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The minimal cone of an algebraic Laurent series

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Abstract
We study the algebraic closure of $\mathbb{K}((x))$, the field of power series in several indeterminates over a field $\mathbb{K}$. In characteristic zero we show that the elements algebraic over $\mathbb{K}((x))$ can be expressed as Puiseux series such that the convex hull of its support is essentially a polyhedral rational cone, strengthening the known results. In positive characteristic we construct algebraic closed fields containing the field of power series and we give examples showing that the results proved in characteristic zero are no longer valid in positive characteristic.

Keywords Power series rings · Support of a Laurent series · Algebraic closure · Orders on a lattice · Henselian valued fields

Mathematics Subject Classification 05E40 · 06A05 · 11J81 · 12J99 · 13F25 · 14B05 · 32B10

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1 Introduction

When $\mathbb{K}$ is a field and $x = (x_1, \ldots, x_n)$ is a vector of n indeterminates, we denote by $\mathbb{K}(\!(x)\!)$ the field of formal power series in n indeterminates. The problem we are studying here concerns the determination of an algebraic closure of $\mathbb{K}(\!(x)\!)$ when $\mathbb{K}$ is an algebraically closed field of any characteristic.

Let us begin with the characteristic zero case. When $n = 1$, the Newton–Puiseux Theorem asserts that the elements that are algebraic over $\mathbb{K}(\!(x)\!)$ are the Puiseux series, i.e. the formal sums of the form $\sum_{k=k_0}^{\infty} a_k x^{k/q}$ for some positive integer q (cf. [26] and [27]).

When $n \geq 2$ there is no known description of the algebraic closure of $\mathbb{K}(\!(x)\!)$. The Abhyankar–Jung Theorem asserts that the roots of a monic polynomial with coefficients in $\mathbb{K}[[x]]$ whose discriminant is a monomial times a unit are Puiseux series (cf. [1,18,21] or [25]). But, in general, polynomials with coefficients in $\mathbb{K}[[x]]$ may not have Puiseux series as roots, as the polynomial $T^2 - (x_1 + x_2)$. Nevertheless, a result of MacDonald asserts that we may express the elements algebraic over $\mathbb{K}(\!(x)\!)$ as Laurent Puiseux series [22]. In order to explain this result let us introduce some terminology.

A *(generalized) series* ξ (with support in $\mathbb{Q}^n$ and coefficients in a field $\mathbb{K}$) is a formal sum $\xi = \sum_{\alpha \in \mathbb{Q}^n} \xi_\alpha x^\alpha$, where $x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and the $\xi_\alpha \in \mathbb{K}$. Its *support* is the set

$$\operatorname{Supp}(\xi) := \{\alpha \in \mathbb{Q}^n | \xi_\alpha \neq 0\}.$$

Such a series is called a *Laurent series* (resp. *Laurent Puiseux series*) if $\operatorname{Supp}(\xi) \subset \mathbb{Z}^n$ (resp. $\operatorname{Supp}(\xi) \subset \frac{1}{k}\mathbb{Z}^n$ for some $k \in \mathbb{N}^*$).

The set of generalized series is a commutative group as we can define the sum of two power series in the usual way. But in general the product of two such series is not well defined. To insure the existence of the product of two generalized series, one has to impose that their support is well-ordered for a total order on $\mathbb{Q}^n$ (see [29] for example). This is the case for example when we consider Laurent series whose supports are included in the translation of a given common strongly convex cone (for example see [3] or [6, Lemma 3.8]). Here, a *strongly convex cone* is a cone that does not contain non-trivial linear subspaces. In particular, for a series ξ whose support is included in a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$, and for $P(x, T) \in \mathbb{K}[[x]][T]$, $P(x, \xi)$ is well defined.

We also recall that a *rational cone* is a finitely generated submonoid of $\mathbb{R}^n$ that is generated by vectors with integer coordinates. Then, MacDonald's Theorem (cf [22, Theorem 3.6]—see also [5]) asserts that the elements that are algebraic over $\mathbb{K}(\!(x)\!)$ can be expressed as Puiseux series with support in the translation of a strongly convex rational cone σ. Moreover MacDonald showed that, for any given $\omega \in {\mathbb{R}_{>0}}^n$ whose coordinates are $\mathbb{Q}$-linearly independent, σ can be chosen in such a way that

$$\forall s \in \sigma \setminus \{\underline{0}\}, \quad s \cdot \omega > 0. \tag{1}$$

Let us remark that, for $q \in \mathbb{N}^*$, a Laurent series $\xi(x_1, \ldots, x_n)$ is algebraic over $\mathbb{K}(\!(x)\!)$ if and only if $\xi(x_1^{1/q}, \ldots, x_n^{1/q})$ is algebraic over $\mathbb{K}(\!(x)\!)$. Therefore, in order to determine an algebraic closure of $\mathbb{K}(\!(x)\!)$ one only needs to determine which are the Laurent series ξ whose support is included in the translation of a rational strongly convex cone σ that are algebraic over $\mathbb{K}(\!(x)\!)$. And by the result of MacDonald, if we fix $\omega \in \mathbb{R}_{>0}{}^n$ whose coordinates are $\mathbb{Q}$-linearly independent, we may even assume that σ satisfies (1).

For such a ω we define the monomial valuation ν_ω in the following way: for $f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha$, we set $\nu_\omega(f) := \min\{\alpha\cdot\omega | f_\alpha \neq 0\}$. This valuation defines a norm $\|\cdot\|_\omega$ on $\mathbb{K}(\!(x)\!)$ by

$$\|f/g\|_\omega := e^{-\nu_\omega(f)+\nu_\omega(g)}.$$

We denote by $\mathbb{L}^\omega$ the completion of $\mathbb{K}(\!(x)\!)$ with respect to $\|\cdot\|_\omega$. Then, we remark that a Laurent series whose support is included in the translation of a cone σ satisfying (1), is necessarily in $\mathbb{L}^\omega$. Therefore in order to determine an algebraic closure of $\mathbb{K}(\!(x)\!)$ one only needs to determine the algebraic closure of $\mathbb{K}(\!(x)\!)$ in $\mathbb{L}^\omega$, its completion for the norm $\|\cdot\|_\omega$. Passing through the completion of a field $\Bbbk$ in order to understand its algebraic closure is a classical process that appears at least in two important situations:

(1) When we want to understand the algebraic closure of $\mathbb{Q}$, we equip $\mathbb{Q}$ with the usual absolute value, and study the algebraic elements of $\mathbb{R}$, its completion, over $\mathbb{Q}$. Indeed the field extension of $\mathbb{R}$ into its algebraic closure $\mathbb{R} \longrightarrow \mathbb{C}$ is the most simple one.
(2) When we want to understand the algebraic closure of $\mathbb{C}(x_1)$, the field of rational functions in one variable, we equip $\mathbb{C}(x_1)$ with the norm $\|\cdot\|$ defined by

$$\forall p, q \in \mathbb{C}[x_1], \quad \|p/q\| := e^{-\mathrm{ord}(p)+\mathrm{ord}(q)}$$

and we study the algebraic closure of $\mathbb{C}(x_1)$ into its completion $\mathbb{C}(\!(x_1)\!)$. Indeed, by the Newton–Puiseux Theorem, the field extension of $\mathbb{C}(\!(x_1)\!)$ into its algebraic closure, the field of Puiseux series, is well described.

It is fascinating that there are similar results between these situations in spite of the fact that the technics used to prove them are quite different. For instance, there is an analogue of the Liouville diophantine approximation Theorem for the elements of $\mathbb{L}^\omega$ that are algebraic over $\mathbb{K}(\!(x)\!)$ (see [16,17,31]). There is also an analogue of Eisenstein's Theorem [13] for the elements of $\mathbb{L}^\omega$ that are algebraic over $\mathbb{K}(\!(x)\!)$ (see [32, Theorem 5.12]) and an analogue of Fabry's Theorem for the elements of $\mathbb{L}^\omega$ that are algebraic over $\mathbb{K}(\!(x)\!)$ (see [6, Theorem 6.4]).

In this paper we investigate necessary conditions for a Laurent series with support in a rational strongly convex cone to be algebraic over $\mathbb{K}(\!(x)\!)$ in any characteristic. We provide conditions in terms of the support of the series. Indeed in the case of the study of the algebraic closure of $\mathbb{C}(x_1)$ into $\mathbb{C}(\!(x_1)\!)$, or the algebraic closure of $\mathbb{K}(x_1)$ into $\mathbb{K}(\!(x_1)\!)$ for a general field $\mathbb{K}$, such conditions have been given, and some questions remain open (as the Dynamical Mordell-Lang Conjecture—cf. [8] or [7]).

In order to explain this we introduce the following definition:

Definition 1.1 Let ξ be a series with support in $\mathbb{Q}^n$ and coefficients in a field $\mathbb{K}$. We set

$$\tau(\xi) := \left\{\omega \in \mathbb{R}_{\geq 0}{}^n \,|\, \exists k \in \mathbb{R},\ \mathrm{Supp}(\xi) \cap \left\{u \in \mathbb{R}^n | u \cdot \omega \leq k\right\} = \emptyset\right\}.$$

For example, if $\mathrm{Supp}(\xi)$ is equal to a cone σ and every unbounded face of σ contains infinitely many elements of $\mathrm{Supp}(\xi)$, then $\tau(\xi)^\vee = \sigma$ (see Definition 2.1 for the dual of a cone). Let us mention that we restrict to vectors $\omega \in \mathbb{R}_{\geq 0}$ since, for a series ξ algebraic over $\mathbb{K}(\!(x)\!)$, $\xi + f(x)$ is algebraic over $\mathbb{K}(\!(x)\!)$ for any $f(x) \in \mathbb{K}[[x]]$.

Remark 1.2 It is straightforward to check that $\tau(\xi)$ is a (non necessarily polyhedral) convex cone (see Lemma 3.1).

Our first main result is that $\tau(\xi)$ is rational when ξ is algebraic over $\mathbb{K}(\!(x)\!)$:

Theorem 1.3 *Let ξ be a Laurent Puiseux series whose support is included in a translation of a strongly convex cone containing $\mathbb{R}_{\geq 0}{}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then the set $\tau(\xi)$ is a strongly convex rational cone.*

From the rationality of $\tau(\xi)$ we can deduce easily the following result:

Corollary 1.4 *Let ξ be a Laurent Puiseux series whose support is included in a translation of a strongly convex cone containing $\mathbb{R}_{\geq 0}{}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then there is $\gamma \in \mathbb{Z}^n$ such that*

$$\mathit{Supp}(\xi) \subset \gamma + \tau(\xi)^\vee.$$

Moreover $\tau(\xi)^\vee$ is the smallest (non necessarily polyhedral) cone having this property.

Now the question is to determine how far is the support of ξ of being equal to a set of the form $\gamma + \tau(\xi)^\vee$. The following result provides an answer to this question:

Theorem 1.5 *Let ξ be a Laurent Puiseux series whose support is included in a translation of a strongly convex cone containing $\mathbb{R}_{\geq 0}{}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then there exist a finite set $C \subset \mathbb{Z}^n$, a Laurent polynomial $p(x)$, and a power series $f(x) \in \mathbb{K}[[x]]$ such that*

$$\mathit{Supp}(\xi + p(x) + f(x)) \subset C + \tau(\xi)^\vee$$

and for every unbounded facet F of $\mathrm{Conv}(C + \tau(\xi)^\vee)$, *we have*

$$\#\left\{\mathit{Supp}(\xi + p(x) + f(x)) \cap F\right\} = +\infty.$$

We will see in Example 5.4 that, in general, the set C cannot be chosen to be one single point. We will also see in Example 5.3 that there is no minimal, maximal or canonical C satisfying Theorem 1.5.

We do not know if this statement can be extended to faces of $\tau(\xi)^\vee$ of smaller dimension. But we have the following result:

Theorem 1.6 *Let ξ be a Laurent Puiseux series whose support is included in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbb{K}$ of characteristic zero. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then, for every $u \in \mathbb{R}^n_{>0}$ in the boundary of $\tau(\xi)$ there exists a Laurent polynomial $p(x)$ such that, if F_u denotes the face defined by u of the convex hull of Supp$(\xi + p(x))$, then*

$$\#\left(F_u \cap Supp(\xi)\right) = +\infty.$$

Let us mention that the cone $\tau(\xi)$ was already considered in [6] where we were not able to prove its rationality and where we gave a very much weaker version of Theorem 1.6.

We will begin by the proof of Theorem 1.3. This proof is not very difficult once we have the right setting, and is essentially based on two tools: the compacity of the space of orders on ${\mathbb{R}_{\geq 0}}^n$, and the construction, for every order $\preceq$ on $\mathbb{Q}^n$, of an algebraically closed field $\mathcal{S}_{\preceq}^{\mathbb{K}}$ containing $\mathbb{K}(\!(x)\!)$. This result of compacity is due to Ewald and Ishida [14] (see also [34]) and is a purely topological result. It will allow us to have a decomposition of ${\mathbb{R}_{\geq 0}}^n$ into a union of finitely many rational strongly convex cones having the following property: for each order $\preceq$, the roots of the minimal polynomial of ξ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in the dual of one of these cones.

