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Unique jet determination of CR maps into Nash sets

Bernhard Lamel [a,b,1], Nordine Mir [a,*,1], Guillaume Rond [c]

[a] Texas A&M University at Qatar, Division of Arts and Sciences, PO Box 23874, Education City, Doha, Qatar

[b] Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria

[c] Université d'Aix-Marseille, Institut de Mathématiques de Marseille (UMR CNRS 7373), Centre de Mathématiques et Informatique, 39 rue F. Joliot Curie, 13013 Marseille, France

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ABSTRACT

Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a Nash set and $\mathscr{E}_{M'}$ the set of points in M' of D'Angelo infinite type. We show that if M is minimal, then, for every point $p \in M$, and for every pair of germs of $\mathscr{C}^\infty$-smooth CR maps $f, g\colon (M,p) \to M'$, there exists an integer $k = k_p$ such that if f and g have the same k-jets at p, and do not send M into $\mathscr{E}_{M'}$, then necessarily $f = g$. Furthermore, the map $p \mapsto k_p$ may be chosen to be bounded on compact subsets of M. As a consequence, we derive the finite jet determination property for pairs of germs of CR maps from minimal real-analytic CR submanifolds in $\mathbb{C}^N$ into Nash subsets in $\mathbb{C}^{N'}$ of D'Angelo finite type, for arbitrary $N, N' \geq 2$.

* Corresponding author.
E-mail addresses: bernhard.lamel@qatar.tamu.edu (B. Lamel), nordine.mir@qatar.tamu.edu (N. Mir), guillaume.rond@univ-amu.fr (G. Rond).

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1. Introduction

A classical theorem by H. Cartan [6] states that if $\Omega \subset \mathbb{C}^N$ is a bounded domain, $H\colon \Omega \to \Omega$ is a biholomorphic map fixing a point $p \in \Omega$, and satisfying $H'(p) = I$, then H is the identity map. Boundary versions of this remarkable theorem have been studied intensively, and have lead to deep results for unique jet determination of (local) CR maps. Unique jet determination of CR diffeomorphisms by jets of higher order has been established under general assumptions involving CR geometric properties of the manifold; we refer the reader to the recent survey by the first two authors [18] for a thorough discussion. The first general results without dimension restrictions appeared only recently in [23,17].

One of the natural obstructions to finite jet determination for arbitrary CR maps arises when the target manifold contains complex varieties. Indeed, if we look at maps collapsing the source space into one of these varieties, they can never be recovered from a jet of finite order at any point.

It is therefore natural to consider maps which do not collapse the source space into the complex varieties in the target. Given a real-analytic set $M' \subset \mathbb{C}^{N'}$, we define its D'Angelo infinite type points $\mathscr{E}_{M'}$ to be the union of all positive dimensional complex varieties contained in M' (see [7,9]). If $\mathscr{E}_{M'} = \emptyset$, then M' is said to be of D'Angelo finite type (see e.g. [14] for a complete overview about various notions of types).

Definition 1.1. Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold and M' as above. We say that a CR map $f\colon M \to M'$ is *non-collapsing* if $f(M) \nsubseteq \mathscr{E}_{M'}$.

The question of whether collapse, as defined above, is the *only* way for unique determination of arbitrary maps to be violated has been open for many years. Here we prove that this is indeed the case for minimal sources (see e.g. [3,5] for this notion) and for real-analytic targets that are Nash, i.e. given by the zero-set of finitely many Nash functions on a Nash submanifold of $\mathbb{C}^{N'}$ (see [2]). Our main result may be stated as follows.

Theorem 1.2. *Let $M \subset \mathbb{C}^N$ be a minimal real-analytic CR submanifold and $M' \subset \mathbb{C}^{N'}$ a Nash set. Then, for every point $p \in M$, there exists an integer $k = k_p$, bounded on compact subsets of M, such that for every pair of germs of non-collapsing $\mathscr{C}^\infty$-smooth CR maps $f, g\colon (M, p) \to M'$, if $j_p^k f = j_p^k g$, then necessarily $f = g$.*

In particular, we do obtain a positive answer to [18, Problem 7.4] for Nash targets containing *no* nontrivial complex-analytic subvarieties.

Corollary 1.3. *Let $M \subset \mathbb{C}^N$ be a minimal real-analytic CR submanifold and $M' \subset \mathbb{C}^{N'}$ a Nash set of D'Angelo finite type. Then, for every point $p \in M$, there exists an integer $k = k_p$, bounded on compact subsets of M, such that for every pair of germs of $\mathscr{C}^\infty$-smooth CR maps $f, g\colon (M, p) \to M'$, if $j_p^k f = j_p^k g$, then necessarily $f = g$.*

As a consequence of known results due to Diederich–Fornæss [10], we get the following global statement in the compact case.

Corollary 1.4. *For every compact real-analytic hypersurface $M \subset \mathbb{C}^N$ and every compact Nash set $M' \subset \mathbb{C}^{N'}$, there exists an integer $\ell = \ell(M, M')$ such that if $f, g\colon (M,p) \to M'$ are two germs of $\mathscr{C}^\infty$-smooth CR maps at some point $p \in M$ with $j_p^\ell f = j_p^\ell g$, it follows that $f = g$.*

Recall from the beginning that biholomorphisms of bounded domains are uniquely determined by their 1-jet at any interior point. In contrast, as follows from the works of Low [19] and Forstnerič [12], if we consider proper holomorphic embeddings between balls in different dimensions, such a uniqueness result does not hold in general. However, assuming enough boundary regularity, Forstnerič [11] proved that any such map is rational with a degree bound only depending on the codimension, yielding as a consequence unique determination at any interior or boundary point. Our last corollary provides a general result for boundary uniqueness of proper maps in any codimension.

Corollary 1.5. *Let $\Omega \subset \mathbb{C}^N$ and $\Omega' \subset \mathbb{C}^{N'}$ be bounded domains with, respectively, smooth real-analytic boundary and smooth Nash boundary. Then there exists an integer ℓ, depending only on $\partial\Omega$ and $\partial\Omega'$, such that if $F, G\colon \Omega \to \Omega'$ are two proper holomorphic mappings extending smoothly up to the boundary near some point $p \in \partial\Omega$ with $j_p^\ell F = j_p^\ell G$, it follows that $F = G$.*

The first general jet determination result in arbitrary dimensions appeared in [17]. In that article, the first two authors dealt with targets which are Nash manifolds under the assumption that the maps under consideration do not possess what we called 2-approximate deformations. This includes target manifolds for which the order of contact with complex curves is at most 1 (such as e.g. strictly pseudoconvex targets). The techniques developed in the present paper allow us to handle non-collapsing CR maps into Nash targets, including, in particular, arbitrary maps into D'Angelo finite type targets, where the order of contact with complex curves can be arbitrary.

Our approach to prove jet determination in arbitrary dimensions is to construct universal families of systems of equations satisfied by the maps under consideration. A natural construction for these systems is given by the *(iterated) refined CR prolongations* introduced in sections 2 and 3, in which a crucial sequence of invariants $(\kappa_{p,f}^\ell)_\ell$ associated to any germ of a CR map f at $p \in M$ is studied. It turns out that vanishing of these invariants for large ℓ allows a bound (depending on ℓ) on the number of systems needed. If we are able to bound ℓ uniformly over the maps under consideration, finite jet determination can be proven, as section 5 shows. We are able to prove such a uniform bound for non-collapsing maps if the target is Nash. One of the properties needed for this step relies on a subtle algebraic property of Nash functions that we call *quasi-finiteness* in section 5. Such a property is a consequence of a new global strong approximation result à la Artin for Nash systems of equations, proven in section 6, that may be of independent interest.

2. The refined CR prolongation of a basic system

2.1. Allowable bases of CR vectors

For a generic real-analytic submanifold $M \subset \mathbb{C}^N$ and $p \in M$, we shall denote by $\mathbb{K}_p(M)$ the quotient field of the ring of germs at p of (complex-valued) real-analytic functions on M. We will need the following result from basic linear algebra whose proof can essentially be found in [15]. In order to formulate it, we say that a vector subspace $S \subset \mathbb{K}_p(M)^{N'}$ is *CR closed* if it is closed under the application of the CR vector fields, that is, $\bar{L}u \in S$ for every $u \in S$ and every real-analytic CR vector field $\bar{L}$ of M near p.

Lemma 2.1. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold, $p \in M$, $m, N' \in \mathbb{Z}_+$ and $A = (A^1, \dots, A^m)$ be m elements of $(\mathbb{K}_p(M))^{N'}$. Assume that the vector subspace generated by $A^1, \dots, A^m$ is of dimension $r < N'$ and that it is CR closed. Then for every choice of an invertible $r \times r$ submatrix $(A^{j_\ell}_{i_k})_{1 \le k, \ell \le r}$ of A, there exists a unique collection of $N' - r$ linearly independent CR vectors $V^1, \dots, V^{N'-r}$ in $(\mathbb{K}_p(M))^{N'}$ such that*

$$ {}^tV^j \cdot A^\gamma = \sum_{\nu=1}^{N'} V^j_\nu A^\gamma_\nu = 0, \ 1 \le j \le N' - r, \ 1 \le \gamma \le m, $$

and such that the $(N' - r) \times (N' - r)$ matrix formed by removing the $i_1, \dots, i_r$ rows from the $(N' \times (N' - r))$ matrix $(V^1, \dots, V^{N'-r})$ equals the identity. In addition, once the above mentioned minor is fixed, there exists a unique universal rational map R such that

$$ (V^1, \dots, V^{N'-r}) = R\left(A^1, \dots, A^m\right). \tag{2.1} $$

The previous lemma associates to every non-zero $r \times r$ minor of A a unique collection of CR vectors in $(\mathbb{K}_p(M))^{N'}$ forming a basis of the annihilator of A and a unique representation through a rational universal map R as in (2.1). Such a collection of CR vectors will be called an *allowable basis of CR vectors* (associated to A). Note that since the number of non-zero minors of A is finite, the collection of all such allowable basis of CR vectors is also finite (and its cardinal is uniformly bounded in terms of m, r, N', independently of M and p). Hence the collection of universal rational representations R as above is also finite as well.

2.2. General setting and some notation

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold. For $p \in M$, we denote by $\mathbb{K}_p^{\mathrm{CR}}(M)$ the subfield of $\mathbb{K}_p(M)$ consisting of the CR ratios. It is well-known (see e.g.

[3,21]) that any such ratio can be identified with the restriction to M of a germ of a meromorphic function at p.

From now until §5, we fix $p_0 \in M$ and shrink M near p_0 in such a way that it is equipped with a fixed basis of real-analytic CR vector fields, $\bar{L} = (\bar{L}_1, \dots, \bar{L}_n)$ where $n = \dim_{\mathrm{CR}} M$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\bar{L}^\alpha = \bar{L}_1^{\alpha_1} \dots, \bar{L}_n^{\alpha_n}$, and $\bar{L}f = (\bar{L}_1 f, \dots, \bar{L}_n f)$ for any map f with components in $\mathbb{K}_p(M)$ for some $p \in M$.

Let $p \in M$ and $\xi = (\xi^1, \dots, \xi^r)$ where each $\xi^j \in (\mathbb{K}_p(M))^{N'}$, with $r \in \mathbb{Z}_+$ arbitrary. We say that ξ is *admissible* if ξ has a *fixed* expression of the form

$$\xi = \mathscr{R}\left((\bar{L}^\alpha \bar{B})_{|\alpha| \le \ell}, \psi(g, \bar{g})\right)$$

for some integer ℓ, some rational holomorphic map $\mathscr{R}$, some vector-valued map B whose components belong to $\mathbb{K}_p^{\mathrm{CR}}(M)$, some germ of a real-analytic CR map $g \colon (M, p) \to \mathbb{C}^{N'}$ and some germ of a real-analytic map ψ near $g(p)$ in $\mathbb{C}^{N'}$. For an admissible ξ, we define

$$\xi_w := \frac{\partial}{\partial w}\left\{\mathscr{R}\left((\bar{L}^\alpha \bar{B})_{|\alpha| \le \ell}, \psi(w, \bar{g})\right)\right\}\Big|_{w=g}$$

where $\xi_w = (\xi_w^1, \dots, \xi_w^r)$ and each ξ_w^j is viewed as a $N' \times N'$ matrix. From the chain rule, it follows that ξ_w is admissible as well. Note also that if ξ is admissible, then, again by the chain rule, $\bar{L}\xi$ is also admissible and one has $(\bar{L}\xi)_w = \bar{L}(\xi_w)$.

Furthermore, for any polynomial map $P(t) = \sum_{\nu \in \mathbb{N}^r} \theta_\nu t^\nu$, $t \in \mathbb{C}^k$, whose coefficients $\theta_\nu \in (\mathbb{K}_p(M))^m$ are admissible ($k, m \in \mathbb{Z}_+$), we define $P_w(t) = \sum_{\nu \in \mathbb{N}^r} \theta_{\nu,w} t^\nu$, whose coefficients remain admissible by the above. To every admissible $\xi = (\xi^1, \dots, \xi^r)$, and to every integer $\ell \ge 1$, we define some associated $(\mathbb{K}_p(M))^{N'}$-valued homogenous holomorphic polynomials $D^\ell(t)$, $t = (t_1, \dots, t_r)$, as follows:

$$D^1(t) := t \cdot \xi, \quad \text{and for } k \ge 1, \qquad D^{k+1}(t) = \frac{1}{k+1}(t \cdot \xi) \cdot D_w^k(t),$$

where $t \cdot \xi = t_1 \xi^1 + \dots + t_r \xi^r \in (\mathbb{K}_p(M))^{N'}$, and where

$$(t \cdot \xi) \cdot D_w^k(t) = (t \cdot \xi)_1 D_{w_1}^k(t) + \dots + (t \cdot \xi)_{N'} D_{w_{N'}}^k(t) \in (\mathbb{K}_p(M))^{N'}.$$

Note that each $D^\ell(t)$ is a homogeneous polynomial map of degree ℓ in t.

For any real-analytic function $\theta(w, \bar{w})$ defined over some open subset of $\mathbb{C}^{N'}$, for multiindices $\alpha, \beta \in \mathbb{N}^{N'}$, we denote $\theta_{\alpha\bar{\beta}}$ for the partial derivative $\theta_{w^\alpha \bar{w}^\beta}$.

