# LOCAL TOPOLOGICAL ALGEBRAICITY
# OF ANALYTIC FUNCTION GERMS

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## Abstract

Mostowski showed that every (real or complex) germ of an analytic set is homeomorphic to the germ of an algebraic set. In this paper we show that every (real or complex) analytic function germ, defined on a possibly singular analytic space, is topologically equivalent to a polynomial function germ defined on an affine algebraic variety.

## 1. Introduction and statement of results

The problem of approximation of analytic objects (sets or mappings) by algebraic ones has attracted many mathematicians; see e.g. [2] and the bibliography therein. Nevertheless there are very few positive results if one requires that the approximation gives a homeomorphism between the approximated object and the approximating one. In this paper we consider two cases of this problem: the local algebraicity of analytic sets and the local algebraicity of analytic functions. The problem can be considered over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

The local topological algebraicity of analytic sets has been established in [12] by Mostowski. More precisely, given an analytic set germ $(V, 0) \subset (\mathbb{K}^n, 0)$, Mostowski shows the existence of a local homeomorphism $\tilde{h} : (\mathbb{K}^{2n+1}, 0) \to (\mathbb{K}^{2n+1}, 0)$ such that, after the embedding $(V, 0) \subset (\mathbb{K}^n, 0) \subset (\mathbb{K}^{2n+1}, 0)$, the image $\tilde{h}(V)$ is algebraic. It is easy to see that Mostowski's proof together with Theorem 2 of [2] (see Theorem 4.2 for a precise statement) gives the following result.

**Theorem 1.1.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be an analytic germ. Then there is a homeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $h(V)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

Mostowski's Theorem seems not to be widely known. Recently Fernández de Bobadilla showed, by a method different from that of Mostowski, the local topological algebraicity of complex hypersurfaces with one-dimensional singular locus; see [3]. We remark that in [12] Mostowski states his results only for $\mathbb{K} = \mathbb{R}$, but his proof works, word by word, for $\mathbb{K} = \mathbb{C}$.

The first purpose of this paper is to present a short proof of Theorem 1.1. We follow closely Mostowski's original approach that is based on two ideas, Płoski's version of Artin approximation (cf. [13]) and Varchenko's theorem stating that the algebraic equisingularity of Zariski implies topological equisingularity. Our proof is shorter, but less elementary. We use a corollary of the Néron Desingularization that we call the Nested Artin-Płoski-Popescu Approximation Theorem. This result is shown in Section 2, and the proof of Theorem 1.1 is given in Section 4. In Section 3 we recall Varchenko's results.

The second purpose of this paper is to show the local algebraicity of analytic functions.

**Theorem 1.2.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ \sigma$ is the germ of a polynomial.*

The proof of Theorem 1.2, given in Section 5, is based on the Nested Artin-Płoski-Popescu Approximation Theorem and a refinement of Varchenko's method.

We end with the following generalization of Theorems 1.1 and 1.2.

**Theorem 1.3.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(V_i, 0) \subset (\mathbb{K}^n, 0)$ be a finite family of analytic set germs and let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ \sigma$ is the germ of a polynomial, and for each $i$, $\sigma^{-1}(V_i)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

**Corollary 1.4.** *Let $g : (V, p) \to (\mathbb{K}, 0)$ be an analytic function germ defined on the germ $(V, p)$ of an analytic space. Then there exists an algebraic affine variety $V_1$, a point $p_1 \in V_1$, the germ of a polynomial function $g_1 : (V_1, p_1) \to (\mathbb{K}, 0)$ and a homeomorphism $\sigma : (V_1, p_1) \to (V, p)$ such that $g_1 = g \circ \sigma$.*

In Section 6 we present examples showing that Theorems 1.1, 1.2 and 1.3 are false if we replace "homeomorphism" by "diffeomorphism". We do not know whether these theorems hold true with "homeomorphism" replaced by "bi-lipschitz homeomorphism".

