REAL ALGEBRAIC SURFACES BIHOLOMORPHICALLY EQUIVALENT BUT NOT ALGEBRAICALLY EQUIVALENT

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ABSTRACT. We answer in the negative the long-standing open question of whether biholomorphic equivalence implies algebraic equivalence for germs of real algebraic manifolds in $\mathbb{C}^n$. More precisely we give an example of two germs of real algebraic surfaces in $\mathbb{C}^2$ that are biholomorphic, but not via an algebraic biholomorphism. In fact we even prove that the components of any biholomorphism between these two surfaces are never solutions of polynomial differential equations. The proof is based on enumerative combinatorics and differential Galois Theory results concerning the nature of the generating series of walks restricted to the quarter plane.

1. INTRODUCTION

Given two germs of smooth real analytic manifolds $(M, 0)$ and $(M', 0)$ in $\mathbb{C}^n$, a general question is to determine if there is a germ of biholomorphism $\Phi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that $\Phi(M) = M'$. The classification of real analytic manifolds up to local biholomorphisms is an old and important problem that goes back to H. Poincaré [Po07], when he showed that real analytic hypersurfaces of $\mathbb{C}^2$ have local invariants. E. Cartan, for germs of real analytic smooth hypersurfaces in $\mathbb{C}^2$ [Ca32], and S. S. Chern and J. K. Moser, for germs of real analytic smooth hypersurfaces in $\mathbb{C}^n$ for $n \geqslant 2$ [CM75], gave a complete description of this classification, when the hypersurfaces are assumed to be Levi non-degenerate. More precisely, S. S. Chern and J. K. Moser first gave a complete classification up to formal biholomorphisms. Then they prove that the unique formal biholomorphism sending such a Levi non-degenerate hypersurface to its normal form is convergent.

Therefore a natural question was to investigate if the formal biholomorphic equivalence implies the convergent biholomorphic equivalence. This question has been widely studied and the reader can consult [Mir13] for a general account of this problem. The first negative answer to this question has been given in [MW83]: the authors considered a particular example of two germs of real algebraic smooth surfaces $(M, 0)$ and $(M', 0)$ that are formally equivalent but not biholomorphically equivalent. This surface M has the following particular property: its tangent space at any point near the origin is totally real, but its tangent space at the origin is a complex line. A surface having this property is called a Bishop surface.

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The question remained open for long for CR manifolds, that is, for manifolds for which the largest $\mathbb{C}$-vector subspace of its tangent space has constant dimension. Recently this question has been answered in the negative in [KS16] where the authors constructed CR manifolds formally biholomorphic but non biholomorphic (let us mention that the components of the formal biholomorphism are solutions of polynomial differential equations).

In this paper we consider the case of real algebraic (not necessarily CR) manifolds in $\mathbb{C}^n$. One can define the notion of algebraic (biholomorphic) equivalence: two germs of smooth real analytic manifolds $(M,0)$ and $(M',0)$ in $\mathbb{C}^n$ are algebraically equivalent if there is a germ of biholomorphism $\Phi:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^n,0)$, defined by algebraic power series, such that $\Phi(M)=M'$. A formal power series $f(x_1,\ldots,x_n)\in\mathbb{C}[\![x_1,\ldots,x_n]\!]$ is called *algebraic* if it satisfies a non trivial relation