2.3. The construction

Our goal in this section is to describe a certain procedure that we call the refined CR prolongation of a basic system: given a family of CR maps on M all satisfying a specific type of system of equations, that we will call a *basic* system, we construct finitely many new basic systems, in a universal way, that the whole initial family of maps has to satisfy and with "better rank properties", as will be explained below.

Let M and p_0 be as §2.2 and Ω a given open subset in $\mathbb{C}^{N'}$. For every $p \in M$, we are given a family of germs of real-analytic CR maps $(M,p) \to \Omega$, that we denote by $\mathscr{F}_p$ and we set $\mathscr{F} := \cup_{p\in M}\mathscr{F}_p$.

We assume that there exist integers $\ell, k_1, k_2, m_1, m_2, N_1$, a $\mathbb{C}^{N_1}$-valued polynomial map $\mathscr{P} = \mathscr{P}(\Lambda, \bar\Lambda, T, \bar T)$, a rational map $\mathscr{Q}$ of its arguments, real-analytic maps $\theta\colon \Omega \to \mathbb{C}^{k_1}$, $\psi\colon \Omega \to \mathbb{C}^{k_2}$, and maps $A = (A_1, \dots, A_{m_1})$ and $B = (B_1, \dots, B_{m_2})$ with components in $\mathbb{K}_p^{\mathrm{CR}}(M)$, such that for every $p \in M$ and every germ $f \in \mathscr{F}_p$, the following system holds in $\mathbb{K}_p(M)$:

$$(\mathbb{X}):\begin{cases} \mathscr{P}\left(A, \bar A, \theta(f,\bar f), \overline{\theta(f,\bar f)}\right) &= 0,\\ \bar L A &= 0,\\ A &= \mathscr{Q}\left(\left(\bar L^\alpha \bar B\right)_{|\alpha|\le \ell}, \psi(f,\bar f)\right). \end{cases} \tag{2.2}$$

We now make the following:

Definition 2.2. Given M and a family $\mathscr{F}$ of maps as above, any system of equations of type $(\mathbb{X})$ satisfied by every map $f \in \mathscr{F}$ is called a *basic system* (associated to $\mathscr{F}$) if the first set of equations in (2.2) is stable under conjugation, i.e. any component of $\overline{\mathscr{P}(\Lambda, \bar\Lambda, \theta(w,\bar w), \overline{\theta}(w,\bar w))}$ appears as one component of the map $\mathscr{P}(\Lambda, \bar\Lambda, \theta(w,\bar w), \overline{\theta(w,\bar w)})$.

Let us therefore now assume that $\mathscr{F}$ satisfies a basic system as above. For every $f \in \mathscr{F}$, using the chain rule, we may rewrite (2.2) as follows:

$$\begin{cases} \mathscr{P}\left(A, \bar A, \theta(f,\bar f), \overline{\theta(f,\bar f)}\right) &= 0,\\ \widehat{\mathscr{Q}}\left((\bar L^\alpha \bar B)_{|\alpha|\le \ell+1}, \bar L \bar f, \psi(f,\bar f), \psi_{\bar w}(f,\bar f)\right) &= 0,\\ A &= \mathscr{Q}\left(\left(\bar L^\alpha \bar B\right)_{|\alpha|\le \ell}, \psi(f,\bar f)\right), \end{cases} \tag{2.3}$$

for some universal polynomial map $\widehat{\mathscr{Q}}$ valued in $\mathbb{C}^{nm_1}$ depending only on $\mathscr{Q}$. Consider the $N' \times N_1$ matrix given by

$$\mathscr{I}_f := \left(\frac{\partial}{\partial w}\left\{\left[\mathscr{P}\left(\mathscr{Q}\left(\left(\bar L^\alpha \bar B\right)_{|\alpha|\le\ell}, \psi(w,\bar f)\right), \bar A, \theta(w,\bar f), \overline{\theta(f,\bar w)}\right)\right]\right\}\Big|_{w=f}\right),$$

and the $N' \times m_1 n$ matrix given

$$\mathscr{J}_f = \left\{\frac{\partial}{\partial w}\left\{\left[\widehat{\mathscr{Q}}\left((\bar L^\alpha \bar B)_{|\alpha|\le\ell+1}, \bar L \bar f, \psi(w,\bar f), \psi_{\bar w}(w,\bar f)\right)\right]\right\}\Big|_{w=f}\right\},$$

whose coefficients both belong to $\mathbb{K}_p(M)$. Let $N' - \kappa_{p,f}(\mathbb{X})$ to be the rank of the $N' \times e(N_1 + m_1 n)$ matrix $(\bar L^\alpha \mathscr{I}_f | \bar L^\alpha \mathscr{J}_f)_{|\alpha|\le N'}$ where $e := \mathrm{card}\{\alpha \in \mathbb{N}^n : |\alpha| \le N'\}$.

Definition 2.3. Given a basic system as in (2.2) and any $f \in \mathscr{F}_p$ with $p \in M$, the integers $N' - \kappa_{p,f}(\mathbb{X})$ and $\kappa_{p,f}(\mathbb{X})$ are called the rank, respectively the degeneracy, of the system $(\mathbb{X})$ with respect to f.

For a fixed $p \in M$ and $f \in \mathscr{F}_p$, let us assume that $\kappa := \kappa_{p,f}(\mathbb{X}) > 0$ and write, in what follows, κ_f for $\kappa_{p,f}(\mathbb{X})$. Then by Lemma 2.1 applied to the column vectors of the matrix $\left(\bar{L}^\alpha(\mathscr{I}_f|\mathscr{J}_f)\right)_{|\alpha|\le N'}$, we may find an allowable basis of CR vectors associated to it, $\mathbf{V} := (V^1, \dots, V^{\kappa_f})$ where each $V^j \in (\mathbb{K}_p(M))^{N'}$ such that,

$$ {}^t\mathbf{V} \cdot (\mathscr{I}_f|\mathscr{J}_f) = 0. \tag{2.4} $$

Furthermore, by the same lemma, we have only a *finite number of choices* for such allowable bases of CR vectors, and we may write

$$ \mathbf{V} = R\left(\left(\bar{L}^\alpha(\mathscr{I}_f|\mathscr{J}_f)\right)_{|\alpha|\le N'}\right), \tag{2.5} $$

where R belongs to a finite family of universal rational maps (independent of f and p). The following notation will be useful.

Definition 2.4. The collection of all allowable bases of CR vectors associated to a given map $f \in \mathscr{F}$ is denoted by $\mathscr{V}_f(\mathbb{X})$, and the collection of all such bases associated to all maps $f \in \mathscr{F}$ is denoted by $\mathscr{V}(\mathbb{X})$.

We now analyse (2.4). Firstly, note that the system ${}^t\mathbf{V} \cdot \mathscr{I}_f = 0$ may be rewritten in the form

$$ {}^t\mathbf{V} \cdot A_w \cdot \mathscr{P}_\Lambda + {}^t\mathbf{V} \cdot \theta_w(f,\bar{f}) \cdot \mathscr{P}_T + {}^t\mathbf{V} \cdot \overline{\theta_{\bar{w}}(f,\bar{f})} \cdot \mathscr{P}_{\bar{T}} = 0, \tag{2.6} $$

where for ease of notation we dropped the arguments $(A, \bar{A}, \theta(f,\bar{f}), \overline{\theta(f,\bar{f})})$ from the derivatives of $\mathscr{P}$. Conjugating (2.6), we get:

$$ \overline{{}^t\mathbf{V} \cdot A_w} \cdot \overline{\mathscr{P}_\Lambda} + {}^t\overline{\mathbf{V}} \cdot \overline{\theta_w(f,\bar{f})} \cdot \overline{\mathscr{P}_T} + {}^t\overline{\mathbf{V}} \cdot \theta_{\bar{w}}(f,\bar{f}) \cdot \overline{\mathscr{P}_{\bar{T}}} = 0. \tag{2.7} $$

Hence (2.6) and (2.7) may be rewritten as

$$ \widetilde{\mathscr{P}}\left(A, \mathbf{V}, {}^t\mathbf{V} \cdot A_w, \bar{A}, \overline{\mathbf{V}}, \overline{{}^t\mathbf{V} \cdot A_w}, \left(\theta_{\gamma\bar{\beta}}(f,\bar{f}), \overline{\theta_{\gamma\bar{\beta}}(f,\bar{f})}\right)_{|\gamma|+|\beta|\le 1}\right) = 0, \tag{2.8} $$

for some universal polynomial $\widetilde{\mathscr{P}}$ depending only on $\mathscr{P}$.

Next, we claim that ${}^t\mathbf{V} \cdot A_w$ is CR. Indeed, from our construction, §2.2 and the fact that each component of $\mathbf{V}$ is CR, the relations ${}^t\mathbf{V} \cdot \mathscr{J}_f = 0$ (given by (2.4)) are equivalent to

$$0 = {}^t\mathbf{V} \cdot (\bar{L}A)_w = {}^t\mathbf{V} \cdot \bar{L}A_w = \bar{L}({}^t\mathbf{V} \cdot A_w). \tag{2.9}$$

In addition, it follows from (2.5) that each $\mathbf{V}$ may be written in the form

$$\mathbf{V} = \mathscr{R}\Bigg((\bar{L}^\alpha \bar{B})_{|\alpha| \le \ell + N' + 1}, (\bar{L}^\alpha \bar{f})_{|\alpha| \le N'+1}, (\bar{L}^\alpha \bar{A})_{|\alpha| \le N'}, \\ \left(\theta_{\nu\bar{\gamma}}(f, \bar{f}), \overline{\theta_{\nu\bar{\gamma}}(f, \bar{f})}\right)_{|\nu|+|\gamma| \le N'+1}, (\psi_{\nu\bar{\gamma}}(f, \bar{f}))_{|\nu|+|\gamma| \le N'+2} \Bigg)$$

where $\mathscr{R}$ belongs to a finite family of universal rational maps (independent of f and p) and depending only on $\mathscr{P}$ and $\mathscr{Q}$. Similarly, it is easy to check that

$${}^t\mathbf{V} \cdot A_w = \widetilde{\mathscr{R}}\Bigg((\bar{L}^\alpha \bar{B})_{|\alpha| \le \ell + N' + 1}, (\bar{L}^\alpha \bar{f})_{|\alpha| \le N'+1}, (\bar{L}^\alpha \bar{A})_{|\alpha| \le N'}, \\ \left(\theta_{\nu\bar{\gamma}}(f, \bar{f}), \overline{\theta_{\nu\bar{\gamma}}(f, \bar{f})}\right)_{|\nu|+|\gamma| \le N'+1}, (\psi_{\nu\bar{\gamma}}(f, \bar{f}))_{|\nu|+|\gamma| \le N'+2} \Bigg) \tag{2.10}$$

where $\widetilde{\mathscr{R}}$ also belongs to a finite family of universal rational maps (independent of f and p) and depending only on $\mathscr{P}$ and $\mathscr{Q}$. Hence, for every $f \in \mathscr{F}$ and for every choice of an allowable basis of CR vectors $\mathbf{V}$ for the basic system $(\mathbb{X})$, if we set:

$$\begin{cases} \mathscr{P}^\# & := (\mathscr{P}, \widetilde{\mathscr{P}}) \\ A^\# & := (A, \mathbf{V}, {}^t\mathbf{V} \cdot A_w) \\ \theta^\#(w, \bar{w}) & := \left(\theta_{\gamma\bar{\beta}}(w, \bar{w}), \overline{\theta_{\gamma\bar{\beta}}(w, \bar{w})}\right)_{|\gamma|+|\beta| \le 1} \\ \psi^\#(w, \bar{w}) & := \left(\left(\theta_{\nu\bar{\gamma}}(w, \bar{w}), \overline{\theta_{\nu\bar{\gamma}}(w, \bar{w})}\right)_{|\nu|+|\gamma| \le N'+1}, (\psi_{w^\nu \bar{w}^\gamma}(w, \bar{w}))_{|\nu|+|\gamma| \le N'+2} \right) \\ B^\# & := (B, f, A) \end{cases} \tag{2.11}$$

we obtain that f satisfies the following new basic system, that we call the *refined CR prolongation of the system associated to* $\mathbb{X}$ *and* $\mathbf{V}$, denoted by $\mathbb{X}^\#(\mathbf{V})$:

$$(\mathbb{X}^\#(\mathbf{V})) : \begin{cases} \mathscr{P}^\# \left(A^\#, \overline{A^\#}, \theta^\#(f, \bar{f}) \right) & = 0, \\ \bar{L}A^\# & = 0, \\ A^\# & = \mathscr{Q}^\# \left(\left(\bar{L}^\alpha \overline{B^\#} \right)_{|\alpha| \le \ell + N' + 1}, \psi^\#(f, \bar{f}) \right). \end{cases} \tag{2.12}$$

Since $B^\#$ is CR, the system $(\mathbb{X}^\#(\mathbf{V}))$ is indeed a basic system, as previously defined, contains the previous system $(\mathbb{X})$, and satisfies the conjugation property on the polynomial $\mathscr{P}^\#$. In the new basic system $(\mathbb{X}^\#(\mathbf{V}))$, the polynomial map $\mathscr{P}^\#$ depends universally on $\mathscr{P}$, the rational map $\mathscr{Q}^\#$ belongs to a finite family (independent of any $f \in \mathscr{F}$) of universal rational maps depending only on $\mathscr{Q}$. We stress here that the rational map $\mathscr{Q}^\#$

indeed depends on a given $f \in \mathscr{F}$. But every $f \in \mathscr{F}$ will satisfy a refined CR prolongation for *some* $\mathscr{Q}^{\#}$ from a universally determined finite family. In particular, the cardinal of all such rational maps is uniformly bounded. To summarize, what we have done is, given a basic system satisfied by a family of maps $f \in \mathscr{F}$, to construct finitely many new basic systems (whose cardinal is uniformly bounded), containing the original one, that depend universally on the original one, and such for each $p \in M$, every $f \in \mathscr{F}_p$ (with $\kappa_{p,f}(\mathbb{X}) > 0$) satisfies one of this new basic systems. Furthermore, for each such f and each associated allowable basis of CR vectors $\mathbf{V}$, by construction, we have

$$\kappa_{p,f}(\mathbb{X}^{\#}(\mathbf{V})) \leq \kappa_{p,f}(\mathbb{X}). \tag{2.13}$$

3. Iterated refined CR prolongations and their properties

In this section, we shall repeatedly apply the construction done in §2 to CR maps valued into a (fixed) real-analytic set $M' \subset \mathbb{C}^{N'}$. We are defining a so-called iterated refined CR prolongation procedure, describe some of its properties, and investigate the relationship between such a construction and the CR geometric properties of M and M'.