The construction of the algebraically closed fields $\mathcal{S}_{\preceq}^{\mathbb{K}}$ has been given in [6] and is based on systematic constructions of algebraically closed valued fields due to Rayner [28].

The proofs of Theorems 1.5 and 1.6 are much more involved. First they require the introduction of intermediate cones that we have to describe and compare with $\tau(\xi)$. Then we need to prove an extension of Dickson's Lemma for general rational cones (see Proposition 4.14) that will help us to show the existence of the finite set C of Theorem 1.6.

Finally we investigate the positive characteristic case. We begin by constructing algebraically closed fields containing $\mathbb{K}(\!(x)\!)$. Each of these fields depends on an order $\preceq$ on $\mathbb{Q}^n$, and their definition extends the definition of $\mathcal{S}_{\preceq}^{\mathbb{K}}$ to the case of a positive characteristic field $\mathbb{K}$. Then we provide several examples showing that Theorems 1.5 and 1.6 as long as Proposition 3.4, that is the key tool to prove Theorem 1.3, are no longer true in the positive characteristic case.

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2 Orders and algebraically closed fields containing $\mathbb{K}(\!(x)\!)$

In this section we introduce the tools needed for the proof of Theorem 1.3.

2.1 The space of orders on $\mathbb{R}_{\geq 0}{}^n$

Definition 2.1 Let us recall that a *cone* $\tau \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ such that for every $t \in \tau$ and $\lambda \geq 0$, $\lambda t \in \tau$. A cone $\tau \subset \mathbb{R}^n$ is *polyhedral* if it has the form

$$\tau = \{\lambda_1 u_1 + \cdots + \lambda_s u_s | \lambda_1, \ldots, \lambda_s \geq 0\}$$

for some given vectors $u_1, \ldots, u_s \in \mathbb{R}^n$. A cone is said to be a *rational cone* if it is polyhedral, and the u_i can be chosen in $\mathbb{Z}^n$.

A cone is *strongly convex* if it does not contain any non trivial linear subspace.

In practice, as almost all the cones that we consider in this paper are polyhedral cones, the term cone will always refer to polyhedral cones (unless stated otherwise).

The *dual* $\sigma^\vee$ of a cone σ is the cone given by

$$\sigma^\vee := \{v \in \mathbb{R}^n | v \cdot u \geq 0, \text{ for all } u \in \sigma\}$$

where $u \cdot v$ stands for the dot product $(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) := u_1 v_1 + \cdots + u_n v_n$.

Remark 2.2 Let ξ be a series and $\omega \in \tau(\xi)$. Then $\text{Supp}(\xi) \subset \gamma + \langle \omega \rangle^\vee$ for some $\gamma \in \mathbb{Z}^n$. Indeed it is enough to choose γ such that $\text{Supp}(\xi) \cap \{u \in \mathbb{R}^n | u \cdot \omega \leq \gamma \cdot \omega\} = \emptyset$.

Definition 2.3 A *preorder* on an abelian group G is a binary relation $\preceq$ such that

(i) $\forall u, v \in G, u \preceq v$ or $v \preceq u$,
(ii) $\forall u, v, w \in G, u \preceq v$ and $v \preceq w$ implies $u \preceq w$,
(iii) $\forall u, v, w \in G, u \preceq v$ implies $u + w \preceq v + w$,

The set of preorders on G is denoted by $\text{ZR}(G)$. The set of orders on G is a subset of $\text{ZR}(G)$ denoted by $\text{Ord}(G)$.

Theorem-Definition 2.4 By [30, Theorem 2.5] for every $\preceq \in \text{ZR}(\mathbb{Q}^n)$ there exist an integer $s \geq 0$ and orthogonal vectors $u_1, \ldots, u_s \in \mathbb{R}^n$ such that

$$\forall u, v \in \mathbb{Q}^n, \; u \preceq v \iff (u \cdot u_1, \ldots, u \cdot u_s) \leq_{\text{lex}} (v \cdot u_1, \ldots, v \cdot u_s).$$

For such a preorder we set $\preceq := \preceq_{(u_1, \ldots, u_s)}$. Such a preorder extends in an obvious way to a preorder on $\mathbb{R}^n$ and the preorders of this form are called *continuous preorders*.

We remark that the orthogonality condition is not essential as, if U_j denotes the linear subspace generated by $u_1, \ldots, u_{j-1}$, and v_j is chosen in $u_j + U_j$ for every $j \geq 2$, then $\preceq_{(u_1, \ldots, u_s)} = \preceq_{(u_1, v_2, \ldots, v_s)}$.

Definition 2.5 Let $A \subset \mathbb{R}^n$ and $\preceq$ be a continuous preorder on $\mathbb{R}^n$. We say that A is $\preceq$-positive if

$$\forall a \in A, \quad a \succeq \underline{0}.$$

Definition 2.6 Let $\preceq$ be a continuous preorder on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$. We say that A is $\preceq$-well-ordered if A is well-ordered with respect to $\preceq$.

Definition 2.7 The set of continuous orders $\preceq$ such that ${\mathbb{R}_{\geq 0}}^n$ is $\preceq$-positive is denoted by Ord_n.

Definition 2.8 Given two preorders $\preceq_1$ and $\preceq_2$, one says that $\preceq_2$ *refines* $\preceq_1$ if

$$\forall u, v \in \mathbb{R}^n,\ u \preceq_2 v \Longrightarrow u \preceq_1 v.$$

Remark 2.9 Let $(u_1, \ldots, u_s)$ be nonzero vectors of $\mathbb{R}^n$. Using Theorem-Definition 2.4 it is easy to check that for a preorder $\preceq$, $\preceq$ refines $\leq_{(u_1,\ldots,u_s)}$ if and only if there exist vectors $u_{s+1}, \ldots, u_{s+k}$ such that $\preceq = \leq_{(u_1,\ldots,u_{s+k})}$.

Lemma 2.10 *Let $\omega \in \mathbb{R}^n$ and σ be a strongly convex cone with $\omega \in \mathrm{Int}(\sigma^\vee)$. Then σ is $\preceq$-positive for every order $\preceq$ refining $\leq_\omega$.*

Proof If $\omega \in \mathrm{Int}(\sigma^\vee)$, we have that $s \cdot \omega > 0$ for every $s \in \sigma \setminus \{0\}$. By Theorem-Definition 2.9, every $\preceq$ refining $\leq_\omega$ is equal to $\leq_{(\omega,v_1,\ldots,v_s)}$ for some vectors v_i. Thus σ is $\preceq$-positive. □

The next easy lemma will be used several times:

Lemma 2.11 [6, Lemma 2.4]. *Let σ_1 and σ_2 be two cones and γ_1 and γ_2 be vectors of $\mathbb{R}^n$. Let us assume that $\sigma_1 \cap \sigma_2$ is full dimensional. Then there exists a vector $\gamma \in \mathbb{Z}^n$ such that*

$$(\gamma_1 + \sigma_1) \cap (\gamma_2 + \sigma_2) \subset \gamma + \sigma_1 \cap \sigma_2.$$

Finally we give the following result, which will be used in the proof of Theorem 1.6 (this is a generalization of [6, Corollary 3.10]):

Lemma 2.12 *Let σ_1, ..., σ_N be strongly convex cones and let $\omega \in \mathbb{R}^n \setminus \{\underline{0}\}$. The following properties are equivalent:*

(i) *We have $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$.*
(ii) *For every order $\preceq \in \mathrm{Ord}(\mathbb{Q}^n)$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive.*

Proof Let us prove that (i) implies (ii). Let $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. We are going to show that for all nonzero vectors v_1, ..., $v_{n-1} \in \langle\omega\rangle^\perp$, with $v_j \in \langle\omega, v_1, \ldots, v_{j-1}\rangle^\perp$ for every j, there is an integer i such that σ_i is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive. Indeed, by Remark 2.9 every preorder refining $\leq_\omega$ is of the form $\leq_{(\omega,v_1,\ldots,v_j)}$ for $1 \leq j \leq n-1$. Therefore ii) is satisfied. So from now on, we fix such vectors v_1, ..., v_{n-1}.

By Lemma 2.10, if $\omega \in \mathrm{Int}(\sigma_i^\vee)$ for some i, then σ_i is $\preceq$-positive for every $\preceq$-refining $\leq_\omega$. In particular it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive. Otherwise, let E_1 denote the set of indices i such that $\omega \in \sigma_i^\vee$. If ω were in the boundary of $\bigcup_{i\in E_1} \sigma_i^\vee$, then ω would belong to some σ_i for $i \notin E_1$ because $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. Thus $\omega \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$.

Since $\omega \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$, there is $\lambda_1 > 0$ such that $\omega + \lambda_1 v_1 \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$. Then two cases may occur:

(1) Assume $\omega+\lambda_1 v_1 \in \mathrm{Int}(\sigma_i^\vee)$ for some $i \in E_1$. Because $i \in E_1$, for $s \in \sigma_i \setminus \{0\}$, either $\omega \cdot s > 0$, or $\omega \cdot s = 0$. In this last case we have $v_1 \cdot s > 0$ since $(\omega+\lambda_1 v_1)\cdot s > 0$ and $\lambda_1 > 0$. Therefore σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1)}$ (In particular it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive).

(2) If $\omega + \lambda_1 v_1 \notin \mathrm{Int}(\sigma_i^\vee)$ for every $i \in E_1$, we denote by E_2 the set of $i \in E_1$ such that $\omega+\lambda_1 v_1 \in \sigma_i^\vee$. As before we necessarily have $\omega+\lambda_1 v_1 \in \mathrm{Int}\left(\bigcup_{i\in E_2} \sigma_i^\vee\right)$. Therefore there is $\lambda_2 > 0$ such that $\omega+\lambda_1 v_1+\lambda_2 v_2 \in \mathrm{Int}\left(\bigcup_{i\in E_2} \sigma_i^\vee\right)$. Once again, if $\omega+\lambda_1 v_1+\lambda_2 v_2 \in \mathrm{Int}(\sigma_i^\vee)$ for some $i \in E_2$, σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1,v_2)}$. Otherwise we repeat the same process until one of the two situations occurs:

(a) there is $j < n-1$ such that $\omega + \lambda_1 v_1 + \cdots + \lambda_j v_j \in \mathrm{Int}(\sigma_i^\vee)$ for some i. Then, we can prove in the same way as (1) that σ_i is $\preceq$-positive for every $\preceq$ refining $\leq_{(\omega,v_1,\ldots,v_j)}$ (hence it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive).
(b) there is no such an index j. Thus we end with $\omega + \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1}$ that belongs to (at least) one $\sigma_i^\vee$. Therefore the cone σ_i is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive, because $\omega \in \sigma_i^\vee$, $\omega + \lambda_1 v_1 \in \sigma_i^\vee$, …, $\omega + \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} \in \sigma_i^\vee$.

This proves that (i) implies (ii).

Now we prove the converse. Assume that for every order $\preceq \in \mathrm{Ord}(\mathbb{Q}^n)$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive.

Let v be a vector with $\|v\| = 1$. By assumption, there is an index i such that σ_i is $\leq_{(\omega,v)}$-positive. Let s_1, …, s_l be generators of σ_i that we assume to be of norm equal to 1. Reordering the s_j, there is an integer $k \geq 0$ such that $s_j \cdot \omega > 0$ for every $j \leq k$, and $s_j \cdot \omega = 0$ for every $j > k$, because σ_i is $\leq_{(\omega,v)}$-positive. Take $\lambda > 0$. When $k > 1$ assume moreover that $\dfrac{\min_{\ell\leq k}\{s_\ell \cdot \omega\}}{2} \geq \lambda$. Then we claim that $\omega + \lambda v \in \sigma_i^\vee$. Indeed, if $j \leq k$ we have

$$(\omega+\lambda v)\cdot s_j = \omega\cdot s_j + \lambda v\cdot s_j \geq \omega\cdot s_j - \lambda\|v\|\|s_j\| \geq \frac{\min_{\ell\leq k}\{s_\ell\cdot\omega\}}{2} > 0.$$

If $j > k$ we have

$$(\omega+\lambda v)\cdot s_j = \lambda v\cdot s_j \geq 0$$

since σ_i is $\leq_{(\omega,v)}$-positive. This implies that $\omega + \lambda v \in \sigma_i^\vee$. Since this is true for every v, we have $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. □

Corollary 2.13 *Let $\omega \in \mathbb{R}_{\geq 0}{}^n \setminus \{\underline{0}\}$ and let σ_1, …, σ_N be strongly convex cones which are $\leq_\omega$-positive. Assume that for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive. Then there is a neighborhood V of ω such that, for every $\omega' \in V$ and every $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, there is an index i such that σ_i is $\preceq'$-positive.*

Proof We have $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$ by the previous lemma. Therefore, the previous lemma shows that we can choose $V = \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. □

The following lemma will be used several times:

Lemma 2.14 *Let ξ be a Laurent series with coefficients in a field $\mathbb{K}$. Assume that $Supp(\xi) \subset \gamma + \sigma$ where $\gamma \in \mathbb{Z}^n$ and σ is a rational cone. Let $\omega \in \sigma^\vee$. Then, for every $t \in \mathbb{R}$, the set*

$$\{u \cdot \omega | u \in Supp(\xi)\} \cap \big] - \infty, t\big]$$

is finite.