**Remark 1.5.** We often identify the germ at the origin of a $\mathbb{K}$-analytic function $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ with its Taylor series, that is, with a convergent power series. We say that a ($\mathbb{K}$-)analytic function or a germ is *Nash* if its graph is semi-algebraic. Thus $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ is the germ of a Nash function if and only if its Taylor series is an algebraic power series (i.e. a power series which is algebraic over $\mathbb{K}[x_1, \ldots, x_n]$ - for instance the power series $u(x)$ such that $u(0) = 1$ and $u(x)^2 = 1 + x$ is an algebraic power series). A *Nash set* is the zero set of finitely many Nash functions.

## 2. Nested Artin-Płoski-Popescu Approximation Theorem

We set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. The ring of convergent power series in $x_1, \ldots, x_n$ is denoted by $\mathbb{K}\{x\}$. If $A$ is a commutative ring, then the ring of algebraic power series with coefficients in $A$ is denoted by $A\langle x \rangle$.

The following result is a corollary of Theorem 11.4 in [16], which itself is a corollary of the Néron-Popescu Desingularization (see [14], [16] or [17] for the proof of this desingularization theorem in whole generality or [15] for a proof in characteristic zero).

**Theorem 2.1.** *Let $f(x, y) \in \mathbb{K}\langle x \rangle[y]^p$ and let us consider a solution $y(x) \in \mathbb{K}\{x\}^m$ of*

$$f(x, y(x)) = 0.$$

*Let us assume that $y_i(x)$ depends only on $(x_1, \ldots, x_{\sigma(i)})$ where $i \longmapsto \sigma(i)$ is an increasing function. Then there exist a new set of variables $z = (z_1, \ldots, z_s)$, an increasing function $\tau$, convergent power series $z_i(x) \in \mathbb{K}\{x\}$ vanishing at $0$ such that $z_1(x), \ldots, z_{\tau(i)}(x)$ depend only on $(x_1, \ldots, x_{\sigma(i)})$, and an algebraic power series vector solution $y(x, z) \in \mathbb{K}\langle x, z \rangle^m$ of*

$$f(x, y(x, z)) = 0,$$

*such that for every $i$,*

$$y_i(x, z) \in \mathbb{K}\langle x_1, \ldots, x_{\sigma(i)}, z_1, \ldots, z_{\tau(i)} \rangle \text{ and } y(x) = y(x, z(x)).$$

**Remark 2.2.** Theorem 2.1 remains valid if we replace "convergent power series" by "formal power series".

For any $i$ we set:

$$A_i = \mathbb{K}\langle x_1, \ldots, x_i \rangle,$$
$$B_i = \mathbb{K}\{x_1, \ldots, x_i\}.$$

We will need at several places the following two lemmas whose proofs are given later (for the definition and properties of an excellent ring see 7.8 [7]

or [11]; a henselian local ring is a local ring satisfying the Implicit Function Theorem; see 18.5 [8]).

**Lemma 2.3.** *Let $B$ be an excellent henselian local subring of $\mathbb{K}[[x_1, \ldots, x_{i-1}]]$ containing $\mathbb{K}\langle x_1, \ldots, x_{i-1}\rangle$ and whose maximal ideal is generated by $x_1, \ldots, x_{i-1}$. Then the ring $A_i \otimes_{A_{i-1}} B$ is noetherian, and its henselization is isomorphic to $B\langle x_i\rangle$.*

**Lemma 2.4.** *Let $B$ be an excellent henselian local subring of $\mathbb{K}[[x_1, \ldots, x_{i-1}]]$ containing $\mathbb{K}\langle x_1, \ldots, x_{i-1}\rangle$ and whose maximal ideal is generated by $x_1, \ldots, x_{i-1}$. Let $I$ be an ideal of $B[x_i]$. Then the henselization of $\frac{B[x_i]_{(x_1, \ldots, x_i)}}{I}$ is isomorphic to $\frac{B\langle x_i\rangle}{I}$.*

*Proof of Theorem* 2.1. By replacing $f(x, y)$ by $f(x, y(0) + y)$ we may assume that $y(0) = 0$.

