\,\widetilde{v} = \sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} (d_{\alpha,\beta}(\widetilde{t}) + \sqrt{-1}\,e_{\alpha,\beta}(\widetilde{t}))z^{\alpha}\overline{z}^{\beta},$$

with $d_{\alpha,\beta}(t)$, $c_{\alpha,\beta}(t) \in \mathbf{R}[t]$, for all $(\alpha,\beta) \in \mathbb{N}^{2p}$. The ideal of $\mathbf{R}[t]$ generated by $d_{\alpha,\beta}(t)$ (resp. $e_{\alpha,\beta}(t)$), $(\alpha,\beta) \in \mathbb{N}^{2p}$ and $\beta \neq 0$, is finitely generated. Thus there exists a finite set J such that $d_{\alpha,\beta}(t)$ (resp. $e_{\alpha,\beta}(t)$), $(\alpha,\beta) \in J$, generate this ideal.

By assumption the system of equations $d_{\alpha,\beta}(t) = e_{\alpha,\beta}(t) = 0$, $(\alpha,\beta) \in J$, has a solution $\widetilde{t} \in \mathrm{Ul}(\mathbf{R})^s$. Thus there exists $\mathcal{E}_2 \in D$ such that $d_{\alpha,\beta}(\widetilde{t}^{(j)}) = e_{\alpha,\beta}(\widetilde{t}^{(j)}) = 0$ for every $j \in \mathcal{E}_2$ and $\beta \neq 0$.

Then we denote $t^{(j)} := \widetilde{t}^{(j)} \in \mathbf{R}$ for any $j \in \mathcal{E}_1 \cap \mathcal{E}_2 \in D$ and $t^{(j)} = 0$, otherwise. Let us denote

$$u_k^{(j)} := \sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} a_{k,\alpha,\beta}(t^{(j)})x^{\alpha}y^{\beta},$$

$$v_k^{(j)} := \sum_{(\alpha,\beta)\in\mathbb{N}^{2p}} b_{k,\alpha,\beta}(t^{(j)})x^{\alpha}y^{\beta},\ \ 1 \le k \le q,$$

and let us denote by u_k and v_k the images of these two sequences in $\mathrm{Ul}(\mathbf{R}[\![x,y]\!])$. Then we see that

$$\begin{cases} f(x,y,u,v) = 0, \\ u_k, v_k \in \mathrm{Ul}(\mathbf{R}[\![x,y]\!]), \\ u_k + \sqrt{-1}\,v_k \in \mathrm{Ul}(\mathbf{C}[\![z]\!]),\ 1 \le k \le q \end{cases}$$

Moreover $u_k - \hat{u}_k^{(j)}, v_k - \hat{v}_k^{(j)} \in (x,y)^c$, which is a contradiction.

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