$$a_0(x)f(x)^d+a_1(x)f(x)^{d-1}\cdots+a_d(x)=0$$

where the $a_i(x)$ are polynomials. The question of whether biholomorphic equivalence implies algebraic equivalence of germs of real algebraic manifolds in whole generality has first been asked in [BER00, 7. Question (b)] (see also [Mir13, Question B]). But this question had already been investigated before: H. Poincaré already studied algebraicity properties of local biholomorphisms between real algebraic hypersurfaces [Po07], and he proved that local biholomorphisms between pieces of 3-spheres in $\mathbb{C}^2$ are necessarily rational mappings (this has been extended to higher dimension by Tanaka [Ta62]). Then an important step was the work of S. M. Webster who proved that biholomorphisms between Levi non-degenerate real algebraic hypersurfaces are necessarily algebraic [We77]. Now the answer is known to be positive in several cases: for example the case of real algebraic hypersurfaces [We77], [BER95], [BMR02], or the case of real algebraic generic manifolds of finite type holomorphically non-degenerate [BER96]. It is known that, for CR manifolds, biholomorphic equivalence implies algebraic equivalence on a Zariski dense subset of points [BRZ01, LM10, Mir12]. See also [Su93, Hu94, SS96, CMS99, Mer01, KLS22, Za99] for other related results, extensions and references, and [Mir13] for a survey about this question.

In this paper we give an example of two Bishop surfaces in $\mathbb{C}^2$ that are biholomorphic but not algebraically biholomorphic. Such surfaces have first been studied by E. Bishop in [Bi65] where he proved that they are locally biholomorphic to a surface defined (locally at 0) by:

$$w=z\overline{z}+\lambda(z^2+\overline{z}^2)+O(|z|^3)\text{ and }\operatorname{Im}(w)=0.$$

The constant λ is a biholomorphic invariant of the germ (M,p), and is called the *Bishop invariant* of (M,p). J. K. Moser and S. M. Webster proved that, for $\lambda\notin\{0,\frac{1}{2},\infty\}$, a Bishop surface admits a (formal) normal form as follows:

$$w=z\overline{z}+(\lambda+\varepsilon w^s)(z^2+\overline{z}^2)\text{ and }\operatorname{Im}(w)=0$$

where $\varepsilon\in\{-1,0,1\}$ and $s\in\mathbb{N}\cup\{\infty\}$ (s is called the *Moser invariant* of (M,p)). They also proved that this normal form can be obtained by a convergent biholomorphism when $\lambda\in]0,\frac{1}{2}[$ but not when $\lambda\in]\frac{1}{2},\infty[$ [MW83]. In fact, when $\lambda\in]0,\frac{1}{2}[$ the automorphic group of the Bishop surface is trivial. Later, J. K. Moser studied the case $\lambda = 0$ and proved that, when $s = \infty$, a Bishop surface is biholomorphically equivalent to the quadric

$$M_\infty = \{(z,w) \in \mathbb{C}^2 \mid w = |z|^2\}.$$

J. K. Moser remarked that the automorphic group of M_∞ is infinite (see Lemma 7 below), making the situation apparently more flexible. The case $\lambda = 0$ and $s < \infty$ has been studied by X. Huang and W. Yin where they prove that a generic Bishop surface with $\lambda = 0$ is no equivalent to an algebraic Bishop surface [HY09].
Here we prove the following theorem:

Theorem 1. *Let M be the Bishop surface defined by*

$$M := \{(z,w) \in \mathbb{C}^2 \mid w - |z|^2(1 + z\overline{z}^2 + z^2\overline{z} + |z|^4)^{-1} = 0 \text{ and } \mathrm{Im}(w) = 0\}.$$

Then the set of biholomorphisms $\Phi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that $\Phi(M_\infty) = M$ is non-empty, but it does not contain any algebraic biholomorphism.
In fact, for every such a Φ, if we set $\Phi(z,w) = (\varphi_1(z,w), \varphi_2(z,w))$ and choose w_0 small enough, the function $z \longmapsto \varphi_2(z, w_0)$ is even hypertranscendental.