Throughout this section, M is a real-analytic generic submanifold of $\mathbb{C}^N$ with $p_0 \in M$ as in §2.2. We also consider a real-analytic subset M' of $\mathbb{C}^{N'}$ given by

$$M' = \{w \in \Omega : \rho(w, \bar{w}) = 0\} \tag{3.1}$$

for some $\mathbb{R}^d$-valued real-analytic function $\rho = (\rho_1, \dots, \rho_d)$ on some open subset $\Omega \subset \mathbb{C}^{N'}$. The family of maps $\mathscr{F}$ we are considering is the family of all germs of real-analytic CR maps $(M, p) \to M'$ with $p \in M$ arbitrary.

3.1. Iterated refined CR prolongations

In what follows, we define for $j \in \{1, \dots, 2n\}$,

$$X_j = \begin{cases} L_j, & \text{if } 1 \leq j \leq n, \\ \bar{L}_{j-n}, & \text{if } n+1 \leq j \leq 2n, \end{cases} \tag{3.2}$$

and set, for every integer $r \geq 0$ and for every smooth map $\varphi \colon M \to \mathbb{C}^{N'}$,

$$X^{(r)}\varphi := (X_{j_1} \dots X_{j_r}\varphi)_{1 \leq j_1, \dots, j_r \leq 2n}\,. \tag{3.3}$$

Using the construction introduced in §2, we now define the iterated refined CR prolongations associated to any map $f \in \mathscr{F}$.

For $p \in M$ and $f \in \mathscr{F}_p$, we have the basic identity (in $\mathbb{K}_p(M)$)

$$(\mathbb{X}): \quad \rho(f, \bar{f}) = 0, \tag{3.4}$$

which we may view, since ρ is real-valued, as a basic system $(\mathbb{X})$ associated to $\mathscr{F}$. Set

$$\kappa^1_{p,f} := \kappa_{p,f}(\mathbb{X}). \tag{3.5}$$

If $\kappa^1_{p,f} = 0$ (which means that f is a holomorphically nondegenerate map in the sense of [15,16]), we also set for every integer $s \geq 2$, $\kappa^s_{p,f} = 0$.

If now $\kappa^1_{p,f} > 0$, then for every $\mathbf{V}^{(1)} \in \mathscr{V}_f(\mathbb{X})$, f satisfies the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}))$, which can be described as follows:

$$\begin{cases} \mathscr{P}^{(1)}\left(\mathbf{V}^{(1)}, \overline{\mathbf{V}^{(1)}}, \left(\rho_{\nu\bar{\gamma}}(f,\bar{f})\right)_{|\nu|+|\gamma|\leq 1}\right) & = 0, \\ \bar{L}\mathbf{V}^{(1)} & = 0, \\ \mathbf{V}^{(1)} = \mathscr{Q}^{(1)}\left(\left(\bar{L}^\alpha \bar{f}\right)_{|\alpha|\leq N'+1}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\leq N'+1}\right), \end{cases} \tag{3.6}$$

where $\mathscr{P}^{(1)}$ is a universal polynomial map, and $\mathscr{Q}^{(1)}$ is a rational map belonging to a finite family of universal rational maps (the family being independent of f, but the particular choice of $\mathscr{Q}^{(1)}$ not). Observe that in order to derive the system (3.6), we have used (2.11)–(2.12) and the fact that ρ is real-valued. Note also that by our construction, the first set of equations of (3.6) is stable under conjugation (following Definition 2.2).

We now set

$$\kappa^2_{p,f} := \min\left\{\kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)})) : \mathbf{V}^{(1)} \in \mathscr{V}_f(\mathbb{X})\right\}. \tag{3.7}$$

If $\kappa^2_{p,f} = 0$, then we set $\kappa^s_{p,f} = 0$ for $s \geq 3$. If not, we then only consider those $\mathbf{V}^{(1)} \in \mathscr{V}_f(\mathbb{X})$ such that $\kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)})) = \kappa^2_{p,f}$ and write

$$\mathscr{V}^1_f(\mathbb{X}) = \left\{\mathbf{V}^{(1)} : \kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)})) = \kappa^2_{p,f}\right\}.$$

For each $\mathbf{V}^{(1)} \in \mathscr{V}^1_f(\mathbb{X})$, consider now the collection $\mathscr{V}_f(\mathbb{X}^\#(\mathbf{V}^{(1)}))$ of allowable bases of CR vectors $\mathbf{V}^{(2)}$ associated to $\mathbb{X}^\#(\mathbf{V}^{(1)})$ as explained in §2. These depend a priori on the chosen allowable basis $\mathbf{V}^{(1)}$. For each $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$ as above, we set:

$$A^{(1)} := \mathbf{V}^{(1)}, \quad A^{(2)} = (A^{(1)}, \mathbf{V}^{(2)}, {}^t\mathbf{V}^{(2)} \cdot A^{(1)}_w).$$

Then f satisfies the basic system $\left(\mathbb{X}^\#(\mathbf{V}^{(1)})\right)^\#(\mathbf{V}^{(2)}) =: (\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}))$ given by

$$\begin{cases} \mathscr{P}^{(2)}\left(A^{(2)}, \overline{A^{(2)}}, \left(\rho_{\nu\bar{\gamma}}(f,\bar{f})\right)_{|\nu|+|\gamma|\leq 2}\right) & = 0, \\ \bar{L}A^{(2)} & = 0, \\ A^{(2)} = \mathscr{Q}^{(2)}\left(\left(\bar{L}^\alpha \bar{f}, \bar{L}^\alpha \overline{A^{(1)}}\right)_{|\alpha|\leq 2N'+2}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\leq 2N'+3}\right), \end{cases} \tag{3.8}$$

where $\mathscr{P}^{(2)}$ is a universal polynomial map and, again, $\mathscr{Q}^{(2)}$ is a rational map belonging to a finite family of universal rational maps (independent of f as before). We call the basic system given by (3.8) the second order CR refined prolongation of $(\mathbb{X})$ with respect to $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)})$. Next, define

$$\kappa^3_{p,f} := \min \left\{ \kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)})) : \mathbf{V}^{(1)} \in \mathscr{V}^1_f(\mathbb{X}),\ \mathbf{V}^{(2)} \in \mathscr{V}_f(\mathbb{X}^\#(\mathbf{V}^{(1)})) \right\}. \tag{3.9}$$

If $\kappa^3_{p,f} = 0$, we set $\kappa^s_{p,f} = 0$ for $s \geq 4$. If not, we consider only those pairs $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)})$ for which $\kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)})) = \kappa^3_{p,f}$ and write

$$\mathscr{V}^2_f(\mathbb{X}) = \left\{ \left(\mathbf{V}^{(1)}, \mathbf{V}^{(2)} \right) : \kappa_{p,f} \left(\mathbb{X}^\# \left(\mathbf{V}^{(1)}, \mathbf{V}^{(2)} \right) \right) = \kappa^3_{p,f} \right\}.$$

For every $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}) \in \mathscr{V}^2_f(\mathbb{X})$, applying again the construction given in §2 to the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}))$ given in (3.8), we obtain a collection of allowable bases of CR vectors $\mathbf{V}^{(3)} \in \mathscr{V}_f(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}))$ such that f satisfies a basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}))^\#(\mathbf{V}^{(3)}) =: (\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \mathbf{V}^{(3)}))$, which we call a third order iterated refined CR prolongation of $(\mathbb{X})$ with respect to $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \mathbf{V}^{(3)})$. Setting

$$A^{(3)} = (A^{(2)}, \mathbf{V}^{(3)}, {}^t\mathbf{V}^{(3)} \cdot A^{(2)}_w),$$

such a basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \mathbf{V}^{(3)}))$ may be written in the form

$$\begin{cases} \mathscr{P}^{(3)} \left(A^{(3)}, \overline{A^{(3)}}, \left(\rho_{\nu\bar\gamma}(f, \bar f) \right)_{|\nu|+|\gamma| \leq 3} \right) & = 0, \\ \bar L A^{(3)} & = 0, \\ A^{(3)} = \mathscr{Q}^{(3)} \left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(2)}} \right)_{|\alpha| \leq 3N'+3}, (\rho_{\nu\bar\gamma}(f, \bar f))_{|\nu|+|\gamma| \leq 3N'+5} \right), \end{cases} \tag{3.10}$$

where $\mathscr{P}^{(3)}$ is a universal polynomial map and, again, $\mathscr{Q}^{(3)}$ is a rational map belonging to a finite family of universal rational maps (the family being independent of f but the particular choice of $\mathscr{Q}^{(3)}$ not).

Now, if for some integer $\ell \geq 2$, we assume that $\kappa^1_{p,f}, \dots, \kappa^\ell_{p,f}$ have been defined as above and are non-zero with associated sets $\mathscr{V}^1_f(\mathbb{X}), \dots, \mathscr{V}^{\ell-1}_f(\mathbb{X})$, and that for every $(\ell-1)$-tuple $(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(\ell-1)}) \in \mathscr{V}^{\ell-1}_f(\mathbb{X})$, the associated $(\ell-1)$-th order refined CR prolongation $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(\ell-1)}))$ has been defined, with associated data $A^{(\ell-1)}$, $\mathscr{P}^{(\ell-1)}$ and $\mathscr{Q}^{(\ell-1)}$ given as follows:

$$\begin{cases} \mathscr{P}^{(\ell-1)} \left(A^{(\ell-1)}, \overline{A^{(\ell-1)}}, \left(\rho_{\nu\bar\gamma}(f, \bar f) \right)_{|\nu|+|\gamma| \leq \ell-1} \right) & = 0, \\ \bar L A^{(\ell-1)} & = 0, \\ A^{(\ell-1)} = \mathscr{Q}^{(\ell-1)} \left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(\ell-2)}} \right)_{|\alpha| \leq (\ell-1)(N'+1)}, (\rho_{\nu\bar\gamma}(f, \bar f))_{|\nu|+|\gamma| \leq (\ell-1)(N'+2)-1} \right). \end{cases} \tag{3.11}$$

Since $\kappa^\ell_{p,f} > 0$, for every $(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(\ell-1)}) \in \mathscr{V}^{\ell-1}_f(\mathbb{X})$, we apply the construction given in §2 to each basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(\ell-1)}))$ to obtain a collection of allowable bases of CR vectors $\mathbf{V}^{(\ell)} \in \mathscr{V}_f(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))$ such that f satisfies the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))^\#(\mathbf{V}^{(\ell)}) =: (\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \mathbf{V}^{(\ell)}))$, which is the ℓ-th order refined CR prolongation of $\mathbb{X}$ with respect to $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots, \mathbf{V}^{(\ell-1)}, \mathbf{V}^{(\ell)})$. As a consequence of the construction done in §2, such a system is of the form

$$\begin{cases} \mathscr{P}^{(\ell)}\left(A^{(\ell)}, \overline{A^{(\ell)}}, \left(\rho_{\nu\bar{\gamma}}(f,\bar{f})\right)_{|\nu|+|\gamma|\le\ell}\right) & = 0, \\ \bar{L}A^{(\ell)} & = 0, \\ A^{(\ell)} = \mathscr{Q}^{(\ell)}\left(\left(\bar{L}^\alpha \bar{f}, \bar{L}^\alpha \overline{A^{(\ell-1)}}\right)_{|\alpha|\le\ell(N'+1)}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\le\ell(N'+2)-1}\right), \end{cases} \tag{3.12}$$

where

$$A^{(\ell)} = (A^{(\ell-1)}, \mathbf{V}^{(\ell)}, {}^t\mathbf{V}^{(\ell)} \cdot A_w^{(\ell-1)}),$$

and $\mathscr{P}^{(\ell)}$ is a universal polynomial map and (yet again) $\mathscr{Q}^{(\ell)}$ is a rational map belonging to a finite family of universal rational maps (the family being independent of f but the particular choice of $\mathscr{Q}^{(\ell)}$ not). We also know from our construction that

$$\begin{cases} A^{(\ell-1)} & = \mathscr{Q}^{(\ell-1)}\left(\left(\bar{L}^\alpha \bar{f}, \bar{L}^\alpha \overline{A^{(\ell-2)}}\right)_{|\alpha|\le(\ell-1)(N'+1)}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\le(\ell-1)(N'+2)-1}\right), \\ & \vdots \\ A^{(1)} & = \mathscr{Q}^{(1)}\left(\left(\bar{L}^\alpha \bar{f}\right)_{|\alpha|\le N'+1}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\le N'+1}\right). \end{cases} \tag{3.13}$$

Then we set:

$$\kappa_{p,f}^{\ell+1} := \min\Big\{\kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)})) : (\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}) \in \mathscr{V}_f^{\ell-1}(\mathbb{X}), \\ \mathbf{V}^{(\ell)} \in \mathscr{V}_f(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))\Big\}.$$

If $\kappa_{p,f}^{\ell+1} = 0$, we set $\kappa_{p,f}^s = 0$ for $s \ge \ell+2$. Otherwise, we define the set $\mathscr{V}_f^\ell(\mathbb{X})$ to be the collection of all ℓ-tuples allowable bases of CR vectors $(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)})$ such that $\kappa_{p,f}^{\ell+1} = \kappa_{p,f}(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}))$.

Hence, for M, M' and $\mathscr{F}$ as above, we have attached to every $p \in M$ and every $f \in \mathscr{F}_p$, a sequence of integers $(\kappa_{p,f}^\ell)_{\ell\in\mathbb{Z}_+}$. Such a sequence depends on our choice of M and associated CR vector fields as well as the chosen (global) defining function ρ for M'. We always assume that all of these have been fixed once and for all. However, note that after fixing these choices, the dependence of the sequence on the base point p is, by unique continuation, pretty straightforward. Indeed, if U is an open subset of M and $g\colon U \to M'$ is a real-analytic CR map, for every integer ℓ, $U \ni p \mapsto \kappa_{p,g}^\ell$ is constant on any connected component of U.