For any $i$ let $l(i)$ be the largest integer such that $y_1(x), \ldots, y_{l(i)}(x) \in \mathbb{K}\{x_1, \ldots, x_i\}$.

For any $i$ let $J_i$ be the kernel of the morphism

$$\varphi_i : \mathbb{K}\langle x_1, \ldots, x_i\rangle[y_1, \ldots, y_{l(i)}] \longrightarrow \mathbb{K}\{x_1, \ldots, x_i\} = B_i$$

defined by $\varphi_i(g(x, y)) = g(x, y(x))$. We define:

$$C_i = \frac{\mathbb{K}\langle x_1, \ldots, x_i\rangle[y_1, \ldots, y_{l(i)}]}{J_i}.$$

Then $C_i$ is a finite type $A_i$-algebra and $C_i$ is a sub-$A_i$-algebra of $C_{i+1}$ since $J_i \subset J_{i+1}$. The morphism $\varphi_i$ induces a morphism $C_i \longrightarrow B_i$ such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}
$$

By Theorem 11.4 in [16] (see also [18]) this diagram may be extended to a commutative diagram as follows:

$$
\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots & \longrightarrow & D_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}
$$

where $D_1$ is a smooth $A_1$-algebra of finite type and $D_i$ is a smooth $D_{i-1} \otimes_{A_{i-1}} A_i$-algebra of finite type for all $i > 1$. We will denote by $D'_{i-1}$ the ring $D_{i-1} \otimes_{A_{i-1}} A_i$ for all $i > 1$ and set $D'_1 = A_1$.

For any $i$ let us write $D_i = \frac{D'_{i-1}[u_{i,1},\ldots,u_{i,q_i}]}{I_i}$. We may make a change of coordinates (of the form $u_{i,j} \longmapsto u_{i,j} + c_{i,j}$ for some $c_{i,j} \in \mathbb{K}$) in such a way that the image of $u_{i,j}$ is in the maximal ideal of $B_i$ for any $i$ and $j$. Thus $D_i \longrightarrow B_i$ factors through the localization morphism $D_i \longrightarrow (D_i)_{\mathfrak{m}_i}$ where $\mathfrak{m}_i = (x_1,\ldots,x_i,u_{i,1},\ldots,u_{i,q_i})$. Let $D_i^h$ be the henselization of $(D_i)_{\mathfrak{m}_i}$. Since $B_i$ is a henselian local ring, the morphism $D_i \longrightarrow B_i$ factors through $D_i^h$ by the universal property of the henselization. Still by this universal property the composition of the morphisms $D_{i-1} \longrightarrow D_i \longrightarrow D_i^h$ factors through $D_{i-1}^h$. Thus we have the following commutative diagram:

$$
\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_1^h & \longrightarrow & D_2^h & \longrightarrow & \cdots & \longrightarrow & D_n^h \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}
$$

We will prove by induction that $D_i^h$ is isomorphic to $\mathbb{K}\langle x_1,\ldots,x_i,z_1,\ldots,z_{\lambda(i)}\rangle$ where $i \longrightarrow \lambda(i)$ is an increasing function and the $z_k$ are new indeterminates.

Since $D_1^h$ is the henselization of $(D_1)_{\mathfrak{m}_1} = \frac{\mathbb{K}\langle x_1\rangle[u_{1,1},\ldots,u_{1,q_1}]}{I}{}_{(x_1,u_{1,1},\ldots,u_{1,q_1})}$, $D_1^h$ is isomorphic to $\frac{\mathbb{K}\langle x_1,u_{1,1},\ldots,u_{1,q_1}\rangle}{I\cdot\mathbb{K}\langle x_1,u_{1,1},\ldots,u_{1,q_1}\rangle}$ by Lemma 2.4. Moreover $D_1$ is smooth over $\mathbb{K}\langle x_1\rangle$ means that the matrix $\left(\frac{\partial f_j}{\partial u_k}(0,0)\right)_{i,j}$, where the $f_j$ are generators of the ideal $I \cdot \mathbb{K}\langle x_1,u_{1,1},\ldots,u_{1,q_1}\rangle$, has maximal rank (by the jacobian criterion for smoothness; see [6, Proposition 22.6.7 (iii)]). Thus by the Implicit Function Theorem the ring $D_1^h$ is isomorphic to $\mathbb{K}\langle x_1,z_1,\ldots,z_{\lambda(1)}\rangle$ for some new indeterminates $z_1,\ldots,z_{\lambda(1)}$. This proves the induction property for $D_1^h$.

