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ABSTRACT

We give examples showing that the usual Artin approximation theorems valid for convergent series over a field are no longer true for convergent series over a commutative Banach algebra. In particular, we construct an example of a commutative integral Banach algebra R such that the ring of formal power series over R is not flat over the ring of convergent power series over R .

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1. Introduction

The classical Artin Approximation Theorem is the following.

Theorem 1.1. [1] *Let $F(x, y)$ be a vector of convergent power series over \mathbb{C} in two sets of variables x and y . Assume given a formal power series solution $\hat{y}(x)$ vanishing at 0,*

$$F(x, \hat{y}(x)) = 0.$$

Then, for any $c \in \mathbb{N}$, there exists a convergent power series solution $y(x)$ vanishing at 0,

$$F(x, y(x)) = 0$$

which coincides with $\hat{y}(x)$ up to degree c ,

$$y(x) \equiv \hat{y}(x) \text{ modulo } (x)^c.$$

The main tools for proving this theorem are the implicit function theorem and the Weierstrass division theorem. But in the case the equations $F(x, y)$ are linear in y , this theorem is equivalent to the faithful flatness of the morphism $\mathbb{C}\{x\} \rightarrow \mathbb{C}[[x]]$ (see [14, Example 1.4] for instance or [4, I.3 Proposition 13]). In fact the faithful flatness of this morphism comes from the fact that $\mathbb{C}\{x\}$ is a Noetherian local ring. And the Noetherianity of $\mathbb{C}\{x\}$ is usually proved by using the Weierstrass division theorem.

Another version of this theorem is as follows.

Theorem 1.2. [2, 17] *Let $F(x, y)$ be a vector of convergent power series over \mathbb{C} in two sets of variables x and y . Then for any integer c there exists an integer β such that for any given approximate solution $\bar{y}(x)$ at order β , $\bar{y}(0) = 0$,*

$$F(x, \bar{y}(x)) \equiv 0 \text{ modulo } (x)^\beta,$$

there exists a formal power series solution $y(x)$ vanishing at 0,

$$F(x, y(x)) = 0$$

which coincides with $\bar{y}(x)$ up to degree c ,

$$y(x) \equiv \bar{y}(x) \text{ modulo } (x)^c.$$

In particular this result implies that, if $F(x, y) = 0$ has approximate solutions at any order, then it has a formal (even convergent by [Theorem 1.1](#)) power series solution.

Let us mention that these results remain valid when we replace \mathbb{C} by a complete valued field, or when we replace the ring of convergent power series over \mathbb{C} by the ring of algebraic power series over a field. In fact these results remain true in the more general setting of excellent Henselian local rings by [12, 13] (see [14] for a review of all these different results).

The aim of this note is to show that these results are no longer true when we replace \mathbb{C} by a commutative Banach algebra over \mathbb{R} or \mathbb{C} . In the first part we construct a commutative Banach algebra R such that $R\{t\} \rightarrow R[[t]]$ is not flat, showing that Artin approximation theorem is not true for linear equations with coefficients in $R\{t\}$.

Let us mention here that $R[[t]]$ is flat over R , for a commutative ring R , if and only if R is coherent (indeed $R[[t]]$ is a direct product of copies of R - see [5, Theorem 2.1]). And there are several known examples of Banach algebras which are not coherent (in fact most of the known Banach algebras are not coherent; see for instance [10] or [7] and the references herein). But the flatness of $R\{t\} \rightarrow R[[t]]$ is a different property that is not related to the coherence of R .

In the second part we provide an example of one polynomial $F(y)$ with coefficients in $R[t]$, where R is the Banach algebra of holomorphic functions over a disc, with the following property: $F(y)$ has approximate solutions up to any order but has no solution in $R[[t]]$. This shows that [Theorem 1.2](#) does not hold for convergent power series over a Banach algebra. Let us mention that this example is a slight modification of an example of Spivakovsky related to a similar problem [16].