We summarize (part of) the above construction in the following precise statement:

Proposition 3.1. *Let M, M' and $\mathscr{F}$ be as above and ℓ a positive integer. Then there exists a universal polynomial map $\mathscr{P}^{(\ell)}$, and 2ℓ collections of sets $\mathbb{S}_1, \dots, \mathbb{S}_\ell$ and $\widetilde{\mathbb{S}}_1, \dots, \widetilde{\mathbb{S}}_\ell$, each consisting of finitely many rational maps (of their arguments) such that the following holds. For every $p \in M$ and every $f \in \mathscr{F}_p$ with $\kappa^\ell_{p,f} > 0$, there exist mappings $A^{(1)}, \dots, A^{(\ell)}$ with components in $\mathbb{K}^{\mathrm{CR}}_p(M)$, such that f satisfies the system*

$$\begin{cases} \mathscr{P}^{(\ell)}\left(A^{(\ell)}, \overline{A^{(\ell)}}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le\ell}\right) = 0, \\ \qquad\qquad\qquad\qquad \bar L A^{(j)} = 0, \ j = 1,\dots,\ell \\ A^{(\ell)} = \mathscr{Q}^{(\ell)}\left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(\ell-1)}}\right)_{|\alpha|\le \ell(N'+1)}, (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le \ell(N'+2)-1}\right) \\ A^{(\ell-1)} = \mathscr{Q}^{(\ell-1)}\left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(\ell-2)}}\right)_{|\alpha|\le (\ell-1)(N'+1)}, (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le (\ell-1)(N'+2)-1}\right), \\ \quad\vdots \\ A^{(1)} = \mathscr{Q}^{(1)}\left(\left(\bar L^\alpha \bar f\right)_{|\alpha|\le N'+1}, (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le N'+1}\right), \end{cases} \tag{3.14}$$

for some $\mathscr{Q}^{(j)} \in \mathbb{S}_j$, $j = 1 \dots, \ell$. Furthermore, the following also holds:

(i) *for every $j = 1, \dots, \ell$, if we denote $r_j := j(j+1)(N'+1)/2$ and use the notation from (3.3), there exists a rational map $\mathscr{T}^{(j)} \in \widetilde{\mathbb{S}}_j$ such that*

$$A^{(j)} = \mathscr{T}^{(j)}\left(X^{r_j}(f,\bar f), (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le r_j}\right); \tag{3.15}$$

(ii) *the rank of the (first two lines of the) system (3.14), as defined in §2.3, equals $N' - \kappa^\ell_{p,f}$;*

(iii) *the first line of the system (3.14) is stable under conjugation and is called the SUC part of the basic system $\mathbb{X}^\#(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(\ell-1)}, \mathbf{V}^{(\ell)})$.*

Proof. What remains to be proven in the proposition is the identity (3.15). Let ℓ, p and f be as in the proposition. We shall now prove (3.15) by induction on j, and that $\mathscr{T}^{(j)}$ depends only on $\mathscr{Q}^{(1)}, \dots, \mathscr{Q}^{(j)}$. The statement for $j = 1$ follows immediately from the last equation in (3.14). Hence, let us assume that (3.15) holds for $j = k$ with $1 \le k \le \ell-1$ and let us prove it for $j = k+1$. Since

$$A^{(k)} = \mathscr{T}^{(k)}\left(X^{r_k}(f,\bar f), (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le r_k}\right),$$

by the chain rule, we have that there exists a rational map $\widetilde{\mathscr{T}}^{(k)}$ depending only on $\mathscr{T}^{(k)}$ such that

$$\begin{aligned} &(\bar L^\alpha \overline{A^{(k)}})_{|\alpha|\le (k+1)(N'+1)} \\ &\quad = \widetilde{\mathscr{T}}^{(k)}\left(X^{r_k+(k+1)(N'+1)}(f,\bar f), (\rho_{\nu\bar\gamma}(f,\bar f))_{|\nu|+|\gamma|\le r_k+(k+1)(N'+1)}\right). \end{aligned} \tag{3.16}$$

Using the formula given by (3.14) for $A^{(k+1)}$ and substituting (3.16) into it yields that

$$A^{(k+1)} = \mathscr{T}^{(k+1)}\left(X^{r_k+(k+1)(N'+1)}(f,\bar{f}), (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\leq r_k+(k+1)(N'+1)}\right),$$

for some rational map $\mathscr{T}^{(k+1)}$ depending only on $\mathscr{Q}^{(k+1)}$ and on $\mathscr{T}^{(k)}$. Since $r_{k+1} = r_k+(k+1)(N'+1)$, we reach the desired result by induction. The proof of the proposition is complete now. □

3.2. *Properties of the sequence* $(\kappa^\ell_{p,f})_{\ell\in\mathbb{Z}_+}$

In order to make in §4 relationships between the above formal construction and the CR geometry of the pair (M, M'), we need to study the sequence $(\kappa^\ell_{p,f})_{\ell\in\mathbb{Z}_+}$ and establish a few of its basic properties.

The first obvious but useful property of the sequence $(\kappa^\ell_{p,f})_{\ell\in\mathbb{Z}_+}$ is the following:

Proposition 3.2. *Let M, M' and $\mathscr{F}$ be as above. For every $p \in M$ and $f \in \mathscr{F}_p$, the sequence $(\kappa^\ell_{p,f})_{\ell\in\mathbb{Z}_+}$ is decreasing.*

Proof. The fact that $\kappa^\ell_{p,f} \geq \kappa^{\ell+1}_{p,f}$ follows from the construction and (2.13). □

Remark 3.3. In view of Proposition 3.2, for every $f \in \mathscr{F}_p$, we may define

$$\kappa^\infty_{p,f} := \lim_{\ell\to+\infty} \kappa^\ell_{p,f}.$$

One of the key properties about the sequence $(\kappa^\ell_{p,f})$ is to understand the consequences on the map f when the sequence stagnates for a certain time. In order to do so, we need to introduce some notation.

Let $\ell \geq 1$, $p \in M$ and $f \in \mathscr{F}_p$ such that $\kappa := \kappa^\ell_{p,f} > 0$. Let $(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)})$ any element in $\mathscr{V}^\ell_f(\mathbb{X})$ and let $t = (t_1, \ldots, t_\kappa) \in \mathbb{C}^\kappa$. Since $\mathbf{V}^{(\ell)}$ is admissible, for every integer $k \geq 1$, we may define, as in §2.2, by induction on k, the following $\mathbb{C}^{N'}$-valued homogeneous (holomorphic) polynomials (in t) of degree k, denoted by $D^k(t)$, with coefficients in $(\mathbb{K}_p(M))^{N'}$:

$$D^1(t) := t\cdot\mathbf{V}^{(\ell)}, \quad \text{and for } k\geq 1,\ D^{k+1}(t) = \frac{1}{k+1}(t\cdot\mathbf{V}^{(\ell)})\cdot D^k_w(t). \tag{3.17}$$

The following result is a crucial property of our construction. It is the bridge linking the iterated refined CR prolongations and CR geometric properties of the pair (M, M').

Proposition 3.4. *Let M, M' and $\mathscr{F}$ be as above, $p \in M$, and $f \in \mathscr{F}_p$. Let $\ell, s \in \mathbb{Z}_+$, with $\ell, s \geq 1$ and assume that $\kappa^\ell_{p,f} = \kappa^{\ell+s}_{p,f} =: \kappa > 0$. Then for every $(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}) \in \mathscr{V}^\ell_f(\mathbb{X})$, if we set $D(t) = D^1(t) + \ldots + D^s(t)$, where each $D^k(t)$ is given by (3.17), the following holds:*

(a) $D(t) \in (\mathbb{K}_p^{CR}(M)[t])^{N'}$;
(b) *in the ring* $\mathbb{K}_p(M)[[t,\bar{t}]]$, *we have the identity*

$$\rho\left(f + D(t), \overline{f + D(t)}\right) = O(|t|^{s+1}). \tag{3.18}$$

Proof. Let $\ell, s \geq 1$ and pick $(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}) \in \mathscr{V}_f^\ell(\mathbb{X})$ and assume that $\kappa := \kappa_{p,f}^\ell = \kappa_{p,f}^{\ell+s} > 0$. Since $\kappa_{p,f}^\ell = \kappa_{p,f}^{\ell+1}$, the allowable basis of CR vectors $\mathbf{V}^{(\ell)}$ for the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))$ happens to be also an allowable basis of CR vectors for the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \mathbf{V}^{(\ell)}))$, since the system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}))$ is the refined CR prolongation associated to the system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))$ and $\mathbf{V}^{(\ell)}$ and therefore contains the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))$ by construction. Using the fact that $\kappa_{p,f}^\ell = \kappa_{p,f}^{\ell+s}$ and proceeding inductively, we get that all the basic systems $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{s \text{ times}}))$ are satisfied and have the same rank $N - \kappa$. We now prove by induction on $k \in \{1, \ldots, s\}$ the following:

Claim – For $1 \leq k \leq s$, the system of equations

$$\begin{cases} \bar{L}(D^1(t)) = \ldots = \bar{L}(D^k(t)) = 0 \\ \rho\left(f + D^1(t) + \ldots + D^k(t), \overline{f + D^1(t) + \ldots D^k(t)}\right) = O(|t|^{k+1}), \end{cases} \tag{3.19}$$

is contained in the set of equations of the basic system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k \text{ times}}))$, the second line of (3.19) is contained in the SUC part of the latter system, and therefore (3.19) is satisfied.

- $k = 1$:

The equations $\bar{L}(D^1(t)) = 0$ are identical to $\bar{L}(\mathbf{V}^{(\ell)}) = 0$, which, by definition are contained in the system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}))$. The identity

$$\rho\left(f + D^1(t), \overline{f + D^1(t)}\right) = O(|t|^2)$$

is equivalent to the system

$$\begin{cases} \rho(f, \bar{f}) = 0, \\ \mathbf{V}^{(\ell)} \cdot \rho_w(f, \bar{f}) = 0 \\ \overline{\mathbf{V}^{(\ell)}} \cdot \rho_{\bar{w}}(f, \bar{f}) = 0. \end{cases} \tag{3.20}$$

Every equation appearing in (3.20) is clearly contained in the SUC part of the system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell)}))$. Indeed this latter is the refined CR prolongation of the system $(\mathbb{X}^\#(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}))$ (as defined in §2.3) whose SUC part always contains the equation $\rho(f, \bar{f}) = 0$.

• Assume now that the claim holds for $k \in \{1, \ldots, s-1\}$ and let us prove it for $k+1$. By the above, we know that the systems $(\mathbb{X}^{\#}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k \text{ times}}))$ and $(\mathbb{X}^{\#}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$ are satisfied and we know that the equations

$$\bar{L}(D^k(t)) = 0, \quad \rho\left(f + D^1(t) + D^k(t), \overline{f + D^1(t) + \ldots D^k(t)}\right) = O(|t|^{k+1}) \tag{3.21}$$

are part of the first mentioned system. From our original construction of the refined CR prolongation in §2.3 and from §2.2, we therefore obtain that the equations

$$\mathbf{V}^{(\ell)} \cdot (\bar{L}D^k(t))_w = \mathbf{V}^{(\ell)} \cdot \bar{L}(D^k_w(t)) = 0 \tag{3.22}$$

are also satisfied, as parts of the equations of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$. Since $\mathbf{V}^{(\ell)}$ is CR, the identity (3.22) is the same as the identity

$$\bar{L}(\mathbf{V}^{(\ell)} \cdot D^k_w(t)) = 0, \tag{3.23}$$

which is clearly identical to $\bar{L}(D^{k+1}(t)) = 0$ and therefore all equations $\bar{L}(D^1(t)) = \ldots = \bar{L}(D^{k+1}(t)) = 0$ are part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$. What remains to be shown is that the equation

$$\rho\left(f + D^1(t) + \ldots + D^{k+1}(t), \overline{f + D^1(t) + \ldots D^{k+1}(t)}\right) = O(|t|^{k+2}), \tag{3.24}$$

is satisfied and is contained in the SUC part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$, which we will do by using some arguments from [16]. We write

$$\rho\left(f + D^1(t) + \ldots + D^{k+1}(t), \overline{f + D^1(t) + \ldots D^{k+1}(t)}\right) = \sum_{i,j \in \mathbb{Z}_+} \frac{R^{i,j}(t, \bar{t})}{i!j!},$$

where each $R^{i,j}$ is a polynomial map in $(t, \bar{t})$, homogeneous of degree i in t, of degree j in $\bar{t}$, and with coefficients in $\mathbb{K}_p(M)$. By the induction assumption, we know that $R^{i,j}(t, \bar{t}) = 0$ for $0 \le i + j \le k$ and that such equations are contained in the SUC part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k \text{ times}}))$. Hence, by the definition of the refined CR prolongation, the equations

$$D^1(t) \cdot R^{i,j}_w(t, \bar{t}) = 0, \quad 0 \le i + j \le k,$$

are satisfied and are contained in the SUC part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$. From the arguments of [16, Lemma 4.2], we have $R^{i+1,j}(t,\bar{t}) = D^1(t) \cdot R_w^{i,j}(t,\bar{t})$ and hence

$$R^{i+1,j}(t,\bar{t}) = 0. \tag{3.25}$$

Furthermore, since each component of ρ is real-valued, it follows that for all integers i,j as above, we have $R^{i,j}(t,\bar{t}) = \overline{R^{j,i}(t,\bar{t})}$. Hence, since the equations (3.25) for all i,j with $0 \leq i+j \leq k$ are contained in the SUC part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)}, \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$, we have $\overline{R^{i,j+1}(t,\bar{t})} = 0$ and such equations also belong to the SUC part of the system $(\mathbb{X}(\mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(\ell-1)} \underbrace{\mathbf{V}^{(\ell)}, \ldots, \mathbf{V}^{(\ell)}}_{k+1 \text{ times}}))$. This proves the desired statement for (3.24).

