

REAL ALGEBRAIC SURFACES BIHOLOMORPHICALLY EQUIVALENT BUT NOT ALGEBRAICALLY EQUIVALENT

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ABSTRACT. We answer in the negative the long-standing open question of whether biholomorphic equivalence implies algebraic equivalence for germs of real algebraic manifolds in \mathbb{C}^n . More precisely we give an example of two germs of real algebraic surfaces in \mathbb{C}^2 that are biholomorphic, but not by means of an algebraic biholomorphism.

1. INTRODUCTION

Given two germs of smooth real analytic manifolds $(M, 0)$ and $(M', 0)$ in \mathbb{C}^n , we say that $(M, 0)$ and $(M', 0)$ are *(biholomorphically) equivalent* if there is a germ of biholomorphism $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $\Phi(M) = M'$. The classification of real analytic manifolds up to local biholomorphisms is an old and important problem that goes back to H. Poincaré [Po07], when he showed that real analytic hypersurfaces of \mathbb{C}^2 have local invariants. E. Cartan, for germs of real analytic smooth hypersurfaces in \mathbb{C}^2 [Ca32], and S. S. Chern and J. K. Moser, for germs of Levi non-degenerate real analytic smooth hypersurfaces in \mathbb{C}^n for $n \geq 2$ [CM75], gave a complete description of this classification. More precisely, S. S. Chern and J. K. Moser first gave a complete classification up to formal biholomorphisms. Then they prove that the unique formal biholomorphism sending such a Levi non-degenerate hypersurface to its normal form is convergent.

Therefore a natural question was to investigate if the formal equivalence implies the convergent equivalence. This question has been widely studied and the reader can consult [Mir13] for a general account of this problem. The first negative answer to this question has been given in [MW83]: the authors considered a particular example of two real algebraic smooth surface germs $(M, 0)$ and $(M', 0)$ that are formally equivalent but not biholomorphically equivalent. This surface M has the following particular property: its tangent space at any point near the origin is totally real, but its tangent space at the origin is a complex line. A surface having this property is called a

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Bishop surface.

Recently this question has also been answered in the negative in [KS16] for CR manifolds, that is, for manifolds for which the largest \mathbb{C} -vector subspace of its tangent space has constant dimension.

Here we consider the case of real algebraic manifolds in \mathbb{C}^n . One can define the notion of *algebraic equivalence*: two germs of smooth real algebraic manifolds $(M, 0)$ and $(M', 0)$ in \mathbb{C}^n are algebraically equivalent if there is a germ of biholomorphism $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$, defined by algebraic power series, such that $\Phi(M) = M'$. A formal power series $f(x_1, \dots, x_n) \in \mathbb{C}[[x_1, \dots, x_n]]$ is called *algebraic* if it satisfies a non trivial relation

$$a_0(x)f(x)^d + a_1(x)f(x)^{d-1} \dots + a_d(x) = 0$$

where the $a_i(x)$ are polynomials. The question of whether biholomorphic equivalence implies algebraic equivalence of germs of real algebraic manifolds in whole generality has first been asked in [BER00, 7. Question (b)] (see also [Mir13, Question B]). But this question had already been investigated before: H. Poincaré studied algebraicity properties of local biholomorphisms between real algebraic hypersurfaces [Po07], and he proved that local biholomorphisms between pieces of 3-spheres in \mathbb{C}^2 are necessarily rational mappings (this has been extended in higher dimension by Tanaka [Ta62]). Then an important step has been made by S. M. Webster who proved that biholomorphisms between Levi non-degenerate real algebraic hypersurfaces are necessarily algebraic [We77]. Now the answer is known to be positive in several cases: for example the case of real algebraic hypersurfaces [We77], [BMR02], or the case of real algebraic generic manifolds of finite type holomorphically non-degenerate [BER96]. It is known that, for CR manifolds, the biholomorphic equivalence implies the algebraic equivalence on a Zariski dense subset of points [BRZ01, LM10]. See also [Su93, Hu94, BR99, SS96, CMS99, Mer01, Mir12, KLS22, Za99] for other related results, extensions and references, and [Mir13] for a survey about this question.