Nevertheless we mention that in the case where R is a complete valuation ring of rank one (in particular a non-Archimedean Banach algebra), Schoutens and Moret-Bailly proved several extensions of [Theorems 1.1](#) and 1.2 (see [15] and [11]).

The note has been motivated by questions from Nefton Pali and Wei Xia.

2. A Banach algebra R such that $R\{t\} \rightarrow R[[t]]$ is not flat

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We begin by the following definition of power series in countable many indeterminates.

Definition 2.1. Let $\mathbb{N}^{(\mathbb{N})}$ be the submonoid of $\mathbb{N}^{\mathbb{N}}$ formed by the sequences whose all but finitely terms are 0. Let $(x_i)_{i \in \mathbb{N}}$ be a countable family of indeterminates. Then $\mathbb{K}[[x_i]]_{i \in \mathbb{N}}$ is the set of series $\sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} a_{\alpha} x^{\alpha}$ where $x^{\alpha} = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \cdots$. This former product is finite since $\alpha_i = 0$ for i large enough. This set is a commutative ring since the sum of sequences $\mathbb{N}^{(\mathbb{N})} \times \mathbb{N}^{(\mathbb{N})} \rightarrow \mathbb{N}^{(\mathbb{N})}$ has finite fibers (see [3, Chapter III, § 2, 11]). Let us mention that this ring is not the (x) -adic completion of $\mathbb{K}[x]$, the ring of polynomials in the x_i (see [18] for instance).

Let x, y, z and w_k for $k \in \mathbb{N}$ be indeterminates. For simplicity we denote by \underline{w} the vector of indeterminates (w_0, w_1, \dots) . We denote by $\mathbb{K}[x, y, z, \underline{w}]$ the ring of polynomials in the indeterminates x, y, z, \underline{w} .

For a polynomial $p = \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k, l, m, \alpha} x^k y^l z^m \underline{w}^{\alpha} \in \mathbb{K}[x, y, z, \underline{w}]$ we set

$$\|p\| := \sum_{k, l, m, \alpha} |a_{k, l, m, \alpha}|.$$

This is well defined because the sum is finite. This defines a norm on $\mathbb{K}[x, y, z, \underline{w}]$.

We denote by $\mathbb{K}\{x, y, z, \underline{w}\}$ the completion of $\mathbb{K}[x, y, z, \underline{w}]$ for this norm. This is the following commutative Banach algebra:

$$\left\{ \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k, l, m, \alpha} x^k y^l z^m \underline{w}^\alpha \mid \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} |a_{k, l, m, \alpha}| < \infty \right\}$$

and the norm of an element $f := \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k, l, m, \alpha} x^k y^l z^m \underline{w}^\alpha$ is

$$\|f\| := \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} |a_{k, l, m, \alpha}|.$$

In particular $\mathbb{K}\{x, y, z, \underline{w}\}$ is a subring of $\mathbb{K}[[x, y, z, w_i]]_{i \in \mathbb{N}}$.

We denote by I the ideal of $\mathbb{K}[x, y, z, \underline{w}]$ generated by the polynomials

$$xw_0 - z^2 \text{ and } yw_k - (k+1)xw_{k+1} \text{ for all } k \geq 0.$$

The ideal $I\mathbb{K}\{x, y, z, \underline{w}\}$ is not closed since it is not finitely generated. Thus, we denote by \bar{I} its closure. This is the set of sums

$$\sum_{k \in \mathbb{N}} f_k(x, y, z, \underline{w})$$

such that $f_k(x, y, z, \underline{w}) \in I\mathbb{K}\{x, y, z, \underline{w}\}$ and $\sum_k \|f_k(x, y, z, \underline{w})\| < \infty$.

Definition 2.2. We denote by R the Banach \mathbb{K} -algebra $\mathbb{K}\{x, y, z, \underline{w}\}/\bar{I}$.