Proposition 3.4 now follows from the claim with $k = s$. $\square$

4. Behavior of the sequence $(\kappa^\ell_{p,f})_{\ell \in \mathbb{Z}_+}$ versus CR geometric properties of the triple (M, M', f)

In this section, we fix M, M' and $\mathscr{F}$ as in previous sections. Recall that this means that M is a fixed real-analytic generic submanifold of $\mathbb{C}^N$ shrunk near a point $p_0 \in M$ as in §2.2 (so that we are using a fixed basis of real-analytic CR vector fields on M). Furthermore M' is a real-analytic subset M' of $\mathbb{C}^{N'}$ given by (3.1) for some fixed $\mathbb{R}^d$-valued real-analytic function $\rho = (\rho_1, \ldots, \rho_d)$ on some open subset $\Omega \subset \mathbb{C}^{N'}$. Recall also that $\mathscr{F}$ denotes the family of all germs of real-analytic CR maps $(M,p) \to M'$ with $p \in M$ arbitrary.

In what follows, our goal is to show how the behaviour of the sequence $(\kappa^\ell_{p,f})_{\ell \in \mathbb{Z}_+}$ associated to any $f \in \mathscr{F}_p$ (defined in §3) is directly connected to CR geometric properties of the triple (M, M', f). Recall that $\mathscr{E}_{M'}$ is the set of D'Angelo infinite type points of M', i.e. the set of points in M' through which there passes a complex-analytic subvariety of positive dimension entirely contained in M'.

Before stating the first result along the above lines, let us introduce some terminology. In what follows, for a positive integer s, a point $q \in M'$ and a m-dimensional complex-submanifold $\Gamma \subset \mathbb{C}^{N'}$ passing through q, we say that Γ is tangent to M' up to order s at q if for one (and hence any) local holomorphic parametrization $\gamma\colon (\mathbb{C}^m, 0) \to (\Gamma, q)$, one has $\rho(\gamma(t), \overline{\gamma(t)}) = O(|t|^{s+1})$. Given an open subset $\widetilde{M}$ of M, a family $(\Gamma_\xi)_{\xi \in \widetilde{M}}$ of m-dimensional complex submanifolds of $\mathbb{C}^{N'}$ is CR if for every $\xi_0 \in \widetilde{M}$ there exists a germ of a real-analytic CR map $(\mathbb{C}^m \times \widetilde{M}, (0, \xi_0)) \ni (t, \xi) \mapsto \gamma_\xi(t) \in \mathbb{C}^{N'}$ such that $(\mathbb{C}^m, 0) \ni t \mapsto \gamma(\xi, t)$ parametrizes Γ_ξ near $\gamma(\xi, 0)$ for ξ near ξ_0.

We have the following:

Proposition 4.1. *Let M, M' and $\mathscr{F}$ be as above, $p \in M$, $f \in \mathscr{F}_p$ and s a positive integer. Assume that the sequence $(\kappa^\ell_{p,f})_{\ell \in \mathbb{Z}_+}$ stagnates at some level $\kappa > 0$, that is, $\kappa^{\ell+s}_{p,f} = \kappa$ for some ℓ. Then there exists a neighbourhood M_p of p in M, and an open dense subset of $\widetilde{M}_p$ of M_p, and a CR family $(\Gamma_\xi)_{\xi \in \widetilde{M}_p}$ of κ-dimensional complex submanifolds of $\mathbb{C}^{N'}$, such that each Γ_ξ is tangent to M' up to order s at $f(\xi)$.*

Proof. By assumption, there exists $\ell \geq 1$ such that $\kappa^\ell_{p,f} = \kappa^{\ell+s}_{p,f}$. Pick $\mathbf{V}^{(\ell)}$ as in Proposition 3.4, and let $D(t)$ be the corresponding associated polynomial. The same proposition states that

$$D(t) \in (\mathbb{K}^{\mathrm{CR}}_p(M)[t])^{N'}, \quad \rho\left(f + D(t), \overline{f + D(t)}\right) = O(|t|^{s+1}) \text{ in } \mathbb{K}_p(M)[\![t, \bar{t}]\!] \tag{4.1}$$

where $t \in \mathbb{C}^\kappa$. Now, since $\mathbf{V}^{(\ell)}$ is formed by κ CR vectors of $(\mathbb{K}^{\mathrm{CR}}_p(M))^{N'}$ of generic rank κ, analogously to [16, §4.2, p. 390], we may find a real-analytic CR map $\widetilde{D}^s \colon (M \times \mathbb{C}^\kappa, (p, 0)) \to (\mathbb{C}^{N'}, 0)$ such that the generic rank of $(\partial_t \widetilde{D}^s)|_{M \times \{0\}}$ equals κ and such that for z in some neighbourhood M_p of p in M and t sufficiently small

$$\rho(f(z) + \widetilde{D}^s(z,t), \overline{f(z) + \widetilde{D}^s(z,t)}) = O(|t|^{s+1}). \tag{4.2}$$

Let $\widetilde{M}_p$ be the open dense subset of M_p consisting of those points z for which the rank of $(\partial_t \widetilde{D}^s)(z, 0)$ is maximal and equals κ. Then for $z \in \widetilde{M}_p$, the map $t \mapsto f(z) + \widetilde{D}^s(z,t)$ parametrizes a κ-dimensional complex submanifold Γ_z of $\mathbb{C}^{N'}$ that is tangent to M' up to order s. Since f and $\widetilde{D}^s$ are CR, the proposition follows. □

As a consequence of Proposition 4.1 and of its proof, we have the following.

Proposition 4.2. *Let M, M' and $\mathscr{F}$ be as above, $p \in M$, $f \in \mathscr{F}_p$. If $\kappa^\infty_{p,f} > 0$, then $f(M) \subset \mathscr{E}_{M'}$.*

Proof. First, let us mention that the condition $f(M) \subset \mathscr{E}_{M'}$ means there is a sufficiently small neighbourhood ω of p in M such that $f(\omega) \subset \mathscr{E}_{M'}$.

Since by Proposition 3.2, the sequence $(\kappa^\ell_{p,f})_{\ell \in \mathbb{Z}_+}$ is decreasing, for some integer ℓ, large enough, we have $\kappa^\ell_{p,f} = \kappa^{\ell+s}_{p,f} = \kappa^\infty_{p,f} = \kappa > 0$ for every integer $s \in \mathbb{Z}_+$. Fix $s \in \mathbb{Z}_+$. It follows from Proposition 4.1 and, more precisely from its proof, that we may find a real-analytic CR map $\widetilde{D}^s \colon (M \times \mathbb{C}^\kappa, (p, 0)) \to (\mathbb{C}^{N'}, 0)$ such that the generic rank of $(\partial_t \widetilde{D}^s)|_{M \times \{0\}}$ equals κ and such (4.2) holds that for $z \in M$ sufficiently close to p and t sufficiently small (depending a priori on s).

In what follows, we identify f with its unique holomorphic extension to a neighbourhood of p in $\mathbb{C}^N$. Writing $E(z, \bar{z}, w, \bar{w}) := \rho(f(z) + w, \overline{f(z) + w})$, we see that E is a real-analytic map in a neighbourhood of $(p, 0)$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$. Using [16, Theorem 5.2], which is a parameter version of a result of [13], there exists a neighbourhood ω_p of p in $\mathbb{C}^N$ and a positive integer ℓ_1 such that for every $z \in \omega_p$ (fixed) and every power series map $S(t) \in (\mathbb{C}\{t\})^{N'}$ satisfying

$$S(0)=0, \quad E(z,\bar{z},S(t),\overline{S(t)})=O(|t|^{\ell_1+1})$$

there exists $\widehat{S}(t)\in(\mathbb{C}\{t\})^{N'}$ (depending on z) such that

$$S(t)=\widehat{S}(t)+O(|t|^2), \quad E(z,\bar{z},\widehat{S}(t),\overline{\widehat{S}(t)})=0.$$

By the above, for every $z\in M\cap\widetilde{\omega}_p$, for some neighbourhood $\widetilde{\omega}_p\subset\omega_p$, we have

$$\widetilde{D}^{\ell_1}(z,0)=0, \quad E(z,\bar{z},\widetilde{D}^{\ell_1}(z,t),\overline{\widetilde{D}^{\ell_1}(z,t)})=O(|t|^{\ell_1+1}),$$

and, therefore, for every $z\in M\cap\widetilde{\omega}_p$, there exists a germ at $0\in\mathbb{C}^\kappa$ of a holomorphic map $t\mapsto\widehat{D}_z(t)$ such that

$$\widehat{D}_z(t)=\widetilde{D}^{\ell_1}(z,t)+O(|t|^2), \quad \rho\left(f(z)+\widehat{D}_z(t),\overline{f(z)+\widehat{D}_z(t)}\right)=0. \tag{4.3}$$

Since the generic rank of $(\partial_t\widetilde{D}^{\ell_1})|_{M\times\{0\}}$ is equal to κ, it follows from both conditions in (4.3) that for a generic point $z\in M\cap\widetilde{\omega}_p$, the κ-dimensional complex submanifold $t\mapsto f(z)+\widehat{D}_z(t)$ is contained in M', i.e. that for z in some dense open subset of $M\cap\widetilde{\omega}_p$, $f(z)\in\mathscr{E}_{M'}$. To reach the final desired conclusion, one needs to invoke the closedness of $\mathscr{E}_{M'}$ in M' (see [8,9]). The proof of the proposition is complete. □

As an immediate consequence we obtain:

Corollary 4.3. *Let M,M' and $\mathscr{F}$ be as above, $p\in M$, $f\in\mathscr{F}_p$. If $\kappa^\infty_{p,f}>0$, then the maximum dimension of real-analytic submanifolds contained in $\mathscr{E}_{M'}$ is greater or equal to the generic rank of f.*

Recall that for $p\in M$, a map $f\in\mathscr{F}_p$ is called *non-collapsing* if $f(M)\not\subset\mathscr{E}_{M'}$. Rephrasing Proposition 4.2, we see that any non-collapsing map germ $f\in\mathscr{F}_p$ must satisfy $\kappa^\ell_{p,f}=0$ for ℓ large enough. Our goal now is to provide some sufficient conditions on the defining function ρ of M' that will guarantee that there exists a *fixed* integer ℓ_0 such that $\kappa^{\ell_0}_{p,f}=0$ for all non-collapsing germs $f\in\mathscr{F}_p$, for every $p\in M$.

To this end, we first recall the notion of regular 1-type (see e.g. [9]). We say that a real-analytic map $\rho\colon\Omega\to\mathbb{R}^d$ is of finite regular 1-type at $q\in M':=\{\rho=0\}$ if

$$T_\rho(q):=\sup\nolimits_\gamma\,\nu_0(\rho\circ\gamma)<\infty$$

where γ ranges over all germs of holomorphic maps $(\mathbb{C},0)\to\Omega$ with $\gamma(0)=q$ and $\gamma'(0)\neq0$, and where $\nu_0\,(\rho\circ\gamma)=\inf\{i+j:\partial^{i+j}_{t^i\bar{t}^j}(\rho\circ\gamma)(0)\neq0\}$. The number $T_\rho(q)$ is called the *regular* 1*-type of ρ at q*. We also introduce the set

$$\mathscr{E}^1_\rho:=\{q\in\Omega\colon T_\rho(q)=\infty\}$$

and note that it follows from [13] and [22] that through any point $q \in \mathscr{E}^1_\rho$ there passes a nonsingular holomorphic curve contained in M'.

Definition 4.4. Let $\rho\colon \Omega \to \mathbb{R}^d$ be a real-analytic map.

(i) We say that ρ is of finite type if $T_\rho := \sup_{q\in M'} T_\rho(q) < \infty$.
(ii) We say that ρ is of quasi-finite type if $I_\rho := \sup_{q\in M'\setminus \mathscr{E}^1_\rho} T_\rho(q) < \infty$.

Remark 4.5. Note that $\rho\colon \Omega \to \mathbb{R}^d$ is of quasi-finite if and only if there exists an integer k_0 such that if $\gamma\colon (\mathbb{C}, 0) \to \mathbb{C}$ is a holomorphic map with $\gamma(0) \in M'$ and $\gamma'(0) \neq 0$ satisfying $\nu_0(\rho \circ \gamma) > k_0$ then there exists a non-singular holomorphic curve $\Gamma \subset M'$ passing through $\gamma(0)$. Note also that the smallest of all such integers k_0 coincides with I_ρ.

It is obvious that if ρ is of finite type, then it is of quasi-finite type and, that in such a situation, $I_\rho \leq T_\rho$. The converse is easily seen to be false by taking for example M' to be a complex submanifold of $\mathbb{C}^{N'}$.

If ρ is submersive at every point of $M' = \{w \in \Omega : \rho(w, \bar{w}) = 0\}$, then M' is a real-analytic submanifold in Ω, and hence saying that ρ is of finite type exactly means that the order of contact of non-singular complex curves with M' is uniformly bounded (see [7,9]). It should be mentioned here that we do not assume that M' is a submanifold and therefore the notion we are considering here is different than the notion of 1-finite type for real-analytic sets, as we are fixing a global real-analytic defining function.

Note that if M' contains a complex-analytic subvariety of positive dimension, then ρ is obviously not of finite type. However, the converse need not hold as the following simple example shows.

Example 4.6. Choose, by the Weierstrass theorem, any entire function $h\colon \mathbb{C} \to \mathbb{C}$ such that for every positive integer n, h has a zero of order n at $(n, 0)$. Then the real-analytic hypersurface $\Sigma \subset \mathbb{C}^2_{z,w}$ given by $\{\rho = 0\}$ where $\rho := \operatorname{Im} w - |h(z)|^2$ is pseudoconvex, everywhere of (D'Angelo) finite type, but ρ is neither of finite type nor of quasi-finite type since for every integer n the order of contact of the non-singular complex curve $w = 0$ with Σ at $(n, 0)$ is exactly n.

Coming back to our original problem, the usefulness of the notion of quasi-finite type is revealed by the following observation.

