RANK THEOREMS IN SINGULAR SPACES AND A DUAL VERSION OF THE ARTIN APPROXIMATION THEOREM

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Abstract. The aim of this paper is to provide rank Theorems for local rings over singular spaces. More precisely, we extend Gabrielov rank Theorem to Weierstrass temperate families, following our previous work. As an application, we provide a dual version of the celebrated Artin's approximation Theorem, extending a Theorem of Rond. The main technical novelty of this work are the development of Kähler differentials and resolution of singularities methods for Weierstrass families.

1. Introduction

In the early sixties, Grothendieck asked the following question [Gro60]:

Consider a morphism $\varphi : \mathbb{C}\{\mathbf{x}\} \longrightarrow \mathbb{C}\{\mathbf{u}\}$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_m)$, and denote by $\widehat{\varphi}$ the induced morphism of formal power series. If φ admits a formal relation (that is, a power series $F \in \operatorname{Ker}(\widehat{\varphi}) \subset \mathbb{C}[\![\mathbf{x}]\!]$), does it admit a convergent relation (that is, $f \in \operatorname{Ker}(\varphi) \subset \mathbb{C}\{\mathbf{x}\}$)?

Let us mention that, in [Ar71], Artin asked independently of Grothendieck, a question about the approximation property with nested solutions. It appeared recently that this question, in the case of linear equations, is equivalent to Grothendieck's question (see [CPR19] for more details).

Gabrielov answered both questions negatively in general [Ga71] and provided a sufficient condition, the notion of regular morphism, in order to provide a positive answer to a reformulation of Grothendieck's question in the form of a *rank Theorem* [Ga73, To90, BCR21]. Let us mention that the answer to this question is positive when the components are algebraic power series, by the nested approximation theorem due to Popescu [Po86] (see also [CPR19]).

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More precisely, consider:

$$\begin{aligned} &\text{the Generic rank:} && \mathrm{r}(\varphi) := \mathrm{rank}_{\mathrm{Frac}(\mathbb{C}\{\mathbf{u}\})}(\mathrm{Jac}(\varphi)), \\ &\text{the Formal rank:} && \mathrm{r}^{\mathcal{F}}(\varphi) := \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[\![\mathbf{x}]\!]}{\mathrm{Ker}(\widehat{\varphi})}\right), \\ &\text{and the Analytic rank:} && \mathrm{r}^{\mathcal{A}}(\varphi) := \dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{\mathbf{x}\}}{\mathrm{Ker}(\varphi)}\right). \end{aligned} \tag{1}$$

Gabrielov's rank Theorem states that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) \implies \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{A}}(\varphi)$. In [BCR22] we extended Gabrielov's rank Theorem to the case of morphisms $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ between rings of Weierstrass temperate power series rings. These rings are Weierstrass families, introduced by Denef and Lipshitz [DL80], which satisfy three extra axioms [BCR22, Def.2.3]. Examples include the family of germs of complex-analytic functions, algebraic power series, Eisenstein power series, and convergent series over a complete non-archimedean field [BCR22, §4].

In fact, the original Gabrielov's rank Theorem is valid over (complex and real) *singular spaces* as well [Ga73, BCR21]. More precisely, it is equally valid for any morphism of local rings

$$\varphi : A \longrightarrow B, \text{ where } A = \frac{\mathbb{C}\{\mathbf{x}\}}{I} \text{ and } B = \frac{\mathbb{C}\{\mathbf{u}\}}{J},$$

where I is radical and J is prime. Note that this is equivalent to providing a germ of a complex analytic morphism $\varphi : (X, 0) \to (Y, 0)$ where X is irreducible and Y is reduced.

The aim of this paper is to extend our W-temperate rank Theorem [BCR22] to singular spaces. While the definition of formal and Weierstrass ranks $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{W}}(\varphi)$ can be easily extended to this setting, see Definition 2.4, the same is not true for the generic rank. Indeed, previous definitions were geometrical and valid only in the complex-analytic and algebraic categories, cf. [Ga73, Iz89, BCR21]. We introduce an algebraic definition of the generic rank $\mathrm{r}(\varphi)$ based on Kähler differentials, see Definition 2.5, extending previous ones. We can now state the main result of the paper:

Theorem 1.1. *Let $\varphi : A \longrightarrow B$ be a morphism of W-temperate local rings, where B is an integral domain, such that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$. Then:*

(1) φ is regular, that is, $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{W}}(\varphi)$;
(2) φ is strongly injective, that is, $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$.

The proof of Theorem 1.1 is given in §§3.1 and 3.2, and commutative algebra variations of it are given in §4, cf. [BCR21, §1.2.2]. The main technical novelties of the paper are the development of Kähler differentials and resolution of singularities methods in the context of Weierstrass families, see §2.2. This allow us to reduce the proof of Theorem 1.1 to [BCR22, Th.1.1].

Remark 1.2. In fact, Theorem 1.1 holds true for an arbitrary Weierstrass family (not necessarily W-temperate) for which [BCR22, Th.1.1] holds true. In particularly, it holds true for the family of convergent power series over a complete non-trivial value field $\mathcal{K}$ of characteristic zero by [BCR22, Cor.1.2].

As an application, we revisit the original question of Grothendieck in order to provide an alternative answer in the form of a dual version of the celebrated Artin's approximation Theorem [Ar68, Ar69, DL80]:

Theorem 1.3. *Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}/_I$ be a morphism of W-temperate rings such that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$, where I is a prime ideal. Let $h_1(\mathbf{u})$, ..., $h_q(\mathbf{u}) \in \mathcal{K}\{\{\mathbf{u}\}\}$ be generators of I. Let F_1, ..., $F_p \in \mathcal{K}[\![\mathbf{x}]\!]$, $\ell_{1,1}$, ..., $\ell_{p,q} \in \mathcal{K}[\![\mathbf{u}]\!]$ and g_1, ..., $g_p \in \mathcal{K}\{\{\mathbf{u}\}\}$ such that*

$$F_i(\varphi(\mathbf{u})) = g_i(\mathbf{u}) + \sum_{j=1}^{q} h_j(\mathbf{u})\ell_{i,j}(\mathbf{u}), \quad \textit{for } i = 1, \ldots, p.$$

Let $c \in \mathbb{N}$. Then there is $F_1^{(c)}, \ldots, F_p^{(c)} \in \mathcal{K}\{\{\mathbf{x}\}\}$ and $\ell_{1,1}^{(c)}, \ldots, \ell_{p,q}^{(c)} \in \mathcal{K}\{\{\mathbf{u}\}\}$ such that, for $i = 1, \ldots, p$:

$$F_i^{(c)}(\varphi(\mathbf{u})) = g_i(\mathbf{u}) + \sum_{j=1}^{q} h_j(\mathbf{u})\ell_{i,j}^{(c)}(\mathbf{u}),$$

$$\textit{and} \;\; F_i^{(c)}(\mathbf{x}) - F_i(\mathbf{x}) \in (\mathbf{x})^c, \;\; \ell_{i,j}^{(c)}(\mathbf{u}) - \ell_{i,j}(\mathbf{u}) \in (\mathbf{u})^c.$$

This result was previously proved for rings of convergent power series over $\mathbb{C}$ in [Ro08]. The proof of Theorem 1.3 is given in §§3.3 and is based on Theorem 1.1.