Proof We can make a translation and assume that $\gamma = 0$. Since σ is a rational cone, by Gordan's Lemma, there exist vectors $v_1, \ldots, v_N \in \sigma \cap \mathbb{Z}^n$ generating $\sigma \cap \mathbb{Z}^n$ as a monoid. Since $\omega \in \sigma^\vee$, we have $v_i \cdot \omega \geq 0$ for every i.

By assumption we have $\sigma = \left\{\sum_{i=1}^N n_i v_i | n_i \in \mathbb{N}\right\}$. Therefore the set $\{u \cdot \omega | u \in \text{Supp}(\xi)\}$ is included in the monoid generated by $v_1 \cdot \omega, \ldots, v_N \cdot \omega$. Since this monoid is finitely generated, the sets $\{u \cdot \omega | u \in \text{Supp}(\xi)\} \cap \big] - \infty, t\big]$ are finite. □

2.2 The space Ord_n as a compact topological space

One important tool for the proof of Theorem 1.3 is the fact that the set of orders Ord_n is a topological compact space for a well chosen topology. This topology has been introduced by Ewald and Ishida [14] (see also [12] for a generalization of this to the sets of preorders on a given group).

Definition 2.15 [14,34]. The set $\text{ZR}(\mathbb{Q}^n)$ is endowed with a topology for which the sets

$$\mathcal{U}_\sigma := \left\{\preceq \in \text{ZR}(\mathbb{Q}^n) \text{ such that } \sigma \text{ is } \preceq\text{-positive}\right\}$$

form a basis of open sets where σ runs over the full dimensional strongly convex rational cones.

Remark 2.16 With this definition we have $\text{Ord}_n = \mathcal{U}_{\mathbb{R}_{\geq 0}{}^n} \cap \text{Ord}(\mathbb{Q}^n)$.

We have the following result:

Theorem 2.17 [14]. *The space* $\text{ZR}(\mathbb{Q}^n)$ *is compact and* $\text{Ord}(\mathbb{Q}^n)$ *is closed in* $\text{ZR}(\mathbb{Q}^n)$. *Moreover every $\mathcal{U}_\sigma$ is compact. Therefore* Ord_n *is compact.*

The following lemma will be useful in the sequel:

Lemma 2.18 *Let $\sigma_1, \ldots, \sigma_N$ be rational cones such that* $\text{Ord}_n \subset \bigcup_{k=1}^N \mathcal{U}_{\sigma_k}$. *Then*

$$\mathbb{R}_{\geq 0}{}^n \subset \bigcup_{k=1}^N \sigma_k^\vee.$$

Proof Let $\omega \in \mathbb{R}_{\geq 0}{}^n$. Let $\preceq \in \text{Ord}_n$ refining $\leq_\omega$. Such a $\preceq$ exists by [6, Lemma 3.18]. Then $\preceq \in \mathcal{U}_{\sigma_k}$ for some k. Since $\preceq$ refines $\leq_\omega$, we have that σ_k is $\leq_\omega$-positive. This means that $\omega \in \sigma_k^\vee$. This proves that $\mathbb{R}_{\geq 0}{}^n \subset \bigcup_{k=1}^N \sigma_k^\vee$. □

2.3 Algebraically closed fields containing $\mathbb{K}((x))$ in characteristic zero

Definition 2.19 Let n be a positive integer and $\preceq \in \mathrm{Ord}_n$.

For a field $\mathbb{K}$ of characteristic zero, we denote by $\mathcal{S}_{\preceq}^{\mathbb{K}}$ the following set:

$$\left\{\xi \text{ series} | \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \preceq\text{-positive rational cone, } \mathrm{Supp}(\xi) \subset (\gamma + \sigma) \cap \frac{1}{k}\mathbb{Z}^n\right\}.$$

We have the following theorem:

Theorem 2.20 [6, Theorem 4.5]. *When $\mathbb{K}$ is an algebraically closed field of characteristic zero, the set $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is an algebraically closed field.*

Definition 2.21 For simplicity we will use the following notation: given a characteristic zero field $\mathbb{K}$, a strongly convex rational cone σ containing $\mathbb{R}_{\geq 0}{}^n$ and $k \in \mathbb{N}^*$, we set

$$\mathcal{S}_{\sigma,k}^{\mathbb{K}} := \left\{\xi \text{ series } | \exists \gamma \in \mathbb{Z}^n, \text{ such that } \mathrm{Supp}(\xi) \subset (\gamma + \sigma) \cap \frac{1}{k}\mathbb{Z}^n\right\}.$$

3 Proofs of Theorem 1.3 and Corollary 1.4

We begin by the following remark:

Lemma 3.1 *Let ξ be a series with support in $\mathbb{Q}^n$. Then $\tau(\xi)$ is a convex (non necessarily polyhedral) cone.*

Proof It is straightforward to see that for $\lambda > 0$ and $\omega \in \tau(\xi)$, $\lambda\omega \in \tau(\xi)$. Thus, we need to prove that for $\omega_1, \omega_2 \in \tau(\xi)$, $\omega_1 + \omega_2 \in \tau(\xi)$. By Remark 2.2, there exist $\gamma_1, \gamma_2 \in \mathbb{Z}^n$, $\sigma_1 \subset \langle\omega_1\rangle^\vee$, $\sigma_2 \subset \langle\omega_2\rangle^\vee$ containing $\mathbb{R}_{\geq 0}{}^n$ such that $\mathrm{Supp}(\xi) \subset (\gamma_1 + \sigma_1) \cap (\gamma_2 + \sigma_2)$. Thus $\mathrm{Supp}(\xi) \subset \gamma + \sigma_1 \cap \sigma_2$ for some $\gamma \in \mathbb{Z}^n$ by Lemma 2.11. This proves the lemma. □

In order to prove Theorem 1.3 we need the following intermediate results:

Lemma 3.2 *Let $\mathbb{K}$ be a characteristic zero field. Let $\xi \in \mathcal{S}_{\sigma,k}^{\mathbb{K}}$ where σ is a strongly convex rational cone containing $\mathbb{R}_{\geq 0}{}^n$, and $k \in \mathbb{N}^*$ (cf.* Definition 2.21*). Let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let us assume that there exists $\sigma_0 \supset \mathbb{R}_{\geq 0}{}^n$ a strongly convex rational cone such that $P(T)$ splits in $\mathcal{S}_{\sigma_0,k}^{\mathbb{K}}$.*

Then

$$\mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi) \neq \emptyset \Longrightarrow \sigma_0^\vee \subset \tau(\xi).$$

Proof Consider a nonzero vector $\omega \in \mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi)$. Since $\xi \in \mathcal{S}_{\sigma,k}^{\mathbb{K}}$, there are $k \in \mathbb{N}$, $\gamma_0 \in \mathbb{Z}^n$, and σ a $\leq_\omega$-positive rational cone, such that

$$\mathrm{Supp}(\xi) \subset (\gamma_0 + \sigma) \cap \frac{1}{k}\mathbb{Z}^n.$$

Since σ is $\leq_\omega$-positive and strongly convex, there exists an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that σ is $\preceq$-positive (see [6, Lemma 3.8]). Thus ξ is a root of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$.

On the other hand, ω is in the interior of $\sigma_0^\vee$, so σ_0 is $\preceq$-positive by Lemma 2.10. Thus the roots of P in $\mathcal{S}_{\sigma_0,k}^{\mathbb{K}}$ are the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ and ξ is one of them. Hence there is some $\gamma \in \mathbb{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma + \sigma_0.$$

Now let $\omega' \in \sigma_0^\vee$. We have $\sigma_0 \subset \langle \omega' \rangle^\vee$. Hence $\omega' \in \tau(\xi)$. This proves the lemma. □

Corollary 3.3 *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $\xi \in \mathcal{S}_{\preceq}^{\mathbb{K}}$ where $\preceq \in \mathrm{Ord}_n$. Let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let σ_i, $i = 1, \ldots, N$, be strongly convex rational cones containing ${\mathbb{R}_{\geq 0}}^n$, and $k \in \mathbb{N}^*$, satisfying the following properties:*

(i) $\displaystyle\bigcup_{i=1}^{N} \sigma_i^\vee = {\mathbb{R}_{\geq 0}}^n$,

(ii) *for every i, and every $\preceq \in \mathcal{U}_{\sigma_i}$, the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ are in $\mathcal{S}_{\preceq,\sigma_i,k}^{\mathbb{K}}$.*

Then, after renumbering the σ_i, there is an integer $l \leq N$ such that

$$\tau(\xi) = \bigcup_{i=1}^{l} \sigma_i^\vee.$$

Proof By Lemma 3.2, we can renumber the σ_i such that $\sigma_i^\vee \subset \tau(\xi)$ for $i \leq l$ and $\mathrm{Int}(\sigma_i^\vee) \cap \tau(\xi) = \emptyset$ for every $i > l$. So we have $\displaystyle\bigcup_{i=1}^{l} \sigma_i^\vee \subset \tau(\xi)$.

Now, suppose that this inclusion is strict: there is an element $\omega \in \tau(\xi)$ such that $\omega \notin \bigcup_{i=1}^{l} \sigma_i^\vee$.

We claim that $\bigcup_{i=1}^{l} \sigma_i^\vee$ is convex. Indeed, assume that it is not. Since the $\sigma_i^\vee$ are convex, this implies that there is $\omega_{i_1} \in \sigma_{i_1}^\vee$, $\omega_{i_2} \in \sigma_{i_2}^\vee$ for some $i_1, i_2 \leq l$, such that $\omega_{i_1} + \omega_{i_2} \notin \bigcup_{i=1}^{l} \sigma_i^\vee$. In this case, the line segment $[\omega_{i_1}, \omega_{i_2}]$ intersects $\bigcup_{i=l+1}^{N} \sigma_i^\vee$. Since $\sigma_{i_1}^\vee$ and $\sigma_{i_2}^\vee$ are full dimensional, we can replace freely ω_{i_1} and ω_{i_2} by any elements close to them. Thus we may assume that $[\omega_{i_1}, \omega_{i_2}]$ intersects $\mathrm{Int}(\sigma_m^\vee)$ for some $m > l$. But this contradicts the fact that $\tau(\xi)$ is convex (see Lemma 3.1).

Therefore, by the Hahn–Banach Theorem there is a hyperplane H separating ω and the convex closed set $\bigcup_{i=1}^{l} \sigma_i^\vee$ in the following sense: one open half space delimited by H, denoted by O, contains ω and $\bigcup_{i=1}^{l} \sigma_i^\vee \subset \mathbb{R}^n \backslash \overline{O}$. Since $\bigcup_{i=1}^{l} \sigma_i^\vee$ is full dimensional, the convex hull $\mathcal{C}$ of ω and $\bigcup_{i=1}^{l} \sigma_i^\vee$ is full dimensional:

$$\mathcal{C} := \left\{ \lambda\omega + (1-\lambda)v \,|\, v \in \bigcup_{i=1}^{l} \sigma_i^\vee, 1 \geq \lambda \geq 0 \right\}.$$

Thus $\mathcal{C} \cap O$ contains an open ball B.

Since $\tau(\xi)$ is convex (see Lemma 3.1), $\mathcal{C} \subset \tau(\xi)$ and $B \subset \tau(\xi)$. Then B intersects one $\sigma_m^\vee$ for $m > l$ because $B \subset O$ and we have assumed $\bigcup_{i=1}^N \sigma_i^\vee = \mathbb{R}_{\geq 0}{}^n$. But because B is open, $B \cap \mathrm{Int}(\sigma_m^\vee) \neq \emptyset$, and this is a contradiction because $B \subset \tau(\xi)$ and $\tau(\xi) \cap \mathrm{Int}(\sigma_m^\vee) = \emptyset$ for $m > l$. Therefore the inclusion is not strict and $\bigcup_{i=1}^l \sigma_i^\vee = \tau(\xi)$. □

Proposition 3.4 *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $P \in \mathbb{K}[[x]][T]$. There is an integer N, strongly convex rational cones $\sigma_1, \ldots, \sigma_N$ containing $\mathbb{R}_{\geq 0}{}^n$, and $k \in \mathbb{N}^*$, such that:*

(i) $\mathrm{Ord}_n \subset \bigcup_{i=1}^N \mathcal{U}_{\sigma_i}$ *and* $\displaystyle\bigcup_{i=1}^N \sigma_i^\vee = \mathbb{R}_{\geq 0}{}^n$,

(ii) *for every* $\preceq \in \mathrm{Ord}_n$, *there is* $j \in \{1, \ldots, N\}$, *such that the roots of* $P(T)$ *in* $\mathcal{S}_{\preceq}^{\mathbb{K}}$ *belong to* $\mathcal{S}_{\sigma_j,k}^{\mathbb{K}}$.