Proposition 4.7. *Let M, $\mathscr{F}$ and ρ be as above. If ρ is of quasi-finite type, then there exists an integer e_0, depending only on M' and N', such that, for every $p \in M$ and every non-collapsing map $f \in \mathscr{F}_p$, we necessarily have $\kappa^{e_0}_{p,f} = 0$. In particular, if ρ is of finite type, then such a property holds true for every $f \in \mathscr{F}_p$ and every $p \in M$.*

Proof. Since ρ is of quasi-finite type, the associated integer I_ρ is finite. Let $p \in M$ and $f \in \mathscr{F}_p$ a non-collapsing map. We claim that $\kappa_{p,f}^{e_0} = 0$ for $e_0 := 1 + N' I_\rho$. Assume by contradiction that this is not the case. It implies that there exists $\ell \geq 1$ and $s > I_\rho$ such that $\kappa_{p,f}^{\ell} = \kappa_{p,f}^{\ell+s} = \kappa > 0$. By Proposition 4.1, there exist a neighbourhood M_p of p in M, and an open dense open subset of $\widetilde{M_p}$ of M_p and a family $(\Gamma_z)_{z\in\widetilde{M_p}}$ of κ-dimensional complex submanifolds of $\mathbb{C}^{N'}$ such that each Γ_z is tangent to M' up to order $s > I_\rho$ at $f(z)$. By the definition of I_ρ, it follows that M' contains a non-singular complex curve through each point $f(z)$ for $z \in \widetilde{M_p}$. Hence $f(\widetilde{M_p}) \subset \mathscr{E}_\rho^1 \subset \mathscr{E}_{M'}$, and therefore, by the closedness of $\mathscr{E}_{M'}$ in M', we get that $f(M_p) \subset \mathscr{E}_{M'}$, a contradiction. The proof of the first part of the proposition is complete. The second part is an obvious consequence of the first. □

Let us emphasize again that the key point in the above proposition lies in the uniformity of the integer e_0 that is independent of all map germs and base points.

It is therefore natural to look for simple conditions implying that ρ is of (quasi-)finite type. We shall give two instances.

Proposition 4.8. *Let $\rho\colon \Omega \to \mathbb{R}^d$ be a real-analytic map defined on some open subset of $\mathbb{C}^N$ and M' the real-analytic set given by the zero set of ρ. If M' is compact, then ρ is of finite type.*

Proof. Let $q \in M'$. It follows from [13] or [16, Theorem 5.2], that there exists a neighbourhood V_q of q in $\mathbb{C}^{N'}$ and an integer m_q such that if $\gamma\colon (\mathbb{C}, 0) \to \Omega$ is non-singular holomorphic curve with $\gamma(0) \in V_q \cap M'$ and if $\nu_0(\rho\circ\gamma) > m_q$, then there exists a non-singular holomorphic curve contained in M' through $\gamma(0)$. But, in view of [10], such an outcome is impossible for a compact real-analytic set M'. Hence

$$\sup_\gamma \Big\{\nu_0(\rho\circ\gamma) : \gamma(0) \in M' \cap V_q,\ \gamma'(0) \neq 0)\Big\} \leq m_q < \infty.$$

The desired conclusion then follows from the compactness assumption on M' and covering M' by finitely many open subsets V_q as above with $q \in M'$. The proof is complete. □

In view of Example 4.6, one cannot replace the compactness condition in Proposition 4.8 by assuming that M' does not contain any complex analytic disc. Interestingly, this is however possible if M' is a real-algebraic set, or more generally, if M' is a Nash set. Indeed, the following statement is a consequence of the more general result, Corollary 6.3, provided in §6.

Proposition 4.9. *Let Ω be a semi-algebraic subset of $\mathbb{C}^{N'}$. Then every Nash map $\rho\colon \Omega \to \mathbb{R}^d$ is of quasi-finite type. In particular, if the Nash set $\{w \in \Omega : \rho(w,\bar{w}) = 0\}$ does not contain any complex-analytic subvariety of positive dimension, then ρ is of finite type.*

Combining Proposition 4.7 and Proposition 4.9, we obtain the following important property.

Proposition 4.10. *Let M, $\mathscr{F}$ be as above, and assume that $M' = \{w \in \Omega : \rho(w, \bar{w}) = 0\}$ where Ω is semi-algebraic and ρ is Nash over Ω. Then there exists an integer e_0, depending only on M' and N', such that, for every $p \in M$ and every non-collapsing map $f \in \mathscr{F}_p$, we necessarily have $\kappa_{p,f}^{e_0} = 0$.*

5. Nash targets, mappings f with $\kappa_f^\infty = 0$, and finite jet determination

In this section, we let $M \subset \mathbb{C}^N$ and $p_0 \in M$, be as in previous sections and assume that M' is a Nash set given by $M' = \{w \in \Omega : \rho(w, \bar{w}) = 0\}$ for some fixed (semi-algebraic) open subset of $\mathbb{C}^{N'}$ and Nash map $\rho \colon \Omega \to \mathbb{R}^d$. As before, $\mathscr{F}$ denotes the family of all germs of real-analytic CR maps from M into M'. Our goal here is to show that if $\mathscr{B} \subset \mathscr{F}$ is the subfamily of map germs for which there exists a fixed integer e_0 such that for every $p \in M$ and every $f \in \mathscr{B}_p$, $\kappa_{p,f}^{e_0} = 0$ then, shrinking M around p_0 if necessary, finite jet determination holds for all maps in $\mathscr{B}$ (see Theorem 5.3). In order to prove this, we will show that $\mathscr{B}$ satisfies property $(*)$ from [17] (finite jet determination then follows). To this end, we shall first recall some notation and notions from [17] and prove the main result of the section given by Proposition 5.2.

5.1. Complexification

Let $M \subset \mathbb{C}^N$ and $p_0 \in M$ be as in previous sections. Denote by n its CR dimension n and by m its codimension in $\mathbb{C}^N$. We fix a basis of real-analytic CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$ on M. Shrinking M near p_0 if necessary, we may choose some polydisc neighbourhood U of p_0, and a real-valued real-analytic map $r = (r_1, \ldots, r_m)$ defined on U such that M is given by

$$M = \left\{ Z \in U : r(Z, \bar{Z}) = 0 \right\}, \tag{5.1}$$

with $\partial r_1 \wedge \ldots \partial r_m \neq 0$ on U. The usual complexification of M is given by

$$\mathscr{M} := \{(Z, \zeta) \in U \times U^* : r(Z, \zeta) = 0\},$$

where $U^* = \{Z \colon \bar{Z} \in U\}$, which (for small enough U) is a complex submanifold of complex dimension $2n + d$ of $U \times U^*$. Furthermore, as in [26,17], we shall consider the second order iterated complexification

$$\mathscr{M}^2 := \{(Z, \zeta, Z^1) \in U \times U^* \times U \colon (Z, \zeta) \in \mathscr{M}, (Z^1, \zeta) \in \mathscr{M}\}.$$

We note that $\mathscr{M}^2$ is a complex submanifold of $U \times U^* \times U \subset \mathbb{C}^{3N}$.

The basis of real-analytic CR vector fields may be written in the form

$$\bar{L}_j = \sum_{\nu=1}^{N} C_{\nu,j}(Z,\bar{Z}) \frac{\partial}{\partial \bar{Z}_\nu}, \quad j = 1,\dots,n, \tag{5.2}$$

where each $C_{\nu,j}$ is real-analytic over U (and depending only on r) and where $Z = (Z_1,\dots,Z_N)$.

5.2. Property $(*)$

We recall property $(*)$ introduced in [17].

Definition 5.1. Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $\mathscr{S}$ a subfamily of $\mathscr{F}$, and $p_0 \in M$. We say that $\mathscr{S}$ satisfies property $(*)_{p_0}$ if there exist a sufficiently small neighbourhood Ω_0 of p_0 in $\mathbb{C}^N$, a positive integer r, a finite family of $\mathbb{C}^{N'}$-valued polynomial maps $\Psi^{(1)},\dots,\Psi^{(L)}$, universal in the sense that they are independent of p_0, M, and a holomorphic map $\Delta(Z,\zeta,Z^1)$, defined on $\Omega_0 \times \Omega_0^* \times \Omega_0$, depending only on M and p_0, such that for every $p \in M_0 := M \cap \Omega_0$ and every $f \in \mathscr{S}_p$, there exists $\ell \in \{1,\dots L\}$ such that

$$\Psi_j^{(\ell)}\left(\Delta(Z,\zeta,Z^1), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\le r}, f_j(Z)\right) = 0,\ j = 1,\dots,N',\ \text{and} \tag{5.3}$$

$$\frac{\partial \Psi_j^{(\ell)}}{\partial T}\left(\Delta(Z,\zeta,Z^1), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\le r}, f_j(Z)\right) \not\equiv 0,\ j = 1,\dots,N', \tag{5.4}$$

for $(Z,\zeta,Z^1) \in \mathscr{M}^2$ sufficiently close to $(p,\bar{p},p)$, and where we write $\Psi^{(\ell)} = (\Psi_1^{(\ell)},\dots,\Psi_{N'}^{(\ell)})$, with T denoting its last argument.

Let e_0 be a fixed positive integer and consider now $\mathscr{F} \supset \mathscr{B}^{e_0} = \cup_{p\in M} \mathscr{B}_p^{e_0}$ where each $\mathscr{B}_p^{e_0}$ consists of those germs $f \in \mathscr{F}_p$ satisfying $\kappa_{p,f}^{e_0} = 0$. Our main result in this section is the following:

Proposition 5.2. *Let M, M', p_0, e_0 and $\mathscr{B}^{e_0}$ be as above. Then $\mathscr{B}^{e_0}$ satisfies property $(*)_{p_0}$.*

Proof. Without loss of generality, we may assume that for every $p \in M$ and every map $f \in \mathscr{B}_p^{e_0}$, we have $\kappa_{p,f}^{e_0} = 0$ and $\kappa_{p,f}^{e_0-1} > 0$. Pick such a map f and point p. By Proposition 3.1, there exists a universal polynomial map $\mathscr{P}^{(e_0)}$, and rational maps $\mathscr{Q}^{(1)},\dots,\mathscr{Q}^{(e_0)}$, each belonging to a finite family of universal rational mappings (independent of p and f, and depending only on e_0) such that

$$\begin{cases} \mathscr{P}^{(e_0)}\left(A^{(e_0)}, \overline{A^{(e_0)}}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le e_0}\right) = 0, \\ \bar L A^{(e_0)} = 0, \\ A^{(e_0)} = \mathscr{Q}^{(e_0)}\left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(e_0-1)}}\right)_{|\alpha|\le e_0(N'+1)}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le e_0(N'+2)-1}\right) \\ A^{(e_0-1)} = \mathscr{Q}^{(e_0-1)}\left(\left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(\ell-2)}}\right)_{|\alpha|\le (e_0-1)(N'+1)}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le (e_0-1)(N'+2)-1}\right), \\ \vdots \\ A^{(1)} = \mathscr{Q}^{(1)}\left(\left(\bar L^\alpha \bar f\right)_{|\alpha|\le N'+1}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le N'+1}\right). \end{cases} \tag{5.5}$$

Furthermore, for every $j = 1, \dots, e_0$, there exists a rational map $\mathscr{T}^{(j)}$, also belonging to a finite family of universal rational maps (independent of p and f, and depending only on e_0) such that for every such $j = 1, \dots, e_0$

$$A^{(j)} = \mathscr{T}^{(j)}\left(X^{r_j}(f,\bar f), \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le r_j}\right), \quad r_j = j(j+1)(N'+1)/2, \tag{5.6}$$

and such that the rank of the (first two lines of the) system (5.5), as defined in §2.3, is equal to N'. These first two lines of (5.5) may be written in the form

$$\widetilde{\mathscr{P}}\left(\overline{A^{(e_0)}}, \left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(e_0-1)}}\right)_{|\alpha|\le e_0(N'+1)}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le e_0(N'+2)}\right) = 0, \tag{5.7}$$

for some universal rational map $\widetilde{\mathscr{P}}$ depending only on $\mathscr{P}^{(e_0)}$ and $\mathscr{Q}^{(e_0)}$ satisfying

$$\operatorname{Rk} \frac{\partial}{\partial w}\left\{\widetilde{\mathscr{P}}\left(\overline{A^{(e_0)}}, \left(\bar L^\alpha \bar f, \bar L^\alpha \overline{A^{(e_0-1)}}\right)_{|\alpha|\le e_0(N'+1)}, \left(\rho_{\nu\bar\gamma}(w,\bar f)\right)_{|\nu|+|\gamma|\le e_0(N'+2)}\right)\right\}\Big|_{w=f} = N'. \tag{5.8}$$