Finally, we equally consider the reverse implication (φ strongly injective $\implies \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$) from Theorem 1.1, which is known to hold true when $\mathcal{K}\{\{\mathbf{x}\}\} = \mathbb{C}\{\mathbf{x}\}$ [EH77, Mi78], or when $\mathcal{K}\{\{\mathbf{x}\}\}$ is the ring of algebraic power series [Ro09]. But this implication is not true for W-Temperate families in general, as shown by the Osgood's example for formal power series:

Example 1.4 (Osgood's Example)**.** Consider the W-temperate family of formal power series over $\mathcal{K}$. Let $\varphi : \mathcal{K}[\![x_1, x_2, x_3]\!] \longrightarrow \mathcal{K}[\![u, v]\!]$ be defined by

$$\varphi(x_1) = u, \;\varphi(x_2) = uv, \;\varphi(x_3) = u\xi(v)$$

where $\xi(v)$ is transcendental over $\mathcal{K}[v]$. Then $\mathrm{r}(\varphi) = 2$ and $\mathrm{r}^{\mathcal{F}}(\varphi) = 3$, cf. [Os1916] or [BCR21, Ex.1.16], but φ is strongly injective because it is a morphism of complete local rings.

Let us remark that, if $\mathcal{K}$ is equipped with the trivial absolute value, $\mathcal{K}[\![\mathbf{x}]\!]$ coincides with the ring of convergent power series over $\mathcal{K}$. Therefore, the reverse implication is no longer true for general rings of convergent power series. But it does hold true provided that we work over *a non trivially valued field*:

Theorem 1.5. *Let $\mathcal{K}$ be a complete value field of characteristic zero whose absolute value is non trivial. Let $\varphi : A \longrightarrow B$ be a morphism of quotients of rings of convergent power series over $\mathcal{K}$, where B is an integral domain. Then*

$$\varphi \text{ strongly injective } \Longrightarrow \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi).$$

Theorem 1.5 has been proved by [EH77] in the case that $A = \mathcal{K}\{\mathbf{x}\}$ and $B = \mathcal{K}\{\mathbf{u}\}$ where $\mathcal{K}$ is a characteristic zero archimedean field, and by [Mi78, Th.1] for morphisms of quotient real or complex-analytic rings. We prove Theorem 1.5 in §§3.4 in a two step process. We first reduce the theorem to the case that $A = \mathcal{K}\{\mathbf{x}\}$ and $B = \mathcal{K}\{\mathbf{u}\}$ via the methods developed in §§2.2. We then modify the proof given in [EH77] to extend it to non-archimedean fields. The fact that $\mathcal{K}$ is non archimedean simplifies the estimates in the proof, while it complicates some of the constructions essentially because the norm is not necessarily surjective over $\mathbb{R}_{\geqslant 0}$.

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2. Weierstrass Temperate families

2.1. **W-Temperate families.** Let $\mathcal{K}$ be an algebraically closed field of characteristic zero. For every $n \in \mathbb{N}$, we denote by $(x_1, \ldots, x_n)$ indeterminates; we will use the compact notation $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ whenever there is no risk of confusion on n. In this note, we will consider two notions of rings of power series over $\mathcal{K}$: *Weierstrass families* (or W-families) introduced by Denef and Lipshitz [DL80], and *Weierstrass temperate families* (or W-temperate families) introduced in our previous work [BCR22]. The definitions of these rings are axiomatic and can be found in [BCR22]. We do not recall them here because we won't use any of the axioms explicitly, but only some of their properties.

Roughly speaking, a Weierstrass family over $\mathcal{K}$ is a sequence of local regular subrings of $(\mathcal{K}[\![x_1, \ldots, x_n]\!])_n$, denoted by $(\mathcal{K}\{\{x_1, \ldots, x_n\}\})_n$, containing the polynomials, stable by permutations of the indeterminates, and for which the Weierstrass division Theorem holds. A W-family satisfies several well-known properties, for example, the local rings are Noetherian, they satisfy a version of Artin's Approximation Theorem [DL80] and they satisfy Noether normalization [BCR22, Prop. 2.8(vi)]. We rely on [BCR22, §2.3] for a complete presentation, as well as for a source of citation for precise results. To this list of properties we add the following one:

Proposition 2.1 (Jacobian Criterion)**.** *Let $(\mathcal{K}\{\{x_1, \ldots, x_n\}\})_n$ be a Weierstrass family. Let $A = \mathcal{K}\{\{x_1, \ldots, x_n\}\}[y_1, \ldots, y_m]_{(\mathbf{x},\mathbf{y})}/I$ and assume that A*

is a regular local domain of dimension d. Then $I = (f_1, \ldots, f_{n+m-d})$, where the f_i belong to $\mathcal{K}\{\{\mathbf{x}\}\}[\mathbf{y}]$, and the rank of the Jacobian matrix $\frac{\partial f_i}{\partial(\mathbf{x},\mathbf{y})}(0,0)$ is equal to $n+m-d$.

Proof. We use the notations and the terminology of [Mat89]. By [Mat89, Theorem 30.8], since $\mathcal{K}$ contains $\mathbb{Q}$, the Weak Jacobian Property holds at the ideal I of $\mathcal{K}\{\{\mathbf{x}\}\}[\mathbf{y}]_{(\mathbf{x},\mathbf{y})}$, that is, there exist $D_1, \ldots, D_{n+m-d} \in \mathrm{Der}_{\mathcal{K}}(\mathcal{K}\{\{\mathbf{x}\}\}[\mathbf{y}]_{(\mathbf{x},\mathbf{y})})$ and $f_1, \ldots, f_{n+m-d} \in I$ such that $\det(D_i f_j(0,0)) \neq 0$ in A. Since the $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_\ell}$ form a $\mathcal{K}$-basis of $\mathrm{Der}_{\mathcal{K}}(\mathcal{K}\{\{\mathbf{x}\}\}[\mathbf{y}]_{(\mathbf{x},\mathbf{y})})$, we may replace the D_i by some of the $\frac{\partial}{\partial x_i}$ and the $\frac{\partial}{\partial y_\ell}$. Then by Theorem [Mat89, Theorem 30.4] we have $I = (f_1, \ldots, f_{n+m-d})$. □

A W-temperate family is a W-family satisfying three additional axioms. We do not state this definition explicitly here. In fact, for this work it is enough to know that W-temperate families satisfy the rank Theorem [BCR22, Thm. 1.1]. A list of examples of W-temperate families is given in [BCR22, §4]; they include algebraic power series, convergent power series (over an algebraically closed complete non-trivial valued field) and Eisenstein power series.

