ARTIN APPROXIMATION OVER BANACH SPACES

GUILLAUME ROND

Abstract. We give examples showing that the usual Artin Approximation theorems valid for convergent series over a field are no longer true for convergent series over a commutative Banach algebra. In particular we construct an example of a commutative integral Banach algebra R such that the ring of formal power series over R is not flat over the ring of convergent power series over R.

1. Introduction

The classical Artin Approximation Theorem is the following:

Theorem 1.1. [1] *Let $F(x,y)$ be a vector of convergent power series over $\mathbb{C}$ in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ vanishing at 0,*

$$F(x,\widehat{y}(x))=0.$$

Then, for any $c\in\mathbb{N}$, there exists a convergent power series solution $y(x)$ vanishing at 0,

$$F(x,y(x))=0$$

which coincides with $\widehat{y}(x)$ up to degree c,

$$y(x)\equiv\widehat{y}(x) \text{ modulo } (x)^c.$$

The main tools for proving this theorem are the implicit function theorem and the Weierstrass division theorem. But in the case the equations $F(x,y)$ are linear in y, this theorem is equivalent to the faithful flatness of the morphism $\mathbb{C}\{x\}\longrightarrow\mathbb{C}[\![x]\!]$ (see [13, Example 1.4] for instance or [4, I. 3 Proposition 13]). In fact the faithful flatness of this morphism comes from the fact that $\mathbb{C}\{x\}$ is a Noetherian local ring. And the Noetherianity of $\mathbb{C}\{x\}$ is usually proved by using the Weierstrass division theorem.
Another version of this theorem is the following one:

Theorem 1.2. [2][16] *Let $F(x,y)$ be a vector of convergent power series over $\mathbb{C}$ in two sets of variables x and y. Then for any integer c there exists*

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an integer β *such that for any given approximate solution* $\overline{y}(x)$ *at order* β, $\overline{y}(0)=0$,

$$F(x,\overline{y}(x)) \equiv 0 \text{ modulo } (x)^{\beta},$$

there exists a formal power series solution $y(x)$ *vanishing at 0,*

$$F(x,y(x))=0$$

which coincides with $\overline{y}(x)$ *up to degree* c*,*

$$y(x) \equiv \overline{y}(x) \text{ modulo } (x)^{c}.$$

In particular this result implies that, if $F(x,y)=0$ has approximate solutions at any order, then it has a formal (even convergent by Theorem 1.1) power series solution.
Let us mention that these results remain valid when we replace $\mathbb{C}$ by a complete valued field, or when we replace the ring of convergent power series over $\mathbb{C}$ by the ring of algebraic power series over a field. In fact these results remain true in the more general setting of excellent Henselian local rings by [12] (see [13] for a review of all these different results).

The aim of this note is to show that these results are no longer true when we replace $\mathbb{C}$ by a commutative Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. In the first part we construct a commutative Banach algebra R such that $R\{t\} \longrightarrow R[\![t]\!]$ is not flat, showing that Artin approximation theorem is not true for linear equations with coefficients in $R\{t\}$.
Let us mention here that $R[\![t]\!]$ is flat over R, for a commutative ring R, if and only if R is coherent (indeed $R[\![t]\!]$ is a direct product of copies of R - see [5, Theorem 2.1]). And there are several known examples of Banach algebras which are not coherent (in fact most of the known Banach algebras are not coherent; see for instance [9] or [8] and the references herein). But the flatness of $R\{t\} \longrightarrow R[\![t]\!]$ is a different property that is not related to the coherence of R.
In the second part we provide an example of one polynomial $F(y)$ with coefficients in $R[t]$, where R is the Banach algebra of holomorphic functions over a disc, with the following property: $F(y)$ has approximate solutions up to any order but has no solution in $R[\![t]\!]$. This shows that Theorem 1.2 does not hold for convergent power series over a Banach algebra. Let us mention that this example is a slight modification of an example of Spivakovsky related to a similar problem [15].
Nevertheless we mention that in the case where R is a complete valuation ring of rank one (in particular a non-archimedean Banach algebra), Schoutens and Moret-Bailly proved several extensions of Theorems 1.1 and 1.2 (see [14] and [11]).
The note has been motivated by questions from Nefton Pali and Wei Xia.