Now let us assume that the induction property is true for $D_{i-1}^h$. By assumption $D_i$ is smooth over $D_{i-1}\otimes_{A_{i-1}}A_i$. Thus $D_i^h$ is smooth over the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$. By the universal property of the henselization the morphism from $D_{i-1}$ to the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$ factors through $D_{i-1}^h$; thus it factors through $D_{i-1}^h\otimes_{A_{i-1}}A_i$. Hence the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$ is isomorphic to the henselization of $D_{i-1}^h\otimes_{A_{i-1}}A_i$. But

$$D_{i-1}^h\otimes_{A_{i-1}}A_i = \mathbb{K}\langle x_1,\ldots,x_{i-1},z_1,\ldots,z_{\lambda(i-1)}\rangle\otimes_{\mathbb{K}\langle x_1,\ldots,x_{i-1}\rangle}\mathbb{K}\langle x_1,\ldots,x_i\rangle.$$

Its henselization is isomorphic to $\mathbb{K}\langle x_1,\ldots,x_i,z_1,\ldots,z_{\lambda(i-1)}\rangle$ by Lemma 2.3. This shows that $D_i^h$ is smooth over $\mathbb{K}\langle x_1,\ldots,x_i,z_1,\ldots,z_{\lambda(i-1)}\rangle$. Hence, by the Implicit Function Theorem as we did for $D_1^h$, $D_i^h$ is isomorphic to $\mathbb{K}\langle x_1,\ldots,x_i,z_1,\ldots,z_{\lambda(i)}\rangle$ for some new indeterminates $z_{\lambda(i-1)+1},\ldots,z_{\lambda(i)}$.

Finally the morphisms $C_i\longrightarrow D_i^h$ define the $y_k(x,z)$ satisfying $f(x,y(x,z))=0$. The power series $z_j(x)$ are defined by the morphisms $D_i^h\longrightarrow B_i$ and the fact that $C_i\longrightarrow B_i$ factors through $D_i^h$ yields $y(x)=y(x,z(x))$. $\qquad\square$

*Proof of Lemma* 2.3. Let $\psi: A_i\otimes_{A_{i-1}}B\longrightarrow B\langle x_i\rangle$ be the morphism defined by $\psi(\sum_j a_j\otimes b_j) = \sum_j a_j b_j$ with $a_j\in A_i$ and $b_j\in B$ for any $j$. The morphism $\psi$ is well defined since $A_i$ and $B$ are subrings of the ring $B\langle x_i\rangle$. The image of $\psi$ is the subring of $B\langle x_i\rangle$ generated by $A_i$ and $B$.

Let us prove that $\psi$ is injective: Let $\sum_j a_j\otimes b_j\in\mathrm{Ker}(\psi)$ with $a_j\in A_i$ and $b_j\in B$ for any $j$. This means that $\sum_j a_j b_j = 0$. Let us write $a_j = \sum_{l\in\mathbb{N}}a_{j,l}x_i^l$ where $a_{j,l}\in A_{i-1}$ for any $j$ and $l$. Thus we have

$$(2.1) \qquad\qquad \sum_j a_{j,l}b_j = 0$$

for any $l\in\mathbb{N}$, and this system of linear equations is equivalent to a finite system by noetherianity. The ring extension $A_{i-1}\longrightarrow B$ is flat since $A_{i-1}\longrightarrow\mathbb{K}[[x_1,\ldots,x_{i-1}]]$ and $B\longrightarrow\mathbb{K}[[x_1,\ldots,x_{i-1}]]$ are faithfully flat (they are completions of local noetherian rings; cf. [11, p. 46, and Theorem 8.14, p. 62]). Thus the solution vector $(b_j)_j$ of (2.1) is a linear combination with