2.2. **Weierstrass morphisms and ranks.** Let $(\mathcal{K}\{\{x_1, \ldots, x_n\}\})_n$ be a Weierstrass family (resp. a W-temperate family). We say that a local ring $A = \mathcal{K}\{\{\mathbf{x}\}\}/_I$ is a Weierstrass local ring (resp. a W-temperate local ring), where I is an ideal of $\mathcal{K}\{\{\mathbf{x}\}\}$. The goal of this section is to extend the definition and results from [BCR22, §§2.2] to the context of morphisms $\varphi : A \longrightarrow B$ between Weierstrass local rings, as well as to prove a resolution of singularities type result for Weierstrass local rings via [Te08], see Proposition 2.10 below. We start by the main definitions:

Definition 2.2. Let $(\mathcal{K}\{\{x_1, \ldots, x_n\}\})_n$ be a Weierstrass family (resp. a W-temperate family), and let $\varphi : A \longrightarrow B$ be a morphism of local rings where $A = \mathcal{K}\{\{\mathbf{x}\}\}/_I$ and $B = \mathcal{K}\{\{\mathbf{u}\}\}/_J$. We call φ a *morphism of Weierstrass local rings* (resp. of W-temperate local rings) if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$, where $\mathfrak{m}_A$ and $\mathfrak{m}_B$ are the maximal ideals of A and B respectively.

Remark 2.3. When $A = \mathcal{K}\{\{\mathbf{x}\}\}$ and $B = \mathcal{K}\{\{\mathbf{u}\}\}$, definition 2.2 is equivalent to [BCR22, Def. 2.3], that is, there exist power series $\varphi_1(\mathbf{u}), \ldots, \varphi_n(\mathbf{u}) \in (\mathbf{u})\mathcal{K}\{\{\mathbf{u}\}\}$ such that

$$\forall f(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}, \quad \varphi(f) = f(\varphi_1(\mathbf{u}), \ldots, \varphi_n(\mathbf{u})).$$

The formal and Weierstrass ranks admit an immediate generalization:

Definition 2.4. Let $\varphi : A \longrightarrow B$ be a morphism of Weierstrass local rings, and denote by $\widehat{\varphi} : \widehat{A} \longrightarrow \widehat{B}$ its extension to the ring of formal power series. We define the formal and Weierstrass ranks:

$$\mathrm{r}^{\mathcal{F}}(\varphi) := \dim\left(\frac{\widehat{A}}{\mathrm{Ker}(\widehat{\varphi})}\right), \quad \mathrm{r}^{\mathcal{W}}(\varphi) := \dim\left(\frac{A}{\mathrm{Ker}(\varphi)}\right).$$

Note that $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{W}}(\varphi)$ are well-defined because ${\widehat{A}}/_{\mathrm{Ker}(\widehat{\varphi})}$ and ${A}/_{\mathrm{Ker}(\varphi)}$ are Noetherian local rings.

We now extend the definition of generic rank via Kähler derivations. Given a local complete $\mathcal{K}$-algebra A, we denote by $\widehat{\Omega}^1_{\mathcal{K}}(A)$ the usual completion of the module of Kähler derivations $\Omega^1_{\mathcal{K}}(A)$ (see [EGA, 20.7.14.2]). The canonical derivation is denoted by $\widehat{d} : A \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(A)$. A morphism $\varphi : A \longrightarrow B$ of complete local rings induces a morphism of B-modules (see [EGA, 20.7.17.1])

$$\widehat{\Omega}^1_{\mathcal{K}}(A) \otimes B \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(B)$$

such that, for every $a \in A$ and $b \in B$, the image of $\widehat{d}(a) \otimes b$ is equal to $b\,\widehat{d}(\varphi(a))$. When B is an integral domain, we also have a morphism of $\mathrm{Frac}(B)$-vector space, denoted by $\widehat{d}\varphi : \widehat{\Omega}^1_{\mathcal{K}}(A) \otimes \mathrm{Frac}(B) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(B) \otimes \mathrm{Frac}(B)$.

Definition 2.5. Let $\varphi : A \longrightarrow B$ be a morphism of Weierstrass local rings where B is an integral domain. We set:

$$\text{the generic rank:}\quad \mathrm{r}(\varphi) := \dim_{\mathrm{Frac}(\widehat{B})} \left(\widehat{d}\varphi \left(\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{B}) \right) \right).$$

Remark 2.6. The following statements are immediate from the definition:

(1) When $A = \mathcal{K}\{\{\mathbf{x}\}\}$ and $B = \mathcal{K}\{\{\mathbf{u}\}\}$, the above definition is equivalent to [BCR22, Eq. (1)].
(2) By definition $Gr(\varphi) = \mathrm{r}(\widehat{\varphi})$.
(3) Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be two morphisms of Weierstrass local rings where B and C are integral domains, and ψ is injective. Then $\widehat{d}\varphi$ induces a $\mathrm{Frac}(\widehat{C})$-morphism $\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{C}) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(\widehat{B}) \otimes \mathrm{Frac}(\widehat{C})$, denoted again by $\widehat{d}\varphi$. It follows from the definition that $\widehat{d}(\psi \circ \varphi) = \widehat{d}\psi \circ \widehat{d}\varphi$.
(4) Let $\varphi : A \longrightarrow B$ be a morphism where B is an integral domain. Then $\mathrm{r}(\varphi)$ equals the generic rank of the induced morphism ${A}/_{\mathrm{Ker}(\varphi)} \longrightarrow B$.

Lemma 2.7. *Let A be an integral Weierstrass local ring. Then:*

$$\dim_{\mathrm{Frac}(\widehat{A})} (\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A})) = \dim A.$$

Proof. Let $A = \mathcal{K}\{\{x_1, \ldots, x_n\}\}/_I$ and note that $\dim A = n - \mathrm{ht}\, I$. Given a system of generators $I = (f_1, \ldots, f_k)$, where $k \geqslant \mathrm{ht}\, I$, we have that:

$$\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A}) = \frac{\sum_{i=1}^n \mathrm{Frac}(\widehat{A}) dx_i}{(df_1, \ldots, df_k)},$$

and it follows from [EGA, Prop. 21.9.5] or [To72, Ch.2, Cor.3.3] that $\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A})$ has dimension equal to $n - \mathrm{ht}\, I = \dim \widehat{A} = \dim A$. □

Corollary 2.8 (cf. [BCR22, Lemma 2.4])**.** *Let $\varphi : A \longrightarrow B$ be a morphism of Weierstrass local rings where B is an integral domain. Then:*

$$\mathrm{r}(\varphi) \leqslant \mathrm{r}^{\mathcal{F}}(\varphi) \leqslant \mathrm{r}^{\mathcal{W}}(\varphi).$$

Proof. Consider the morphism of complete local rings $\widehat{\varphi} : \widehat{A} \longrightarrow \widehat{B}$. Note that $\widehat{d}\varphi(\widehat{d}a) = 0$ for all $a \in \mathrm{Ker}(\widehat{\varphi})$. In particular:

$$\mathrm{r}(\varphi) = \mathrm{r}(\widehat{\varphi}) = \dim_{\mathrm{Frac}(\widehat{B})}\left(\widehat{d}\varphi\left(\widehat{\Omega}^1_{\mathcal{K}}\left(\widehat{A}/\mathrm{Ker}(\widehat{\varphi})\right) \otimes \mathrm{Frac}(\widehat{B})\right)\right)$$