In order to denote that two series f and $g \in \mathbb{K}\{x, y, z, \underline{w}\}$ have the same image in R , we write $f \equiv_R g$. The norm of the image \bar{f} of an element $f \in \mathbb{K}\{x, y, z, \underline{w}\}$ is

$$\|\bar{f}\| = \inf_{g \in \bar{I}} \|f + g\| = \inf_{g \in \bar{I}} \|f + g\|.$$

Now we denote by $R\{t\}$ the ring of convergent series in the indeterminate t with coefficients in R . We have the following result.

Proposition 2.3. *The linear equation*

$$(x - yt)f(t) = z^2 \tag{2.1}$$

has a unique solution $f(t)$ in $R[[t]]$ and this solution is not convergent.

From this we will deduce the following result.

Theorem 2.4. *The Banach \mathbb{K} -algebra R is an integral domain and the morphism $R\{t\} \rightarrow R[[t]]$ is not flat.*

2.1. Proofs of Proposition 2.3 and Theorem 2.4

We begin by giving the following key result.

Lemma 2.5. *x is not a zero divisor in R .*

Proof. First of all, we will determine a subset of $\mathbb{K}\{x, y, z, \underline{w}\}$ such that every element of $\mathbb{K}\{x, y, z, \underline{w}\}$ is equal modulo \bar{I} to a unique series of this subset.

First we remark that

$$M_1 := yw_i w_j \equiv_R (i+1)xw_{i+1} w_j \equiv_R \frac{i+1}{j} yw_{i+1} w_{j-1} =: M_2 \tag{2.2}$$

for all integers i and j with $i < j$. If $j = i + 1$ these two monomials are equal, otherwise the largest index of a monomial w_j appearing in the expression of M_2 is strictly less than for M_1 .

Now we have, for $i > 0$:

$$z^2 w_i \equiv_R x w_0 w_i \equiv_R \frac{1}{i} y w_0 w_{i-1}. \quad (2.3)$$

A well-chosen composition of these operations transforms any monomial of the form $Cx^a z^k y^l w_0^{n_0} \dots w_i^{n_i}$ into a monomial of the form $rCx^{a'} z^{k'} y^l w_0^{n_0} \dots w_j^{n_j}$ where j is minimal and $r \in (0, 1]$.

By repeating these two operations we may reduce every monomial to a constant times one of the following monic monomials:

$$\begin{cases} z^\varepsilon x^a y^l w_0^{n_0} & \text{with } a > 1, l, n_0 \geq 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon y^l w_i^{n_i} & \text{with } l > 0, i > 0, n_i > 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon y^l w_i^{n_i} w_{i+1}^{n_{i+1}} & \text{with } l > 0, i \geq 0, n_i, n_{i+1} > 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon w_0^{n_0} \dots w_i^{n_i} & \text{with } n_i > 0 \text{ with } \varepsilon \in \{0, 1\}. \end{cases} \quad (2.4)$$

We denote by E the subset of $\mathbb{K}[x, y, z, \underline{w}]$ of polynomials that are sums of monomials of (2.4) (up to multiplicative constants), and by \bar{E} the closure of E in $\mathbb{K}\{x, y, z, \underline{w}\}$, that is the set of convergent power series whose non zero monomials are those of (2.4) (up to multiplicative constants). We have shown that every polynomial is equivalent to a polynomial of E modulo I . To prove the unicity we proceed as follows.

We set

$$\begin{aligned} F_0 &:= xw_0 - z^2, & F_{k+1} &:= yw_k - (k+1)xw_{k+1} \text{ for } k \geq 0 \\ G_{k,l} &:= (l+1)yw_k w_{l+1} - (k+1)yw_l w_{k+1} \text{ for all } k < l. \end{aligned}$$