Proof By Theorem 2.20 for every order $\preceq \in \mathrm{Ord}_n$ there is an element $\gamma_{\preceq} \in \mathbb{Z}^n$, and a $\preceq$-positive strongly convex rational cone $\sigma_{\preceq}$ such that the roots of P can be expanded as series in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ with support in $\gamma_{\preceq} + \sigma_{\preceq}$.

In particular we have $\mathrm{Ord}_n \subset \mathcal{U}_{\mathbb{R}_{\geq 0}{}^n} \subset \bigcup_{\preceq} \mathcal{U}_{\sigma_{\preceq}}$. Hence, by Theorem 2.17, we can extract from this family of cones $\sigma_{\preceq}$, a finite number of cones, denoted by $\sigma_{\preceq_1}$, $\ldots$, $\sigma_{\preceq_N}$, such that $\mathrm{Ord}_n \subset \bigcup_{i=1}^N \mathcal{U}_{\sigma_{\preceq_i}}$. Therefore, by Lemma 2.18, we have that $\mathbb{R}_{\geq 0}{}^n \subset \bigcup_{i=1}^N \sigma_{\preceq_i}^\vee$. Because the $\sigma_{\preceq_i}$ contain $\mathbb{R}_{\geq 0}{}^n$, we have $\mathbb{R}_{\geq 0}{}^n = \bigcup_{i=1}^N \sigma_{\preceq_i}^\vee$. Moreover these cones satisfy the following property:

> $\forall \preceq \in \mathrm{Ord}_n$, $\exists \gamma_{\preceq} \in \mathbb{Z}^n$, $\exists i \in \{1, \ldots, N\}$, such that the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in $\gamma_{\preceq} + \sigma_{\preceq_i}$.

Assume that the same integer $i \in \{1, \ldots, N\}$ satisfies the previous property for two orders $\preceq_1$ and $\preceq_2 \in \mathrm{Ord}_n$. That is, the roots of P in $\mathcal{S}_{\preceq_1}^{\mathbb{K}}$ (resp. in $\mathcal{S}_{\preceq_2}^{\mathbb{K}}$) have support in $\gamma_{\preceq_1} + \sigma_i$ (resp. in $\gamma_{\preceq_2} + \sigma_i$). Then the roots of P in $\mathcal{S}_{\preceq_2}^{\mathbb{K}}$ are elements of $\mathcal{S}_{\preceq_1}^{\mathbb{K}}$, thus the roots of P in $\mathcal{S}_{\preceq_2}^{\mathbb{K}}$ coincide with its roots in $\mathcal{S}_{\preceq_1}^{\mathbb{K}}$. Therefore we may assume that the element $\gamma_{\preceq}$ does depend only on i. □

Proof of Theorem 1.3 First, by replacing each of the x_i by some power of x_i, we may assume that ξ is a Laurent series. By Proposition 3.4, there exist strongly convex rational cones $\sigma_1, \ldots, \sigma_N$ satisfying (i) and (ii) of Corollary 3.3. Therefore, by Corollary 3.3, we have that $\tau(\xi)$ is a strongly convex rational cone. This proves Theorem 1.3. □

Remark 3.5 For a formal power series $f \in \mathbb{K}[[x]]$ we denote by $\mathrm{NP}(f)$ its Newton polyhedron. Let p be a vertex of $\mathrm{NP}(f)$. The set of vectors $v \in \mathbb{R}^n$ such that $p + \lambda v \in \mathrm{NP}(f)$ for some $\lambda \in \mathbb{R}_{\geq 0}$ is a rational strongly convex cone. Such a cone is called the *cone of the Newton polyhedron of f associated with the vertex p*. We have the following generalization of Abhyankar–Jung Theorem that provides in an effective way some cones satisfying Corollary 3.3:

Theorem 3.6 (Strong form of the Abhyankar–Jung Theorem) [15, Théorème 3] [4, Theorem 7.1] [25, Theorem 6.2]. *Let $\mathbb{K}$ be a characteristic zero field. Let $P(Z) \in$*

$\mathbb{K}[[x]][Z]$ *be a monic polynomial and let* Δ *be its discriminant. Let* $\mathrm{NP}(\Delta)$ *denote the Newton polyhedron of* Δ*. Then the set of cones of* $\mathrm{NP}(\Delta)$ *satisfies the properties of Corollary* 3.3.

Therefore, if ξ is integral over $\mathbb{K}[[x]]$, that is $P(T)$ is a monic polynomial in T, we may replace the use of Corollary 3.3 (thus Proposition 3.4 and thus Theorems 2.17 and 2.20) by Theorem 3.6.

Now we are able to prove Corollary 1.4:

Proof of Corollary 1.4 By replacing $\mathbb{K}$ by its algebraic closure, we may assume that $\mathbb{K}$ is algebraically closed. Since $\tau(\xi)$ is rational, let $\omega_1, \ldots, \omega_s \in \mathbb{Z}^n$ be generators of $\tau(\xi)$. Thus we have $\tau(\xi)^\vee = \bigcap_{i=1}^s \langle\omega_i\rangle^\vee$. Therefore, we have $\mathrm{Supp}(\xi) \subset \gamma + \tau(\xi)^\vee$ for some $\gamma \in \mathbb{Z}^n$ by Remark 2.2 and Lemma 2.11.

On the other hand if σ is a cone (not necessarily finitely generated) such that $\mathrm{Supp}(\xi) \subset \gamma + \sigma$ for some $\gamma \in \mathbb{Z}^n$, then we have $\sigma^\vee \subset \tau(\xi)$ by the definition of $\tau(\xi)$, that is, $\tau(\xi)^\vee \subset \sigma$. ☐

4 Proof of Theorems 1.5 and 1.6

4.1 Preliminary results

Definition 4.1 For a Laurent series ξ we set

$$\tau_0'(\xi) = \left\{\omega \in \mathbb{R}_{\geq 0}{}^n \setminus \{\underline{0}\} | \# \left(\mathrm{Supp}(\xi) \cap \left\{u \in \mathbb{R}^n | u \cdot \omega \leq k\right\}\right) < \infty, \forall k \in \mathbb{R}\right\},$$
$$\tau_1'(\xi) = \left\{\omega \in \mathbb{R}_{\geq 0}{}^n \setminus \{\underline{0}\} | \# \left(\mathrm{Supp}(\xi) \cap \left\{u \in \mathbb{R}^n | u \cdot \omega \leq k\right\}\right) = \infty, \forall k \in \mathbb{R}\right\}.$$

We have the following lemma:

Lemma 4.2 *Let* ξ *be a Laurent series with support in a translation of a strongly convex cone containing* $\mathbb{R}_{\geq 0}{}^n$*. We have* $\tau_0'(\xi) \subset \tau(\xi) \subset \overline{\tau_0'(\xi)}$.

Proof We have $\tau_0'(\xi) \subset \tau(\xi)$ by definition.

Let $\omega \in \tau(\xi)$. Then by Remark 2.2, $\mathrm{Supp}(\xi) \subset \gamma + \langle\omega\rangle^\vee$ for some $\gamma \in \mathbb{Z}^n$.

On the other hand, by hypothesis, $\mathrm{Supp}(\xi)$ is included in $\gamma' + \sigma$ where $\gamma' \in \mathbb{Z}^n$ and σ is a strongly convex cone such that $\mathbb{R}_{\geq 0}{}^n \subset \sigma$. Thus, by Lemma 2.11, $\mathrm{Supp}(\xi)$ is included in a translation of the strongly convex cone $\sigma \cap \langle\omega\rangle^\vee$.

We have $\omega \in \langle\omega\rangle^{\vee\vee} \subset \left(\sigma \cap \langle\omega\rangle^\vee\right)^\vee$, and $\left(\sigma \cap \langle\omega\rangle^\vee\right)^\vee$ is full dimensional. Thus there exists a sequence $(\omega_k)_k$ of vectors in $\mathrm{Int}\left(\left(\sigma \cap \langle\omega\rangle^\vee\right)^\vee\right)$ that converges to ω.

We have to prove that the ω_k belong to $\tau_0'(\xi)$. For $u \in (\sigma \cap \langle\omega\rangle^\vee) \setminus \{\underline{0}\}$, we have $u \cdot \omega_k \neq 0$ because $\omega_k \in \mathrm{Int}\left(\left(\sigma \cap \langle\omega\rangle^\vee\right)^\vee\right)$. This shows that $\sigma \cap \langle\omega\rangle^\vee \cap \langle\omega_k\rangle^\perp = \{\underline{0}\}$. Therefore, because $\mathrm{Supp}(\xi)$ is included in a translation of $\sigma \cap \langle\omega\rangle^\vee$, for all k we have:

$$\omega_k \in \left\{\omega' \in \mathbb{R}^n | \# \left(\mathrm{Supp}(\xi) \cap \left\{u \in \mathbb{R}^n | u \cdot \omega' \leq k\right\}\right) < \infty, \forall k \in \mathbb{R}\right\}.$$

Moreover, because $\omega \in \tau(\xi) \subset \mathbb{R}_{\geq 0}{}^n$ and $\mathbb{R}_{\geq 0}{}^n \subset \sigma$, we have $\mathbb{R}_{\geq 0}{}^n = (\mathbb{R}_{\geq 0}{}^n)^\vee \subset \sigma \cap \langle\omega\rangle^\vee$. Therefore the ω_k are in $\mathbb{R}_{\geq 0}{}^n$, and they are nonzero for k large enough because $(\omega_k)_k$ converges to ω which is nonzero. This shows that $\omega_k \in \tau'_0(\xi)$ for k large enough, therefore $\omega \in \overline{\tau'_0(\xi)}$. $\square$

Corollary 4.3 *Under the hypothesis of Theorem 1.6, we have*

$$\tau(\xi) = \overline{\tau'_0(\xi)}.$$

Proof By Lemma 4.2 we have $\tau'_0(\xi) \subset \tau(\xi) \subset \overline{\tau'_0(\xi)}$. Since $\tau(\xi)$ is closed (it is a rational cone, thus a polyhedral cone, by Theorem 1.3) we have $\tau(\xi) = \overline{\tau'_0(\xi)}$. $\square$

Definition 4.4 In the rest of this section we consider the following setting: ξ is a Laurent series with support included in the translation of a strongly convex rational cone, and ξ is algebraic over $\mathbb{K}[[x]]$ where $\mathbb{K}$ is a field of characteristic zero. From now on we enlarge $\mathbb{K}$ in order to assume that $\mathbb{K}$ is algebraic closed. We denote by $P \in \mathbb{K}[[x]][T]$ the minimal polynomial of ξ and, for any order $\preceq \in \mathrm{Ord}_n$, $\xi_1^{\preceq}, \ldots, \xi_d^{\preceq}$ denote the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$. We set

$$\tau_0(\xi) := \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \backslash \{\underline{0}\} | \text{ for all } \preceq \text{ that refines } \leq_\omega, \ \exists i \text{ such that } \xi = \xi_i^{\preceq}\right\},$$
$$\tau_1(\xi) := \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \backslash \{\underline{0}\} | \xi \neq \xi_i^{\preceq}, \text{ for all } \preceq \text{ that refines } \leq_\omega, \ \forall i = 1, \ldots, d\right\},$$

Remark 4.5 These sets were introduced in [6], but only for $\omega \in {\mathbb{R}_{>0}}^n$. In this case it was proved that $\tau_0(\xi) \cap {\mathbb{R}_{>0}}^n = \tau'_0(\xi) \cap {\mathbb{R}_{>0}}^n$ and $\tau_1(\xi) \cap {\mathbb{R}_{>0}}^n = \tau'_1(\xi) \cap {\mathbb{R}_{>0}}^n$ (see [6, Lemmas 5.8, 5.11]). Taking into account all the $\omega \in {\mathbb{R}_{\geq 0}}^n$ changes the situation. In particular we do not have $\tau_0(\xi) = \tau'_0(\xi)$ in general (see Example 4.12).

Proposition 4.6 *We have* $\tau_1(\xi) = \tau'_1(\xi)$ *and* $\tau'_0(\xi) \subset \tau_0(\xi)$.

Proof The proof of the equality $\tau_1(\xi) = \tau'_1(\xi)$ is exactly the proof of [6, Lemma 5.11]. Let us prove $\tau'_0(\xi) \subset \tau_0(\xi)$. Let $\omega \in \tau'_0(\xi)$, in particular:

$$\#\left(\mathrm{Supp}(\xi) \cap \left\{u \in \mathbb{R}^n | u \cdot \omega \leq k\right\}\right) < \infty, \ \forall k \in \mathbb{R}, \tag{2}$$

and let us consider an order $\preceq$ that refines $\leq_\omega$.