2. A Banach algebra R such that $R\{t\} \longrightarrow R[\![t]\!]$ is not flat

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We begin by the following definition of power series in countable many indeterminates:

Definition 2.1. Let $\mathbb{N}^{(\mathbb{N})}$ be the submonoid of $\mathbb{N}^{\mathbb{N}}$ formed by the sequences whose all but finitely terms are 0. Let $(x_i)_{i\in\mathbb{N}}$ be a countable family of indeterminates. Then $\mathbb{K}[\![x_i]\!]_{i\in\mathbb{N}}$ is the set of series $\sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} a_\alpha x^\alpha$ where $x^\alpha = x_0^{\alpha_0}\cdots x_n^{\alpha_n}\cdots$. This former product is finite since $\alpha_i = 0$ for i large enough. This set is a commutative ring since the sum of sequences $\mathbb{N}^{(\mathbb{N})}\times\mathbb{N}^{(\mathbb{N})} \longrightarrow \mathbb{N}^{(\mathbb{N})}$ has finite fibers (see [3, Chapter III, § 2, 11]). Let us mention that this ring is not the (x)-adic completion of $\mathbb{K}[x]$, the ring of polynomials in the x_i (see [17] for instance).

Let x, y, z and w_k for $k \in \mathbb{N}$ be indeterminates. For simplicity we denote by $\underline{w}$ the vector of indeterminates $(w_0, w_1, \ldots)$. We denote by $\mathbb{K}[x, y, z, \underline{w}]$ the ring of polynomials in the indeterminates x, y, z, $\underline{w}$.
For a polynomial $p = \sum_{k\in\mathbb{N},l\in\mathbb{N},m\in\mathbb{N},\alpha\in\mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha}x^k y^l z^m \underline{w}^\alpha \in \mathbb{K}[x, y, z, \underline{w}]$ we set

$$\|p\| := \sum_{k,l,m,\alpha} |a_{k,l,m,\alpha}|.$$

This is well defined because the sum is finite. This defines a norm on $\mathbb{K}[x, y, z, \underline{w}]$.
We denote by $\mathbb{K}\{x, y, z, \underline{w}\}$ the completion of $\mathbb{K}[x, y, z, \underline{w}]$ for this norm. This is the following commutative Banach algebra:

$$\left\{ \sum_{k\in\mathbb{N},l\in\mathbb{N},m\in\mathbb{N},\alpha\in\mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha}x^k y^l z^m \underline{w}^\alpha \mid \sum_{k\in\mathbb{N},l\in\mathbb{N},m\in\mathbb{N},\alpha\in\mathbb{N}^{(\mathbb{N})}} |a_{k,l,m,\alpha}| < \infty \right\}$$

and the norm of an element $f := \sum_{k\in\mathbb{N},l\in\mathbb{N},m\in\mathbb{N}\alpha\in\mathbb{N}^{(\mathbb{N})}} a_{k,l,m,\alpha}x^k y^l z^m \underline{w}^\alpha$ is

$$\|f\| := \sum_{k\in\mathbb{N},l\in\mathbb{N},m\in\mathbb{N},\alpha\in\mathbb{N}^{(\mathbb{N})}} |a_{k,l,m,\alpha}|.$$

In particular $\mathbb{K}\{x, y, z, \underline{w}\}$ is a subring of $\mathbb{K}[\![x, y, z, w_i]\!]_{i\in\mathbb{N}}$.
We denote by I the ideal of $\mathbb{K}[x, y, z, \underline{w}]$ generated by the polynomials

$$xw_0 - z^2 \text{ and } yw_k - (k+1)xw_{k+1} \text{ for all } k \geq 0.$$

The ideal $I\mathbb{K}\{x, y, z, \underline{w}\}$ is not closed since it is not finitely generated. Thus, we denote by $\overline{I}$ its closure. This is the set of sums

$$\sum_{k\in\mathbb{N}} f_k(x, y, z, \underline{w})$$

such that $f_k(x, y, z, \underline{w}) \in I\mathbb{K}\{x, y, z, \underline{w}\}$ and $\sum_k \|f_k(x, y, z, \underline{w})\| < \infty$.