Since $A^{(e_0)}$ and $A^{(e_0-1)}$ have components in $\mathbb{K}_p^{CR}(M)$, complexifying (5.7) and the vector fields given in (5.2) yields the identity

$$\widehat{\mathscr{P}}\Big(\widehat\Delta(Z,\zeta), \overline{A^{(e_0)}}(\zeta), \left(\partial^\delta \bar f(\zeta), \partial^\delta \overline{A^{(e_0-1)}}(\zeta)\right)_{|\delta|\le e_0(N'+1)}, \left(\rho_{\nu\bar\gamma}(f(Z), \bar f(\zeta))\right)_{|\nu|+|\gamma|\le e_0(N'+2)}\Big) = 0, \tag{5.9}$$

for $(Z,\zeta) \in \mathscr{M}$ near $(p,\bar p)$ i.e. in the field of fractions of $\mathbb{C}\{Z-p, \zeta-\bar p\}/\mathscr{M}$. Here $\widehat{\mathscr{P}}$ is some universal polynomial map depending on $\widetilde{\mathscr{P}}$ and $\widehat\Delta$ is a holomorphic map (constructed from the complexification of the coefficients of the CR vector fields) defined on $U \times U^*$ and therefore depending only on M (and p_0). In the same vein, taking complex conjugates in (5.6) and complexifying, we also have for $j \in \{e_0-1, e_0\}$ and $(Z^1,\zeta) \in \mathscr{M}$ near $(p,\bar p)$

$$\overline{A^{(j)}}(\zeta) = \widetilde{\mathscr{T}}^{(j)}\left(\widetilde{\Delta}(Z^1,\zeta), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\leq r_j}, (\rho_{\nu\bar{\gamma}}(f(Z^1),\bar{f}(\zeta)))_{|\nu|+|\gamma|\leq r_j}\right), \tag{5.10}$$

for some universal rational map $\widetilde{\mathscr{T}}^{(j)}$ and some map $\widetilde{\Delta}$ holomorphic on $U\times U^*$ (depending on M). Furthermore, differentiating (5.10) for $j = e_0 - 1$, we get that for $(Z^1,\zeta)\in\mathscr{M}$ near $(p,\bar{p})$

$$\begin{aligned}&\left(\partial^\alpha \overline{A^{(e_0-1)}}(\zeta)\right)_{|\delta|\leq e_0(N'+1)}\\ &\quad= \mathscr{U}\left(\check{\Delta}(Z^1,\zeta), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\leq r_{e_0}}, (\rho_{\nu\bar{\gamma}}(f(Z^1),\bar{f}(\zeta)))_{|\nu|+|\gamma|\leq r_{e_0}}\right),\end{aligned} \tag{5.11}$$

for some universal rational map $\mathscr{U}$ and some map $\check{\Delta}$ holomorphic on $U\times U^*$ (depending on M). Substituting (5.11) and (5.10) (for $j = e_0$) into (5.9), and using the fact that ρ is a polynomial map, we obtain an identity of the form

$$\Theta\left(\Delta(Z,\zeta,Z^1), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\leq r_{e_0}}, f(Z)\right) = 0, \tag{5.12}$$

for $(Z,\zeta,Z^1)\in\mathscr{M}^2$ near $(p,\bar{p},p)$, for some universal ratio of complex-algebraic maps Θ, depending only ρ and, for some holomorphic map Δ on $U\times U^*\times U$ depending only on M and p_0. Furthermore, from our construction, we also have

$$\mathrm{Rk}\left\{\frac{\partial\Theta}{\partial w}\left(\Delta(Z,\zeta,Z^1), \left(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta)\right)_{|\mu|\leq r_{e_0}}, w\right)\right\}\Big|_{w=f(Z)} = N', \tag{5.13}$$

where the rank is understood as the generic rank over the manifold $\mathscr{M}^2$ near $(p,\bar{p},p)$. Proposition 5.2 now follows using the same arguments as those that can be found at the end of the proof of [17, Theorem 3.3] relying on [17, Lemma 6.1]. □

The interest in establishing property $(*)_{p_0}$ for the finite jet determination problem lies in the following result proved in [17].

Theorem 5.3. *Let $M\subset\mathbb{C}^N$ be a real-analytic generic submanifold, $p_0\in M$ and $\mathscr{S}$ be a subfamily of $\mathscr{F}$ satisfying $(*)_{p_0}$. If M is minimal at p_0, there exists a neighbourhood M_{p_0} of p_0 in M and an integer $K>0$, such that for every $q\in M_{p_0}$, if f,g are two elements of $\mathscr{S}_q$ satisfying $j_q^K f = j_q^K g$, then $f=g$.*

Combining Theorem 5.3 with Propositions 5.2 and 4.10, we therefore obtain:

Theorem 5.4. *Let $M\subset\mathbb{C}^N$ be a generic real-analytic submanifold, minimal at a point $p_0\in M$ and $M'\subset\mathbb{C}^{N'}$ be a Nash set given by $M'=\{w\in\Omega : \rho(w,\bar{w})=0\}$ for some semi-algebraic open subset $\Omega\subset\mathbb{C}^{N'}$ and Nash map $\rho\colon\Omega\to\mathbb{R}^d$. Then there exists a neighbourhood M_{p_0} of p_0 in M and an integer $K>0$, such that for every $q\in M_{p_0}$, if $f,g\colon(M,q)\to M'$ are two germs of non-collapsing real-analytic CR maps satisfying $j_q^K f = j_q^K g$, then $f=g$.*

5.3. Proof of Theorem 1.2

We first note that all germs of non-collapsing $\mathscr{C}^\infty$-smooth CR maps are automatically real-analytic according to [20] (the main result in [20] is stated for real-algebraic targets, but the proof applies for Nash targets as well). Next, observe that the Nash set M' can be written as a finite union $\cup_{i\in I} M'_i$ where each M'_i is open in M' and given by the zero set of a Nash map over some semi-algebraic open subset of $\mathbb{C}^{N'}$ (see [2]). Hence, when M is generic, the desired conclusion follows after applying Theorem 5.4 to every M'_i. The nongeneric case may be reduced to the generic one using standard arguments (see e.g. [17]).

6. Global strong approximation results for real-algebraic or Nash systems

For a field $\Bbbk$ and indeterminates $x = (x_1, \dots, x_m)$, we denote by $\Bbbk[\![x]\!]$ the ring of formal power series in the indeterminates x_i over $\Bbbk$. We denote by $\Bbbk\langle x\rangle$ the subring of algebraic power series, that is, the formal power series that are algebraic over $\Bbbk[x]$. The ideal of the ring of power series $\Bbbk[\![x]\!]$ generated by the x_i is denoted by (x). For a power series $h(x)$, $h(x) \in (x)^c$ if and only if the coefficients of all the monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, for $\alpha_1 + \dots + \alpha_n < c$, in the expansion of $h(x)$ are zero.

The aim of this section is to prove Corollary 6.3, which is a global strong Artin approximation theorem for Nash systems. Before that, we start by proving Theorem 6.1 which is a global version of [13, Theorem 1.3] in the case of polynomial equations. This can be seen as a CR version of the strong Artin approximation Theorem [1, Theorem 6.1]. The proof of this result is a bit different from the one given in [13], and is based on a reduction to Proposition 6.2, which is a global version of [13, Theorem 1.1] for polynomial equations. This reduction is based on arguments due to [4] and involving ultraproducts. Corollary 6.3 is completely new: its proof is based on a reduction to Theorem 6.1 involving an induction on the height of the ideal of equations, based on the Noetherianity of the ring of Nash functions.

Theorem 6.1. *Let $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_p)$, $u = (u_1, \dots, u_N)$, $v = (v_1, \dots, v_N)$ and let $f \in \mathbb{R}[x, y, u, v]^d$.*

There is a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that for every $c \in \mathbb{N}$, for every $u(x, y)$, $v(x, y) \in \mathbb{R}[\![x, y]\!]^N$ with

$$\forall k \in \{1, \dots, d\},\ \ f_k(x, y, u(x, y), v(x, y)) \in (x, y)^{\beta(c)},$$

and

$$\forall i, j,\ \ \frac{\partial u_i}{\partial x_j}(x,y) - \frac{\partial v_i}{\partial y_j}(x,y) = \frac{\partial u_i}{\partial y_j}(x,y) + \frac{\partial v_i}{\partial x_j}(x,y) = 0, \tag{6.1}$$

there exist $\widetilde{u}(x, y)$, $\widetilde{v}(x, y) \in \mathbb{R}\langle x, y\rangle^N$

$$\forall k \in \{1, \dots, d\},\ f_k(x, y, \widetilde{u}(x, y), \widetilde{v}(x, y)) = 0,$$
$$\forall i, j,\ \frac{\partial \widetilde{u}_i}{\partial x_j}(x, y) - \frac{\partial \widetilde{v}_i}{\partial y_j}(x, y) = \frac{\partial \widetilde{u}_i}{\partial y_j}(x, y) + \frac{\partial \widetilde{v}_i}{\partial x_j}(x, y) = 0 \tag{6.2}$$

and

$$\forall i,\ \widetilde{u}_i(x, y) - u_i(x, y), \widetilde{v}_i(x, y) - v_i(x, y) \in (x, y)^c.$$

Proof. First we prove the existence of formal power series $\widetilde{u}(x, y)$, $\widetilde{v}(x, y) \in \mathbb{R}[\![x, y]\!]^N$ satisfying the conclusion of the theorem. The proof is done by contradiction: assume that such a β does not exist, that is, there exists c and, for every $\ell \in \mathbb{N}$, $u_\ell(x, y)$, $v_\ell(x, y) \in \mathbb{R}[\![x, y]\!]^N$ such that

$$\forall k,\ f_k(x, y, u_\ell(x, y), v_\ell(x, y)) \in (x, y)^\ell,$$
$$\forall i, j,\ \frac{\partial u_{\ell,i}}{\partial x_j}(x, y) - \frac{\partial v_{\ell,i}}{\partial y_j}(x, y) = \frac{\partial u_{\ell,i}}{\partial y_j}(x, y) + \frac{\partial v_{\ell,i}}{\partial x_j}(x, y) = 0,$$

but there is no $\widetilde{u}(x, y)$, $\widetilde{v}(x, y) \in \mathbb{R}[\![x, y]\!]^N$ such that

$$\forall k \in \{1, \dots, m\},\ f_k(x, y, \widetilde{u}(x, y), \widetilde{v}(x, y)) = 0,$$
$$\forall i, j,\ \frac{\partial \widetilde{u}_i}{\partial x_j}(x, y) - \frac{\partial \widetilde{v}_i}{\partial y_j}(x, y) = \frac{\partial \widetilde{u}_i}{\partial y_j}(x, y) + \frac{\partial \widetilde{v}_i}{\partial x_j}(x, y) = 0$$

and

$$\forall i,\ \widetilde{u}_i(x, y) - u_{\ell,i}(x, y),\ \widetilde{v}_i(x, y) - v_{\ell,i}(x, y) \in (x, y)^c.$$

Let U be a non principal ultrafilter on $\mathbb{N}$. (For the details concerning ultrafilters and ultrapowers, see [4] and the references therein.) For a ring A we denote by $\mathrm{Ul}(A)$ the ultrapower $\left(\prod_{k\in\mathbb{N}} A\right)/D$. We denote by $\underline{u}$ and $\underline{v}$ the images of the sequences $(u_\ell(x, y))_\ell$ and $(v_\ell(x, y))_\ell$ in $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])^N$. We remark that $\mathrm{Ul}(\mathbb{R})$ is a real closed field (cf. [13, 2.2] for example). We identify the ring $\mathrm{Ul}(\mathbb{R})[x, y]$ with a subring of $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])$ by identifying x (resp. y) with the image of the constant sequence $(x)_\ell$ (resp. $(y)_\ell$). We denote again by f the image of the constant sequence $(f)_\ell$ in $\mathrm{Ul}(\mathbb{R}[x, y, u, v])$ that belongs to $\mathrm{Ul}(\mathbb{R})[x, y, u, v]$.

Because D is non principal, we have that $f_k(x, y, \underline{u}, \underline{v}) \in (x, y)^\ell$ for every k and every $\ell \in \mathbb{N}$. We denote by $(x, y)^\infty$ the intersection of all the powers of (x, y) in $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])$, we define $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])_{\mathrm{sep}} := \mathrm{Ul}(\mathbb{R}[\![x, y]\!])/(x, y)^\infty$ and we denote by π the quotient map from $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])$ to $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])_{\mathrm{sep}}$. The restriction of π to $\mathrm{Ul}(\mathbb{R})[x, y]$ is injective and, as a $\mathrm{Ul}(\mathbb{R})[x, y]$-algebra, $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])_{\mathrm{sep}}$ is isomorphic to $\mathrm{Ul}(\mathbb{R})[\![x, y]\!]$ by [4, Lemma 3.4]. Thus we identify $\mathrm{Ul}(\mathbb{R}[\![x, y]\!])_{\mathrm{sep}}$ with $\mathrm{Ul}(\mathbb{R})[\![x, y]\!]$ and therefore the restriction of π to $\mathrm{Ul}(\mathbb{R})[x, y]$ is the identity map.

Hence, by using Proposition 6.2 given below with $\mathbf{R} = \mathrm{Ul}(\mathbb{R})$, there is u', $v' \in \mathrm{Ul}(\mathbb{R})\langle x,y\rangle^N$ such that $f_k(x,y,u',v') = 0$ in $\mathrm{Ul}(\mathbb{R})\langle x,y\rangle$ for every k,

$$\forall i,j,\ \frac{\partial u_i'}{\partial x_j}(x,y) - \frac{\partial v_i'}{\partial y_j}(x,y) = \frac{\partial u_i'}{\partial y_j}(x,y) + \frac{\partial v_i'}{\partial x_j}(x,y) = 0$$

and

$$\forall i,\ u_i' - \pi(\underline{u}_i), v_i' - \pi(\underline{v}_i) \in (x,y)^c.$$

We remark that $\mathrm{Ul}(\mathbb{R}[\![x,y]\!])$ is a Henselian local ring (cf. [4, p. 193]) and $\mathrm{Ul}(\mathbb{R})\langle x,y\rangle$ is the Henselization of $\mathrm{Ul}(\mathbb{R})[x,y]$ (see [25, Lemma 2.29] for example). Therefore there is a unique $\mathrm{Ul}(\mathbb{R})[x,y]$-morphism

$$\varphi : \mathrm{Ul}(\mathbb{R})\langle x,y\rangle \longrightarrow \mathrm{Ul}(\mathbb{R}[\![x,y]\!]).$$

We have that $\pi\circ\varphi_{|\mathrm{Ul}(\mathbb{R})[x,y]}$ is the identity, therefore, by the unicity of the Henselization, $\pi\circ\varphi_{|\mathrm{Ul}(\mathbb{R})\langle x,y\rangle}$ is also the identity. We also denote by φ the induced morphism of modules $\mathrm{Ul}(\mathbb{R})\langle x,y\rangle^N \longrightarrow \mathrm{Ul}(\mathbb{R}[\![x,y]\!])^N$. In particular this shows that

$$\forall k,\ f_k(x,y,\varphi(u'),\varphi(v')) = 0 \text{ in } \mathrm{Ul}(\mathbb{R}[\![x,y]\!])$$

and, because $\pi\circ\varphi(u') = u'$ equals $\pi(\underline{u})$ modulo $(x,y)^c$ (resp. $\pi\circ\varphi(v') = v'$ equals $\pi(\underline{v})$ modulo $(x)^c$), we have $\varphi(u')-\underline{u}$, $\varphi(v')-\underline{v} \in (x,y)^c$. Let us denote by $(u_\ell'')_\ell$ and $(v_\ell'')_\ell$ two sequences of $\left(\mathbb{R}[\![x,y]\!]^N\right)^{\mathbb{N}}$ whose images in $\mathrm{Ul}(\mathbb{R}[\![x,y]\!])^N$ equal $\varphi(u')$ and $\varphi(v')$. Since D is non principal, we have

$$u_{\ell,i}'' - u_{\ell,i} \in (x,y)^\ell \text{ for every } \ell\in E,\ \forall i = 1,\dots,N,$$

where E is an infinite subset of $\mathbb{N}$. This contradicts our initial assumption and proves the existence of $\widetilde{u}(x,y)$, $\widetilde{v}(x,y) \in \mathbb{R}[\![x,y]\!]^N$. Finally we apply Proposition 6.2 with $\mathbf{R} = \mathbb{R}$ to see that we can choose $\widetilde{u}(x,y)$, $\widetilde{v}(x,y) \in \mathbb{R}\langle x,y\rangle^N$. □