Now, set $C = \widehat{A}/\mathrm{Ker}(\widehat{\varphi})$ to simplify the notation. Since the morphism $C \longrightarrow \widehat{B}$ is injective, we conclude that C is an integral domain and that $\mathrm{Frac}(\widehat{B})$ is a field extension of $\mathrm{Frac}(C)$. Therefore:

$$\dim_{\mathrm{Frac}(\widehat{B})}\left(\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}/\mathrm{Ker}(\widehat{\varphi})) \otimes \mathrm{Frac}(\widehat{B})\right) = \dim_{\mathrm{Frac}(C)}\left(\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}/\mathrm{Ker}(\widehat{\varphi})) \otimes \mathrm{Frac}(C)\right)$$

Finally, it follows from Lemma 2.7 that

$$\mathrm{r}(\varphi) = \mathrm{r}(\widehat{\varphi}) \leqslant \dim\left(\widehat{A}/\mathrm{Ker}(\widehat{\varphi})\right) = \mathrm{r}^{\mathcal{F}}(\varphi).$$

Next, by Artin approximation Theorem, $\mathrm{Ker}(\varphi)\widehat{A}$ is a prime ideal, cf. [BCR22, Cor. 2.11], and by [Mat89, Theorem 9.4] the height of $\mathrm{Ker}(\varphi)A$ is less than or equal to the height of $\mathrm{Ker}(\widehat{\varphi})$. We conclude that $\mathrm{r}^{\mathcal{F}}(\varphi) \leqslant \mathrm{r}^{\mathcal{W}}(\varphi)$. □

Proposition 2.9 (cf. [BCR22, Prop. 2.7])**.** *Let $\varphi : A \longrightarrow B$ be a morphism of Weierstrass local rings where B is an integral domain. The ranks $\mathrm{r}(\varphi)$, $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{W}}(\varphi)$ are preserved if we compose φ with:*

(1) a morphism $\sigma : B \longrightarrow \mathcal{K}\{\{u_1, \dots, u_m\}\}$ such that $\mathrm{r}(\sigma) = \dim(B)$,
(2) a finite morphism $\tau : C \longrightarrow A$ of Weierstrass rings.

Proof. Let us prove (1). By Corollary 2.8, $\dim(B) = \mathrm{r}(\sigma) \leqslant \mathrm{r}^{\mathcal{F}}(\sigma) \leqslant \mathrm{r}^{\mathcal{W}}(\sigma) \leqslant \dim(B)$, and we conclude that σ and $\widehat{\sigma}$ are injective morphisms. Therefore $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{W}}(\varphi)$ are preserved after composition with σ. Finally, $\mathrm{r}(\varphi)$ is preserved by Remark 2.6(2).

Let us prove that $\mathrm{r}(\varphi \circ \tau) = \mathrm{r}(\varphi)$ under hypothesis (2). First of all, we may replace A by $A/\mathrm{Ker}(\varphi)$, and C by $C/\tau^{-1}(\mathrm{Ker}(\varphi))$, in order to assume that τ is injective and finite. In particular we may assume that A and C are integral domains and we have $\dim C = \dim A$. We claim that the induced morphism

$$\widehat{d}\varphi : \widehat{\Omega}^1_{\mathcal{K}}(\widehat{C}) \otimes \mathrm{Frac}(\widehat{A}) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A})$$

is surjective.

First we assume the claim. Since τ is finite, then $\widehat{\tau}$ is finite (see [Ho61]). So the induced morphism $\widehat{C}/\mathrm{Ker}(\widehat{\tau}) \longrightarrow \widehat{A}$ is finite and injective. Therefore, we have

$$\dim(\widehat{C}) - \mathrm{ht}(\mathrm{Ker}(\widehat{\tau})) = \dim(\widehat{C}/\mathrm{Ker}(\widehat{\tau})) = \dim(\widehat{A}) = \dim(A) = \dim(C)$$

so $\mathrm{Ker}(\widehat{\tau})$ is a height 0 prime ideal. Since $\widehat{C}$ is an integral domain, we have $\mathrm{Ker}(\widehat{\tau}) = (0)$, so $\widehat{\tau}$ is finite and injective. In particular, $\mathrm{Frac}(\widehat{C}) \to \mathrm{Frac}(\widehat{A})$ is a field extension. Thus, we get that:

$$\dim_{\mathrm{Frac}(\widehat{C})}\left(\widehat{\Omega}^1_{\mathcal{K}}(\widehat{C}) \otimes \mathrm{Frac}(\widehat{C})\right) \geqslant \dim_{\mathrm{Frac}(\widehat{A})}\left(\widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A})\right),$$

and by Lemma 2.7 we conclude that $\widehat{d}\varphi : \widehat{\Omega}^1_{\mathcal{K}}(\widehat{C}) \otimes \mathrm{Frac}(\widehat{A}) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}}(\widehat{A}) \otimes \mathrm{Frac}(\widehat{A})$ is an isomorphism. We conclude by Remark 2.6(2) that $\mathrm{r}(\varphi \circ \tau) = \mathrm{r}(\varphi)$. It remains to prove the claim.

Indeed, let $a \in \widehat{A}$. There exists and integer $k > 0$ and $f_1, \ldots, f_d \in \widehat{C}$ such that

$$a^k + \varphi(f_1)a^{k-1} + \cdots + \varphi(f_k) = 0.$$

We choose k to be the degree of a over $\widehat{C}$, so a does not satisfy such an integral relation of degree $k' < k$. Therefore, we have

$$\left[ka^{k-1} + \varphi(f_1)(k-1)a^{k-1} + \cdots + \varphi(f_{k-1})\right] \widehat{d}a + \left[a^k \widehat{d}\varphi(\widehat{d}f_{k-1}) + \cdots + \widehat{d}\varphi(\widehat{d}f_k)\right] = 0.$$

Since k is the degree of a over $\widehat{C}$, the coefficient multiplying $\widehat{d}a$ is non-zero. We conclude $\widehat{d}\tau$ is surjective, as claimed.

Then, we prove that $\mathrm{r}^{\mathcal{W}}(\varphi) = \mathrm{r}^{\mathcal{W}}(\varphi \circ \tau)$. Indeed, the morphism $C/_{\mathrm{Ker}(\varphi \circ \tau)} \longrightarrow A/_{\mathrm{Ker}(\varphi)}$ is finite an injective, so both rings have the same dimension.