In this paper we give an example of two algebraic Bishop surfaces in \mathbb{C}^2 that are biholomorphic but not algebraically biholomorphic, showing that the answer to this question is negative in general. Such surfaces have first been studied by E. Bishop in [Bi65] where he proved that they are locally biholomorphic to a surface defined (locally at 0) by:

$$w = z\bar{z} + \gamma(z^2 + \bar{z}^2) + O(|z|^3) \text{ and } \text{Im}(w) = 0.$$

The constant γ is a biholomorphic invariant of the germ (M, p) , and is called the *Bishop invariant* of (M, p) . J. K. Moser and S. M. Webster proved that, for $\gamma \notin \{0, \frac{1}{2}, \infty\}$, a Bishop surface admits a (formal) normal form as follows:

$$w = z\bar{z} + (\gamma + \varepsilon w^s)(z^2 + \bar{z}^2) \text{ and } \text{Im}(w) = 0$$

where $\varepsilon \in \{-1, 0, 1\}$ and $s \in \mathbb{N}^* \cup \{\infty\}$. They also proved that this normal form can be obtained by a convergent biholomorphism when $\gamma \in]0, \frac{1}{2}[$ (see [MW83, Theorem 1]). Here we prove the following:

Theorem 1. *Let M_γ be the Bishop surface defined by*

$$M_\gamma := \{(z, w) \in \mathbb{C}^2 \mid w = |z|^2 + \gamma(\bar{z}^2 + z^3\bar{z})\}.$$

Assume that $\gamma \in]0, \frac{1}{2}[$ is a transcendental number. Then M is biholomorphically equivalent to its normal form, but not by means of an algebraically biholomorphic map.

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2. ALGEBRAIC POWER SERIES

In this part we review some results about algebraic power series that we will use in the proof of Theorem 1. The reader can consult [Ro18a] and the references therein for more details about the ring of algebraic power series.

Let \mathbb{K} be a field and x denote the vector of indeterminates (x_1, \dots, x_n) . A formal power series $f(x_1, \dots, x_n) \in \mathbb{K}[[x_1, \dots, x_n]]$ is called *algebraic* if it satisfies a non trivial relation

$$a_0(x)f(x)^d + a_1(x)f(x)^{d-1} \dots + a_d(x) = 0$$

where the $a_i(x)$ are polynomials. The set of algebraic power series forms a subring of $\mathbb{K}[[x]]$ denoted by $\mathbb{K}\langle x \rangle$. This is a Noetherian local ring, and, when $\mathbb{K} = \mathbb{C}$ every algebraic power series is a convergent power series. This ring satisfies the Implicit Function Theorem in the following sense: let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be two tuples of variables. Let $f_1(x, y), \dots, f_m(x, y) \in \mathbb{K}\langle x, y \rangle$ such that

$$f_1(0, 0) = \dots = f_m(0, 0), \text{ and the matrix } \frac{\partial f}{\partial y}(0, 0) \text{ is invertible.}$$

Then there is a unique vector of algebraic power series $y(x) = (y_1(x), \dots, y_m(x)) \in \mathbb{K}\langle x \rangle^m$ such that

$$f_1(x, y(x)) = \dots = f_m(x, y(x)) \text{ and } y_1(0) = \dots = y_m(0) = 0.$$

In fact $\mathbb{K}\langle x \rangle$ is the smallest subring of $\mathbb{K}[[x]]$ containing $\mathbb{K}[x]$ and satisfying the Implicit Function Theorem.

We mention that the composition of algebraic power series is again an algebraic power series. Moreover if a power series $f(x)$ with complex coefficients is algebraic, then its conjugate $\bar{f}(x)$ is also algebraic.