Then we consider the following monomial order: We define

$$x^a y^k z^l w_1^{\alpha_1} \dots w_n^{\alpha_n} > x^{a'} y^{k'} z^{l'} w_1^{\alpha'_1} \dots w_n^{\alpha'_n}$$

if

$$\begin{aligned} a + k + l + \sum_i \alpha_i &> a' + k' + l' + \sum_i \alpha'_i, \text{ or } a + k + l + \sum_i \alpha_i = a' + k' + l' + \sum_i \alpha'_i \\ \text{and } (l, a, k, \alpha_n, \dots, \alpha_0) &>_{\text{lex}} (l', a', k', \alpha'_n, \dots, \alpha'_0) \end{aligned}$$

where $>_{\text{lex}}$ denotes the lexicographic order. That is, we first compare the total degree of two monomials, then we order the indeterminates as

$$z > x > y > w_l > w_k \text{ for all } l > k.$$

We claim that $\{F_j, G_{k,l}\}_{j,k,l \in \mathbb{N}, l > k}$ is a Gröbner basis of I for this order. In order to prove this, we only need to compute the S-polynomials of the elements of this set of polynomials, and then their reduction (see [6] for the terminology). This is Buchberger's Algorithm which is very classical in the Noetherian case. The case of polynomial rings in countably many indeterminates works identically, cf. [8, Proposition 1.13] for instance. The only S-polynomials we have to consider are those of polynomials whose leading terms are not coprime, that is, for $l > k$,

$$S(F_{k+1}, F_{l+1}); S(G_{k,l}, F_{l+1}); S(G_{k,l}, F_k).$$

We have $S(F_{k+1}, F_{l+1}) = G_{k,l}$. Moreover

$$S(G_{k,l}, F_{l+1}) = y(yw_k w_l - (k+1)xw_l w_{k+1}).$$

This leading term of $S(G_{k,l}, F_{l+1})$ is $-(k+1)xyw_lw_{k+1}$, and it is equal to $y(F_{k+1}w_l - yw_kw_l)$. Therefore $S(G_{k,l}, F_{l+1}) = F_{k+1}yw_l$.

Finally we have

$$\begin{aligned} S(G_{k,l}, F_k) &= kx((l+1)yw_kw_{l+1} - (k+1)yw_lw_{k+1}) + (l+1)yw_{l+1}(yw_{k-1} - kxw_k) \\ &= (l+1)y^2w_{k-1}w_{l+1} - k(k+1)xyw_lw_{k+1}. \end{aligned}$$

Its leading term is $-k(k+1)xyw_lw_{k+1}$ and it is divisible by the leading term of F_{k+1} . The remainder of the division of $S(G_{k,l}, F_k)$ by F_{k+1} is

$$(l+1)y^2w_{k-1}w_{l+1} - ky^2w_kw_l = yG_{k-1,l}.$$

Therefore the reductions of these S-polynomials is always zero, hence the family $\{F_j, G_{k,l}\}_{j,k,l \in \mathbb{N}, l > k}$ is a Gröbner basis of I . Thus, the initial ideal of I is generated by the monomials

$$z^2, xw_{k+1}, yw_kw_{l+1} \text{ for } 0 \leq k < l.$$

Therefore every polynomial of $\mathbb{K}[x, y, z, \underline{w}]$ is equivalent modulo I to a unique polynomial of E .

Now let $f \in \mathbb{K}\{x, y, z, \underline{w}\}$. We can write $f = \sum_{n \in \mathbb{N}} C_n x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n}$ where the C_n are in \mathbb{K}^* . In particular $\sum_n |C_n| < \infty$. For every $n \in \mathbb{N}$, there is a unique $(a'_n, b'_n, c'_n, \alpha'_n)$ and a unique $r_n \in (0, 1]$ such that

$$C_n x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n} - r_n C_n x^{a'_n} y^{b'_n} z^{c'_n} \underline{w}^{\alpha'_n} \in I$$

and $x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n}$ has one the forms given in (2.4). Now, for every $n \in \mathbb{N}$, we set

$$g_n := \sum_{k=0}^{n-1} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} \underline{w}^{\alpha'_k} + \sum_{k \geq n} C_k x^{a_k} y^{b_k} z^{c_k} \underline{w}^{\alpha_k}.$$

In particular we have that $P_n := f - g_n \in I$ and the sequence $(g_n)_n$ converges in $\mathbb{K}\{x, y, z, \underline{w}\}$ to the series $g = \sum_{k \in \mathbb{N}} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} \underline{w}^{\alpha'_k} \in \mathbb{K}\{x, y, z, \underline{w}\}$. Therefore the sequence $(P_n)_n$ converges in $\mathbb{K}\{x, y, z, \underline{w}\}$, and its limits is in \bar{I} .

