

ARTIN APPROXIMATION OVER BANACH SPACES

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ABSTRACT. We give examples showing that the usual Artin Approximation theorems valid for convergent series over a field are no longer true for convergent series over a commutative Banach algebra. In particular we construct an example of a commutative integral Banach algebra R such that the ring of formal power series over R is not flat over the ring of convergent power series over R .

1. INTRODUCTION

The classical Artin Approximation Theorem is the following:

Theorem 1.1. [Ar68] *Let $F(x, y)$ be a vector of convergent power series over \mathbb{C} in two sets of variables x and y . Assume given a formal power series solution $\widehat{y}(x)$ vanishing at 0,*

$$F(x, \widehat{y}(x)) = 0.$$

Then, for any $c \in \mathbb{N}$, there exists a convergent power series solution $y(x)$ vanishing at 0,

$$F(x, y(x)) = 0$$

which coincides with $\widehat{y}(x)$ up to degree c ,

$$y(x) \equiv \widehat{y}(x) \text{ modulo } (x)^c.$$

The main tools for proving this theorem are the implicit function theorem and the Weierstrass division theorem. But in the case the equations $F(x, y)$ are linear in y , this theorem is equivalent to the faithful flatness of the morphism $\mathbb{C}\{x\} \rightarrow \mathbb{C}[[x]]$ (see [Ro17, Example 1.4] for instance or [Bo85, I. 3 Proposition 13]). In fact the faithful flatness of this morphism comes from the fact that $\mathbb{C}\{x\}$ is a Noetherian local ring. And the Noetherianity of $\mathbb{C}\{x\}$ is usually proved by using the Weierstrass division theorem.

Another version of this theorem is the following one:

Theorem 1.2. [Ar69][Wa75] *Let $F(x, y)$ be a vector of convergent power series over \mathbb{C} in two sets of variables x and y . Then for any integer c there*

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exists an integer β such that for any given approximate solution $\bar{y}(x)$ at order β , $\bar{y}(0) = 0$,

$$F(x, \bar{y}(x)) \equiv 0 \text{ modulo } (x)^\beta,$$

there exists a formal power series solution $y(x)$ vanishing at 0,

$$F(x, y(x)) = 0$$

which coincides with $\bar{y}(x)$ up to degree c ,

$$y(x) \equiv \bar{y}(x) \text{ modulo } (x)^c.$$

In particular this result implies that, if $F(x, y) = 0$ has approximate solutions at any order, then it has a formal (even convergent by Theorem 1.1) power series solution.

Let us mention that these results remain valid when we replace \mathbb{C} by a complete valued field, or when we replace the ring of convergent power series over \mathbb{C} by the ring of algebraic power series over a field. In fact these results remain true in the more general setting of excellent Henselian local rings by [Po86] (see [Ro17] for a review of all these different results).

The aim of this note is to show that these results are no longer true when we replace \mathbb{C} by a commutative Banach algebra over \mathbb{R} or \mathbb{C} . In the first part we construct a commutative Banach algebra R such that $R\{t\} \rightarrow R[[t]]$ is not flat, showing that Artin approximation theorem is not true for linear equations with coefficients in $R\{t\}$.

Let us mention here that $R[[t]]$ is flat over R , for a commutative ring R , if and only if R is coherent (indeed $R[[t]]$ is a direct product of copies of R - see [Ch60, Theorem 2.1]). And there are several known examples of Banach algebras which are not coherent (in fact most of the known Banach algebras are not coherent; see for instance [MVR76] or [Hi90] and the references herein). But the flatness of $R\{t\} \rightarrow R[[t]]$ is a different property that is not related to the coherence of R .

In the second part we provide an example of one polynomial $F(y)$ with coefficients in $R[t]$, where R is the Banach algebra of holomorphic functions over a disc, with the following property: $F(y)$ has approximate solutions up to any order but has no solution in $R[[t]]$. This shows that Theorem 1.2 does not hold for convergent power series over a Banach algebra. Let us mention that this example is a slight modification of an example of Spivakovsky related to a similar problem [Sp94].

Nevertheless we mention that in the case where R is a complete valuation ring of rank one (in particular a non-archimedean Banach algebra), Schoutens and Moret-Bailly proved several extensions of Theorems 1.1 and 1.2 (see [Sc88] and [MB12]).

The note has been motivated by questions from Nefton Pali and Wei Xia.