Definition 2.2. We denote by R the Banach $\mathbb{K}$-algebra $\mathbb{K}\{x,y,z,\underline{w}\}/\overline{I}$.

In order to denote that two series f and $g\in\mathbb{K}\{x,y,z,\underline{w}\}$ have the same image in R, we write $f\equiv_R g$. The norm of the image $\overline{f}$ of an element $f\in\mathbb{K}\{x,y,z,\underline{w}\}$ is

$$\|\overline{f}\|=\inf_{g\in\overline{I}}\|f+g\|=\inf_{g\in I}\|f+g\|.$$

Now we denote by $R\{t\}$ the ring of convergent series in the indeterminate t with coefficients in R. We have the following result:

Proposition 2.3. *The linear equation*

$$(x-yt)f(t)=z^2 \tag{2.1}$$

has a unique solution $f(t)$ in $R[\![t]\!]$ and this solution is not convergent.

From this we will deduce the following result:

Theorem 2.4. *The Banach $\mathbb{K}$-algebra R is an integral domain and the morphism $R\{t\}\longrightarrow R[\![t]\!]$ is not flat.*

2.1. **Proofs of Proposition 2.3 and Theorem 2.4.** We begin by giving the following key result:

Lemma 2.5. *x is not a zero divisor in R.*

Proof. First of all, we will determine a subset of $\mathbb{K}\{x,y,z,\underline{w}\}$ such that every element of $\mathbb{K}\{x,y,z,\underline{w}\}$ is equal modulo $\overline{I}$ to a unique series of this subset. First we remark that

$$M_1:=yw_iw_j\equiv_R(i+1)xw_{i+1}w_j\equiv_R\frac{i+1}{j}yw_{i+1}w_{j-1}=:M_2 \tag{2.2}$$

for all integers i and j with $i<j$. If $j=i+1$ these two monomials are equal, otherwise the largest index of a monomial w_j appearing in the expression of M_2 is strictly less than for M_1.
Now we have, for $i>0$:

$$z^2w_i\equiv_R xw_0w_i\equiv_R\frac{1}{i}yw_0w_{i-1}. \tag{2.3}$$

A well chosen composition of these operations transforms any monomial of the form $Cx^az^ky^lw_0^{n_0}\cdots w_i^{n_i}$ into a monomial of the form $rCx^{a'}z^{k'}y^lw_0^{n'_0}\cdots w_j^{n'_j}$ where j is minimal and $r\in(0,1]$.
By repeating these two operations we may reduce every monomial to a constant times one of the following monic monomials:

$$\begin{cases} z^\varepsilon x^ay^lw_0^{n_0} & \text{with } a>1, l, n_0\geq 0 \text{ and } \varepsilon\in\{0,1\},\\ z^\varepsilon y^lw_i^{n_i} & \text{with } l>0, i>0, n_i>0 \text{ and } \varepsilon\in\{0,1\},\\ z^\varepsilon y^lw_i^{n_i}w_{i+1}^{n_{i+1}} & \text{with } l>0, i\geq 0, n_i, n_{i+1}>0 \text{ and } \varepsilon\in\{0,1\},\\ z^\varepsilon w_0^{n_0}\ldots w_i^{n_i} & \text{with } n_i>0 \text{ with } \varepsilon\in\{0,1\}, \end{cases} \tag{2.4}$$

We denote by E the subset of $\mathbb{K}[x,y,z,\underline{w}]$ of polynomials that are sums of monomials of (2.4) (up to multiplicative constants), and by $\overline{E}$ the closure

of E in $\mathbb{K}\{x, y, z, \underline{w}\}$, that is the set of convergent power series whose non zero monomials are those of (2.4) (up to multiplicative constants). We have shown that every polynomial is equivalent to a polynomial of E modulo I. To prove the unicity we proceed as follows.
We set

$$F_0 := xw_0 - z^2, F_{k+1} := yw_k - (k+1)xw_{k+1} \text{ for } k \geq 0$$

$$G_{k,l} := (l+1)yw_kw_{l+1} - (k+1)yw_lw_{k+1} \text{ for all } k < l.$$

Then we consider the following monomial order: We define

$$x^ay^kz^lw_1^{\alpha_1}\cdots w_n^{\alpha_n} > x^{a'}y^{k'}z^{l'}w_1^{\alpha_1'}\cdots w_n^{\alpha_n'}$$