The following remarks will be useful in the sequel:

Remark 2. Let $\Phi : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ be a bi-analytic automorphism induced by algebraic power series. Then the inverse Φ^{-1} is also induced by algebraic power series. This comes from the Inverse Function Theorem that is equivalent to the Implicit Function Theorem.

Remark 3. Since the ring $\mathbb{K}\langle x \rangle$ is a Noetherian local ring, the ring extension $\mathbb{K}\langle x \rangle \longrightarrow \mathbb{K}[[x]]$ is faithfully flat (see [Mat89, Theorem 8.14] for example). This implies that any system of linear equations

$$a_{j,1}(x)y_1 + \cdots + a_{j,p}(x)y_p = b_j(x), \text{ for } j = 1, \dots, q,$$

where the $a_{j,k}(x)$ and $b_j(x) \in \mathbb{K}\langle x \rangle$, that has a formal power series solution

$$(f_1(x), \dots, f_p(x)) \in \mathbb{K}[[x]]^p,$$

has also a solution in $\mathbb{K}\langle x \rangle^p$ (see [Ro18c, Example 1.4] for example).

Remark 4. Let $\mathbb{K} \longrightarrow \mathbb{L}$ be an algebraic field extension. Then $\mathbb{K}[[x]] \cap \mathbb{L}\langle x \rangle = \mathbb{K}\langle x \rangle$. Indeed, if $f(x) \in \mathbb{K}[[x]] \cap \mathbb{L}\langle x \rangle$, then $f(x)$ is algebraic over $\mathbb{L}\langle x \rangle$. But $\mathbb{K}[[x]] \longrightarrow \mathbb{L}[[x]]$ is an algebraic extension of rings. Therefore, $f(x)$ is algebraic over $\mathbb{K}[[x]]$.

Lemma 5. *Let \mathbb{K} be field and c be finite over \mathbb{K} of degree $d > 1$. Assume that c is separable over \mathbb{K} . Let $x = (x_1, \dots, x_n)$ be a tuple of indeterminates, and let $\varphi(x) \in \mathbb{K}[c]\langle x \rangle$. Let us write*

$$\varphi(x) = \varphi_0(x) + c\varphi_1(x) + \cdots + c^{d-1}\varphi_{d-1}(x)$$

where the φ_j are formal power series with coefficients in \mathbb{K} . Then, for every j , $\varphi_j(x) \in \mathbb{K}\langle x \rangle$.

Proof. The power series $\varphi(x)$ is algebraic over $\mathbb{K}[c][x]$. But $\mathbb{K}[c][x]$ is a finite extension of $\mathbb{K}[x]$, thus $\varphi(x)$ is algebraic over $\mathbb{K}[x]$. Let $P(x, y) \in \mathbb{K}[x, y]$ be nonzero with $P(x, \varphi(x)) = 0$.

Let us denote by $c_1 := c, c_2, \dots, c_d$ the conjugates of c over \mathbb{K} in a given algebraic closure $\overline{\mathbb{K}}$ of \mathbb{K} . Let us set, for $j = 1, \dots, d$

$$\varphi(c_j, x) := \varphi_0(x) + c_j\varphi_1(x) + \cdots + c_j^{d-1}\varphi_{d-1}(x).$$

The $\varphi(c_j, x)$ are algebraic power series since $P(x, \varphi(c_j, x)) = 0$ for every j . Indeed, if σ is a \mathbb{K} -automorphism of $\overline{\mathbb{K}}$ sending $c = c_1$ on c_j , we have

$$0 = P(x, \varphi(c, x)) = \sigma(P(x, \varphi(c, x))) = P(x, \varphi(\sigma(c), x)) = P(x, \varphi(c_j, x))$$

since the coefficients of P belong to \mathbb{K} . We have

$$\begin{pmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{d-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{d-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & c_d & c_d^2 & \cdots & c_d^{d-1} \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \vdots \\ \varphi_{d-1}(x) \end{pmatrix} = \begin{pmatrix} \varphi(c_1, x) \\ \varphi(c_2, x) \\ \vdots \\ \varphi(c_d, x) \end{pmatrix}$$