2. A BANACH ALGEBRA R SUCH THAT $R\{t\} \rightarrow R[[t]]$ IS NOT FLAT

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We begin by the following definition of power series in countable many indeterminates:

Definition 2.1. Let $\mathbb{N}^{(\mathbb{N})}$ be the subgroup of $\mathbb{N}^{\mathbb{N}}$ formed by the sequences such that all but finitely terms are 0. Let $(x_i)_{i \in \mathbb{N}}$ be a countable family of indeterminates. Then $\mathbb{K}[[x_i]]_{i \in \mathbb{N}}$ is the set of series $\sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} a_\alpha x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdots$. This former product is finite since $\alpha_i = 0$ for i large enough. This set is a commutative ring since the product of sequences $\mathbb{N}^{(\mathbb{N})} \times \mathbb{N}^{(\mathbb{N})}$ has finite fibers (see [Bo70, Chapter III, §, 11]). Let us mention that this ring is not the (x) -adic completion of $\mathbb{K}[x]$, the ring of polynomials in the x_i (see [Ye11] for instance).

Let x, y, z and w_k for $k \in \mathbb{Z}_{\geq 0}$ be indeterminates. For simplicity we denote by \underline{w} the vector of indeterminates (w_0, w_1, \dots) . We denote by $\mathbb{K}[x, y, z, \underline{w}]$ the ring of polynomials in the indeterminates x, y, z, \underline{w} .

For a polynomial $p = \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha} x^k y^l z^m \underline{w}^\alpha \in \mathbb{K}[x, y, z, \underline{w}]$ we set

$$\|p\| := \sum_{k,l,m,\alpha} |a_{k,l,m,\alpha}|.$$

This is well defined because the sum is finite. This defines a norm on $\mathbb{K}[x, y, z, \underline{w}]$.

We denote by $\mathbb{K}\{x, y, z, \underline{w}\}$ the completion of $\mathbb{K}[x, y, z, \underline{w}]$ for this norm. This is the following commutative Banach algebra:

$$\left\{ \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha} x^k y^l z^m \underline{w}^\alpha \mid \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} |a_{k,l,m,\alpha}| < \infty \right\}$$

and the norm of an element $f := \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha} x^k y^l z^m \underline{w}^\alpha$ is

$$\|f\| := \sum_{k \in \mathbb{N}, l \in \mathbb{N}, m \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}} |a_{k,l,m,\alpha}|.$$

In particular $\mathbb{K}\{x, y, z, \underline{w}\}$ is a subring of $\mathbb{K}[[x, y, z, w_i]]_{i \in \mathbb{N}}$.

We denote by I the ideal of $\mathbb{K}[x, y, z, \underline{w}]$ generated by the polynomials

$$xw_0 - z^2 \text{ and } yw_k - (k+1)xw_{k+1} \text{ for all } k \geq 0.$$

The ideal $I\mathbb{K}\{x, y, z, \underline{w}\}$ is not closed since it is not finitely generated. Thus, we denote by \bar{I} its closure. This is the set of sums

$$\sum_{k \in \mathbb{N}} f_k(x, y, z, \underline{w})$$

such that $f_k(x, y, z, \underline{w}) \in I\mathbb{K}\{x, y, z, \underline{w}\}$ and $\sum_k \|f_k(x, y, z, \underline{w})\| < \infty$.

Definition 2.2. We denote by R the Banach \mathbb{K} -algebra $\mathbb{K}\{x, y, z, \underline{w}\}/\bar{I}$.

In order to denote that two series f and $g \in \mathbb{K}\{x, y, z, \underline{w}\}$ have the same image in R , we write $f \equiv_R g$. The norm of the image \bar{f} of an element $f \in \mathbb{K}\{x, y, z, \underline{w}\}$ is

$$\|\bar{f}\| = \inf_{g \in \bar{I}} \|f + g\| = \inf_{g \in I} \|f + g\|.$$

Now we denote by $R\{t\}$ the ring of convergent series in the indeterminate t with coefficients in R . We have the following result:

Proposition 2.3. *The linear equation*

$$(2.1) \quad (x - yt)f(t) = z^2$$

has a unique solution $f(t)$ in $R[[t]]$ and this solution is not convergent.