The proof that $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi \circ \tau)$ is similar. Indeed, since τ is finite, $\widehat{\tau}$ is also finite by the Weierstrass division Theorem (see for instance [BCR21, Cor. 1.10] for this claim), and we conclude as before. □

Proposition 2.10. *Let B be a Weierstrass local ring. There exists a morphism of Weierstrass local rings $\sigma : B \longrightarrow \mathcal{K}\{\{u_1, \ldots, u_m\}\}$ such that*

$$m = \dim(B) = \mathrm{r}(\sigma) = \mathrm{r}^{\mathcal{F}}(\sigma) = \mathrm{r}^{\mathcal{W}}(\sigma).$$

Proof. By [DL80, Remark 1.3 (10)] or [Ro18, Proposition 2.20], B is an excellent Noetherian ring of characteristic zero. Therefore, by [Te08, Theorem 3.4.3], there exists a resolution of singularities of $\mathrm{Spec}(B)$, that is, a blowup of an ideal $J \subset B$

$$X \longrightarrow \mathrm{Spec}(B)$$

such that X is nonsingular. We denote by $p \in X$ a closed point of X and U an affine open subset of X containing p. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}(U)$ corresponding to p. Then $\mathcal{O}(U)_{\mathfrak{m}_p} \simeq B[a_1, \ldots, a_q]_{\mathfrak{m}}$ where the a_i belong to $\mathrm{Frac}(B)$ and $\mathfrak{m}$ is a maximal ideal of $B[a_1, \ldots, a_q]$. After rewriting the a_i as $a_i^{(0)} + \varepsilon_i$ where $a_i^{(0)} \in \mathcal{K}$, and replacing a_i by ε_i, we may assume that the a_i belong to $\mathfrak{m}$. We denote by B' the ring $\mathcal{K}\{\{\mathbf{u}, a_1, \ldots, a_q\}\}/I$ where $\mathcal{O}(U)_{\mathfrak{m}_p} = \mathcal{K}\{\{\mathbf{u}\}\}[\mathfrak{a}]/I$ (we set $\mathfrak{a} = (a_1, \ldots, a_q)$). We have $\widehat{B}' = \widehat{\mathcal{O}}(U)_{\mathfrak{m}_p}$. Since $\mathcal{O}(U)_{\mathfrak{m}_p}$ is regular of dimension $d = \dim(B)$, by the Jacobian Criterion we have

$$B[a_1, \ldots, a_q]_{\mathfrak{m}} \simeq \frac{\mathcal{K}\{\{u_1, \ldots, u_m\}\}[a_1, \ldots, a_q]_{(\mathbf{u}, \mathfrak{a})}}{(f_1, \ldots, f_{m+q-d})}$$

where the f_j belong to $\mathcal{K}\{\{\mathbf{u}\}\}[\mathfrak{a}]$, and the Jacobian matrix $\left(\frac{\partial f_j}{\partial(\mathbf{u}, \mathfrak{a})}\right)(0, 0)$ has rank $m + q - d$. Therefore, by the implicit function Theorem [BCR22, Prop. 2.8(i)], we have that $B' \simeq \mathcal{K}\{\{v_1, \ldots, v_d\}\}$ for some new indeterminates v_1,

$\ldots, v_d$. The induced morphism $B \longrightarrow B'$ is denoted by σ. The morphism $\pi : X \longrightarrow \mathrm{Spec}(B)$ is a blowup, it is birational, and it induces an isomorphism

$$\Omega^1_{\mathcal{K}}(B) \otimes_{\mathcal{K}} \mathrm{Frac}(B) \simeq \Omega^1_{\mathcal{K}}(\mathcal{O}(U)_{\mathfrak{m}_p}) \otimes_{\mathcal{K}} \mathrm{Frac}(B).$$

In fact

$$\Omega^1_{\mathcal{K}}(\mathcal{O}(U)_{\mathfrak{m}_p}) \simeq \left[\Omega^1_{\mathcal{K}}(B) + \sum_{i=1}^{q} \mathcal{O}(U)_{\mathfrak{m}_p} da_i\right] \otimes_B \mathcal{O}(U)_{\mathfrak{m}_p}$$

and the da_i belong to $\Omega^1_{\mathcal{K}}(B) \otimes \mathrm{Frac}(B)$. Therefore we have

$$\widehat{\Omega}^1_{\mathcal{K}}(\widehat{\mathcal{O}}(U)_{\mathfrak{m}_p}) \simeq \left[\widehat{\Omega}^1_{\mathcal{K}}(\widehat{B}) + \sum_{i=1}^{q} \widehat{\mathcal{O}}(U)_{\mathfrak{m}_p} \widehat{d}a_i\right] \otimes_{\widehat{B}} \widehat{\mathcal{O}}(U)_{\mathfrak{m}_p}$$

so

$$\widehat{d}\sigma : \widehat{\Omega}^1_{\mathcal{K}}(\widehat{B}) \otimes \mathrm{Frac}(\widehat{B}') \simeq \widehat{\Omega}^1_{\mathcal{K}}(\widehat{B}') \otimes \mathrm{Frac}(\widehat{B}').$$

This proves that $m = \mathrm{r}(\sigma)$. Moreover we have $m = \mathrm{r}^{\mathcal{F}}(\sigma) = \mathrm{r}^{\mathcal{W}}(\sigma)$. Indeed σ factors as $B \longrightarrow \mathcal{O}(U)_{\mathfrak{m}_p} \longrightarrow B'$ where the first morphism is a birational morphism between integral domains, and the second morphism is injective (indeed the injective morphism $\mathcal{O}(U)_{\mathfrak{m}_p} \longrightarrow \widehat{\mathcal{O}}(U)_{\mathfrak{m}_p} = \widehat{B}'$ factors through it). Therefore σ is injective. In the same way, $\widehat{\sigma}$ is injective. $\square$

3. Proofs of the Theorems

3.1. **Proof of Theorem 1.1(1).** By Noether normalization [BCR22, Prop. 2.8(vi)] and Propositions 2.9 and 2.10, we may assume that $B = \mathcal{K}\{\{\mathbf{u}\}\}$ and $A = \mathcal{K}\{\{\mathbf{x}\}\}$. We conclude by [BCR22, Thm 1.1].

3.2. **Proof of Theorem 1.1(2).** We only need to prove that $\widehat{\varphi}(\widehat{A}) \cap B \subset \varphi(A)$, since the reverse inclusion is trivial. So let $f \in \widehat{\varphi}(\widehat{A}) \cap B$, and let $\widehat{g} \in \widehat{A}$ be such that $\widehat{\varphi}(\widehat{g}) = f$.

We have that $A = \mathcal{K}\{\{\mathbf{x}\}\}/I$ for some ideal I of $\mathcal{K}\{\{\mathbf{x}\}\}$. We define $A' = \mathcal{K}\{\{\mathbf{x}, t\}\}/I \cdot \mathcal{K}\{\{\mathbf{x}, t\}\}$ for some new indeterminate t. And we define a new morphism $\varphi' : A' \longrightarrow B$ by $\varphi'_{|A} = \varphi$ and $\varphi'(t) = f$. We have that $t - \widehat{g} \in \mathrm{Ker}(\widehat{\varphi}')$, therefore the canonical injection

$$\psi : \widehat{A}/\mathrm{Ker}(\widehat{\varphi}) \longrightarrow \widehat{A}'/\mathrm{Ker}(\widehat{\varphi}')$$

is an isomorphism, so $\mathrm{r}^{\mathcal{F}}(\varphi') = \mathrm{r}^{\mathcal{F}}(\varphi)$. By Corollary 2.8 we have