Let us denote by M the left-hand Vandermonde matrix. This matrix is invertible and the entries of its inverse belong to $\overline{\mathbb{K}}$. Therefore, by Remark 4, $\varphi_0(x), \dots, \varphi_{d-1}(x)$ belong to $\mathbb{K}\langle x \rangle$. \square

Lemma 6. *Let $x = (x_1, \dots, x_n)$, \mathbb{K} be a characteristic zero field, \mathbb{k} be a subfield of \mathbb{K} and $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \in \mathbb{K}\langle x \rangle$. Then there is a field $\mathbb{L} \subset \mathbb{K}$, finitely generated over \mathbb{k} , such that*

$$\forall \alpha \in \mathbb{N}^n, f_\alpha \in \mathbb{L}.$$

Proof. This result is well known but we did not find an explicit reference. This result is an easy corollary of [Gi69]: let \mathbb{K}_0 be the field extension of \mathbb{k} generated by the coefficients of a nonzero polynomial $P(x, y)$ such that $P(x, f(x)) = 0$. So \mathbb{K}_0 is a finitely generated field extension of \mathbb{k} . Moreover f is algebraic over $\mathbb{K}_0[x]$, so by [Gi69, Theorem 1.1], all the f_α are algebraic over \mathbb{K}_0 . Then by [Gi69, Theorem 2.4] or [CK08, Theorem 1.1], the f_α belong to a finite field extension \mathbb{L} of \mathbb{K}_0 . In particular $\mathbb{K}_0 \rightarrow \mathbb{L}$ is a finitely generated field extension. So $\mathbb{k} \rightarrow \mathbb{L}$ is finitely generated. \square

Lemma 7 (Eisenstein's Theorem). *(see [Ro18b, Lemma 11] or [Ro24] for example) Let $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_r)$ be two tuples of indeterminates and \mathbb{K} be a field extension of $\overline{\mathbb{Q}}$ generated by algebraically independent elements $\gamma_1, \dots, \gamma_r$. Let $f \in \mathbb{K}\langle x \rangle^m$ be a tuple of algebraic power series. Then there exist a polynomial $S(t) \in \overline{\mathbb{Q}}[t]$ and, for every $\alpha \in \mathbb{N}^n$, polynomials $R_{j,\alpha}(t) \in \overline{\mathbb{Q}}[t]$ such that*

$$\forall j = 1, \dots, m, f_j(x) = F_j(\gamma_1, \dots, \gamma_r, x) \text{ with } F_j(t, x) = \sum_{\alpha \in \mathbb{N}^n} \frac{R_{j,\alpha}(t)}{S(t)^{|\alpha|}} x^\alpha$$

where $|\alpha| := \alpha_1 + \dots + \alpha_n$.

3. PROOF OF THE MAIN THEOREM

Let $\gamma \in]0, 1/2[$. By [MW83, Theorem 1], there is a convergent biholomorphic map germ $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $\Phi(M_\gamma)$ is the germ of some algebraic set $N_{\gamma,\varepsilon,s}$ as defined in the introduction. In particular there is a convergent biholomorphic map Φ sending M_γ into the real hyperplane $\text{Im}(w) = 0$. If we denote by $\varphi_1(z, w)$ and $\varphi_2(z, w) \in \mathbb{C}\{z, w\}$ the components of Φ , this means that

$$\varphi_2(z, w) - \overline{\varphi_2}(\bar{z}, \bar{w}) = 0$$

when $w = |z|^2 + \gamma(\bar{z}^2 + z^3\bar{z})$. This is equivalent to

$$\varphi_2(z, |z|^2 + \gamma(\bar{z}^2 + z^3\bar{z})) - \overline{\varphi_2}(\bar{z}, |z|^2 + \gamma(\bar{z}^2 + z^3\bar{z})) = 0.$$