From this we will deduce the following result:

Theorem 2.4. *The Banach \mathbb{K} -algebra R is an integral domain and the morphism $R\{t\} \rightarrow R[[t]]$ is not flat.*

2.1. Proofs of Proposition 2.3 and Theorem 2.4. We begin by giving the following key result:

Lemma 2.5. *We have*

$$(0 : x)_R = (0).$$

Proof. First one remarks that every series in R has a representation as a sum

$$(2.2) \quad xp(x, y, z, w_0) + q(y, z, \underline{w})$$

where $p(x, y, z, w_0) \in \mathbb{K}\{x, y, z, w_0\}$, and $q(y, z, \underline{w}) \in \mathbb{K}\{y, z, \underline{w}\}$. This representation is not unique in general, for instance

$$3yw_1w_3 \equiv_R 2yw_2^2,$$

but we can make it unique as follows:

First we remark that

$$(2.3) \quad M_1 := yw_iw_j \equiv_R (i+1)xw_{i+1}w_j \equiv_R \frac{i+1}{j}yw_{i+1}w_{j-1} =: M_2$$

for all integers i and j with $i < j$. If $j = i+1$ these two monomials are equal, otherwise the largest index of a monomial w_j appearing in the expression of M_2 is strictly less than for M_1 .

Now we have, for $i > 0$:

$$(2.4) \quad z^2w_i \equiv_R xw_0w_i \equiv_R \frac{1}{i}yw_0w_{i-1}.$$

A well chosen composition of these operations transforms any monomial into a monomial $cx^a z^k y^l w_0^{n_0} \cdots w_i^{n_i}$ where i is minimal and $c \in (0, 1]$. Hence these two operations replace elements of $\mathbb{K}\{x, y, z, \underline{w}\}$ by elements of $\mathbb{K}\{x, y, z, \underline{w}\}$.

By repeating these two operations we may reduce every monomial to a monomial of one of the following forms:

$$(2.5) \quad \begin{cases} z^\varepsilon x^a y^l w_0^{n_0} & \text{with } a > 1, l, n_0 \geq 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon y^l w_i^{n_i} & \text{with } l > 0, i > 0, n_i > 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon y^l w_i^{n_i} w_{i+1}^{n_{i+1}} & \text{with } l > 0, i \geq 0, n_i, n_{i+1} > 0 \text{ and } \varepsilon \in \{0, 1\}, \\ z^\varepsilon w_0^{n_0} \dots w_i^{n_i} & \text{with } n_i > 0 \text{ with } \varepsilon \in \{0, 1\}, \end{cases}$$

The unicity comes from the following observation:

We set

$$F_0 := xw_0 - z^2, F_{k+1} := yw_k - (k+1)xw_{k+1} \quad \text{for } k \geq 0$$

$$G_{k,l} := (l+1)yw_k w_{l+1} - (k+1)yw_l w_{k+1} \quad \text{for all } k < l.$$

Then we consider the following monomial order: We define

$$x^a y^k z^l w_1^{\alpha_1} \dots w_n^{\alpha_n} > x^{a'} y^{k'} z^{l'} w_1^{\alpha'_1} \dots w_n^{\alpha'_n}$$

if

$$a+k+l+\sum_i \alpha_i > a'+k'+l'+\sum_i \alpha'_i, \quad \text{or } a+k+l+\sum_i \alpha_i = a'+k'+l'+\sum_i \alpha'_i$$

$$\text{and } (l, a, k, \alpha_n, \dots, \alpha_0) >_{lex} (l', a', k', \alpha'_n, \dots, \alpha'_0)$$

where $>_{lex}$ denotes the lexicographic order. That is, we first compare the total degree of two monomials, then we order the indeterminates as

$$z > x > y > w_l > w_k \quad \text{for all } l > k.$$

We claim that $\{F_j, G_{k,l}\}_{j,k,l \in \mathbb{N}, l > k}$ is a Gröbner basis of $I \cap \mathbb{K}[x, y, z, w]$ for this order. In order to prove this, we only need to compute the S-polynomials of the elements of this set of polynomials, and then their reduction (see [CLO97] for the terminology). The only S-polynomials we have to consider are those of polynomials whose leading terms are not coprime, that is, for $l > k$,

$$S(F_{k+1}, F_{l+1}); S(G_{k,l}, F_{l+1}); S(G_{k,l}, F_k).$$

We have $S(F_{k+1}, F_{l+1}) = G_{k,l}$. Moreover

$$S(G_{k,l}, F_{l+1}) = y(yw_k w_l - (k+1)xw_l w_{k+1}).$$

This leading term of $S(G_{k,l}, F_{l+1})$ is $-(k+1)xyw_l w_{k+1}$, and it is equal to $y(F_{k+1}w_l - yw_k w_l)$. Therefore $S(G_{k,l}, F_{l+1}) = F_{k+1}yw_l$.