Therefore, every power series of $\mathbb{K}\{x, y, z, \underline{w}\}$ can be written as a sum of a power series in \bar{I} and a convergent power series whose monomials are as in (2.4) (up to multiplicative constants).

We remark that, by repeating (2.2) $\lfloor \frac{j-i}{2} \rfloor$ times, we have

$$yw_i w_j \equiv_R r y w_{i + \lfloor \frac{j-i}{2} \rfloor} w_{j - \lfloor \frac{j-i}{2} \rfloor}$$

for some constant $r \in (0, 1]$. Moreover applying (2.3) reduces by 2 the degree in z of a monomial. Therefore, a monomial of the form

$$Cx^a y^b z^c w_1^{\alpha_1} \cdots w_j^{\alpha_j}$$

of total degree $d = a + b + c + \sum_k \alpha_k$, is not equal to a monomial involving only the indeterminates

$$x, y, z, \text{ and } w_i \text{ for } i < \frac{j - \frac{c}{2}}{2^d}.$$

Moreover (2.2) and (2.3) transforms monomials into monomials of the same degree since I is generated by homogeneous binomials. Therefore, given a monomial M among those of (2.4) (up to some multiplicative constant), there is finitely many monomials that are equal to M modulo I .

Now let $f \in \bar{E} \cap \bar{I}$, $f = \sum_{(a,b,c,\alpha)} f_{(a,b,c,\alpha)} x^a y^b z^c \underline{w}^\alpha$. Let us fix such (a, b, c, α) that $x^a y^b z^c \underline{w}^\alpha$ is one of the monic monomials of (2.4). There is only a finite number of distinct monomials that are equal to $f_{(a,b,c,\alpha)} x^a y^b z^c \underline{w}^\alpha$ modulo I . Let us denote them by

$$C_1 x^{a_1} y^{b_1} z^{c_1} \underline{w}^{\alpha_1}, \dots, C_N x^{a_N} y^{b_N} z^{c_N} \underline{w}^{\alpha_N}.$$

We can remark that there is only a finite number of F_l that have a monomial that divides at least one of the following monic monomials

$$x^a y^b z^c \underline{w}^\alpha, x^{a_1} y^{b_1} z^{c_1} \underline{w}^{\alpha_1}, \dots, x^{a_N} y^{b_N} z^{c_N} \underline{w}^{\alpha_N}. \quad (2.5)$$

We denote them by F_{l_1}, \dots, F_{l_p} . Because $f \in \bar{I}$, we can write $f = \sum_{l \in \mathbb{N}} f_l F_l$ where the f_l are in $\mathbb{K}[[x, y, z, \underline{w}]]$. For every $i \in \{1, \dots, p\}$ we remove from f_{l_i} all the monomials that do not divide one of the monomials (2.5), and we denote by f'_{l_i} the resulting polynomial. Then we have that

$$P := \sum_{i=1}^p f'_{l_i} F_{l_i} \in I.$$

By construction the coefficients of the monomials (2.5) in the expansion of P are the corresponding coefficients in the expansion of f , that is

$$f_{(a, b, c, \alpha)}, 0, \dots, 0$$

respectively. Therefore, the coefficient of $x^a y^b z^c \underline{w}^\alpha$ in the expansion of the unique $Q \in E$ such that $Q \equiv_R P$, is equal to $f_{(a, b, c, \alpha)}$ because no other monomial than those listed in (2.5) (up to some multiplicative constants) is equivalent to a monomial of the form $C x^a y^b z^c \underline{w}^\alpha$ where $C \in \mathbb{K}^*$. But $Q = 0$ since $P \in I$, thus $f_{(a, b, c, \alpha)} = 0$. Hence $f = 0$ and $\bar{E} \cap \bar{I} = 0$.