Let $(u_l)_l$ be a sequence of elements of $\mathrm{Supp}(\xi)$ such that $u_l \succeq u_{l+1}$ for every $l \in \mathbb{N}$. Then $u_l \geq_\omega u_{l+1}$, that is $u_l \cdot \omega \geq u_{l+1} \cdot \omega$, for every $l \in \mathbb{N}$. Therefore by (2), this sequence contains only finitely many distinct terms. Therefore $u_{l+1} = u_l$ for l large enough because $\preceq$ is an order. This shows that $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. Thus by [6, Corollary 4.6] ξ is an element of $\mathcal{S}_{\preceq}^{\mathbb{K}}$. This shows that $\omega \in \tau_0(\xi)$. $\square$

Proposition 4.7 *The sets* $\tau_0(\xi)$ *and* $\tau_1(\xi)$ *are open subsets of* ${\mathbb{R}_{\geq 0}}^n$.

Proof Let us consider the cones σ_i given by Proposition 3.4. In particular, for every $\omega \in {\mathbb{R}_{\geq 0}}^n \backslash \{\underline{0}\}$, the set of orders $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ is included in $\bigcup_{i=1}^N \mathcal{U}_{\sigma_i}$. The set $\mathcal{T}_\omega = \{\sigma_1, \ldots, \sigma_N\}$ satisfies the following property:

> For any order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$, there is $\sigma \in \mathcal{T}_\omega$, σ being $\preceq$-positive, such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ are in $\mathcal{S}_{\sigma,k}^{\mathbb{K}}$ for some $k \in \mathbb{N}^*$.

Moreover, let us choose $\mathcal{T}_\omega$ to be minimal among the sets of cones having this property. Then Corollary 2.13 implies that, for every $\omega' \in \mathbb{R}_{\geq 0}{}^n \setminus \{\underline{0}\}$ close enough to ω, and for any order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, there is $\sigma \in \mathcal{T}_\omega$ such that the roots of P in $\mathcal{S}^{\mathbb{K}}_{\preceq'}$ are in $\mathcal{S}^{\mathbb{K}}_{\sigma,k}$ for some $k \in \mathbb{N}^*$. Since $\mathcal{T}_\omega$ is minimal with this property, for every ω' close enough to ω, for every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$ and for every $i = 1, \ldots, d$, there is an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that $\xi_i^{\preceq'} = \xi_{j_i}^{\preceq}$ for some j_i.

If $\omega \in \tau_0(\xi)$ then ξ is equal to some $\xi_i^{\preceq}$ for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for every $\omega' \in \mathbb{R}_{\geq 0}{}^n$ close enough to ω and every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, $\xi = \xi_j^{\preceq'}$ for some j. Thus $\omega' \in \tau_0(\xi)$. This proves that $\tau_0(\xi)$ is open in $\mathbb{R}_{\geq 0}{}^n$.

If $\omega \in \tau_1(\xi)$ then $\xi \neq \xi_i^{\preceq}$ for every i and for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for $\omega' \in \mathbb{R}_{\geq 0}{}^n$ close enough to ω and every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, $\xi \neq \xi_j^{\preceq'}$ for every j. Hence $\omega' \in \tau_1(\xi)$ and $\tau_1(\xi)$ is open. □

Corollary 4.8 *We have*

$$\overline{\tau_0'(\xi)} \cap \tau_1'(\xi) = \emptyset.$$

Proof The sets $\tau_0(\xi)$ and $\tau_1(\xi)$ are disjoint and open in $\mathbb{R}_{\geq 0}{}^n$. Thus $\overline{\tau_0(\xi)} \cap \tau_1(\xi) = \emptyset$. This proves the corollary because $\tau_0'(\xi) \subset \tau_0(\xi)$ and $\tau_1'(\xi) = \tau_1(\xi)$ by Proposition 4.6. □

Lemma 4.9 *We have*

$$\overline{\tau_0'(\xi)} = \overline{\tau_0(\xi) \cap \mathbb{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

Proof The set $\tau_0(\xi)$ is open. Therefore every $w \in \tau_0(\xi) \cap (\mathbb{R}_{\geq 0}{}^n \setminus \mathbb{R}_{>0}{}^n)$ can be approximated by elements of $\tau_0(\xi) \cap \mathbb{R}_{>0}{}^n$. Hence

$$\overline{\tau_0(\xi) \cap \mathbb{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

By [6, Lemma 5.8] $\tau_0'(\xi) \cap \mathbb{R}_{>0}{}^n = \tau_0(\xi) \cap \mathbb{R}_{>0}{}^n$. We have that $\tau_0'(\xi)$ is convex (the proof is exactly the same as the proof of [6, Lemma 5.9]). Thus we have

$$\overline{\tau_0'(\xi) \cap \mathbb{R}_{>0}{}^n} = \overline{\tau_0'(\xi)}$$

by [9, Prop. 16–Cor. 1; II.2.6]. Hence

$$\overline{\tau_0'(\xi)} = \overline{\tau_0'(\xi) \cap \mathbb{R}_{>0}{}^n} = \overline{\tau_0(\xi) \cap \mathbb{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

□

Corollary 4.10 *For every $f \in \mathbb{K}[[x]]^*$ we have*

$$\tau_0(\xi + f) = \tau_0(\xi),\ \ \tau_1(\xi + f) = \tau_1(\xi),\ \ \tau(\xi + f) = \tau(\xi),$$
$$\tau_0(f\xi) \supset \tau_0(\xi),\ \ \tau_1(f\xi) \supset \tau_1(\xi),\ \ \tau(f\xi) \supset \tau(\xi).$$

Proof The minimal polynomial of $\xi + f$ is $Q(T) := P(T - f)$. Thus, for a given $\preceq \in \mathrm{Ord}_n$, the roots of $Q(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ are $\xi_1^{\preceq} + f, \dots, \xi_d^{\preceq} + f$. This shows that

$$\tau_0(\xi + f) = \tau_0(\xi),\ \tau_1(\xi + f) = \tau_1(\xi).$$

Lemma 4.9 and Corollary 4.3 imply that $\tau(\xi + f) = \tau(\xi)$.

Now, the polynomial $R(T) := f^d P(T/f)$ vanishes at $f\xi$. On the other hand, if $\overline{R}(T)$ is a polynomial with $\overline{R}(f\xi) = 0$, then $\overline{R}(fT)$ is a polynomial vanishing at ξ. This shows that $P(T)$ divides $\overline{R}(fT)$. Thus, the minimal polynomial of $f\xi$ has degree d and divides $R(T)$, thus it is of the form $\frac{1}{g}R(T) = \frac{f^d}{g}P(T/f)$ for some $g \in \mathbb{K}[[x]]$, $g \neq 0$.

Therefore, for a given $\preceq \in \mathrm{Ord}_n$, the roots in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ of the minimal polynomial of $f\xi$ are $f\xi_1^{\preceq}, \dots, f\xi_d^{\preceq}$. This shows that

$$\tau_0(f\xi) \supset \tau_0(\xi),\ \tau_1(f\xi) \supset \tau_1(\xi),\ \overline{\tau_0'(f\xi)} \supset \overline{\tau_0'(\xi)}.$$

This proves the corollary. □

Example 4.11 Let $\xi = \sum_{i \geq 0} \left(\frac{x_1}{x_2}\right)^i = \frac{x_2}{x_2 - x_1}$. Here $\tau(\xi)$ is the cone generated by $(1, 0)$ and $(1, 1)$. But $\tau((x_2 - x_1)\xi) = {\mathbb{R}_{\geq 0}}^n$. Therefore we do not have $\tau(f\xi) = \tau(\xi)$ in general.

Example 4.12 We can see on a basic example that $\tau_0'(\xi + f) \neq \tau_0'(\xi)$ in general: let $n = 2$ and fix $\xi = \sum_{k \in \mathbb{N}} x_1^k$ and $f = 1 - \xi$. Then $\tau_0'(\xi) = \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$ but $\tau_0'(\xi + f) = {\mathbb{R}_{\geq 0}}^2$. This also shows that $\tau_0(\xi) \neq \tau_0'(\xi)$ in general.

4.2 Proof of Theorem 1.6

First, by replacing each of the x_i by some power of x_i, we may assume that ξ is a Laurent series.

Here we denote by σ the face of $\tau(\xi)^\vee$ defined by u. We set

$$H_u(t) := \{v \in \mathbb{R}^n | v \cdot u = t\},\ H_u(t)^+ = \{v \in \mathbb{R}^n | v \cdot u \geq t\}. \tag{3}$$

The vector u is in the boundary of $\tau_0'(\xi)$ because $\tau(\xi) = \overline{\tau_0'(\xi)}$ by Corollary 4.3. Hence by Corollary 4.8 we have $u \notin \tau_1'(\xi)$. Thus, we have $u \in \tau_0'(\xi)$ or $u \in {\mathbb{R}_{\geq 0}}^n \backslash (\tau_0'(\xi) \cup \tau_1'(\xi))$. Assume that $u \in \tau_0'(\xi)$. By Proposition 4.7, $\tau_0'(\xi) \cap {\mathbb{R}_{>0}}^n$ is open. Thus, because u is in the boundary of $\tau_0'(\xi)$, we have $u \in {\mathbb{R}_{\geq 0}}^n \backslash {\mathbb{R}_{>0}}^n$, which contradicts the hypothesis. Therefore $u \notin \tau_0'(\xi)$. Thus we use the following lemma whose proof is given below:

Lemma 4.13 *Let $u \notin \tau_0'(\xi) \cup \tau_1'(\xi)$. Then there exist a Laurent polynomial $p_\sigma(x)$ and a real number t_σ such that*

$$\mathit{Supp}(\xi + p_\sigma(x)) \subset H_u(t_\sigma)^+ \text{ and } \#\,(\mathit{Supp}(\xi + p_\sigma(x)) \cap H_u(t_\sigma)) = +\infty. \tag{4}$$

Now we denote by $p(x)$ the sum of distinct monomials that appear in all the $p_\sigma(x)$ (there is a finite number of faces σ), and Theorem 1.6 is proved.

Proof of Lemma 4.13 Because $u \notin \tau'_0(\xi) \cup \tau'_1(\xi)$, the following set is non empty and bounded from above :

$$E_\sigma := \left\{t \in \mathbb{R} | \# \left(\mathrm{Supp}(\xi) \cap \left\{v \in \mathbb{R}^n | v \cdot u < t\right\}\right) < \infty\right\}.$$

Let us set $t_\sigma := \sup E_\sigma$. By Lemma 2.14, the set $\{v \cdot u | v \in \mathrm{Supp}(\xi)\} \cap]-\infty, t]$ is finite for every t (here u belongs to the closure of $\tau(\xi)$). Thus, we may order the elements of $\{v \cdot u | v \in \mathrm{Supp}(\xi)\}$ as $t_0 < t_1 < \cdots$, and necessarily t_σ is one of these elements. Therefore the set $\mathrm{Supp}(\xi) \cap \{v \in \mathbb{R}^n | v \cdot u = t_\sigma\}$ is infinite and $\mathrm{Supp}(\xi) \cap \{v \in \mathbb{R}^n | v \cdot u < t_\sigma\}$ is finite. So we denote by $-p_\sigma(x)$ the sum of the monomials of ξ whose exponents belong to $\{v \in \mathbb{R}^n | v \cdot u < t_\sigma\}$ and (4) is satisfied (that is, we remove from ξ the monomials that are in $\{v \in \mathbb{R}^n | v \cdot u < t_\sigma\}$). □

4.3 Proof of Theorem 1.5

We begin by giving a strengthened version of Lemma 2.11 that we will need in the proof of Theorem 1.5:

Proposition 4.14 (Dickson's Lemma) *Let $\sigma_1, \ldots, \sigma_k$ be convex rational cones such that $\sigma := \bigcap_{j=1}^k \sigma_j$ is a full dimensional convex rational cone. Let $\gamma_1, \ldots, \gamma_k \in \mathbb{Z}^n$. Then there exists a finite set $C \subset \mathbb{Z}^n$ such that*

$$\bigcap_{j=1}^k (\gamma_j + \sigma_j) \cap \mathbb{Z}^n = C + \sigma \cap \mathbb{Z}^n.$$

Proof Up to a translation we may assume that $\gamma_j \in \sigma \cap \mathbb{Z}^n$ for every j because σ is full dimensional. Let $u_1, \ldots, u_s$ be vectors with integer coordinates generating $\sigma \cap \mathbb{Z}^n$. Then the ring R_σ of polynomials in $x_1, \ldots, x_n$ with support in $\sigma \cap \mathbb{Z}^n$ is isomorphic to $\mathbb{K}[U_1, \ldots, U_s]/I$ for some binomial ideal I. This is well known and this can be described as follows (for instance see [11, Proposition 1.1.9] for details):
for any linear relation $L := \{\sum_{i=1}^s \lambda_i u_i = 0\}$ with $\lambda_i \in \mathbb{Z}$ we consider the binomial

$$B_L := \prod_{i | \lambda_i \geq 0} U_i^{\lambda_i} - \prod_{i | \lambda_i < 0} U_i^{-\lambda_i}.$$