A function $F(z)$ is called *hypertranscendental* if it is not solution of a polynomial differential equation

$$P\left(z, F(z), \frac{\partial F}{\partial z}, \ldots, \frac{\partial^N F}{\partial z^N}\right) = 0$$

where $P \in \mathbb{C}(z)[X_0, \ldots, X_N]$ for some integer N. In particular a hypertranscendental series is not algebraic.
In fact our result is more general since we prove the same result for any Bishop surface defined by $\mathrm{Im}(w) = 0$ and one of the following equations:

- $|z|^2 - w(1 + z\overline{z}^2 + z^2\overline{z} + |z|^4) = 0$
- $|z|^2 - w(1 + z + \overline{z} + |z|^4) = 0$
- $|z|^2 - w(1 + z + \overline{z} + z\overline{z}^2 + z^2\overline{z}) = 0$
- $|z|^2 - w(1 + z^2 + \overline{z}^2 + z\overline{z}^2 + z^2\overline{z}) = 0$
- $|z|^2 - w(1 + z^2 + \overline{z}^2 + z\overline{z}^2 + z^2\overline{z} + |z|^4) = 0$
- $|z|^2 - w(1 + z + \overline{z} + z^2 + \overline{z}^2 + |z|^4) = 0$
- $|z|^2 - w(1 + z + \overline{z} + z^2 + \overline{z}^2 + z\overline{z}^2 + z^2\overline{z}) = 0$

Our proof is based on the fact that the functional equations satisfied by the generating series of the some walks restricted to the quarter plane involve a polynomial (its kernel) that can be interpreted, in some cases, as the equation of a Bishop surface. And we use the non trivial fact that the solutions of these equations are transcendental. This kind of idea appeared in [Ro20] where we give a negative answer to a question of N. Mir.

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2. Generating series of walks restricted to the quarter plane

We consider the following situation: We fix a set of *steps* $\mathcal{S} \subset \{-1,0,1\}^2 \backslash \{(0,0)\}$. For every $(i,j) \in \mathbb{N}^2$ and $n \in \mathbb{N}$, we denote by $a_{i,j,n}$ the number of walks with steps in $\mathcal{S}$ of length n, starting at the origin and ending at (i,j), and staying in the quarter plane $\mathbb{N}^2$. The associated generating series Q is defined as follows:

$$Q(x,y,t) := \sum_{i,j,n\in\mathbb{N}} a_{i,j,n}x^iy^jt^n \in \mathbb{Z}[\![x,y,t]\!].$$

The reader may consult [BMM09] or [DHRS18] for a general presentation of the study of these generating series. We recall here that $Q(x,y,t)$ is the solution of a functional equation of the form

$$xy = \left(xy - t\sum_{(a,b)\in\mathcal{S}} x^{a+1}y^{b+1}\right)Q(x,y,t) + R(x,t) + S(y,t) \tag{2.1}$$

where $R(x,t)$ and $S(y,t)$ can be expressed in term of $Q(x,y,t)$. Since $a_{0,0,0} = 1$, we have that $Q(0,0,0) = 1$. The polynomial

$$K_{\mathcal{S}}(x,y,t) := xy - t\sum_{(a,b)\in\mathcal{S}} x^{a+1}y^{b+1}$$

is called the *kernel* of the equation.
By the division theorem of formal power series (see [Ro20]), this equation has a unique solution $(Q(x,y,t),\, R(x,t)+S(y,t))$, that is a unique solution whose second term is a power series without a monomial divisible by xy, and this solution is convergent. We can easily show that the real algebraic surface $M_{\mathcal{S}}$ defined by

$$K_{\mathcal{S}}(z,\overline{z},w) = 0 \text{ and } \operatorname{Im}(w) = 0$$

is smooth if and only if $(-1,-1)\in\mathcal{S}$, and in this case the germ $(M_{\mathcal{S}},0)$ is a Bishop surface with a Bishop invariant $\lambda = 0$ and Moser invariant $s=\infty$ (see [Ro20, Lemma 3]).