$$\mathrm{r}(\varphi) \leqslant \mathrm{r}(\varphi') \leqslant \mathrm{r}^{\mathcal{F}}(\varphi') = \mathrm{r}^{\mathcal{F}}(\varphi),$$

thus, $\mathrm{r}(\varphi') = \mathrm{r}^{\mathcal{F}}(\varphi')$, and $\mathrm{r}^{\mathcal{F}}(\varphi') = \mathrm{r}^{\mathcal{W}}(\varphi')$ by Theorem 1.1(1). Because B and $\widehat{B}$ are integral domains, $\mathrm{Ker}(\varphi')$ and $\mathrm{Ker}(\widehat{\varphi}')$ are prime ideals. By Artin approximation for Weierstrass rings, we have that $\mathrm{Ker}(\varphi')\widehat{A}'$ is a prime ideal, cf. [BCR22, Cor. 2.11]. Since it has the same height as that of $\mathrm{Ker}(\widehat{\varphi}')$, we have

$$\mathrm{Ker}(\varphi')\widehat{A}' = \mathrm{Ker}(\widehat{\varphi}').$$

In particular, if we denote by $\rho : A/_{\mathrm{Ker}(\varphi)} \longrightarrow A'/_{\mathrm{Ker}(\varphi')}$ the canonical injection, then $\widehat{\rho} = \psi$ is an isomorphism. Therefore, $\widehat{\rho}$ induces an isomorphism between $\widehat{A}/\widehat{\mathfrak{m}}_A$ and $\widehat{B}/\widehat{\mathfrak{m}}_B$. But $\widehat{A}/\widehat{\mathfrak{m}}_A \simeq A/\mathfrak{m}_A$ and $\widehat{B}/\widehat{\mathfrak{m}}_B \simeq B/\mathfrak{m}_B$, hence ρ induces an isomorphism between $A/\mathfrak{m}_A$ and $B/\mathfrak{m}_B$. Then, by the Weierstrass preparation Theorem [BCR22, Prop. 2.8(v)], [Mal67], c.f. [BCR21, §1.2.5], ρ is also an isomorphism. We set $g := \rho^{-1}(t) \in A$. Then $\widehat{\rho}(g - \widehat{g}) = 0$, so $g - \widehat{g} \in \mathrm{Ker}(\widehat{\varphi})$, thus, $f = \varphi(g)$. This proves that $\widehat{\varphi}(\widehat{A}) \cap B \subset \varphi(A)$.

3.3. **Proof of Theorem 1.3.** Recall that, by [Ro05, Théorème 3.1] and Artin Approximation Theorem [DL80], there is a constant $c_0 \in \mathbb{N}$ such that, for every $c \in \mathbb{N}$ and every $\ell_1, \dots, \ell_q \in \mathbb{K}[\![\mathbf{u}]\!]$ such that $\sum_{j=1}^q h_j(\mathbf{u})\ell_j(\mathbf{u}) \in (\mathbf{u})^{c+c_0}$, there exist $\ell'_1, \dots, \ell'_q \in \mathbb{K}\{\{\mathbf{u}\}\}$ such that $\sum_j h_j(\mathbf{u})\ell'_j(\mathbf{u}) = 0$ and, for every j, $\ell'_j(\mathbf{u}) - \ell_j(\mathbf{u}) \in (\mathbf{u})^c$.

First we start by the case where $p = 1$. By Theorem 1.1(2), there is $F^{(0)} \in \mathcal{K}\{\{\mathbf{x}\}\}$ such that $F^{(0)}(\varphi_1(\mathbf{u}), \dots, \varphi_n(\mathbf{u})) = g_1(\mathbf{u})$ modulo I. Therefore, by replacing F by $F - F^{(0)}$ we may assume that $g_1(\mathbf{u}) = 0$. The ideals $\mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\widehat{\varphi})$ are prime ideals of $\mathcal{K}\{\{\mathbf{x}\}\}$ and $\mathcal{K}[\![\mathbf{x}]\!]$ respectively, and the equality $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{W}}(\varphi)$ is equivalent to the equality of the heights of these two ideals. Since $\mathcal{K}\{\{\mathbf{x}\}\}$ is Noetherian, the height of $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ equals the height of $\mathrm{Ker}(\varphi)$. Now, by [BCR22, Cor. 2.11] $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is again a prime ideal, so the equality $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{W}}(\varphi)$ is equivalent to the equality $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!] = \mathrm{Ker}(\widehat{\varphi})$. It is well known that, since $\mathcal{K}\{\{\mathbf{x}\}\}$ is Noetherian, $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is the closure of $\mathrm{Ker}(\varphi)$ in $\mathcal{K}[\![\mathbf{x}]\!]$ for the $(\mathbf{x})$-adic topology. Therefore, for every c, there is $F^{(c)}(\mathbf{x})$ such that $F^{(c)}(\varphi(\mathbf{u})) \in I$ and $F^{(c)} - F \in (\mathbf{x})^{c+c_0}$.

In particular, we have $F^{(c)}(\varphi(\mathbf{u})) - F(\varphi(\mathbf{u})) \in (\mathbf{u})^c$. So there exist $\overline{\ell}_1^{(c)}, \dots, \overline{\ell}_q^{(c)} \in \mathcal{K}\{\{\mathbf{u}\}\}$ such that $\sum_{j=1}^q h_j(\mathbf{u})(\ell_j(\mathbf{u}) - \overline{\ell}_j(\mathbf{u})) \in (\mathbf{u})^c$. We conclude by the first remark.

Now assume that $p > 1$. We define a new morphism

$$\psi : \mathcal{K}\{\{\mathbf{x}, \mathbf{y}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}, \mathbf{z}\}\}/_I$$

such that, for every $F \in \mathcal{K}\{\{\mathbf{x}, \mathbf{y}\}\}$, $\psi(F) = F(\varphi_1(\mathbf{u}), \dots, \varphi_n(\mathbf{u}), z_1, \dots, z_p)$, and $\mathbf{y} = (y_1, \dots, y_p)$ and $\mathbf{z} = (z_1, \dots, z_p)$ are vectors of new indeterminates. We have $\mathrm{r}(\psi) = \mathrm{r}(\varphi) + p = \mathrm{r}^{\mathcal{F}}(\varphi) + p = \mathrm{r}^{\mathcal{F}}(\psi)$. We set $F = \sum_{i=1}^p y_i F_i(\mathbf{x})$ and $g = \sum_{i=1}^p z_i g_i(\mathbf{u})$. Then

$$F(\psi_1, \dots, \psi_{n+p}) = g \text{ mod. } I.$$

Therefore, by the case $p = 1$, there is $\widetilde{F} \in \mathcal{K}\{\{\mathbf{x}, \mathbf{y}\}\}$ such that

$$\widetilde{F}(\psi_1, \dots, \psi_{n+p}) = g \text{ mod. } I$$

and $\widetilde{F} - F \in (\mathbf{x}, \mathbf{y})^{c+c_0+1}$. Let us write $\widetilde{F} = \sum_{d\in\mathbb{N}} \widetilde{F}^{(d)}(\mathbf{x}, \mathbf{y})$ where $\widetilde{F}^{(d)}$ is homogeneous of degree d with respect to the y_k. We have

$$\widetilde{F}(\psi_1, \dots, \psi_{n+p}) = \sum_{d\in\mathbb{N}} \widetilde{F}^{(d)}(\varphi_1(\mathbf{u}), \dots, \varphi_n(\mathbf{u}), \mathbf{z}) = \sum_{i=1}^p g_i(\mathbf{u}) z_i \text{ mod. } I.$$