Therefore every series of $\mathbb{K}\{x, y, z, \underline{w}\}$ is equivalent modulo \bar{I} to a unique series of \bar{E} .

Now take $f \in \mathbb{K}\{x, y, z, \underline{w}\}$ such that $x \equiv_R 0$. We can write $f = xp(x, y, z, w_0) + q(y, z, \underline{w})$ and assume that the monomials in the expansion of $xp(x, y, z, w_0) + q(y, z, \underline{w})$ are only those of (2.4). Then

$$x^2 p(x, y, z, w_0) + xq(y, z, \underline{w}) \equiv_R 0.$$

The representation of $x^2 p(x, y, z, w_0) + xq(y, z, \underline{w})$ as a sum of monomials as in (2.4) has the form

$$x^2 p(x, y, z, w_0) + xq(y, z, w_0, 0) + \bar{q}(y, z, \underline{w}) = 0 \quad (2.6)$$

where $\bar{q}(y, z, \underline{w})$ is the series obtained from $xq(y, z, \underline{w}) - xq(y, z, w_0, 0)$ by replacing the monomials as follows (using the two previous operations (2.2) and (2.3)):

$$\left\{ \begin{array}{ll} xz^e y^l w_i^{n_i} & \mapsto \frac{1}{i} z^e y^{l+1} w_{i-1} w_i^{n_i-1}, \text{ if } i > 0 \\ xz^e y^l w_i^{n_i} w_{i+1}^{n_{i+1}} & \mapsto \frac{1}{i+1} z^e y^{l+1} w_i^{n_i+1} w_{i+1}^{n_{i+1}-1}, \text{ if } i > 0 \\ xz^e w_0^{n_0} \dots w_i^{n_i} & \mapsto Cz^e y w_j^{m_j} w_{j+1}^{m_{j+1}} \text{ or } Cz^e y w_j^{m_j} \\ \text{for } i > 0 \text{ and } n_i > 0 & \text{for some } C \in \mathbb{K}, |C| \leq 1, j \geq 0. \end{array} \right. \quad (2.7)$$

Indeed for the third monomial we have

$$xz^e w_0^{n_0} \dots w_i^{n_i} \equiv_R \frac{1}{i+1} z^e y w_0^{n_0} \dots w_{i-1}^{n_{i-1}+1} w_i^{n_i-1}$$

and this monomial on the right side can be transformed into a monomial of the form $Cz^e y w_j^{m_j} w_{j+1}^{m_{j+1}}$ or $Cz^e y w_j^{m_j}$ for some $C \in \mathbb{K}, |C| \leq 1$, and $j \geq 0$, by using the two operations (2.2) and (2.3) on monomials.

This shows that the three types of monomials that we obtain after multiplication by x are all distinct, that is the map defined by (2.7) is injective. By (2.6) we have $\bar{q}(y, z, \underline{w}) = 0$, therefore $q(y, z, \underline{w}) - q(y, z, w_0, 0) = 0$.

Moreover, again by (2.6), we have

$$x^2p(x, y, z, w_0) + xq(y, z, w_0, 0) = 0.$$

This shows that $x^2p(x, y, z, w_0) + xq(y, z, \underline{w}) = 0$. Therefore x is not a zero divisor in R . \square

Proof of Proposition 2.3. Let $f(t) \in R[[t]]$ such that

$$(x - yt)f(t) = z^2.$$

By writing $f = \sum_{k=0}^{\infty} f_k t^k$ with $f_k \in R$ for every k , we have

$$\begin{aligned} xf_0 &= z^2 \\ xf_k - yf_{k-1} &= 0 \quad \forall k \geq 1. \end{aligned}$$