To such a kernel $K_{\mathcal{S}}$ is usually associated a group of birational automorphisms $G(\mathcal{S})$ that preserves $K_{\mathcal{S}}$. By [KR12, Theorem 1], this group is finite if and only if the series Q is D-finite (or holonomic). In fact we have the following stronger result:

Theorem 2. [KR12][DHRS18] *If $G(\mathcal{S})$ is infinite then $R(x,t)$ and $S(y,t)$ are x-hypertranscendental and y-hypertranscendental. That is, for $t_0\in\mathbb{C}$ fixed small enough, $R(x,t_0)$ (resp. $S(y,t_0)$) is not a solution, as a function of x (resp. of y), of a polynomial differential equation. In particular, in this case, $R(x,t)$ and $S(y,t)$ are transcendental convergent power series.*

There are 56 such walks and their list can be found in [BMM09, Table 4] for instance.
Among these 56 walks with infinite group, only 7 satisfy the following two properties:

(1) $(-1,-1)\in\mathcal{S}$
(2) Their kernel satisfies the following symmetry:

$$K_{\mathcal{S}}(z,\overline{z},w) = K_{\mathcal{S}}(\overline{z},z,w). \tag{2.2}$$

These correspond to the following sets $\mathcal{S}$:

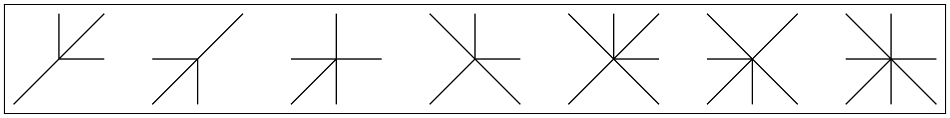

FIGURE 1

Remark 3. In general, even if the sum $R(x,t) + S(y,t)$ is uniquely determined, $R(x,t)$ and $S(y,t)$ are not uniquely determined. But if $\mathcal{S}$ satisfies the symmetry (2.2), then $R(x,t)+S(y,t)$ in (2.1) can be written in a unique way as $R(x,t)+R(y,t)$ where $R(x,t)$ is a convergent power series.

Sketch of Proof of Theorem 2. For the convenience of the reader we provide here the main ideas of the proof of Theorem 2.
First the equation $K_{\mathcal{S}}(x,y,t) = 0$ defines an algebraic curve in $\mathbb{C}^2_{x,y}$, as soon as $t \in \mathbb{C}$ is fixed. The Zariski closure of this curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ can be shown to be an elliptic curve E_t.
Now the series $Q(x,y,t)$ is a convergent power series, so for $|t|$ small enough, $R(x,t)$ and $S(y,t)$ define analytic functions on pieces of E_t. Equation (2.1) allows us to prove that these two functions can be extended as meromorphic functions on the whole E_t.
Let $\pi : \mathbb{C} \longrightarrow E_t$ be the uniformization of the elliptic curve E_t. We can lift these two functions as meromorphic functions r_x and s_y as they satisfy the following equations (see [KR12, Theorem 4])

$$\forall u \in \mathbb{C}, \quad r_x(u+\omega_1) = r_x(u) \text{ and } s_y(u+\omega_1) = s_y(u)$$

where ω_1 is one of the periods of the elliptic curve. Moreover these two functions satisfy some difference equations of the form

$$\tau \circ r_x - r_x = b_1 \text{ and } \tau \circ s_y - s_y = b_2 \tag{2.3}$$

where b_1 and b_2 are two elliptic functions on $\mathbb{C}$, and τ is a translation map on $\mathbb{C}$. This translation map is explicitly defined from the group $G(\mathcal{S})$.
Now the space of meromorphic differential forms on E_t is a one dimensional vector space over $\mathbb{C}(E_t)$. One can prove that there is a derivation δ on $\mathbb{C}(E_t)$ such that $\tau \circ \delta = \delta \circ \tau$. Thus $(\mathbb{C}(E_t), \delta, \tau)$ is a *differential field.* This allows us to use differential Galois Theory to prove the following (see [DHRS18, Proposition 3.6]): if f is meromorphic on $\mathbb{C}$ and satisfies an equation $\tau(f) - f = b$ for some $b \in \mathbb{C}(E_t)$, and if f is a solution of a differential equation, then there exist constants $c_1, \ldots, c_d$ and $g \in \mathbb{C}(E_t)$ such that