Finally we have

$$\begin{aligned} S(G_{k,l}, F_k) &= kx((l+1)yw_k w_{l+1} - (k+1)yw_l w_{k+1}) + (l+1)yw_{l+1}(yw_{k-1} - kw_k) \\ &= (l+1)y^2 w_{k-1} w_{l+1} - k(k+1)xyw_l w_{k+1}. \end{aligned}$$

The leading term of it is $-k(k+1)xyw_l w_{k+1}$ and it is divisible by the leading term of F_{k+1} . The remainder of the division of $S(G_{k,l}, F_k)$ by F_{k+1} is

$$(l+1)y^2 w_{k-1} w_{l+1} - ky^2 w_k w_l = yG_{k-1,l}$$

Thore the reductions of these S-polynomials is always zero, hence $\{F_j, G_{k,l}\}_{j,k,l \in \mathbb{N}, l > k}$ is a Gröbner basis of $I \cap \mathbb{K}[x, y, z, \underline{w}]$. Thus, the initial ideal of $I \cap \mathbb{K}[x, y, z, \underline{w}]$ is generated by the monomials

$$z^2, xw_{k+1}, yw_k w_{l+1} \text{ for } 0 \leq k < l.$$

Therefore every polynomial of $\mathbb{K}[x, y, z, \underline{w}]$ can be uniquely written as a finite sum of monomials as in (2.5). Therefore, every power series of $\mathbb{K}[[x, y, z, \underline{w}]]$ can be uniquely written as a formal sum of monomials as in (2.5). This proves the claim.

Thus, if we assume that the monomials of $xp(x, y, z) + q(y, z, \underline{w})$ are only of the types of (2.5), the representation (2.2) is unique.

Now take $xp(x, y, z, w_0) + q(y, z, \underline{w}) \in (0 : x)_R$, and assume that the monomials in the expansion of $xp(x, y, z, w_0) + q(y, z, \underline{w})$ are only those of (2.5). Then

$$x^2 p(x, y, z, w_0) + xq(y, z, \underline{w}) \equiv_R 0.$$

The representation of $x^2 p(x, y, z, w_0) + xq(y, z, \underline{w})$ as a sum of monomials as in (2.5) has the form

$$(2.6) \quad x^2 p(x, y, z, w_0) + xq(y, z, w_0, 0) + \bar{q}(y, z, \underline{w}) = 0$$

where $\bar{q}(y, z, \underline{w})$ is the series obtained from $xq(y, z, \underline{w}) - xq(y, z, w_0, 0)$ by replacing the monomials as follows (using the two previous operations (2.3) and (2.4)):

$$(2.7) \quad \begin{cases} xz^\varepsilon y^l w_i^{n_i} & \mapsto \frac{1}{i} z^\varepsilon y^{l+1} w_{i-1} w_i^{n_i-1}, \text{ if } i > 0 \\ xz^\varepsilon y^l w_i^{n_i} w_{i+1}^{n_{i+1}} & \mapsto \frac{1}{i+1} z^\varepsilon y^{l+1} w_i^{n_i+1} w_{i+1}^{n_{i+1}-1}, \text{ if } i > 0 \\ xz^\varepsilon w_0^{n_0} \dots w_i^{n_i} & \mapsto Cz^\varepsilon y w_j^{m_j} w_{j+1}^{m_{j+1}} \text{ or } Cz^\varepsilon y w_j^{m_j} \\ \text{for } i > 0 \text{ and } n_i > 0 & \text{for some } C \in \mathbb{K}, |C| \leq 1, j \geq 0 \end{cases}$$

Indeed for the third monomial we have

$$xz^\varepsilon w_0^{n_0} \dots w_i^{n_i} \equiv_R \frac{1}{i+1} z^\varepsilon y w_0^{n_0} \dots w_{i-1}^{n_{i-1}+1} w_i^{n_i-1}$$

and this monomial on the right side can be transformed into a monomial of the form $Cz^\varepsilon y w_j^{m_j} w_{j+1}^{m_{j+1}}$ or $Cz^\varepsilon y w_j^{m_j}$ for some $C \in \mathbb{K}$, $|C| \leq 1$, and $j \geq 0$, by using the two operations (2.3) and (2.4) on monomials.

This shows that the three types of monomials that we obtain after multiplication by x are all distinct, that is the map defined by (2.7) is injective. By (2.6) we have $\bar{q}(y, z, \underline{w}) = 0$, therefore $q(y, z, \underline{w}) - q(y, z, w_0, 0) = 0$.