coefficients in $B$ of solution vectors in $A_{i-1}$ (cf. [11, Theorem 7.6, p. 49]). Thus $(b_j)_j = \sum_k b'_k (a'_{j,k})_j$ where $b'_k \in B$ and, for any $k$, $(a'_{j,k})_j$ are vectors with entries in $A_{i-1}$ which are solutions of (2.1). This means that

$$\sum_j a_j \otimes b_j = \sum_{j,k} a_j \otimes b'_k a'_{j,k} = \sum_k \sum_j a_j a'_{j,k} \otimes b'_k = \sum_k (\sum_l (\sum_j a_{j,l} a'_{j,k}) x_i^l) \otimes b'_k = 0.$$

Thus $\mathrm{Ker}(\psi) = (0)$.

Obviously $\mathrm{Im}(\psi)$ contains $B[x_i]$ whose henselization is $B\langle x_i \rangle$ by Lemma 2.4, thus $\psi$ induces a surjective morphism between the henselization of $A_i \otimes_{A_{i-1}} B$ and $B\langle x_i \rangle$. This surjective morphism is also injective since $\psi$ is injective and $A_i \otimes_{A_{i-1}} B$ is a domain. (Indeed if $y \neq 0$ is in the henselization of $A_i \otimes_{A_{i-1}} B$, then $y$ is a root of a non-zero polynomial with coefficients in $A_i \otimes_{A_{i-1}} B$. Since $A_i \otimes_{A_{i-1}} B$ is a domain and $y \neq 0$ we may assume that this polynomial has a non-zero constant term denoted by $a$. If the image of $y$ in $B\langle x_i \rangle$ is zero, then $\psi(a) = 0$ which is a contradiction.)

On the other hand, $B\langle x_i \rangle$ is the henselization of $B[x_i]$ which is noetherian; thus $B\langle x_i \rangle$ is noetherian (cf. [8, Théorème 18.6.6]). This proves that the henselization of $A_i \otimes_{A_{i-1}} B$ is noetherian. Hence $A_i \otimes_{A_{i-1}} B$ is noetherian (cf. [8, Théorème 18.6.6]). □

*Proof of Lemma* 2.4. The elements of the henselization of a local ring $A$ are algebraic over $A$ by construction. Thus the henselization of $\frac{B[x_i]}{I}$ is a subring of $\frac{B\langle x_i \rangle}{I}$.

On the other hand let us prove first that $B\langle x_i \rangle$ is the henselization of $B[x_i]_{(x_1,\ldots,x_i)}$. If $y \in B\langle x_i \rangle$, then $y$ is a root of a polynomial $P(Y)$ with coefficients in $B[x_i]$. By the Artin Approximation Theorem (see [16, Theorem 11.3]), $y$ may be approximated by elements which are in the henselization of $B[x_i]$. Since $P(Y)$ has only a finite number of roots, this means that $y$ is in the henselization of $B[x_i]_{(x_1,\ldots,x_i)}$. Thus $B\langle x_i \rangle$ is the henselization of $B[x_i]_{(x_1,\ldots,x_i)}$. Now the morphism $B[x_i] \longrightarrow \frac{B[x_i]}{I}$ induces a morphism $B\langle x_i \rangle \longrightarrow \left( \frac{B[x_i]}{I}_{(x_1,\ldots,x_i)} \right)^h$ of $B[x_i]$-algebras by the universal property of the henselization. It is clear that the kernel of this morphism is generated by $I$ thus we get an injective morphism $\frac{B\langle x_i \rangle}{I} \longrightarrow \left( \frac{B[x_i]}{I}_{(x_1,\ldots,x_i)} \right)^h$ of $B[x_i]$-algebras. Since $\left( \frac{B[x_i]}{I}_{(x_1,\ldots,x_i)} \right)^h$ is a subring of $\frac{B\langle x_i \rangle}{I}$, this shows that the morphism $\frac{B\langle x_i \rangle}{I} \longrightarrow \left( \frac{B[x_i]}{I}_{(x_1,\ldots,x_i)} \right)^h$ is an isomorphism. □

## 3. Algebraic equisingularity of Zariski

Notation: Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Then we denote $x^i = (x_1, \ldots, x_i) \in \mathbb{C}^i$.