Therefore, since I is generated by series that do not depend on the z_i, $\widetilde{F}^{(d)}(\varphi_1(\mathbf{u}),\ldots,\varphi_n(\mathbf{u}),\mathbf{z})\in I$ for every $d\neq 1$ and we may write

$$\widetilde{F}(\mathbf{x},\mathbf{y})=\sum_{i=1}^{p}\widetilde{F}_i(\mathbf{x})y_i.$$

Then we have $\widetilde{F}_i(\mathbf{x})-F_i(\mathbf{x})\in(\mathbf{x})^{c+c_0}$ and

$$\widetilde{F}_i(\varphi_1(\mathbf{u}),\ldots,\varphi_n(\mathbf{u}))=g_i(\mathbf{u}) \text{ mod. } I,\quad \text{for } i=1,\ldots,p.$$

Then, we use the first remark to conclude as we did for $p=1$.

3.4. **Proof of Theorem 1.5.** Under the hypothesis of the Theorem, by [BCR22, Prop.4.5], the family of convergent power series over $\mathcal{K}$ is a Weierstrass family.

We start by reducing the Theorem to the case that A and B are rings of convergent power series. Indeed, we may replace A by ${}^{A}\!/_{\mathrm{Ker}(\varphi)}$ and assume that φ is injective. By Noether normalization [BCR22, Prop. 2.8(vi)] and Proposition 2.10 there exist an injective finite morphism $\tau:\mathcal{K}\{\{x_1,\ldots,x_n\}\}\longrightarrow A$ and a morphism $\sigma: B\longrightarrow\mathcal{K}\{\{u_1,\ldots,u_m\}\}$ with $\mathrm{r}(\sigma)=\dim(B)$. Note that τ and σ are strongly injective by Theorem 1.1(2) and Remark 1.2. Therefore $\sigma\circ\varphi\circ\tau$ is strongly injective, $\mathrm{r}(\sigma\circ\varphi\circ\tau)=\mathrm{r}(\varphi)$, $\mathrm{r}^{\mathcal{F}}(\sigma\circ\varphi\circ\tau)=\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{W}}(\sigma\circ\varphi\circ\tau)=\mathrm{r}^{\mathcal{W}}(\varphi)$. It is enough to prove the Theorem when $A=\mathcal{K}\{\{\mathbf{x}\}\}$ and $B=\mathcal{K}\{\{\mathbf{u}\}\}$.

We may assume that $\mathcal{K}$ is a characteristic zero field that is non-archimedean, since the archimedean case is proven in [EH77]. The proof is made by contradiction. Assume that there exists an injective morphism $\varphi:\mathcal{K}\{\{\mathbf{x}\}\}\to\mathcal{K}\{\{\mathbf{u}\}\}$ such that $\mathrm{r}(\varphi)<\dim(\mathcal{K}\{\{\mathbf{x}\}\})$ and such that φ is not strongly injective. We consider the following two Lemmas:

Lemma 3.1 (see [EH77, §4])**.** *If such a φ exists, we may assume that all the $\varphi_i(\mathbf{u})$ depend only on $u_1,\ \ldots,\ u_{n-1}$, where $n=\dim\mathcal{K}\{\{\mathbf{u}\}\}$.*

Lemma 3.2 (cf. [EH77, Lemma 4.2])**.** *If $\varphi_1(\mathbf{u}),\ \ldots,\ \varphi_n(\mathbf{u})$ depend only on $u_1,\ \ldots,\ u_{n-1}$, there exists a sequence of polynomials $(P_d)_{d\in\mathbb{N}}$ in $\mathcal{K}[x_1,\ldots,x_n]$, such that*

- $\deg(P_d)\leqslant d$ *for every d,*
- *the maximum of the absolute values of the coefficients of P_d equals 1 for every d,*
- *there exists $\rho>0$ such that $\lim\limits_{d\to\infty}\|P_d(\varphi(\mathbf{u}))\|_\rho^{\frac{1}{d}}=0$.*

The proof of Lemma 3.1 is given in the paragraph before [EH77, Lemma 4.2] and is based only on the fact that $\mathcal{K}$ is a field of characteristic zero. The proof of Lemma 3.2 is given in [EH77, Lemma 4.2] when $\mathcal{K}$ is an archimedean field and we provide a proof in the non-archimedean case below.

We can now finish the proof of the Theorem. Indeed, since the absolute value of $\mathcal{K}$ is non trivial, there is $t\in\mathcal{K}$ such that $|t|>1$. For $d\in\mathbb{N}$, we

define $m(d)\in\mathbb{N}$ by
$$|t|^{m(d)}\leqslant \frac{1}{\|P_d(\varphi(\mathbf{u}))\|_\rho}<|t|^{m(d)+1}.$$
We set
$$f(\mathbf{x}):=\sum_{d\geqslant 2} t^{m(d!)}P_{d!}(\mathbf{x})x_1^{d!}.$$
First, we remark that, for $d\neq e$, the supports of $P_{d!}(\mathbf{x})x_1^{d!}$ and $P_{e!}(\mathbf{x})x_1^{e!}$ are disjoints, because $d!\leqslant \operatorname{ord}(P_{d!}(\mathbf{x})x_1^{d!})\leqslant deg(P_{d!}(\mathbf{x})x_1^{d!})\leqslant 2d!$. In particular $f(\mathbf{x})$ is a well defined power series. Moreover, we have that $f(\mathbf{x})$ is a divergent power series. Indeed, each $P_{d!}$ has a coefficient whose absolute value is 1. Therefore $t^{m(d!)}$ is the coefficient of a monomial of $f(\mathbf{x})$ of degree $k(d)\in[d!,2d!]$. And we have
$$(|t|^{m(d!)})^{\frac{1}{k(d)}}\geqslant\left(\frac{1}{|t|\|P_{d!}(\varphi(\mathbf{u}))\|_\rho}\right)^{\frac{1}{2d!}}\xrightarrow[d\to+\infty]{}+\infty$$
Next, let us prove that $\varphi(f)=f(\varphi(\mathbf{u}))$ is convergent. Indeed, for $\rho_1<\rho$, we have
$$\|f(\varphi(\mathbf{u}))\|_{\rho_1}\leqslant\sum_{d\geqslant 2}|t|^{m(d!)}\|P_{d!}(\varphi(\mathbf{u}))\|_{\rho_1}\|\varphi_1(\mathbf{u})\|_{\rho_1}^{d!}\leqslant\sum_{d\geqslant 2}\|\varphi_1(\mathbf{u})\|_{\rho_1}^{d!},$$
and the right hand side sum is finite if ρ_1 is small enough. This proves that $\varphi(f)$ is convergent and φ is not strongly injective.