Moreover, again by (2.6), we have

$$x^2 p(x, y, z, w_0) + xq(y, z, w_0, 0) = 0.$$

This shows that $x^2 p(x, y, z, w_0) + xq(y, z, \underline{w}) = 0$. This proves that

$$(0) = (0 : x)_R.$$

□

Proof of Proposition 2.3. Let $f(t) \in R[[t]]$ such that

$$(x - yt)f(t) = z^2.$$

By writing $f = \sum_{k=0}^{\infty} f_k t^k$ with $f_k \in R$ for every k , we have

$$x f_0 = z^2$$

$$x f_k - y f_{k-1} = 0 \quad \forall k \geq 1.$$

Thus

$$x f_0 = z^2 = x w_0$$

so $x(f_0 - w_0) = 0$ and $f_0 = w_0$ by Lemma 2.5. Then we will prove by induction on k that $f_k = k! w_k$ for every k . Assume that this is true for an integer $k \geq 0$. Then we have

$$x f_{k+1} - y f_k = k! y w_k = (k+1)! x w_{k+1}.$$

Hence $x(f_{k+1} - (k+1)! w_{k+1}) = 0$ and $f_{k+1} = (k+1)! w_{k+1}$ by Lemma 2.5. Therefore the only solution of

$$(x - yt)f(t) = z^2$$

is the series $\sum_{k=0}^{\infty} k! w_k t^k$, and this one is divergent because $\|w_k\| = 1$. This holds because in every element of I , the monomial w_k has coefficient 0. \square

Now we can give the proof of Theorem 2.4:

Proof of Theorem 2.4. Since x is not a zero divisor in R by Lemma 2.5, the localization morphism

$$R \longrightarrow R_{1/x}$$

is injective. But $R_{1/x}$ is isomorphic to $\mathbb{K}\{x, y, z\}_{1/x}$ since in $R_{1/x}$ we have

$$w_0 = z^2/x \text{ and } \forall k \geq 0, w_k = \frac{1}{k!} y^k z^2 x^{k+1}.$$

But $\mathbb{K}\{x, y, z\}_{1/x}$ is an integral domain (this is a localization of the integral domain $\mathbb{K}\{x, y, z\}$), therefore so is R .

Now assume that the morphism $R\{t\} \longrightarrow R[[t]]$ is flat. By [Ma89, Theorem 7.6] there exist an integer $s \geq 1$, and convergent series

$$a_1(t), \dots, a_s(t), b_1(t), \dots, b_s(t) \in R\{t\}$$

such that

$$(2.8) \quad (x - yt)a_i(t) - z^2 b_i(t) = 0 \text{ for every } i,$$

and formal power series

$$h_1(t), \dots, h_s(t) \in R[[t]]$$

such that

$$f(t) = \sum_{i=1}^s a_i(t) h_i(t), \quad 1 = \sum_{i=1}^s b_i(t) h_i(t).$$

Indeed the vector $(f(t), 1)$ is a solution of the linear equation

$$(x - yt)f(t) - z^2g(t) = 0$$

with $f(t) := \sum_{k=0}^{\infty} k! w_k t^k$.

Then

$$\tilde{g}(t) := \sum_{i=1}^s b_i(t)h_i(0) = 1 + t\varepsilon(t)$$

for some $\varepsilon(t) \in R\{t\}$. Since 1 is a unit of R , $1 + t\varepsilon(t)$ is a unit in $R\{t\}$.

Set $\tilde{f}(t) := \sum_i a_i(t)h_i(0)$. By (2.8), $(\tilde{f}(t), \tilde{g}(t))$ is a solution of the equation

$$(x - yt)\tilde{f}(t) - z^2\tilde{g}(t) = 0.$$

Since $\tilde{g}(t)$ is a unit in $R\{t\}$ we have

$$(x - yt)\tilde{f}(t)\tilde{g}(t)^{-1} = z^2.$$

This contradicts Theorem 2.3. Therefore $R\{t\} \rightarrow R[[t]]$ is not flat. \square

3. AN EXAMPLE CONCERNING THE STRONG ARTIN APPROXIMATION THEOREM

Let n be a positive integer, $x = (x_1, \dots, x_n)$ and $\rho > 0$. We set $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Then

$$B_\rho^n := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mid \|f\|_\rho := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| \rho^{|\alpha|} < \infty \right\}$$

is a Banach space equipped with the norm $\|\cdot\|_\rho$. Of course $\mathbb{K}[x] \subset B_\rho^n$.