This is also equivalent to the existence of two power series $\ell_1(z, \bar{z}, w, \bar{w})$, $\ell_2(z, \bar{z}, w, \bar{w}) \in \mathbb{C}\{z, \bar{z}, w, \bar{w}\}$ such that

(3.1)

$$\begin{aligned} \varphi_2(z, w) - \overline{\varphi_2}(\bar{z}, \bar{w}) &= \ell_1(z, \bar{z}, w, \bar{w}) (w - |z|^2 - \gamma(\bar{z}^2 + z^3\bar{z})) \\ &\quad + \ell_2(z, \bar{z}, w, \bar{w}) (\bar{w} - |z|^2 - \gamma(z^2 + \bar{z}^3z)) \end{aligned}$$

When $\gamma > 1/2$ is such that the solutions of the equation $\gamma T^2 - T + \gamma = 0$ are not roots of unity (one says that γ is *exceptional*), Moser and Webster

proved that there is a formal biholomorphic map germ Φ satisfying Equation (3.1), but that this formal map is not convergent (see [MW83, Theorem 6.1]).

Now let us fix $\gamma_1 \in]0, 1/2[$ to be a transcendental number. The proof is done by contradiction : assume that there exists a convergent biholomorphic map germ $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ whose components, $\varphi_1(z, w)$, $\varphi_2(z, w) \in \mathbb{C}\langle z, w \rangle$ are algebraic power series and satisfy (3.1). Since $\varphi_2(z, w)$, $\bar{\varphi}_2(\bar{z}, \bar{w})$, $w - |z|^2 - \gamma_1(\bar{z}^2 + z^3\bar{z})$ and $\bar{w} - |z|^2 - \gamma(z^2 + \bar{z}^3z)$ are algebraic power series, by Remark 3 we may assume that $\ell_k(z, \bar{z}, w, \bar{w}) \in \mathbb{C}\langle z, \bar{z}, w, \bar{w} \rangle$ for $k = 1, 2$.

By Lemma 6 the coefficients of the algebraic series $\varphi_2(z, w)$ and the $\ell_k(z, \bar{z}, w, \bar{w})$, for $k = 1, 2$, belong to a finitely generated field extension \mathbb{L} of $\mathbb{K} := \overline{\mathbb{Q}}(\gamma_1)$. Let $\xi_1, \dots, \xi_p \in \mathbb{C}$ be generators of \mathbb{L} : $\mathbb{L} = \mathbb{K}(\xi_1, \dots, \xi_p)$. Since $i \in \overline{\mathbb{Q}} \subset \mathbb{K}$, we may replace the ξ_k by their real and imaginary parts in order to assume that \mathbb{L} is a finitely generated field extension of \mathbb{K} generated by real numbers. Take a maximal algebraically independent subfamily of $\{\xi_1, \dots, \xi_p\}$ and denote the elements of this family by $\gamma_2, \dots, \gamma_r$. Thus \mathbb{L} is a finite field extension of $\mathbb{L}_1 := \mathbb{K}(\gamma_2, \dots, \gamma_r)$. By the primitive element theorem, this finite extension is generated by one algebraic element that we write as $a + ib$ where $a, b \in \mathbb{R}$. So the coefficients of the algebraic series are in $\mathbb{L}_1[a, b]$. But, it is well known that $\mathbb{L}_1[a, b] = \mathbb{L}_1[a + \lambda b]$ for all but finitely many $\lambda \in \mathbb{L}_1$. So we may find $\lambda \in \mathbb{Q}$ such that $c := a + \lambda b \in \mathbb{R}$ is finite over \mathbb{L}_1 and the coefficients of the algebraic series belong to $\mathbb{K}(\gamma_2, \dots, \gamma_r)[c]$.