Then I is the ideal generated by the B_L for L running over the $\mathbb{Z}$-linear relations between the u_i. Moreover, for $\gamma \in \sigma \cap \mathbb{Z}^n$, the isomorphism $R_\sigma \longrightarrow \mathbb{K}[U]/I$ sends x^γ onto U^{α_γ} where $\alpha_\gamma \in \mathbb{Z}^s_{\geq 0}$ is defined by $\gamma = \sum_{i=1}^s \alpha_{\gamma,i} u_i$.
Because the γ_j belong to σ, we have

$$\bigcap_{j=1}^k (\gamma_j + \sigma_j) \subset \bigcap_{j=1}^k \sigma_j = \sigma.$$

Therefore the set of monomials x^u for $u \in \bigcap_{j=1}^k (\gamma_j + \sigma_j) \cap \mathbb{Z}^n$, is equal to the set of monomials of a monomial ideal of R_σ. By Gordan's Lemma, this ideal is generated by a finite number of monomials. If C denotes the set of exponents of these generators, we have $\bigcap_{j=1}^k (\gamma_j + \sigma_j) \cap \mathbb{Z}^n = C + \sigma \cap \mathbb{Z}^n$. □

Proof of Theorem 1.5 As in the proof of Theorem 1.6 we may replace each of the x_i by some power of x_i, and assume that ξ is a Laurent series.
By [24, Proposition 1.3], because $\tau(\xi)^\vee$ is a strongly convex rational cone, for each nonzero face $\sigma \subset \tau(\xi)^\vee$, there is a vector u_σ in the boundary of $\tau(\xi)$ such that

$$\sigma = \langle u_\sigma \rangle^\perp \cap \tau(\xi)^\vee.$$

In fact, as seen in the proof of [24, Proposition 1.3], we can freely choose u_σ in the relative interior of $\sigma^\perp \cap \tau(\xi)$, where $\sigma^\perp \cap \tau(\xi)$ is a face of dimension n-dim(σ) of $\tau(\xi)$. Thus, when σ is a facet of $\tau(\xi)^\vee$, $\sigma^\perp \cap \tau(\xi)$ is a half-line that is generated by one vector with integer coordinates. Therefore, when σ is a facet of $\tau(\xi)^\vee$, we can choose $u_\sigma \in \mathbb{Z}^n$.

From now on, σ will always denote a facet of $\tau(\xi)^\vee$. We have

$$\tau(\xi)^\vee = \bigcap_{\sigma \text{ facet of } \tau(\xi)^\vee} H_{u_\sigma}(0)^+$$

where the $H_{u_\sigma}(t)^+$ are defined in (3). The vectors u_σ are in the boundary of $\tau'_0(\xi)$ because $\tau(\xi) = \overline{\tau'_0(\xi)}$ by Corollary 4.3. Hence by Corollary 4.8 we have $u_\sigma \notin \tau'_1(\xi)$ for any facet σ. Thus for every facet σ of $\tau(\xi)^\vee$ we have $u_\sigma \in \tau'_0(\xi)$ or $u_\sigma \in {\mathbb{R}_{\geq 0}}^n \setminus (\tau'_0(\xi) \cup \tau'_1(\xi))$. We will reduce to the situation where none of the u_σ are in $\tau'_0(\xi)$:

Let σ be a facet of $\tau(\xi)^\vee$ for which $u_\sigma \in \tau'_0(\xi)$. By Proposition 4.7, $\tau'_0(\xi) \cap {\mathbb{R}_{>0}}^n$ is open. Thus, because u_σ is in the boundary of $\tau'_0(\xi)$, we have $u_\sigma \in {\mathbb{R}_{\geq 0}}^n \setminus {\mathbb{R}_{>0}}^n$. In particular at least one of the coordinates of u_σ is zero, hence $\langle u_\sigma \rangle^\perp$ contains at least one line generated by one vector with integer coordinates. Therefore there exists $f_\sigma(x) \in \mathbb{K}[[x]]$ with support in $\langle u_\sigma \rangle^\perp \cap {\mathbb{R}_{\geq 0}}^n$ and such that

$$\# \left\{ \text{Supp}(\xi + f_\sigma(x)) \cap \langle u_\sigma \rangle^\perp \cap {\mathbb{R}_{\geq 0}}^n \right\} = +\infty.$$

Moreover we can do this simultaneously for every facet σ of $\tau(\xi)^\vee$ such that $u_\sigma \in \tau'_0(\xi)$, hence there exists $f(x) \in \mathbb{K}[[x]]$ such that for every such facet σ:

$$\# \left\{ \text{Supp}(\xi + f(x)) \cap \langle u_\sigma \rangle^\perp \cap {\mathbb{R}_{\geq 0}}^n \right\} = +\infty. \tag{5}$$

By Corollary 4.10 $\tau(\xi) = \tau(\xi + f(x))$. But $u_\sigma \notin \tau'_0(\xi + f(x))$ by (5). Therefore, we replace ξ with $\xi + f(x)$. This does not change $\tau(\xi)$, but this allows us to assume that $u_\sigma \in {\mathbb{R}_{\geq 0}}^n \setminus (\tau'_0(\xi) \cup \tau'_1(\xi))$. Therefore we may assume that none of the u_σ is in $\tau'_0(\xi)$.

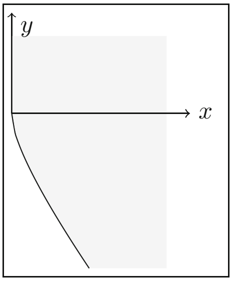


Fig. 1 Example 5.1

Then we apply Lemma 4.13 to see, as in the proof of Theorem 1.6, that modulo a finite number of monomials and a formal power series $f(x) \in \mathbb{K}[[x]]$, the support of ξ is included in $\bigcap_{\sigma \text{ facet of } \tau(\xi)^\vee} H_{u_\sigma}(t_\sigma)^+ \cap \mathbb{Z}^n$. Moreover each $H_{u_\sigma}(t_\sigma)$ contains infinitely many monomials of ξ, i.e there is a Laurent polynomial $p(x)$ such that

$$\text{Supp}(\xi + f(x) + p(x)) \subset \bigcap_{\sigma \text{ facet of } \tau(\xi)^\vee} H_{u_\sigma}(t_\sigma)^+ \cap \mathbb{Z}^n$$
$$\text{and } \#\left(\text{Supp}(\xi + f(x) + p(x)) \cap H_{u_\sigma}(t_\sigma)\right) = +\infty \quad \forall \sigma.$$

For every σ facet of $\tau(\xi)^\vee$ we have $H_{u_\sigma}(t_\sigma)^+ = \gamma_\sigma + H_{u_\sigma}(0)^+$ for any $\gamma_\sigma \in H_\sigma(t_\sigma)$. But, since $H_{u_\sigma}(t_\sigma) \cap \mathbb{Z}^n \neq \emptyset$, we may fix $\gamma_\sigma \in \mathbb{Z}^n$. Since $u_\sigma \in \mathbb{Z}^n$, the cone $H_{u_\sigma}^+$ is rational. Thus, by Corollary 4.14 there is a finite set $C \subset \mathbb{Z}^n$ such that

$$\bigcap_{\sigma \text{ facet of } \tau(\xi)^\vee} H_{u_\sigma}(t_\sigma)^+ \cap \mathbb{Z}^n = C + \bigcap_{\sigma \text{ facet of } \tau(\xi)^\vee} H_{u_\sigma}(0)^+ \cap \mathbb{Z}^n = C + \tau(\xi)^\vee \cap \mathbb{Z}^n.$$

Because the sum of two convex sets is a convex set, we have

$$\text{Conv}(C + \tau(\xi)^\vee) = \text{Conv}(C) + \tau(\xi)^\vee$$

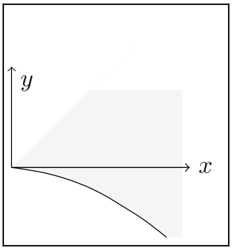
is an unbounded convex polytope. Moreover each unbounded facet of $\text{Conv}(C + \tau(\xi)^\vee)$ is the intersection of $\text{Conv}(C + \tau(\xi)^\vee)$ with one H_{u_σ} for some facet σ of $\tau(\xi)^\vee$. Therefore every unbounded facet of $\text{Conv}(C + \tau(\xi)^\vee)$ contains infinitely many elements of $\text{Supp}(\xi + f(x) + p(x))$. □

5 Some examples

Example 5.1 Let $E := \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} | y \geq -x - \sqrt{x}\}$ and let ξ be a Laurent series whose support is $\mathbb{Z}^2 \cap E$ as follows (Fig. 1):

Here $\tau(\xi)$ is the cone generated by $(1, 0)$ and $(1, 1)$, but $\text{Supp}(\xi) \subset \sigma$ where σ is the cone $\{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} | y > -x\}$. Since $\sigma \subsetneq \tau(\xi)^\vee$, ξ is not algebraic over $\mathbb{K}((x, y))$ by Corollary 1.4.

Fig. 2 Example 5.2



Moreover $\tau_1'(\xi)$ is equal to the rational cone generated by (1, 0) and (1, 1) minus the origin. So $\tau_1'(\xi)$ is not open. In this case $\mathbb{R}_{\geq 0}{}^n \setminus \{\underline{0}\} = \tau_0'(\xi) \cup \tau_1'(\xi)$.

Example 5.2 We consider the set

$$E := \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} | y \geq \ln(x+1)\}.$$

We rotate it by an angle of $-\pi/4$ and denote this set by Γ. We denote a Laurent series whose support is $\Gamma \cap \mathbb{Z}^2$ by ξ (see Fig. 2).

Then $\tau(\xi)^\vee$ is the cone generated by $(1, -1)$ and $(0, 1)$, so it is rational, but ξ is not algebraic as Theorem 1.5 is not satisfied.

Moreover $\tau(\xi)$ is generated by (1, 0) and (1, 1). Thus the vector (1, 1) is in the boundary of $\tau(\xi)$ but here $(1, 1) \in \tau_0'(\xi)$. Thus $\tau_0'(\xi)$ is closed.

Example 5.3 Let σ be the cone generated by the vectors (1, 0), (0, 1) and $(1, -1)$. Then the series $\xi := \sum_{k=0}^{\infty}(xy^{-1})^k$ has support in σ and it is straightforward to see that $\sigma = \tau(\xi)^\vee$. Let $N \in \mathbb{Z}^*$ and set $p_N(x, y) := \sum_{k=0}^{N-1}(xy^{-1})^k$ (when $N > 0$) or $p_N(x, y) = \sum_{k=N}^{0}(xy^{-1})^k$ (when $N < 0$). Let C_N denote the point $(N, -N)$. Then, we have

$$C_N \in \operatorname{Supp}(\xi - p_N(x, y)) \subset C_N + \sigma.$$

This shows that there is no canonical choice for C_N in Theorem 1.5, neither a minimal or maximal C_N.

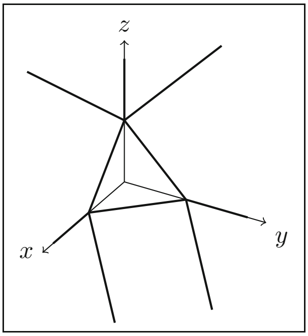
Example 5.4 Let C be the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and let σ be the cone generated by the vectors $(1, -1, 1)$, $(-1, 1, 1)$, and $(1, 1, -1)$. We can construct a Laurent series ξ, algebraic over $\mathbb{K}[[x, y, z]]$, with support in $\operatorname{Conv}(C) + \sigma$, such that all the unbounded faces of $\operatorname{Conv}(C) + \sigma$ contain infinitely many monomials of ξ as follows:

We fix an algebraic series $G(T)$ not in $\mathbb{K}(T)$. We remark that, for $a, b, c \in \mathbb{Z}$, the series $G(x^a y^b z^z)$ is algebraic over $\mathbb{K}(x, y, z)$, and it is a formal sum of monomials of the form $x^{ka} y^{kb} z^{kc}$ with $k \in \mathbb{N}$. Thus its support is included in the half line generated by the vector (a, b, c).