$$\delta(b)^d + c_1\delta(b)^{d-1} + \cdots + c_d b = \tau(g) - g. \tag{2.4}$$

Then the idea is to look at the poles of such a b: if q_0 is a pole of b of order m, then any $\tau^k(q_0)$ is also a pole of b of order m, when $k \in \mathbb{Z}$, because τ and δ commute. In particular, if b_1 has a pole q_0 of order $m \geqslant 1$, but one of the $\tau^k(q_0)$ is not a pole of order $\geqslant m$ of b_1, then b_1 does not satisfy an equation of the form (2.4), thus r_x is hypertranscendental. This is done by determining explicitly the poles of b_1 and b_2. This last part is a bit technical and depends explicitly on $\mathcal{S}$. $\square$

3. Proof of Theorem 1

In fact we will prove that Theorem 1 is true for every Bishop surface $M_{\mathcal{S}} \subset \mathbb{C}^2_{z,w}$ defined by

$$K_{\mathcal{S}}(z,\overline{z},w)=0 \text{ and } \operatorname{Im}(w)=0$$

where $\mathcal{S}$ is one of the sets given in Figure 1. That is, $K_{\mathcal{S}}(z,\overline{z},w)$ equals one of the polynomials given after Theorem 1. Let M_∞ be the Bishop surface $\mathbb{C}^2_{z,w}$ defined by

$$|z|^2-w=0 \text{ and } \operatorname{Im}(w)=0$$

or, equivalently, by

$$|z|^2-\frac{1}{2}(w+\overline{w})=0 \text{ and } \operatorname{Im}(w)=0.$$

Definition 4. Let $H:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C}^2,0)$ be the germ of a holomorphic map. We say that H is *triangular* if H has the form:

$$H:\begin{array}{ccc}(\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C}^2,0)\\ (z,w) & \longmapsto & (z,h(z,w))\end{array}$$

with $h(z,w)=\mu w+\varepsilon(z,w)$ where $\mu\in\mathbb{C}^*$ and $\operatorname{ord}(\varepsilon)\geqslant 2$.

We have the following lemma:

Lemma 5. *The set of germs of triangular biholomorphic maps H such that $H(M_{\mathcal{S}})=M_\infty$ is non-empty. This is the set of maps*

$$(z,w)\longmapsto(z,h(z,w))$$

with $h(z,w)=2R(z,w)+if(w)$ where $R(z,w)$ is the series defined in Remark 3 and $f(w)$ is a convergent power series with real coefficients with $f(0)=0$. Moreover $h(z,w)$ is z-hypertranscendental. In particular it is not algebraic.

Proof. Since H is biholomorphic and $\dim(M_\infty)=\dim(M_{\mathcal{S}})$, we have $H(M_{\mathcal{S}})=M_\infty$ if and only if $H(M_{\mathcal{S}})\subset M_\infty$. Then we notice that $H(M_{\mathcal{S}})\subset M_\infty$ if and only if there exist two formal power series $k(z,w,\overline{z},\overline{w})$ and $\ell(z,w,\overline{z},\overline{w})$ such that (here $w=u+iv$) :

$$|z|^2-\frac{1}{2}h(z,w)-\frac{1}{2}\overline{h(z,w)}+vk(z,w,\overline{z},\overline{w})+K_{\mathcal{S}}(z,\overline{z},w)\ell(z,w,\overline{z},\overline{w})=0.$$