Thus

$$xf_0 = z^2 = xw_0$$

so $x(f_0 - w_0) = 0$ and $f_0 = w_0$ by Lemma 2.5. Then we will prove by induction on k that $f_k = k! w_k$ for every k . Assume that this is true for an integer $k \geq 0$. Then we have

$$xf_{k+1} = yf_k = k! yw_k = (k+1)! xw_{k+1}.$$

Hence $x(f_{k+1} - (k+1)! w_{k+1}) = 0$ and $f_{k+1} = (k+1)! w_{k+1}$ by Lemma 2.5. Therefore, the only solution of

$$(x - yt)f(t) = z^2$$

is the series $\sum_{k=0}^{\infty} k! w_k t^k$, and this one is divergent because $\|w_k\| = 1$. This holds because in every element of I , the monomial w_k has coefficient 0. \square

Now we can give the proof of Theorem 2.4.

Proof of Theorem 2.4. Since x is not a zero divisor in R by Lemma 2.5, the localization morphism

$$R \rightarrow R_{1/x}$$

is injective. But $R_{1/x}$ is isomorphic to $\mathbb{K}\{x, y, z\}_{1/x}$ since in $R_{1/x}$ we have

$$w_0 = z^2/x \text{ and } \forall k \geq 0, w_k = \frac{1}{k!} y^k z^2 x^{k+1}.$$

But $\mathbb{K}\{x, y, z\}_{1/x}$ is an integral domain (this is a localization of the integral domain $\mathbb{K}\{x, y, z\}$), therefore so is R .

Now assume that the morphism $R\{t\} \rightarrow R[[t]]$ is flat. By [9, Theorem 7.6] applied to the linear equation $(x - yt)F - z^2G = 0$, there exist an integer $s \geq 1$, and convergent series

$$a_1(t), \dots, a_s(t), b_1(t), \dots, b_s(t) \in R\{t\}$$

such that

$$(x - yt)a_i(t) - z^2b_i(t) = 0 \text{ for every } i, \quad (2.8)$$

and formal power series

$$h_1(t), \dots, h_s(t) \in R[[t]]$$

such that

$$f(t) = \sum_{i=1}^s a_i(t)h_i(t), 1 = \sum_{i=1}^s b_i(t)h_i(t).$$

Indeed the vector $(f(t), 1)$ is a solution of the linear equation

$$(x - yt)f(t) - z^2g(t) = 0$$

with $f(t) := \sum_{k=0}^{\infty} k! w_k t^k$.
Then

$$\tilde{g}(t) := \sum_{i=1}^s b_i(t)h_i(0) = 1 + t\varepsilon(t)$$

for some $\varepsilon(t) \in R\{t\}$. Since 1 is a unit of R , $1 + t\varepsilon(t)$ is a unit in $R\{t\}$.

Set $\tilde{f}(t) := \sum_i a_i(t)h_i(0)$. By (2.8), $(\tilde{f}(t), \tilde{g}(t))$ is a solution of the equation

$$(x - yt)\tilde{f}(t) - z^2\tilde{g}(t) = 0.$$

Since $\tilde{g}(t)$ is a unit in $R\{t\}$ we have

$$(x - yt)\tilde{f}(t)\tilde{g}(t)^{-1} = z^2.$$

This contradicts Theorem 2.3. Therefore $R\{t\} \rightarrow R[[t]]$ is not flat. □

3. An example concerning the strong Artin approximation theorem

Let n be a positive integer, $x = (x_1, \dots, x_n)$ and $\rho > 0$. We set $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Then

$$B_\rho^n := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mid \|f\|_\rho := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| \rho^{|\alpha|} < \infty \right\}$$

is a Banach space equipped with the norm $\|\cdot\|_\rho$. Of course $\mathbb{K}[x] \subset B_\rho^n$.

Remark 3.1. We do not have

$$B_\rho^n[[t]] \cap \mathbb{K}\{x, t\} = B_\rho^n\{t\}.$$

For instance, the power series

$$f = \sum_{k \in \mathbb{N}} x_1^{k!} t^k$$

is a convergent power series in (x, t) , belongs to $B_2^n[[t]]$, but

$$\sum_k \|x_1^{k!}\| 2\tau^k = \sum_k 2^{k!} \tau^k = \infty$$

for every $\tau > 0$. Therefore $f \notin B_2^n\{t\}$.