Proposition 6.2. *Let* $\mathbf{R}$ *be a real closed field and let* $f \in \mathbf{R}[x,y,u,v]^d$*. Assume given a formal solution* $\widehat{u}$, $\widehat{v} \in \mathbf{R}[\![x,y]\!]^N$

$$\forall k,\ f_k(x,y,\widehat{u}(x,y),\widehat{v}(x,y)) = 0$$

with

$$\forall i,j,\ \frac{\partial \widehat{u}_i}{\partial x_j}(x,y) - \frac{\partial \widehat{v}_i}{\partial y_j}(x,y) = \frac{\partial \widehat{u}_i}{\partial y_j}(x,y) + \frac{\partial \widehat{v}_i}{\partial x_j}(x,y) = 0, \tag{6.3}$$

and let $c\in\mathbb{N}$*. Then there exist* u', $v' \in \mathbf{R}\langle x,y\rangle^N$ *such that*

$$\forall k,\ f_k(x,y,u'(x,y),v'(x,y))=0$$

with

$$\forall i,j,\ \frac{\partial u'_i}{\partial x_j}(x,y)-\frac{\partial v'_i}{\partial y_j}(x,y)=\frac{\partial u'_i}{\partial y_j}(x,y)+\frac{\partial v'_i}{\partial x_j}(x,y)=0$$

and

$$\forall i,\ u'_i-\widehat{u}_i, v'_i-\widehat{v}_i\in(x,y)^c.$$

Proof. The proof is similar to the proof of [13, Theorem 1.1]: the idea is to reduce the theorem to the classical Artin Approximation Theorem for polynomial equations [1]. We set $\mathbf{C}:=\mathbf{R}+\sqrt{-1}\mathbf{R}$ and we write the $f_k(x,y,u,v)$ as $\rho_k(t,\overline{t},\zeta,\overline{\zeta})\in\mathbf{C}[t,\overline{t},\zeta,\overline{\zeta}]$ with

$$t=x+\sqrt{-1}y,\ \overline{t}=x-\sqrt{-1}y,\ \zeta=u+\sqrt{-1}v,\ \overline{\zeta}=u-\sqrt{-1}v.$$

We set $\widehat{\zeta}:=\widehat{u}+\sqrt{-1}\widehat{v}$. It is well known that the Cauchy-Riemann Equations (6.3) are satisfied if and only if $\widehat{\zeta}\in\mathbf{C}[\![t]\!]^N$. We define the morphisms

$$\begin{array}{rccc} \widehat{\gamma}^*: & \mathbf{C}[t,\overline{t},\zeta,\overline{\zeta}] & \longrightarrow & \mathbf{C}[\![t,\overline{t}]\!] \\ & h(t,\overline{t},\zeta,\overline{\zeta}) & \longmapsto & h(t,\overline{t},\widehat{\zeta},\overline{\widehat{\zeta}}) \\ \widehat{\zeta}^*: & \mathbf{C}[t,\zeta] & \longrightarrow & \mathbf{C}[\![t]\!] \\ & h(t,\zeta) & \longmapsto & h(t,\widehat{\zeta}) \end{array}$$

Then, exactly as proved in [13], we have

$$\operatorname{Ker}(\widehat{\gamma}^*)=\operatorname{Ker}(\widehat{\zeta}^*)\mathbf{C}[t,\overline{t},\zeta,\overline{\zeta}]+\overline{\operatorname{Ker}(\widehat{\zeta}^*)}\mathbf{C}[t,\overline{t},\zeta,\overline{\zeta}] \tag{6.4}$$

Then, we apply the Artin Approximation Theorem [1] to $\operatorname{Ker}(\widehat{\zeta}^*)$:

For $c\in\mathbb{N}$, there is $\zeta'\in\mathbf{C}\langle t\rangle^N$ such that $\zeta'-\widehat{\zeta}^*\in(t)^c$ and, for every $s\in\operatorname{Ker}(\widehat{\zeta}^*)$, $s(t,\zeta'(t))=0$. Therefore, by (6.4), we have $\rho_k(t,\overline{t},\zeta'(t),\overline{\zeta}'(t))=0$ for every k. This proves the result by defining u' and v' as $u'(x,y)+\sqrt{-1}v'(x,y)=\zeta'$. □

For an open subset $\Omega\subset\mathbb{R}^k$, we denote by $N(\Omega)$ the ring of Nash functions on Ω, that is, the real analytic functions on Ω whose germ at every point of Ω is algebraic over the field of rational functions.

If $f=(f_1,\dots,f_d)\in N(\Omega)^d$, we denote by $\langle f\rangle$ the ideal of $N(\Omega)$ generated by the f_i. We remark that, for $c\in\mathbb{N}\cup\{\infty\}$ and $u(x,y)$, $v(x,y)\in\mathbb{R}[\![x,y]\!]^N$, we have

$$\forall k\in\{1,\dots,d\},\ f_k(u(x,y),v(x,y))\in(x,y)^c\Longleftrightarrow\forall g\in\langle f\rangle,\ g(u(x,y),v(x,y))\in(x,y)^c.$$

By convention, $h(x,y)\in(x,y)^\infty$ if $h(x,y)=0$.

So, from now on, if we set $I := \langle f \rangle$, we will write $I(u(x,y),v(x,y)) \in (x,y)^c$ for:

$$\forall k \in \{1,\dots,d\},\ f_k(u(x,y),v(x,y)) \in (x,y)^c.$$

Corollary 6.3. *Let Ω be an open subset of $\mathbb{C}^N$ such that, for every $V \subset W$ real algebraic subsets of $\mathbb{C}^N$, the set $(W \setminus V) \cap \Omega$ has finitely many connected components. Let $\rho : \Omega \longrightarrow \mathbb{R}^d$ be a real-analytic map whose components are algebraic. Then there is a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that the following holds:*

For every $c \in \mathbb{N}$ and every germ of a holomorphic map $\gamma : (\mathbb{C}^p, 0) \longrightarrow \mathbb{C}^N$ such that $\rho \circ \gamma(t) \in (t)^{\beta(c)}$, then there is a germ of a holomorphic algebraic map $\widetilde{\gamma} : (\mathbb{C}^p, 0) \longrightarrow \mathbb{C}^N$ such that $\widetilde{\gamma} - \gamma \in (t)^c$ with $\rho \circ \widetilde{\gamma} \equiv 0$.

Proof. First we remark that if the statement is true for an integer $c+1$, then it is true for the integer c because $(t)^{c+1} \subset (t)^c$. So we may assume that $c \geq 1$. In this case if $\widetilde{\gamma} - \gamma \in (t)^c$ then $\widetilde{\gamma}(0) = \gamma(0)$.

We write $\rho = (\rho_1, \dots, \rho_d)$, and denote by $f_1, \dots, f_{2d}$ the real and imaginary parts of the ρ_i. We set

$$t = x + \sqrt{-1}y,\ \overline{t} = x - \sqrt{-1}y,\ \zeta = u + \sqrt{-1}v,\ \overline{\zeta} = u - \sqrt{-1}v$$

and we write $\gamma(t) = u(x,y) + \sqrt{-1}v(x,y)$ where $u(x,y)$ and $v(x,y)$ are real valued. It is well known that γ is holomorphic (that is, depends only on t and not on $\overline{t}$) if and only if the Cauchy-Riemann Equations (6.1) are satisfied. Therefore we need to prove Theorem 6.1 for the f_i and we set $\widetilde{\gamma} = \widetilde{u} + \sqrt{-1}\widetilde{v}$.

Let $I = \langle f \rangle$ denote the ideal of $N(\Omega)$ generated by $f_1, \dots, f_{2d}$. By [24, Theorem 2.1], under the condition on Ω given in the statement of the corollary, the ring $N(\Omega)$ is Noetherian. Moreover the height of an ideal of $N(\Omega)$ is less or equal to $2N$ (cf. [24, Prop. 2.2, Cor. 2.12]). The proof of the corollary will be done by a decreasing induction on the height of I. If I is a maximal ideal, then $I = \langle u_1 - a_1, \dots, u_N - a_N, v_1 - b_1, \dots, v_N - b_N \rangle$ where $a + \sqrt{-1}b \in \Omega$ (cf. [24, Cor. 2.12]). In this case, the corollary is trivial.

Now let $h < 2N$ and assume that the corollary has been proved for the ideals of height $> h$. By the Noetherianity of $N(\Omega)$, we can write

$$I = Q_1 \cap \cdots \cap Q_s$$

where the Q_k are primary ideals. Moreover if h denotes the height of I, the heights of the Q_k are larger than or equal to h. If we set $P_k = \sqrt{Q_k}$, the P_k are prime ideals. For every k, let $\ell_k \in \mathbb{N}$ be such that $P_k^{\ell_k} \subset Q_k$. We may assume that $\ell_k = \ell$ for every k by replacing the ℓ_k by a larger integer.

Assume that the corollary has been proved for every prime ideal of height h. Then let β be a function satisfying the corollary for all the P_k. Let c and $u(x,y), v(x,y) \in \mathbb{R}[\![x,y]\!]^N$ such that

$$I(u(x,y),v(x,y)) \in (x,y)^{\ell s\beta(c)}, \tag{6.5}$$

and satisfying the Cauchy-Riemann Equations (6.2). Then we claim that there is $k_0 \in \{1,\dots,s\}$ such that $Q_{k_0}(u(x,y),v(x,y)) \in (x,y)^{\ell\beta(c)}$. Indeed, if this were not the case, for every k we may choose $g_k \in Q_k$ such that $g_k(u(x,y),v(x,y)) \notin (x,y)^{\ell\beta(c)}$. Therefore $\prod_{k=1}^{s} g_k(u(x,y),v(x,y)) \notin (x,y)^{\ell s\beta(c)}$ but $\prod_{k=1}^{s} g_k \in I$ contradicting (6.5).

Thus, we have $P_{k_0}(u(x,y),v(x,y)) \in (x,y)^{\beta(c)}$, and by assumption on β, there is $\widetilde{u}$, $\widetilde{v} \in N(\Omega)$ such that $P_{k_0}(\widetilde{u},\widetilde{v}) = 0$ and

$$\forall i,\ \widetilde{u}_i(x,y) - u_i(x,y), \widetilde{v}_i(x,y) - v_i(x,y) \in (x,y)^c.$$

Since $I \subset P_{k_0}$ we have $I(\widetilde{u},\widetilde{v}) = 0$, and the corollary is proved for I. Therefore, we are reduced to prove the corollary for prime ideals.

Assume that I is a prime ideal. We denote by I' the prime ideal of $\mathbb{R}[u,v]$ defined by $I' = I \cap \mathbb{R}[u,v]$. By Theorem 6.1, the corollary is satisfied for I', and we denote by β' a function for which this corollary is satisfied for I'.

By [24, Prop. 2.9], we can write $I'N(\Omega) = I \cap I_2$ where I_2 is an ideal of $N(\Omega)$ of height h such that $I = I_2$ or $I_2 \not\subset I$. If $I = I_2$, then we can apply Theorem 6.1 to I' and the corollary is proved for I. If $I_2 \not\subset I$, we set $J = I + I_2$. Since I is prime, the height of J is larger than or equal to $h+1$ and the inductive hypothesis applies to J. We denote by β a function for which Corollary 6.3 is satisfied for J. Now let $c \in \mathbb{N}$ and assume that

$$I(u(x,y),v(x,y)) \in (x,y)^{\beta'(\beta(c))+\beta(c)}.$$

We consider the following two cases:

(1) If $J(u(x,y),v(x,y)) \in (x,y)^{\beta(c)}$, then there is $\widetilde{u}$, $\widetilde{v} \in N(\Omega)$ such that $J(\widetilde{u},\widetilde{v}) = 0$, satisfying the Cauchy-Riemann equations (6.2), and

$$\forall i,\ \widetilde{u}_i(x,y) - u_i(x,y),\ \widetilde{v}_i(x,y) - v_i(x,y) \in (x,y)^c.$$

Since $I \subset J$, we have $I(\widetilde{u},\widetilde{v}) = 0$ and the corollary is proved.

(2) If $J(u(x,y),v(x,y)) \notin (x,y)^{\beta(c)}$, then $I_2(u(x,y),v(x,y)) \notin (x,y)^{\beta(c)}$ because $I(u(x,y),v(x,y)) \in (x,y)^{\beta(c)}$. Since $I' \subset I$ and $I(u(x,y),v(x,y)) \in (x,y)^{\beta'(\beta(c))+\beta(c)} \subset (x,y)^{\beta'(\beta(c))}$, there is $\widetilde{u}, \widetilde{v} \in N(\Omega)$ such that $I'(\widetilde{u},\widetilde{v}) = 0$, satisfying the Cauchy-Riemann equations (6.2), and

$$\forall i,\ \widetilde{u}_i(x,y) - u_i(x,y),\ \widetilde{v}_i(x,y) - v_i(x,y) \in (x,y)^{\beta(c)}. \tag{6.6}$$

Now, by (6.6), because $I_2(u(x,y),v(x,y)) \notin (x,y)^{\beta(c)}$ we have $I_2(\widetilde{u},\widetilde{v}) \neq 0$. Since $I\, I_2 \subset I'$, we have $I(\widetilde{u},\widetilde{v}) = 0$. Therefore the corollary is proved for I with respect to the function $\beta' \circ \beta + \beta$. $\square$

Examples of open subsets Ω satisfying the assumption of Corollary include open subsets definable in a o-minimal structure, such as semi-algebraic open subsets or subanalytic bounded open subsets (see [24]).

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