if

$$a+k+l+\sum_i \alpha_i > a'+k'+l'+\sum_i \alpha_i', \text{ or } a+k+l+\sum_i \alpha_i = a'+k'+l'+\sum_i \alpha_i'$$

$$\text{and } (l, a, k, \alpha_n, \ldots, \alpha_0) >_{lex} (l', a', k', \alpha_n', \ldots, \alpha_0')$$

where $>_{lex}$ denotes the lexicographic order. That is, we first compare the total degree of two monomials, then we order the indeterminates as

$$z > x > y > w_l > w_k \text{ for all } l > k.$$

We claim that $\{F_j, G_{k,l}\}_{j,k,l\in\mathbb{N},\, l>k}$ is a Gröbner basis of I for this order. In order to prove this, we only need to compute the S-polynomials of the elements of this set of polynomials, and then their reduction (see [6] for the terminology). This is Buchberger's Algorithm which is very classical in the Noetherian case. The case of polynomial rings in countably many indeterminates works identically, cf. [7, Proposition 1.13] for instance. The only S-polynomials we have to consider are those of polynomials whose leading terms are not coprime, that is, for $l > k$,

$$S(F_{k+1}, F_{l+1});\ S(G_{k,l}, F_{l+1});\ S(G_{k,l}, F_k).$$

We have $S(F_{k+1}, F_{l+1}) = G_{k,l}$. Moreover

$$S(G_{k,l}, F_{l+1}) = y(yw_kw_l - (k+1)xw_lw_{k+1}).$$

This leading term of $S(G_{k,l}, F_{l+1})$ is $-(k+1)xyw_lw_{k+1}$, and it is equal to $y(F_{k+1}w_l - yw_kw_l)$. Therefore $S(G_{k,l}, F_{l+1}) = F_{k+1}yw_l$.
Finally we have

$$S(G_{k,l}, F_k) = kx((l+1)yw_kw_{l+1} - (k+1)yw_lw_{k+1}) + (l+1)yw_{l+1}(yw_{k-1} - kxw_k)$$

$$= (l+1)y^2w_{k-1}w_{l+1} - k(k+1)xyw_lw_{k+1}.$$

Its leading term is $-k(k+1)xyw_lw_{k+1}$ and it is divisible by the leading term of F_{k+1}. The remainder of the division of $S(G_{k,l}, F_k)$ by F_{k+1} is

$$(l+1)y^2w_{k-1}w_{l+1} - ky^2w_kw_l = yG_{k-1,l}$$

Therefore the reductions of these S-polynomials is always zero, hence the family $\{F_j, G_{k,l}\}_{j,k,l\in\mathbb{N},\ l>k}$ is a Gröbner basis of I. Thus, the initial ideal of I is generated by the monomials

$$z^2, xw_{k+1}, yw_k w_{l+1} \text{ for } 0 \le k < l.$$

Therefore every polynomial of $\mathbb{K}[x,y,z,\underline{w}]$ is equivalent modulo I to a unique polynomial of E.

Now let $f \in \mathbb{K}\{x,y,z,\underline{w}\}$. We can write $f = \sum_{n\in\mathbb{N}} C_n x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n}$ where the C_n are in $\mathbb{K}^*$. In particular $\sum_n |C_n| < \infty$. For every $n \in \mathbb{N}$, there is a unique $(a'_n, b'_n, c'_n, \alpha'_n)$ and a unique $r_n \in (0,1]$ such that

$$C_n x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n} - r_n C_n x^{a'_n} y^{b'_n} z^{c'_n} \underline{w}^{\alpha'_n} \in I$$

and $x^{a_n} y^{b_n} z^{c_n} \underline{w}^{\alpha_n}$ has one the forms given in (2.4). Now, for every $n \in \mathbb{N}$, we set

$$g_n := \sum_{k=0}^{n-1} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} \underline{w}^{\alpha'_k} + \sum_{k\ge n} C_k x^{a_k} y^{b_k} z^{c_k} \underline{w}^{\alpha_k}.$$

In particular we have that $P_n := f - g_n \in I$ and the sequence $(g_n)_n$ converges in $\mathbb{K}\{x,y,z,\underline{w}\}$ to the series $g = \sum_{k\in\mathbb{N}} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} \underline{w}^{\alpha'_k} \in \mathbb{K}\{x,y,z,\underline{w}\}$. Therefore the sequence $(P_n)_n$ converges in $\mathbb{K}\{x,y,z,\underline{w}\}$, and its limits is in $\overline{I}$.