We provide two examples based on an example of Spivakovsky concerning the extension of Theorem 1.2 to the nested case (see [16]).

Example 3.2. Let $n = 1$ and set

$$F(x, t, y_1, y_2) := xy_1^2 - (x + t)y_2^2 \in B_\rho\{t\}[y_1, y_2].$$

Let

$$\sqrt{1+t} = 1 + \sum_{n \geq 1} a_n t^n \in \mathbb{Q}\{t\}$$

be the unique power series such that $(\sqrt{1+t})^2 = 1+t$ and whose value at the origin is 1. For every $c \in \mathbb{N}$ we set $y_2^{(c)}(t) := x^c$ and $y_1^{(c)}(t) := x^c + \sum_{n=1}^c a_n x^{c-n} t^n \in B_\rho\{t\}$. Then

$$F(x, t, y_1^{(c)}(t), y_2^{(c)}(t)) \in (t)^{c+1}.$$

On the other hand the equation $f(x, t, y_1(t), y_2(t)) = 0$ has no solution $(y_1(t), y_2(t)) \in B_\rho\{t\}^2$ but $(0, 0)$. Indeed let us denote by T_0 the Taylor map at 0:

$$T_0 : B_\rho\{t\} \rightarrow \mathbb{K}[[x, t]].$$

If $f(x, t, y_1(t), y_2(t)) = 0$ then

$$xT_0(y_1(t))^2 - (x+t)T_0(y_2(t))^2 = 0.$$

But since $\mathbb{K}[[x, t]]$ is a unique factorization domain, this equality implies that $T_0(y_1(t)) = T_0(y_2(t)) = 0$, hence $y_1(t) = y_2(t) = 0$.

This shows that there is no $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $y(t) \in B_\rho\{t\}^2$ and every $k \in \mathbb{N}$ with

$$F(x, t, y(t)) \in (t)^{\beta(k)}$$

there exists $\tilde{y}(t) \in B_\rho\{t\}^2$ such that

$$F(x, t, \tilde{y}(t)) = 0$$

and $\tilde{y}(t) - y(t) \in (t)^k$.

Example 3.3. We can modify a little bit the previous example to construct a F as before that does not depend on t . We set

$$G(x, y_1, y_2, y_3) := xy_1^2 - (x + y_3)y_2^2 \in B_\rho[y_1, y_2, y_3].$$

For every $c \in \mathbb{N}$ we set $y_2^{(c)}(t) := x^c, y_1^{(c)}(t) := x^c + \sum_{n=1}^c a_n x^{c-n} t^n$ and $y_3^{(c)}(t) := t \in B_\rho\{t\}$. Then

$$G(x, y_1^{(c)}(t), y_2^{(c)}(t), y_3^{(c)}(t)) \in (t)^c.$$

Now if $\tilde{y}(t) \in B_\rho\{t\}^3$ satisfies $G(x, \tilde{y}(t)) = 0$ and

$$\tilde{y}(t) - y(t) \in (t)^2$$

then $\tilde{y}_3(t) = x + t + \varepsilon(t)$ with $\varepsilon(t) \in (t^2)$. Thus $x + \tilde{y}_3(t)$ is an irreducible power series in x and t , and it is coprime with x . By the same argument based on the Taylor map as in [Example 3.2](#), the relation

$$x\tilde{y}_1(t)^2 - (x + t + \varepsilon(t))\tilde{y}_2(t)^2 = 0$$

implies that $\tilde{y}_1(t) = \tilde{y}_2(t) = 0$.

This shows that there is no $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $y(t) \in B_\rho\{t\}^3$ and every $k \in \mathbb{N}$ with

$$G(x, y(t)) \in (t)^{\beta(k)}$$

there exists $\tilde{y}(t) \in B_\rho\{t\}^3$ such that

$$G(x, \tilde{y}(t)) = 0$$

and $\tilde{y}(t) - y(t) \in (t)^k$.

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