If d denotes the degree of c over \mathbb{L}_1 , we can write

$$\varphi_2(z, w) = \sum_{j=0}^{d-1} c^j \varphi_{2,j}(z, w) \text{ and } \ell_k(z, \bar{z}, w, \bar{w}) = \sum_{j=0}^{d-1} c^j \ell_{k,j}(z, \bar{z}, w, \bar{w})$$

where the $\varphi_{2,j}$ and the $\ell_{k,j}$ are power series with coefficients in \mathbb{L}_1 . By Lemma 5, the $\varphi_{2,j}$ and the $\ell_{k,j}$ are algebraic power series with coefficients in \mathbb{L}_1 . Moreover, because (3.1) is a linear equation, because $c \in \mathbb{R}$, and because the coefficients of $w - |z|^2 - \gamma(\bar{z}^2 + z^3\bar{z})$ and $\bar{w} - |z|^2 - \gamma(z^2 + \bar{z}^3z)$ are in \mathbb{L}_1 , $(\varphi_2, \ell_1, \ell_2)$ is a solution of (3.1) if and only if all the $(\varphi_{2,j}, \ell_{1,j}, \ell_{2,j})$ are solutions of (3.1). Moreover, since Φ is a germ of biholomorphism, the vanishing order of $\varphi_2(z, w)$ is 1. So the vanishing order of $\varphi_{2,j}$, for some j (let us say j_0), is also 1, since $1, c, \dots, c^{d-1}$ are \mathbb{L}_1 -linearly independent. Therefore we may replace $(\varphi_2, \ell_1, \ell_2)$ by $(\varphi_{2,j_0}, \ell_{1,j_0}, \ell_{2,j_0})$ and assume that the coefficients of φ_2 and the ℓ_k are in \mathbb{L}_1 .

Now, by Lemma 7, we can write

$$(3.2) \quad \varphi_2(z, w) = \sum_{\alpha \in \mathbb{N}^2} \frac{R_\alpha(\gamma_1, \dots, \gamma_r)}{S(\gamma_1, \dots, \gamma_r)^{|\alpha|}} z^{\alpha_1} w^{\alpha_2}$$

$$(3.3) \quad \text{and } \ell_k(z, \bar{z}, w, \bar{w}) = \sum_{(\alpha, \beta) \in \mathbb{N}^4} \frac{R_{k, \alpha, \beta}(\gamma_1, \dots, \gamma_r)}{S(\gamma_1, \dots, \gamma_r)^{|\alpha| + |\beta|}} z^{\alpha_1} w^{\alpha_2} \bar{z}^{\beta_1} \bar{w}^{\beta_2}$$

for some polynomials $R_\alpha, S, R_{k,\alpha,\beta} \in \overline{\mathbb{Q}}[t_1, \dots, t_r]$. We set $t = (t_1, \dots, t_r)$ and

$$U(t, z, w) := \sum_{\alpha \in \mathbb{N}^2} \frac{R_\alpha(t_1, \dots, t_r)}{S(t_1, \dots, t_r)^{|\alpha|}} z^{\alpha_1} w^{\alpha_2}$$

and

$$V_k(t, z, \bar{z}, w, \bar{w}) = \sum_{(\alpha, \beta) \in \mathbb{N}^4} \frac{R_{k,\alpha,\beta}(t_1, \dots, t_r)}{S(t_1, \dots, t_r)^{|\alpha|+|\beta|}} z^{\alpha_1} w^{\alpha_2} \bar{z}^{\beta_1} \bar{w}^{\beta_2}.$$

Let $P(t, z, w, u) \in \overline{\mathbb{Q}}(t)[z, w, u]$ and $Q_k(t, z, \bar{z}, w, \bar{w}, v) \in \overline{\mathbb{Q}}(t)[z, \bar{z}, w, \bar{w}, v]$ be irreducible polynomials such that

(3.4)

$$P(\gamma_1, \dots, \gamma_r, z, w, \varphi_2(z, w)) = 0, \quad Q_k(\gamma_1, \dots, \gamma_r, z, \bar{z}, w, \bar{w}, \ell_k(z, \bar{z}, w, \bar{w})) = 0.$$