Therefore, every power series of $\mathbb{K}\{x,y,z,\underline{w}\}$ can be written as a sum of a power series in $\overline{I}$ and a convergent power series whose monomials are as in (2.4) (up to multiplicative constants).

We remark that, by repeating (2.2) $\lfloor \frac{j-i}{2} \rfloor$ times, we have

$$yw_i w_j \equiv_R ryw_{i+\lfloor \frac{j-i}{2}\rfloor} w_{j-\lfloor \frac{j-i}{2}\rfloor}$$

for some constant $r \in (0,1]$. Moreover applying (2.3) reduces by 2 the degree in z of a monomial. Therefore, a monomial of the form

$$Cx^a y^b z^c w_1^{\alpha_1} \cdots w_j^{\alpha_j}$$

of total degree $d = a+b+c+\sum_k \alpha_k$, is not equal to a monomial involving only the indeterminates

$$x, y, z, \text{ and } w_i \text{ for } i < \frac{j-\frac{c}{2}}{2^d}.$$

Moreover (2.2) and (2.3) transforms monomials into monomials of the same degree since I is generated by homogeneous binomials. Therefore, given a monomial M among those of (2.4) (up to some multiplicative constant), there is finitely many monomials that are equal to M modulo I.

Now let $f \in \overline{E} \cap \overline{I}$, $f = \sum_{(a,b,c,\alpha)} f_{(a,b,c,\alpha)} x^a y^b z^c \underline{w}^\alpha$. Let us fix such (a,b,c,α) such that $x^a y^b z^c \underline{w}^\alpha$ is one of the monic monomials of (2.4). There is only a finite number of distinct monomials that are equal to $f_{(a,b,c,\alpha)} x^a y^b z^c \underline{w}^\alpha$ modulo I. Let us denote them by

$$C_1 x^{a_1} y^{b_1} z^{c_1} \underline{w}^{\alpha_1}, \ldots, C_N x^{a_N} y^{b_N} z^{c_N} \underline{w}^{\alpha_N}.$$

We can remark that there is only a finite number of F_l that have a monomial that divides at least one of the following monic monomials

$$x^a y^b z^c \underline{w}^\alpha, x^{a_1} y^{b_1} z^{c_1} \underline{w}^{\alpha_1}, \dots, x^{a_N} y^{b_N} z^{c_N} \underline{w}^{\alpha_N}. \tag{2.5}$$

We denote them by $F_{l_1}, \dots, F_{l_p}$. Because $f \in \overline{I}$, we can write $f = \sum_{l \in \mathbb{N}} f_l F_l$ where the f_l are in $\mathbb{K}[\![x, y, z, \underline{w}]\!]$. For every $i \in \{1, \dots, p\}$ we remove from f_{l_i} all the monomials that do not divide one of the monomials (2.5), and we denote by f'_{l_i} the resulting polynomial. Then we have that

$$P := \sum_{i=1}^{p} f'_{l_i} F_{l_i} \in I.$$

By construction the coefficients of the monomials (2.5) in the expansion of P are the corresponding coefficients in the expansion of f, that is

$$f_{(a,b,c,\alpha)}, 0, \dots, 0$$

respectively. Therefore, the coefficient of $x^a y^b z^c \underline{w}^\alpha$ in the expansion of the unique $Q \in E$ such that $Q \equiv_R P$, is equal to $f_{(a,b,c,\alpha)}$ because no other monomial than those listed in (2.5) (up to some multilplicative constants) is equivalent to a monomial of the form $Cx^a y^b z^c \underline{w}^\alpha$ where $C \in \mathbb{K}^*$. But $Q = 0$ since $P \in I$, thus $f_{(a,b,c,\alpha)} = 0$. Hence $f = 0$ and $\overline{E} \cap \overline{I} = 0$.
Therefore every series of $\mathbb{K}\{x, y, z, \underline{w}\}$ is equivalent modulo $\overline{I}$ to a unique series of $\overline{E}$.