Proof of Lemma 3.2. We repeat the initial argument from [EH77]. The set of polynomials of $\mathcal{K}[x_1,\ldots,x_n]$ of degree $\leqslant d$, denoted by $\Lambda_d(n)$, is a $\mathcal{K}$-vector space of dimension $\binom{d+n}{n}$. For $P\in\Lambda_d(n)$, we have $P(\varphi_1,\ldots,\varphi_n)\in\mathcal{K}\{\{u_1,\ldots,u_{n-1}\}\}$ and its coefficients are linear combinations of the coefficients of P. Therefore we may find $P\in\Lambda_d(n)\smallsetminus\{0\}$ such that $\binom{d+n}{n}$ given coefficients of $P(\varphi)$ are zero. For any integer D, a power series with $n-1$ indeterminates has $\binom{D+n-1}{D}$ monomials of degree $\leqslant D$. Therefore, if $\binom{D+n-1}{D}<\binom{d+n}{d}$, we may assume that $\operatorname{ord}(P(\varphi))>D$. In particular, as seen in [EH77], there exists a constant $L>0$, such that for d large enough, there exists $P_d\in\Lambda_d(n)$ with
$$\operatorname{ord}(P_d(\varphi_1,\ldots,\varphi_n))\geqslant Ld^{\frac{n}{n-1}}. \tag{2}$$
By multiplying P_d by a well chosen constant, we may assume that, for every d, the maximum of the absolute values of the coefficients of P_d is equal to 1. From now on we fix this sequence of polynomials $(P_d)_{d\in\mathbb{N}}$.

We recall that a power series $f=\sum_{\alpha\in\mathbb{N}^n}f_\alpha\mathbf{x}^\alpha$ is convergent if and only if there exists $r>0$ such that $\|f\|_r:=\sum_{\alpha\in\mathbb{N}^n}|f_\alpha|r^{|\alpha|}<+\infty$. As the $\varphi_i(\mathbf{u})$ are convergent, there exists $r>0$ and a positive constant $M>0$ such that
$$\forall i,\quad \|\varphi_i(\mathbf{u})\|_r<M. \tag{3}$$

Because the norm is not trivial, for every $\varepsilon > 0$, the set $]0, \varepsilon[\, \cap |\mathcal{K}|$ is non empty. Therefore, we may assume that there exists $c, C \in \mathcal{K}$ such that $|c| = r$ and $|C| = M$.

For $f = \sum_{\alpha\in\mathbb{N}^{n-1}} f_\alpha \mathbf{u}^\alpha$ and $g = \sum_{k\in\mathbb{N}} g_k z^k$, where z is a new indeterminate, we write $f \lll g$ if, for every k, $\sum_{|\alpha|=k} |f_\alpha| \leqslant |g_k|$. Using this notation, (3) gives $\varphi_i(\mathbf{u}) \lll \frac{C}{1-{z}/{c}}$ for all i. For a power series $f = \sum_{\alpha\in\mathbb{N}^{n-1}} f_\alpha \mathbf{u}^\alpha$, we define $\Delta(f) := \sum_{k\in\mathbb{N}}(\sum_{|\alpha|=k} |f_\alpha|) z^\alpha$. We can check that, for two power series f and g, $\Delta(fg) \leqslant \Delta(f)\Delta(g)$. This implies that, for every $a \in \mathbb{N}^n$, we have

$$\varphi_1(\mathbf{u})^{\alpha_1} \cdots \varphi_n(\mathbf{u})^{\alpha_n} \lll \left(\frac{C}{1-{z}/{c}}\right)^{|\alpha|}.$$

Since $\mathcal{K}$ is non-archimedean, $|q| = 1$ for every $q \in \mathbb{Q}$, so the absolute value of the coefficient of z^k in the expansion of the term in the right hand side is $|{M^k}/{r^k}|$. Therefore, for every $\alpha \in \mathbb{N}^n$, we have $\varphi_1(\mathbf{u})^{\alpha_1} \cdots \varphi_n(\mathbf{u})^{\alpha_n} \lll \frac{C^{|\alpha|}}{1-{z}/{c}}$. Hence, since $\mathcal{K}$ is non-archimedean and since the absolute values of the coefficients of P_d are less than 1, we have for all $d \in \mathbb{N}$, $P_d(\varphi_1(\mathbf{u}), \dots, \varphi_n(\mathbf{u})) \lll \frac{C^d}{1-{z}/{c}}$. By (2), we have

$$P_d(\varphi(\mathbf{u})) \lll C^d \sum_{k \geqslant L d^{\frac{n}{n-1}}} \frac{z^k}{c^k} = \left(\frac{z}{c}\right)^{\lceil L d^{\frac{n}{n-1}} \rceil} \frac{C^d}{1-{z}/{c}}.$$

So, for $\rho < r$, $\|P_d(\varphi(\mathbf{u}))\|_\rho \leqslant \frac{\left(\frac{\rho}{r}\right)^{\lceil L d^{\frac{n}{n-1}} \rceil} M^d}{1-{\rho}/{r}}$, and $\lim\limits_{d\to\infty} \|P_d(\varphi(\mathbf{u}))\|_\rho^{\frac{1}{d}} = 0$. $\square$

4. Variations of the rank Theorem

The following are variations of the rank Theorem 1.1:

Theorem 4.1. *The following statements hold true:*

- *(ii) Let $\varphi : A \longrightarrow B$ be a morphism temperate rings, where B is an integral domain. Assume that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$. If $f \in B$ is algebraic (resp. integral) over $\widehat{A}$ then f is algebraic (resp. integral) over A.*
- *(iii) Let $f \in \mathcal{K}\{\{\mathbf{x}, t\}\}$, where t is a single indeterminate. Assume that there is a non-zero $g \in \mathcal{K}[\![\mathbf{x}, t]\!]$, such that $fg \in \mathcal{K}[\![\mathbf{x}]\!][t]$. Then there is a non-zero $h \in \{\{\mathbf{x}, t\}\}$ such that $fh \in \mathcal{K}\{\{\mathbf{x}\}\}[t]$.*
- *(iv) Let $f \in \mathcal{K}\{\{\mathbf{x}, z\}\}$ where z is a single indeterminate and $n \geqslant 2$. Set*

$$A := \mathcal{K}[\![\mathbf{x}]\!] \quad \text{and} \quad B := \frac{\mathcal{K}[\![\mathbf{x}, z]\!]}{(x_1 - x_2 z)}.$$

If the image of f in B is integral over A, then f is integral over $\mathcal{K}\{\{\mathbf{x}\}\}$.

Proof. In [BCR21] we proved that Theorem 1.1(1) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in the case where $\mathcal{K}\{\{\mathbf{x}\}\} = \mathbb{C}\{\mathbf{x}\}$ (see [BCR21, Th. 1.7 & Cor. 1.9]). The proofs provided in [BCR21] are still valid, word by word in the case general case of temperate family of rings. The reader only needs to change the use [BCR21, Lemma 2.3] by the Noether normalisation [BCR22, Prop. 2.8(vi)] and Propositions 2.10 and 2.9. □

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