This is equivalent to

$$z\overline{z}-\frac{1}{2}h(z,u)-\frac{1}{2}\overline{h}(\overline{z},u)+K_{\mathcal{S}}(z,\overline{z},u)\ell(z,\overline{z},u)=0. \tag{3.1}$$

for some $\ell(z,\overline{z},u)\in\mathbb{C}[\![z,\overline{z},u]\!]$. But (3.1) is exactly (2.1) whose unique solution is the generating series $Q=-\ell$ that counts the number of walks restricted to the quarter plane by the length and by the end point, and whose set of elementary steps is $\mathcal{S}$. Indeed, in (2.1) the series have real coefficients, and since the kernel satisfies the symmetry (2.2), we have $\overline{S}(x,t)=S(x,t)=R(x,t)$. Thus, by unicity we have

$$\frac{1}{2}(h(z,u)+\overline{h}(\overline{z},u))=R(z,u)+R(\overline{z},u)$$

is a z-hypertranscendental power series (with real coefficients) as explained before. Thus $h(z,u)=2R(z,u)+if(u)$ is a z-hypertranscendental power series, where $f(u)$ is a convergent power series with real coefficients.

Now, by (3.1), since $K_S=|z|^2-w(1+\eta(z,w))$ where $\operatorname{ord}(\eta)\geqslant 1$ and $\ell(0,0,0)=-1$, we have that $h(z,w) = (1+ic)w + \varepsilon(z,w)$ with $\mathrm{ord}(\varepsilon) \geqslant 2$ and $c \in \mathbb{R}$. This proves the lemma. □

Remark 6. The map $H_0 : (z,w) \longmapsto (z, 2R(z,w))$ is a triangular biholomorphic map germ with $H_0(M_{\mathcal{S}}) = M_\infty$, and this is exactly the germ of formal holomorphic map given in [Mo85, Proposition 2.1].

Proof of Theorem 1. Let $\Phi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that $\Phi(M_{\mathcal{S}}) = M_\infty$. Thus $g := \Phi \circ H_0^{-1} \in \mathrm{Aut}(M_\infty, 0)$, the group of biholomorphism germs of $(\mathbb{C}^2, 0)$ preserving M_∞. We need the use the following lemma (this statement appears in several works without a proof, so we provide a proof below for the sake of completeness):

Lemma 7. [Mo85, 2.11] *The automorphism group* $\mathrm{Aut}(M_\infty, 0)$ *of* M_∞ *is the set of automorphisms of the form*

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C}^2, 0) \\ z & \longmapsto & a(w)\dfrac{z - wb(w)}{1 - z\bar{b}(w)} \\ w & \longmapsto & a(w)\overline{a}(w)w \end{array}$$

where $a(w)$ *and* $b(w)$ *are power series with* $a(0) \neq 0$ *and* $b(0) = 0$.

So, the components of g have the form

$$g_1(z,w) = a(w)\frac{z - wb(w)}{1 - z\bar{b}(w)} \text{ and } g_2(z,w) = a(w)\overline{a}(w)w$$

as in Lemma 7. We write $\Phi(z,w) = (\varphi_1(z,w), \varphi_2(z,w))$. Since $\Phi = g \circ H_0$, we have

$$\varphi_2(z,w) = 2R\left(a(w)\frac{z - wb(w)}{1 - z\bar{b}(w)}, a(w)\overline{a}(w)w\right).$$

Let $\rho > 0$ such that the convergent power series a, b (resp. R) converge on the disc of radius ρ (resp. ball of radius ρ). Let $w_0 \in \mathbb{C}$, $|w_0| < \rho$. Set $\alpha := a(w_0)$, $\beta := b(w_0)$, and set $F(z) := \varphi_2(z, w_0) = 2R\left(\alpha\dfrac{z - \beta w_0}{1 - z\overline{\beta}}, \alpha\overline{\alpha}w_0\right)$.
Then we have

$$\frac{\partial F}{\partial z}(z) = 2\alpha\frac{1 - \beta\overline{\beta}w_0}{(1 - z\overline{\beta})^2}\frac{\partial R}{\partial z}\left(\alpha\frac{z - \beta w_0}{1 - z\overline{\beta}}, \alpha\overline{\alpha}w_0\right).$$