Up to multiplication by an element of $\overline{\mathbb{Q}}(t)$ we may assume that each of the three polynomials P, Q_1 and Q_2 has a monomial (in the indeterminates $z, \bar{z}, w, \bar{w}, u$ and v) whose coefficient in $\overline{\mathbb{Q}}(t)$ equals 1. We have that

(3.5)

$$P(t, z, w, U(t, z, w)), \quad Q_k(t, z, \bar{z}, w, \bar{w}, V_k(t, z, \bar{z}, w, \bar{w})) \in \overline{\mathbb{Q}}(t)[[z, \bar{z}, w, \bar{w}]]$$

and, by (3.4), the coefficients of these three power series vanish when we set $t = \underline{\gamma} = (\gamma_1, \dots, \gamma_r)$. The coefficients of these three power series are rational functions whose denominators are products of a finite number of polynomials in t : the polynomial $S(t)$ and the (finitely many) denominators of the coefficients of P, Q_1 and Q_2 . The numerators of the three series in (3.5) are polynomials in t_1, \dots, t_r that vanish at $\underline{\gamma}$. Since the γ_i are algebraically independent over $\overline{\mathbb{Q}}$, these numerators are identically zero. So these series in (3.5) are identically zero.

Let \mathcal{V} be the common zero locus in \mathbb{R}^r of $S(t)$ and of the denominators of the coefficients of P, Q_1 and Q_2 . The set \mathcal{V} is a proper algebraic subset of \mathbb{R}^r and $\underline{\gamma} \in \mathbb{R}^r \setminus \mathcal{V}$. And for $\underline{\delta} \in \mathbb{R}^r \setminus \mathcal{V}$, the series $U(\underline{\delta}, z, w)$, the $V(\underline{\delta}, z, \bar{z}, w, \bar{w})$ and the polynomials $P(\underline{\delta}, z, w, u)$ and $Q(\underline{\delta}, z, \bar{z}, w, \bar{w}, v)$ are well defined. Moreover, because the series in (3.5) are zero, $U(\underline{\delta}, z, w)$ and the $V_k(\underline{\delta}, z, \bar{z}, w, \bar{w})$ are algebraic power series since they are roots of $P(\underline{\delta}, z, w, u)$ and of the $Q_k(\underline{\delta}, z, \bar{z}, w, \bar{w}, v)$ (which are nonzero polynomials because P, Q_1 and Q_2 have a coefficient equal to 1). We can expand the following power series:

$$\begin{aligned} & U(t, z, w) - \overline{U}(t, \bar{z}, \bar{w}) - V_1(t, z, \bar{z}, w, \bar{w}) (w - |z|^2 - t_1(\bar{z}^2 + z^3\bar{z})) \\ & - V_2(t, z, \bar{z}, w, \bar{w}) (\bar{w} - |z|^2 - t_1(z^2 + \bar{z}^3z)) = \sum_{\beta \in \mathbb{N}^4} F_\beta(t_1, \dots, t_r) z^{\beta_1} \bar{z}^{\beta_2} w^{\beta_3} \bar{w}^{\beta_4} \end{aligned}$$

where the $F_\beta(t)$ are rational functions in $\overline{\mathbb{Q}}(t_1, \dots, t_r)$, whose denominators do not vanish on $\mathbb{R}^r \setminus \mathcal{V}$. Since $(\varphi_2, \ell_1, \ell_2)$ is a solution of (3.1), we have that

$$\forall \beta \in \mathbb{N}^4, \quad F_\beta(\underline{\gamma}) = 0.$$

Since $\gamma_1, \dots, \gamma_r$ are algebraically independent over $\overline{\mathbb{Q}}$, we have that, for every β , $F_\beta(t) = 0$.