Remark 3.1. We do not have

$$B_\rho^n[[t]] \cap \mathbb{K}\{x, t\} = B_\rho^n\{t\}.$$

For instance, the power series

$$f = \sum_{k \in \mathbb{N}} x_1^{k!} t^k$$

is a convergent power series in (x, t) , belongs to $B_2^n[[t]]$, but

$$\sum_k \|x_1^{k!}\|_2 \tau^k = \sum_k 2^{k!} \tau^k = \infty$$

for every $\tau > 0$. Therefore $f \notin B_2^n\{t\}$.

We provide two examples based on an example of Spivakovsky concerning the extension of Theorem 1.2 to the nested case (see [Sp94]).

Example 3.2. Let $n = 1$ and set

$$F(x, t, y_1, y_2) := xy_1^2 - (x + t)y_2^2 \in B_\rho\{t\}[y_1, y_2].$$

Let

$$\sqrt{1+t} = 1 + \sum_{n \geq 1} a_n t^n \in \mathbb{Q}\{t\}$$

be the unique power series such that $(\sqrt{1+t})^2 = 1+t$ and whose value at the origin is 1. For every $c \in \mathbb{N}$ we set $y_2^{(c)}(t) := x^c$ and $y_1^{(c)}(t) := x^c + \sum_{n=1}^c a_n x^{c-n} t^n \in B_\rho\{t\}$. Then

$$F(x, t, y_1^{(c)}(t), y_2^{(c)}(t)) \in (t)^{c+1}.$$

On the other hand the equation $f(x, t, y_1(t), y_2(t)) = 0$ has no solution $(y_1(t), y_2(t)) \in B_\rho\{t\}^2$ but $(0, 0)$. Indeed let us denote by T_0 the Taylor map at 0:

$$T_0 : B_\rho\{t\} \longrightarrow \mathbb{K}\llbracket x, t \rrbracket.$$

If $f(x, t, y_1(t), y_2(t)) = 0$ then

$$xT_0(y_1(t))^2 - (x+t)T_0(y_2(t))^2 = 0.$$

But since $\mathbb{K}\llbracket x, t \rrbracket$ is a unique factorization domain, this equality implies that $T_0(y_1(t)) = T_0(y_2(t)) = 0$, hence $y_1(t) = y_2(t) = 0$.

This shows that there is no $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that for every $y(t) \in B_\rho\{t\}^2$ and every $k \in \mathbb{N}$ with

$$F(x, t, y(t)) \in (t)^{\beta(k)}$$

there exists $\tilde{y}(t) \in B_\rho\{t\}^2$ such that

$$F(x, t, \tilde{y}(t)) = 0$$

and $\tilde{y}(t) - y(t) \in (t)^k$.

Example 3.3. We can modify a little bit the previous example to construct a F as before that does not depend on t . We set

$$G(x, y_1, y_2, y_3) := xy_1^2 - (x + y_3)y_2^2 \in B_\rho[y_1, y_2, y_3].$$

For every $c \in \mathbb{N}$ we set $y_2^{(c)}(t) := x^c$, $y_1^{(c)}(t) := x^c + \sum_{n=1}^c a_n x^{c-n} t^n$ and $y_3^{(c)}(t) := t \in B_\rho\{t\}$. Then

$$G(x, y_1^{(c)}(t), y_2^{(c)}(t), y_3^{(c)}(t)) \in (t)^c.$$

Now if $\tilde{y}(t) \in B_\rho\{t\}^3$ satisfies $G(x, \tilde{y}(t)) = 0$ and

$$\tilde{y}(t) - y(t) \in (t)^2$$

then $\tilde{y}_3(t) = x + t + \varepsilon(t)$ with $\varepsilon(t) \in (t^2)$. Thus $x + \tilde{y}_3(t)$ is an irreducible power series in x and t , and it is coprime with x . By the same argument based on the Taylor map as in Example 3.2, the relation

$$x\tilde{y}_1(t)^2 - (x + t + \varepsilon(t))\tilde{y}_2(t)^2 = 0$$

implies that $\tilde{y}_1(t) = \tilde{y}_2(t) = 0$.

This shows that there is no $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $y(t) \in B_\rho\{t\}^3$ and every $k \in \mathbb{N}$ with

$$G(x, y(t)) \in (t)^{\beta(k)}$$

there exists $\tilde{y}(t) \in B_\rho\{t\}^3$ such that

$$G(x, \tilde{y}(t)) = 0$$

and $\tilde{y}(t) - y(t) \in (t)^k$.

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