Then we set

$$\xi = G(x) + G(y) + zG(z) + zG\left(\frac{xz}{y}\right) + zG\left(\frac{yz}{x}\right) + (x+y)G\left(\frac{xy}{z}\right).$$

Fig. 3 Example 5.4



Then ξ is algebraic over $\mathbb{K}((x, y, z))$, its support is $\mathrm{Conv}(C)+\sigma$ and all the unbounded faces of $\mathrm{Conv}(C)+\sigma$ contain infinitely many monomials of ξ (see Fig. 3). Therefore $\tau(\xi)^\vee = \sigma$. Moreover we can see that there is no $\gamma \in \mathbb{R}^n$ such that $\mathrm{Supp}(\xi) \subset \gamma + \sigma$ and every face of $\gamma+\sigma$ contains infinitely many monomials of ξ, even after removing monomials of ξ belonging to ${\mathbb{R}_{\geq 0}}^3$. Indeed, if it were the case, the four unbounded edges of $\mathrm{Conv}(C)+\sigma$ that are not included in ${\mathbb{R}_{\geq 0}}^3$ would intersect at one point and this is clearly not the case. Thus we cannot assume that the finite set C of Theorem 1.5 is a single point.

6 The positive characteristic case

In positive characteristic, unlike the characteristic zero case, we cannot express roots of polynomials as Puiseux series with support in rational strongly convex cones. This already appears in the univariate case, since it has been noticed by Chevalley [10] that none of the roots of the polynomial $T^p - x_1^{p-1}T - x_1^{p-1}$ can be expressed as Puiseux series, when $p > 0$ denotes the characteristic of the base field. This shows that the Newton–Puiseux Theorem is no more valid in positive characteristic. Then Abhyankar noticed that for such a polynomial, the roots can be expressed as generalized series with support in $\mathbb{Q}$ with the additional property that their support is well-ordered [2]. Here such a root can be written as $\sum_{k\in\mathbb{N}^*} x_1^{1-\frac{1}{p^k}}$. The determination of the algebraic closure of $\mathbb{K}((x_1))$ for $n = 1$, when $\mathbb{K}$ is a positive characteristic field, was finally achieved very recently (see [19,20]).

For $n \geq 2$, this problem has recently been investigated by Saavedra [33]. He generalized Macdonald's Theorem to the positive characteristic case as follows:

Theorem 6.1 [33, Theorem 5.3]. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Let $\omega \in {\mathbb{R}_{>0}}^n$ be a vector whose coordinates are $\mathbb{Q}$-linearly independent. The set*

$$\mathcal{S}_{\omega}^{\mathbb{K}} = \Big\{ \xi \text{ series } | \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \text{ a } \leq_\omega \text{-positive rational cone,}$$
$$Supp(\xi) \subset (\gamma + \sigma) \cap \frac{1}{k} G(p) \text{ and } Supp(\xi) \text{ is } \leq_\omega \text{-well-ordered} \Big\},$$

where

$$G(p) = \bigcup_{\ell \in \mathbb{N}} \frac{1}{p^\ell} \mathbb{Z}^n,$$

is an algebraically closed field.

We give here a positive characteristic version of $\mathcal{S}_{\preceq}^{\mathbb{K}}$:

Definition 6.2 We fix an order $\preceq \in \mathrm{Ord}_n$ and a field $\mathbb{K}$ of characteristic $p > 0$. We set

$$\mathcal{S}_{\preceq}^{\mathbb{K}} := \{ \xi \text{ series } | \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \text{ a } \preceq \text{-positive rational cone containing } {\mathbb{R}_{\geq 0}}^n,$$
$$\text{such that } \mathrm{Supp}(\xi) \subset (\gamma + \sigma) \cap \frac{1}{k} G(p), \text{ and } \mathrm{Supp}(\xi) \text{ is } \preceq \text{-well-ordered} \}.$$

Then the following result, extending Theorem 6.1 is the positive characteristic analogue of Theorem 2.20:

Theorem 6.3 *Let $\preceq \in \mathrm{Ord}_n$ and $\mathbb{K}$ be an algebraically closed field of positive characteristic. Then the set $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is an algebraically closed field containing $\mathbb{K}((x))$.*

In order to prove this theorem we will use the notion of field-family introduced by Rayner:

Definition 6.4 [28]. A family $\mathcal{F}$ of subsets of an ordered abelian group $(G, \preceq)$ is said to be a field-family with respect to G if we have the following.

(1) Every element of $\mathcal{F}$ is a well-ordered subset of G.
(2) The elements of the members of $\mathcal{F}$ generate G as an abelian group.
(3) $\forall (A, B) \in \mathcal{F}^2$, $A \cup B \in \mathcal{F}$.
(4) $\forall A \in \mathcal{F}$ and $B \subset A$, $B \in \mathcal{F}$.
(5) $\forall (A, \gamma) \in \mathcal{F} \times G$, $\gamma + A \in \mathcal{F}$.
(6) $\forall A \in \mathcal{F}$, if A is $\preceq$-positive, the semigroup generated by A belongs to $\mathcal{F}$.

Theorem 6.5 [28, Theorem 2]. *If $\mathcal{F}$ is a field-family with respect to G then the set*

$$\left\{ \sum_{g \in G} a_g x^g | \{g | a_g \neq 0\} \in \mathcal{F} \right\}$$

is a Henselian valued field.

For $\preceq\in \mathrm{Ord}_n$ we set

$$\mathcal{F}_{\preceq} := \big\{A \subset \mathbb{Q}^n | \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \text{ a } \preceq\text{-positive rational cone containing } {\mathbb{R}_{\geq 0}}^n,$$
$$A \subset (\gamma + \sigma) \cap \frac{1}{k} G(p), \text{ and } A \text{ is } \preceq\text{-well-ordered}\,\big\}.$$

Proposition 6.6 *The set $\mathcal{F}_{\preceq}$ is a field-family with respect to $(\mathbb{Q}^n, \preceq)$.*

Proof It is straightforward to verify that $\mathcal{F}_{\preceq}$ satisfies the items (1), (2), (4) and (5) of Definition 6.4. For (3), if $A, B \in \mathcal{F}_{\preceq}$, we have

$$A \subset (\gamma_A + \sigma_A) \cap \frac{1}{k_A} G(p), \quad B \subset (\gamma_B + \sigma_B) \cap \frac{1}{k_B} G(p)$$

for some $\gamma_A, \gamma_B \in \mathbb{Z}^n$, σ_A and σ_B $\preceq$-positive rational cones containing ${\mathbb{R}_{\geq 0}}^n$ and k_A, $k_B \in \mathbb{N}^*$. We can replace k_A and k_B by their least common multiple and assume that $k_A = k_B$. We can also replace σ_A and σ_B by the cone σ gerenated by σ_A and σ_B. Since σ_A and σ_B are $\preceq$-positive and rational, σ is also $\preceq$-positive and rational. Finally we may assume that $\gamma_A = \gamma_B$ by Lemma 2.11. Moreover A and B are $\preceq$-well-ordered, thus $A \cup B$ is $\preceq$-well-ordered. This shows that (3) is satisfied.

Therefore we only prove (6) here. The proof is done by induction on n. In fact we will prove by induction on n, the following claim:

Claim: *For $A \subset (\gamma+\sigma)$, where $\gamma \in \mathbb{Z}^n$, σ is a a strongly convex rational cone, and A is $\preceq$-positive and $\preceq$-well-ordered, there exists a $\preceq$-positive rational cone $\sigma' \supset \sigma$ such that $A \subset \sigma'$.*

This claim, along with the following theorem, proves the proposition:

Theorem 6.7 [23, Theorem 3.4, p. 206]. *Let A be a well-ordered subset of an ordered group $(G, \preceq)$. If A is $\preceq$-positive, the semigroup generated by A is well-ordered.*

Let us consider a set A as in the claim.

If $n = 1$, there is only two orders on $\mathbb{Q}$. Both cases are symmetric, thus we may assume that $\preceq$ is the usual order $\leq$ on $\mathbb{Q}$ and $\sigma = \mathbb{R}_{\geq 0}$. Therefore we may assume that $\gamma = 0$ as $A \subset \mathbb{Q}_{\geq 0}$. In this case $\mathcal{U}_\sigma = \{\leq\}$. Since A is $\leq$-positive and $\leq$-well-ordered, $\langle A \rangle \subset \mathbb{Q}_{\geq 0}$ is also $\leq$-well-ordered by Theorem 6.7. This settles the case $n = 1$.

So from now on, assume that $n > 1$ and that the result is satisfied for $n - 1$.

We know that there exist nonzero vectors $(u_1, \ldots, u_s) \in (\mathbb{R}^n)^s$ and $(q_1, \ldots, q_r) \in (\mathbb{Q}^n)^r$ such that $\preceq=\leq_{(u_1,\ldots,u_s)}$ and $\sigma = \langle q_1, \ldots, q_r \rangle$.

Assume first that $\gamma \succeq \underline{0}$. Then $A \subset \sigma' = \langle \gamma, q_1, \ldots, q_r \rangle$ and σ' is a $\preceq$-positive rational cone. Hence A is included in $\sigma' \cap \frac{1}{k} G(p)$, and the claim is proved.

Now assume that $\gamma \prec \underline{0}$. By replacing σ by the cone generated by σ and $-\gamma$, we may assume that $\underline{0} \in \gamma + \sigma$. We define $a := \min(A \setminus \{\underline{0}\})$ and we set

$$H := \{u \in \mathbb{R}^n \text{ such that } u \cdot u_1 = a \cdot u_1\},$$
$$H^+ := \{u \in \mathbb{R}^n \text{ such that } u \cdot u_1 \geq a \cdot u_1\}.$$

By assumption, $a \succ \underline{0}$. Hence $a \cdot u_1 \geq 0$ because $\preceq=\leq_{(u_1,\ldots,u_s)}$.

Case 1: If $a \cdot u_1 > 0$ we define σ' to be the closure of the cone spanned by $H \cap (\gamma + \sigma)$. A vector $u \in \sigma'$ is not in the cone spanned by $H \cap (\gamma + \sigma)$ if and only if $\mathbb{R}_{\geq 0}u$ is the limit of halflines of the form $\mathbb{R}_{\geq 0}v_n$ with $v_n \in H \cap (\gamma + \sigma)$ and $(v_n)_n$ is not bounded. Therefore, we may assume that $v_n = \gamma + \lambda_n v'$ where $\lambda_n > 0$ and $v' \in \sigma$ is orthogonal to u_1. Therefore σ' is generated by the vertices of $H \cap (\gamma + \sigma)$ and the generators $\sigma \cap \langle u_1 \rangle^{\perp}$, in particular σ' is a rational cone.

Moreover σ' is a $\preceq$-positive cone, because $u \succeq 0$ for every $u \in H \cap (\gamma + \sigma)$, and $(\gamma + \sigma) \cap H^+ \subset \sigma'$. Finally it is clear that σ' is strongly convex: if $u, -u \in \sigma'$, since σ' is $\preceq$-positive, $u \in \sigma' \cap \langle u_1 \rangle^{\perp} = \sigma \cap \langle u_1 \rangle^{\perp}$; but σ is strongly convex, thus $u = 0$.

Now, if $a' \in A$, we have $a' \cdot u_1 \geq a \cdot u_1$, thus there is $1 \geq \lambda > 0$ such that $\lambda a' \in H$. But we have

$$\lambda a' = \gamma + \lambda(a' - \gamma) - \gamma \in \gamma + \sigma$$

because $-\gamma \in \sigma$ and $a' - \gamma \in \sigma$. Therefore, $\lambda a' \in \sigma'$ and $A \subset \sigma'$. Thus the claim is proved in this case.

Case 2: Assume that $a \cdot u_1 = 0$. We denote the set $A \cap H \cap \mathbb{Q}^n$ by B, and we set $a_1 := \min(A \setminus B)$. Since $A \subset \{\delta \in \mathbb{Q}^n | \delta \succeq \underline{0}\}$ and $a_1 \notin H$, we have $a_1 \cdot u_1 > 0$. By Case 1, there exists a strongly convex rational $\preceq$-positive cone σ_1 containing σ such that $A \setminus B \subset \sigma_1$.

We have that $H \cap \mathbb{Q}^n$ is a $\mathbb{Q}$-vector space of dimension $d < n$. We set $V := (H \cap \mathbb{Q}^n) \otimes_{\mathbb{Q}} \mathbb{R}$. We set $\sigma_2 := \sigma \cap V$ and we denote by $\preceq_V$ the restriction of $\preceq$ to V. Then σ_2 is a strongly convex rational $\preceq_V$-positive cone. If σ_2 is not full dimensional, we replace σ_2 by a strongly convex rational $\preceq_V$-positive cone that is full dimensional. Therefore, by Lemma 2.11, we have that $B \subset \gamma_2 + \sigma_2$ for some $\gamma_2 \in V$. Therefore, by the inductive hypothesis, there is a strongly convex rational cone $\preceq_V$-positive σ_3 such that $\sigma_2 \subset \sigma_3$ and $B \subset \sigma_3$.

Now we set $\sigma' := \sigma_1 + \sigma_3$. This cone is rational and $\preceq$-positive, thus it is strongly convex. Moreover it contains A, therefore the claim is proved. □

Proof of Theorem 6.3 By Proposition 6.6 and Theorem 6.5, the set $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is a Henselian valued field.