Lemma 8. *There is a finite set $E \subset \overline{\mathbb{Q}}$ such that*

$$\forall \delta_1 \in \mathbb{R} \setminus E, \exists \delta_2, \dots, \delta_r \in \mathbb{R} \text{ such that } (\delta_1, \dots, \delta_r) \in \mathbb{R}^r \setminus \mathcal{V}.$$

Proof. Let $\delta_1 \in \mathbb{R}$. Assume that for all $\delta_2, \dots, \delta_r \in \mathbb{R}$, $(\delta_1, \dots, \delta_r) \in \mathcal{V}$. Let $A(t)$ be one polynomial among the following polynomials: $S(t)$ and the (finitely many) denominators of the coefficients of P and Q . By assumption $A(\delta_1, t_2, \dots, t_r)$ vanishes on \mathbb{R}^{r-1} . Thus $A(\delta_1, t_2, \dots, t_r) = 0$, so $A(t)$ is divisible by $(t - \delta_1)$. But because $\mathbb{C}[t_1, \dots, t_r]$ is a unique factorization domain, there is only finitely many $\delta_1 \in \mathbb{R}$ such that $t - \delta_1$ divides $A(t)$. We denote by E the set of these elements. Moreover, since $A(t) \in \overline{\mathbb{Q}}[t]$, $t - \delta_1$ divides $A(t)$ only when $\delta_1 \in \overline{\mathbb{Q}}$; indeed, $t - \delta_1$ divides $A(t)$ if and only if δ_1 is a root of every coefficient (in $\overline{\mathbb{Q}}[t_1]$) of $A(t)$ seen as a polynomial in t_2, \dots, t_r – but these coefficients are polynomials in $\overline{\mathbb{Q}}[t_1]$, so their roots are in $\overline{\mathbb{Q}}$. Now if $t - \delta_1$ does not divide $A(t)$, there exists $(\delta_2, \dots, \delta_r) \in \mathbb{R}^{r-1}$ such that $A(\underline{\delta}) \neq 0$, so $(\delta_1, \dots, \delta_r) \in \mathbb{R}^r \setminus \mathcal{V}$. \square

So finally, for $\delta_1 \in \mathbb{R} \setminus E$, there is $\delta_2, \dots, \delta_r \in \mathbb{R}$ such that

$$\varphi'_2(z, w) := U(\underline{\delta}, z, w) \text{ and } \ell'_k(z, \bar{z}, w, \bar{w}) := V_k(\underline{\delta}, z, \bar{z}, w, \bar{w}), \text{ for } k = 1, 2,$$

are well defined algebraic power series satisfying

(3.6)

$$\begin{aligned} \varphi'_2(z, w) - \overline{\varphi}'_2(\bar{z}, \bar{w}) &= \ell'_1(z, \bar{z}, w, \bar{w}) (w - |z|^2 - \delta_1(\bar{z}^2 + z^3\bar{z})) \\ &\quad + \ell'_2(z, \bar{z}, w, \bar{w}) (\bar{w} - |z|^2 - \delta_1(z^2 + \bar{z}^3z)) \end{aligned}$$

and the linear term $L(z, w)$ of $\varphi'_2(z, w)$ is nonzero. Let $\varphi'_1(z, w)$ be a homogeneous linear polynomial which is linearly independent with $L(z, w)$. Then the map

$$\Phi' : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0),$$

whose components are φ'_1 and φ'_2 , is a local algebraic biholomorphic germ by Remark 2. Since (3.6) is satisfied, we have that $\Phi'(M_{\delta_1}) \subset \{\text{Im}(w) = 0\}$. Moreover Φ' is a (convergent) biholomorphism since algebraic power series are convergent power series. But if we choose $\delta_1 > 1/2$ to be exceptional (that is possible since E is finite), such a Φ' cannot exist by [MW83, Theorem 6.1]. Therefore, by contradiction, M_γ is biholomorphically equivalent to $N_{\gamma, \varepsilon, s}$ for some $(\varepsilon, s) \in \{1, 0, -1\} \times (\mathbb{N}^* \cup \{\infty\})$, but these two germs are not equivalent by means of an algebraic biholomorphic map.

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