Now take $f \in \mathbb{K}\{x, y, z, \underline{w}\}$ such that $x \equiv_R 0$. We can write $f = xp(x, y, z, w_0) + q(y, z, \underline{w})$ and assume that the monomials in the expansion of $xp(x, y, z, w_0) + q(y, z, \underline{w})$ are only those of (2.4). Then

$$x^2 p(x, y, z, w_0) + xq(y, z, \underline{w}) \equiv_R 0.$$

The representation of $x^2 p(x, y, z, w_0) + xq(y, z, \underline{w})$ as a sum of monomials as in (2.4) has the form

$$x^2 p(x, y, z, w_0) + xq(y, z, w_0, 0) + \overline{q}(y, z, \underline{w}) = 0 \tag{2.6}$$

where $\overline{q}(y, z, \underline{w})$ is the series obtained from $xq(y, z, \underline{w}) - xq(y, z, w_0, 0)$ by replacing the monomials as follows (using the two previous operations (2.2) and (2.3)):

$$\left\{ \begin{array}{ccc} xz^\varepsilon y^l w_i^{n_i} & \longmapsto & \frac{1}{i} z^\varepsilon y^{l+1} w_{i-1} w_i^{n_i - 1}, \text{if } i > 0 \\ xz^\varepsilon y^l w_i^{n_i} w_{i+1}^{n_{i+1}} & \longmapsto & \frac{1}{i+1} z^\varepsilon y^{l+1} w_i^{n_i+1} w_{i+1}^{n_{i+1}-1}, \text{ if } i > 0 \\ xz^\varepsilon w_0^{n_0} \dots w_i^{n_i} & \longmapsto & Cz^\varepsilon y w_j^{m_j} w_{j+1}^{m_{j+1}} \text{ or } Cz^\varepsilon y w_j^{m_j} \\ \text{for } i > 0 \text{ and } n_i > 0 & & \text{for some } C \in \mathbb{K}, |C| \leq 1, j \geq 0 \end{array} \right. \tag{2.7}$$

Indeed for the third monomial we have

$$xz^\varepsilon w_0^{n_0} \dots w_i^{n_i} \equiv_R \frac{1}{i+1} z^\varepsilon y w_0^{n_0} \cdots w_{i-1}^{n_{i-1}+1} w_i^{n_i - 1}$$

and this monomial on the right side can be transformed into a monomial of the form $Cz^\varepsilon y w_j^{m_j} w_{j+1}^{m_{i+1}}$ or $Cz^\varepsilon y w_j^{m_j}$ for some $C \in \mathbb{K}$, $|C| \leq 1$, and $j \geq 0$,

by using the two operations (2.2) and (2.3) on monomials.
This shows that the three types of monomials that we obtain after multiplication by x are all distinct, that is the map defined by (2.7) is injective. By (2.6) we have $\overline{q}(y,z,\underline{w})=0$, therefore $q(y,z,\underline{w})-q(y,z,w_0,0)=0$.
Moreover, again by (2.6), we have

$$x^2p(x,y,z,w_0)+xq(y,z,w_0,0)=0.$$

This shows that $x^2p(x,y,z,w_0)+xq(y,z,\underline{w})=0$. Therefore x is not a zero divisor in R. □