Assume that $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is not algebraically closed. Then, by [28, Lemma 4] there exists $a \in \mathcal{S}_{\preceq}^{\mathbb{K}}$ such that $T^p - T - a$ is irreducible in $\mathcal{S}_{\preceq}^{\mathbb{K}}[T]$. Let us write

$$a = a^+ + a^-$$

where $\mathrm{Supp}(a^-) \subset \{b \in \mathbb{Q}^n | b \prec \underline{0}\}$ and $\mathrm{Supp}(a^+) \subset \{b \in \mathbb{Q}^n | b \succeq \underline{0}\}$. Because the map $b \longmapsto b^p$ is an additive map, if ξ^+ is a root of $T^p - T - a^+$ and ξ^- is root of $T^p - T - a^-$, then $\xi^+ + \xi^-$ is a root of $T^p - T - a$. We will prove that $T^p - T - a^+$ and $T^p - T - a^-$ admit a root in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ contradicting the fact that $T^p - T - a$ is irreducible.

Since $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is a Henselian valued field,

$$\mathfrak{O} := \left\{ \xi \in \mathcal{S}_{\preceq}^{\mathbb{K}} | \forall b \in \mathrm{Supp}(\xi), b \succeq \underline{0} \right\}$$

is a Henselian local ring with maximal ideal

$$\mathfrak{m} := \left\{ \xi \in \mathcal{S}_{\preceq}^{\mathbb{K}} | \forall b \in \mathrm{Supp}(\xi), b \succ \underline{0} \right\}.$$

The polynomial $T^p - T - a^+ \in \mathfrak{O}[T]$ has a root modulo $\mathfrak{m}$ since $\mathbb{K}$ is algebraically closed (here $\mathfrak{O}/\mathfrak{m} = \mathbb{K}$). Moreover the derivative of this polynomial is -1. Thus this polynomial satisfies Hensel's Lemma and admits a root ξ^+ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$.

In order to prove that $T^p - T - a^-$ has a root in $\mathcal{S}_{\preceq}^{\mathbb{K}}$, we follow the proofs of [28, Theorem 3], and [33, Theorem 5.3]. We write $a^- = \sum_{q \in \mathbb{Q}^n} a_q^- x^q$ and we define

$$\xi^- := \sum_{q \in \mathbb{Q}^n} \left(\sum_{i=1}^{\infty} \left(a_{p^i q}^- \right)^{\frac{1}{p^i}} \right) x^q.$$

We can verify that ξ^- is well defined: for a given $q \in \mathrm{Supp}(a^-)$, the sequence $(p^i q)_i$ is strongly decreasing for the order $\preceq$ since $q \prec \underline{0}$. Therefore $a_{p^i q}^- = 0$ for i large enough because $\mathrm{Supp}(a^-)$ is $\preceq$-well-ordered. Hence the sum $\sum_{i=1}^{\infty} \left(a_{p^i q}^- \right)^{\frac{1}{p^i}}$ is in fact a finite sum.

Then we remark that

$$\mathrm{Supp}(\xi^-) \subset \bigcup_{i \in \mathbb{N}^*} \frac{1}{p^i} \mathrm{Supp}(a^-),$$

thus $\mathrm{Supp}(\xi^-)$ is $\preceq$-well-ordered by [33, Lemma 5.2].

Finally we claim that $\mathrm{Supp}(\xi^-)$ is contained in the translation of a rational $\preceq$-positive cone. In order to prove this we assume that $\mathrm{Supp}(a^-) \subset \gamma + \sigma$ where $\gamma \in \mathbb{Z}^n$ and σ is a rational $\preceq$-positive cone, and we denote by $\gamma_1, \ldots, \gamma_s \in \mathbb{Z}^n$ some generators of σ.

First we assume that $\gamma \succeq \underline{0}$. Let $\alpha \in \mathrm{Supp}(\xi^-)$, $\alpha = \frac{1}{p^i}\alpha'$ with $\alpha' \in \mathrm{Supp}(a^-)$. We have

$$\frac{1}{p^i}\alpha' + \gamma = \frac{1}{p^i}(\alpha' - \gamma) + \left(\frac{1}{p^i} + 1 \right) \gamma \in \sigma_1$$

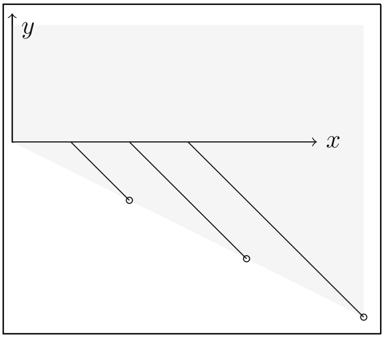
where σ_1 is the cone generated by the γ_i and γ. Thus $\alpha \in -\gamma + \sigma_1$, which proves the claim because σ_1 is rational and $\preceq$-positive.

Now assume that $\gamma \prec \underline{0}$ and consider $\alpha \in \mathrm{Supp}(\xi^-)$ written $\alpha = \frac{1}{p^i}\alpha'$ as before. Then $\frac{1}{p^i}(\alpha' - \gamma) \in \sigma$ because $\alpha' \in \gamma + \sigma$. Thus

$$\frac{1}{p^i}\alpha' - \gamma = \frac{1}{p^i}(\alpha' - \gamma) + \left(1 - \frac{1}{p^i} \right) (-\gamma) \in \sigma_2$$

where σ_2 is the cone generated by the γ_i and $-\gamma$. Thus $\alpha \in \gamma + \sigma_2$, which proves the claim because σ_2 is rational and $\preceq$-positive.

Fig. 4 Example 6.8



Moreover an easy computation shows that ξ^- is a root of $T^p - T - a^-$. This proves the theorem. □

6.1 Examples

We do not know if Theorem 1.3 remains valid for elements of $\mathcal{S}_{\preceq}^{\mathbb{K}}$ when $\mathbb{K}$ is positive characteristic field, but all the other results proved before in characteristic zero are non longer true in positive characteristic, as shown by the following examples:

Example 6.8 Let $\mathbb{K}$ be a field of characteristic $p > 0$. Set $f = \sum_{k=1}^{\infty} t^{1-\frac{1}{p^k}}$. The series f is algebraic over $\mathbb{K}(t)$ because $f^p - t^{p-1} f - t^{p-1} = 0$. Thus $g := \sum_{k=1}^{\infty} \left(\frac{x}{y}\right)^{1-\frac{1}{p^k}}$ is algebraic over $\mathbb{K}(x, y)$. We set $\xi = \sum_{k=1}^{\infty} (xg)^k$. Because $\xi = \frac{xg}{1-xg}$, ξ is rational over the field extension of $\mathbb{K}(x, y)$ by g. Hence ξ is algebraic over $\mathbb{K}(x, y)$.
We see that all the monomials of $(xg)^k$ are of the form $x^{k-l} y^l$ for $l \in \mathbb{Q}_{\geq 0}$. Therefore the support of ξ is included in the cone σ generated by $(2, -1)$ and $(0, 1)$ (see Fig. 4). Moreover the support of $(xg)^k$ contains a sequence of points converging to $(2k, -k)$. But $(2k, -k)$ does not belong to the support of ξ since $(1, -1)$ does not belong to the support of g. Hence $\tau(\xi) = \sigma^\vee$ is generated by $(1, 0)$ and $(1, 2)$.

But the conclusions of Theorem 1.5 and 1.6 do not hold in this case: there is no hyperplane $H_\lambda = \{(x, y) \in \mathbb{R}^2 | x + 2y = \lambda\}$ containing infinitely many elements of $\text{Supp}(\xi)$ such that $H_\lambda^- := \{(x, y) \in \mathbb{R}^2 | x + 2y < \lambda\}$ contains only finitely many elements of $\text{Supp}(\xi)$.

Here $\tau_0'(\xi) = \emptyset$. This shows that Lemma 4.2 is not valid in general for generalized series with exponents in $\mathbb{Q}^n$ that are algebraic over $\mathbb{K}((x))$, for a positive characteristic field $\mathbb{K}$.

Example 6.9 We set $a = \sum_{i=1}^{\infty} x^i y^{-1} \in \mathbb{F}_2((x, y))$ and $P(T) = T^2 + T + a$. For $i \in \mathbb{N}^*$ we also denote $P_i(T) = T^2 + T + x^i y^{-1}$. We consider an order $\preceq \in \text{Ord}_2$.

The roots of P_i in $\mathcal{S}_{\preceq}^{\mathbb{F}_2}$ are

$$\begin{cases} \xi_i^{(1)} \text{ and } \xi_i^{(1)} + 1, \text{ with } \xi_i^{(1)} = \sum_{k=1}^{\infty} x^{i2^{-k}} y^{-2^{-k}} \text{ when } (-i, 1) \succ \underline{0} \\ \xi_i^{(2)} \text{ and } \xi_i^{(2)} + 1, \text{ with } \xi_i^{(2)} = -\sum_{k=1}^{\infty} x^{i2^k} y^{-2^k} \text{ when } (i, -1) \succ \underline{0} \end{cases}$$

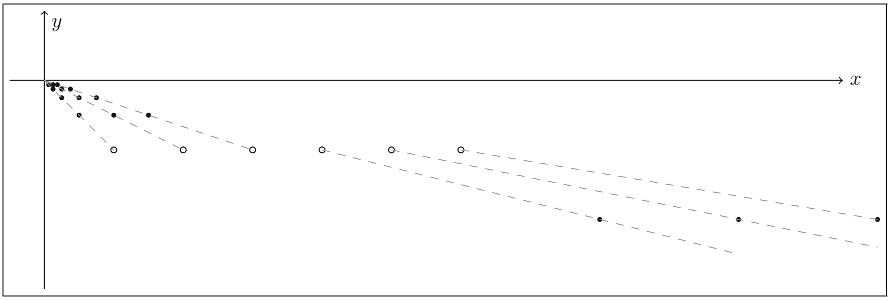


Fig. 5 Example 6.9

Let $i_0 = \sup\{i \in \mathbb{N}^* | (-i, 1) \succ \underline{0}\} \in \mathbb{N}^* \cup \{\infty\}$. Therefore the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{F}_2}$ are $\xi_{\preceq}$ and $\xi_{\preceq} + 1$ where

$$\xi_{\preceq} = \sum_{i=1}^{i_0} \xi_i^{(1)} + \sum_{i_0+1}^{\infty} \xi_i^{(2)}.$$

We can replace $P(T)$ by $\widetilde{P}(T) := T^2 + yT + \sum_{i=1}^{\infty} x^i \in \mathbb{K}[[x]][T]$ and remark that $\widetilde{P}(yT) = y^2 P(T)$, thus the roots of $P_1(T)$ are obtained from those of $P(T)$ by multiplication by y. This proves that Proposition 3.4 is no longer valid in positive characteristic (Fig. 5).

Example 6.10 Let $\mathbb{K} = \mathbb{F}_2$ be the field with two elements. The series

$$a(x, y) = x \sum_{k=1}^{\infty} \left(\frac{x}{y}\right)^{1-2^{-k}}$$

is algebraic over $\mathbb{F}_2(x, y)$. Thus the roots of $T^2 + T + a$ are algebraic over $\mathbb{F}_2(x, y)$. One of these roots is

$$\xi = -\sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \left(\frac{x^{2-2^{-k}}}{y^{1-2^{-k}}}\right)^{2^{\ell}} \in \mathcal{S}_{\preceq}^{\mathbb{F}_2}$$

where $\preceq \in \mathrm{Ord}_2$ is such that $(1, -1) \succ \underline{0}$. The support of ξ is given on Fig. 6 below. Thus, $\tau(\xi)^{\vee}$ is the cone generated by $(2, -1)$ and $(0, 1)$. Here $\tau_0(\xi)$ is not open since $(1, 1) \in \tau_0(\xi)$: here $\tau_0(\xi)$ is equal to the cone generated by $(1, 0)$ and $(1, 1)$ minus the origin. Thus Proposition 4.7 is no longer valid in positive characteristic. We remark that $\tau_0'(\xi) = \emptyset$.
On the following picture, the small circles indicate the terms of the support of $a(x, y)$, while the bullets indicate the terms of the support of ξ:

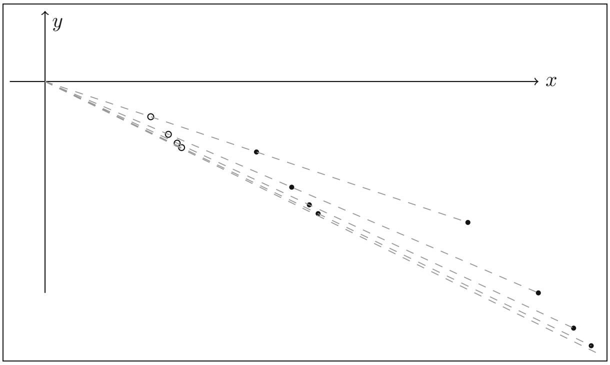


Fig. 6 Example 6.10

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