Let us set, for every $n \in \mathbb{N}$, $G_n(z) := 2\dfrac{\partial^n R}{\partial z^n}\left(\alpha\dfrac{z - \beta w_0}{1 - z\overline{\beta}}, \alpha\overline{\alpha}w_0\right)$. Then, by induction on n, we have

$$\forall n \geqslant 1, \quad \frac{\partial^n F}{\partial z^n}(z) = \sum_{k=1}^{n} \sigma_{n,k}(z) G_k(z)$$

for some rational functions $\sigma_{k,n}(z) \in \mathbb{C}(z)$ with $\sigma_{n,n}(z) = \left(\alpha\dfrac{1 - \beta\overline{\beta}w_0}{(1 - z\overline{\beta})^2}\right)^n \neq 0$. If $F(z)$ was solution of a polynomial differential equation

$$P\left(z, F(z), \frac{\partial F}{\partial z}(z), \dots, \frac{\partial^N F}{\partial z^N}(z)\right) = 0$$

where $P \in \mathbb{C}(z)[X_0, \ldots, X_N]$ for some integer N, then we would have

$$P\left(z, G_0(z), \sigma_{1,1}(z)G_1(z), \ldots, \sum_{k=1}^{N} \sigma_{N,k}(z)G_k(z)\right) = 0 \tag{3.2}$$

The map

$$\widetilde{P} \longmapsto \widetilde{P}\left(z, X_0, \sigma_{1,1}(z)X_1, \ldots, \sum_{k=1}^{N} \sigma_{N,k}(z)X_k\right)$$

is an automorphism of $\mathbb{C}(x)[X_0, \ldots, X_N]$, thus the polynomial

$$Q(z, X_0, \ldots, X_N) := P\left(z, X_0, \sigma_{1,1}(z)X_1, \ldots, \sum_{k=1}^{N} \sigma_{N,k}(z)X_k\right)$$

is nonzero. But the map $z \longmapsto \alpha\dfrac{z - \beta w_0}{1 - z\overline{\beta}}$ is invertible and its inverse is

$$z \longmapsto \sigma(z) := \frac{z + \alpha\beta w_0}{\alpha + \overline{\beta}z}.$$

Thus we would obtain

$$Q\left(\sigma(z), R(z, \alpha\overline{\alpha}w_0), \ldots, \frac{\partial^N R}{\partial z^N}(z, \alpha\overline{\alpha}w_0)\right) = 0$$

But this is a contradiction with the fact that $R(z, w)$ is z-transcendental. Therefore $F(z) = \varphi_2(z, w_0)$ is z-transcendental.

□

Proof of Lemma 7. First, a direct computation shows that such automorphisms preserve M_∞.
Now let $g : (z, w) \longrightarrow (Z(z, w), W(z, w))$ be a biholomorphic map preserving M_∞. Thus $|Z|^2 = W + \overline{W}$ and $\mathrm{Im}(W) = 0$. Since W is holomorphic in z and w, and z is a complex coordinate, this implies that W is real valued and depends only on w. Since g is a biholomorphism, $W'(0) \neq 0$, so $W(w) = wh(w)$ for some non vanishing holomorphic function h. Since W is real, the coefficients of h are real, so $h(w) = a(w)\overline{a}(w)$ for some analytic function $a(w)$ with $a(0) \neq 0$.
We set $\widetilde{Z}(z, w) := \frac{Z(z,w)}{a(w)}$. Thus $|\widetilde{Z}(z, w)|^2 = w$. In particular the linear part of $Z(z, w)$ is of the form αz where $|\alpha| = 1$.
Now let us consider the restriction of $\widetilde{Z}(z, w)_{|w=w_0}$ where w_0 is a nonzero constant. This function map is 1-1 in restriction to the circle $|z|^2 = w_0$. Thus $\widetilde{Z}(z, w)_{|w=w_0}$ is a Möbius transform of the form $\alpha\frac{z - w_0 b}{1 - z\overline{b}}$ for some $b \in \mathbb{C}$. Since b and θ depend on w analytically, this proves the result by replacing $a(w)$ by $e^{i\theta(w)}a(w)$. □

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