Proof of Proposition 2.3. Let $f(t)\in R[\![t]\!]$ such that

$$(x-yt)f(t)=z^2.$$

By writing $f=\sum_{k=0}^{\infty}f_kt^k$ with $f_k\in R$ for every k, we have

$$xf_0=z^2$$

$$xf_k-yf_{k-1}=0 \quad \forall k\geq 1.$$

Thus

$$xf_0=z^2=xw_0$$

so $x(f_0-w_0)=0$ and $f_0=w_0$ by Lemma 2.5. Then we will prove by induction on k that $f_k=k!\,w_k$ for every k. Assume that this is true for an integer $k\geq 0$. Then we have

$$xf_{k+1}=yf_k=k!\,yw_k=(k+1)!\,xw_{k+1}.$$

Hence $x(f_{k+1}-(k+1)!\,w_{k+1})=0$ and $f_{k+1}=(k+1)!\,w_{k+1}$ by Lemma 2.5. Therefore the only solution of

$$(x-yt)f(t)=z^2$$

is the series $\sum_{k=0}^{\infty}k!\,w_kt^k$, and this one is divergent because $\|w_k\|=1$. This holds because in every element of I, the monomial w_k has coefficient 0. □

Now we can give the proof of Theorem 2.4:

Proof of Theorem 2.4. Since x is not a zero divisor in R by Lemma 2.5, the localization morphism

$$R\longrightarrow R_{1/x}$$

is injective. But $R_{1/x}$ is isomorphic to $\mathbb{K}\{x,y,z\}_{1/x}$ since in $R_{1/x}$ we have

$$w_0=z^2/x \text{ and } \forall k\geq 0, w_k=\frac{1}{k!}y^kz^2x^{k+1}.$$

But $\mathbb{K}\{x,y,z\}_{1/x}$ is an integral domain (this is a localization of the integral domain $\mathbb{K}\{x,y,z\}$), therefore so is R.

Now assume that the morphism $R\{t\}\longrightarrow R[\![t]\!]$ is flat. By [10, Theorem

7.6] applied to the linear equation $(x-yt)F - z^2G = 0$, there exist an integer $s \geq 1$, and convergent series

$$a_1(t), \dots, a_s(t), b_1(t), \dots, b_s(t) \in R\{t\}$$

such that

$$(x-yt)a_i(t) - z^2 b_i(t) = 0 \text{ for every } i, \tag{2.8}$$

and formal power series

$$h_1(t), \dots, h_s(t) \in R[\![t]\!]$$

such that

$$f(t) = \sum_{i=1}^{s} a_i(t)h_i(t), \ 1 = \sum_{i=1}^{s} b_i(t)h_i(t).$$

Indeed the vector $(f(t), 1)$ is a solution of the linear equation

$$(x-yt)f(t) - z^2 g(t) = 0$$

with $f(t) := \sum_{k=0}^{\infty} k!\, w_k t^k$.
Then

$$\widetilde{g}(t) := \sum_{i=1}^{s} b_i(t)h_i(0) = 1 + t\varepsilon(t)$$

for some $\varepsilon(t) \in R\{t\}$. Since 1 is a unit of R, $1 + t\varepsilon(t)$ is a unit in $R\{t\}$.
Set $\widetilde{f}(t) := \sum_i a_i(t)h_i(0)$. By (2.8), $(\widetilde{f}(t), \widetilde{g}(t))$ is a solution of the equation

$$(x-yt)\widetilde{f}(t) - z^2\widetilde{g}(t) = 0.$$

Since $\widetilde{g}(t)$ is a unit in $R\{t\}$ we have

$$(x-yt)\widetilde{f}(t)\widetilde{g}(t)^{-1} = z^2.$$

This contradicts Theorem 2.3. Therefore $R\{t\} \longrightarrow R[\![t]\!]$ is not flat. □

3. An Example concerning the strong Artin approximation theorem

Let n be a positive integer, $x = (x_1, \dots, x_n)$ and $\rho > 0$. We set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

$$B_\rho^n := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mid ||f||_\rho := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| \rho^{|\alpha|} < \infty \right\}$$

is a Banach space equipped with the norm $|| \cdot ||_\rho$. Of course $\mathbb{K}[x] \subset B_\rho^n$.

Remark 3.1. We do not have

$$B_\rho^n[\![t]\!] \cap \mathbb{K}\{x, t\} = B_\rho^n\{t\}.$$

For instance, the power series

$$f=\sum_{k\in\mathbb{N}} x_1^{k!}t^k$$

is a convergent power series in (x,t), belongs to $B_2^n[\![t]\!]$, but

$$\sum_k \|x_1^{k!}\|_2\tau^k=\sum_k 2^{k!}\tau^k=\infty$$

for every $\tau>0$. Therefore $f\notin B_2^n\{t\}$.

We provide two examples based on an example of Spivakovsky concerning the extension of Theorem 1.2 to the nested case (see [15]).

Example 3.2. Let $n=1$ and set

$$F(x,t,y_1,y_2):=xy_1^2-(x+t)y_2^2\in B_\rho\{t\}[y_1,y_2].$$

Let

$$\sqrt{1+t}=1+\sum_{n\geq 1}a_nt^n\in\mathbb{Q}\{t\}$$

be the unique power series such that $(\sqrt{1+t})^2=1+t$ and whose value at the origin is 1. For every $c\in\mathbb{N}$ we set $y_2^{(c)}(t):=x^c$ and $y_1^{(c)}(t):=x^c+\sum_{n=1}^c a_nx^{c-n}t^n\in B_\rho\{t\}$. Then

$$F(x,t,y_1^{(c)}(t),y_2^{(c)}(t))\in(t)^{c+1}.$$

On the other hand the equation $f(x,t,y_1(t),y_2(t))=0$ has no solution $(y_1(t),y_2(t))\in B_\rho\{t\}^2$ but $(0,0)$. Indeed let us denote by T_0 the Taylor map at 0:

$$T_0:B_\rho\{t\}\longrightarrow\mathbb{K}[\![x,t]\!].$$

If $f(x,t,y_1(t),y_2(t))=0$ then

$$xT_0(y_1(t))^2-(x+t)T_0(y_2(t))^2=0.$$

But since $\mathbb{K}[\![x,t]\!]$ is a unique factorization domain, this equality implies that $T_0(y_1(t))=T_0(y_2(t))=0$, hence $y_1(t)=y_2(t)=0$.

This shows that there is no $\beta:\mathbb{N}\longrightarrow\mathbb{N}$ such that for every $y(t)\in B_\rho\{t\}^2$ and every $k\in\mathbb{N}$ with

$$F(x,t,y(t))\in(t)^{\beta(k)}$$

there exists $\widetilde{y}(t)\in B_\rho\{t\}^2$ such that

$$F(x,t,\widetilde{y}(t))=0$$

and $\widetilde{y}(t)-y(t)\in(t)^k$.

Example 3.3. We can modify a little bit the previous example to construct a F as before that does not depend on t. We set

$$G(x,y_1,y_2,y_3):=xy_1^2-(x+y_3)y_2^2\in B_\rho[y_1,y_2,y_3].$$

For every $c\in\mathbb{N}$ we set $y_2^{(c)}(t):=x^c$, $y_1^{(c)}(t):=x^c+\sum_{n=1}^c a_nx^{c-n}t^n$ and $y_3^{(c)}(t):=t\in B_\rho\{t\}$. Then

$$G(x,y_1^{(c)}(t),y_2^{(c)}(t),y_3^{(c)}(t))\in(t)^c.$$

Now if $\widetilde{y}(t)\in B_\rho\{t\}^3$ satisfies $G(x,\widetilde{y}(t))=0$ and

$$\widetilde{y}(t)-y(t)\in(t)^2$$

then $\widetilde{y}_3(t)=x+t+\varepsilon(t)$ with $\varepsilon(t)\in(t^2)$. Thus $x+\widetilde{y}_3(t)$ is an irreducible power series in x and t, and it is coprime with x. By the same argument based on the Taylor map as in Example 3.2, the relation

$$x\widetilde{y}_1(t)^2-(x+t+\varepsilon(t))\widetilde{y}_2(t)^2=0$$

implies that $\widetilde{y}_1(t)=\widetilde{y}_2(t)=0$.

This shows that there is no $\beta:\mathbb{N}\longrightarrow\mathbb{N}$ such that for every $y(t)\in B_\rho\{t\}^3$ and every $k\in\mathbb{N}$ with

$$G(x,y(t))\in(t)^{\beta(k)}$$

there exists $\widetilde{y}(t)\in B_\rho\{t\}^3$ such that

$$G(x,\widetilde{y}(t))=0$$

and $\widetilde{y}(t)-y(t)\in(t)^k$.

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E-mail address: guillaume.rond@univ-amu.fr

Université Publique, France