**3.1. Assumptions.** Let $V$ be an analytic hypersurface in a neighborhood of the origin in $\mathbb{C}^l \times \mathbb{C}^n$ and let $W = V \cap (\mathbb{C}^l \times \{0\})$. Suppose there are given complex pseudopolynomials[1]

$$F_i(t, x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t, x^{i-1}) x_i^{p_i - j}, \qquad i = 0, \ldots, n,$$

$t \in \mathbb{C}^l$, $x^i \in \mathbb{C}^i$, with complex analytic coefficients $a_{i-1,j}$, that satisfy

(1) $V = F_n^{-1}(0)$.
(2) $F_{i-1}(t, x^{i-1}) = 0$ if and only if $F_i(t, x^{i-1}, x_i) = 0$ considered as an equation on $x_i$ with $(t, x^{i-1})$ fixed, has fewer roots than for generic $(t, x^{i-1})$.
(3) $F_0 \equiv 1$.
(4) There are positive reals $\delta_k > 0$, $k = 1, \ldots, l$, and $\varepsilon_j > 0$, $j = 1, \ldots, n$, such that $F_i$ are defined on the polydiscs $U_i := \{|t_k| < \delta_k, |x_j| < \varepsilon_j, k = 1, \ldots, l, j = 1, \ldots, i\}$.
(5) All roots of $F_i(t, x^{i-1}, x_i) = 0$, for $(t, x^{i-1}) \in U_{i-1}$, lie inside the circle of radius $\varepsilon_i$.
(6) Either $F_i(t, 0) \equiv 0$ or $F_i \equiv 1$ (and in the latter case $F_k \equiv 1$ for all $k \leq i$).

We may take as $F_{i-1}$ the Weierstrass polynomial associated to the reduced discriminant of $F_i$ or a generalized discriminant (see the next section).

We shall denote $V_i = F_i^{-1}(0) \subset U_i$. For the parameter $t$ fixed we write $V_t := V \cap (\{t\} \times \mathbb{C}^n)$, $V_{i,t} := V_i \cap (\{t\} \times \mathbb{C}^i)$, and $U_{i,t} = U_i \cap (\{t\} \times \mathbb{C}^i)$. We identify $W$ and $U_0$.

**Theorem 3.1** ([20, Theorem 1], [21, Theorem 1]). *Under the above assumptions $V$ is topologically equisingular along $W$ with respect to the family of sections $V_t = V \cap (\{t\} \times \mathbb{C}^n)$. This means that for all $t \in W$ there is a homeomorphism $h_t : U_{n,0} \to U_{n,t}$ such that $h_t(V_0) = V_t$ and $h_t(0) = 0$.*

**3.2. Remarks on Varchenko's proof of Theorem 3.1.** As Varchenko states in Remark 1 of [20] a stronger result holds, the family $V_t$ is topologically trivial, in the sense that the homeomorphisms $h_t$ depend continuously on $t$.

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[1] A pseudopolynomial is a polynomial in $x_i$ with coefficients that are analytic in the other variables. The pseudopolynomials $F_i$ that we construct later are moreover distinguished polynomials in $x$, i.e. are of the form $x_i^p + \sum_{j=1}^{p} a_j(x^{i-1}) x_i^{p-j}$ where $a_j(0) = 0$ for all $j$. They depend analytically on $t$, which is considered as a parameter.

The details of the proof of Theorem 3.1 (with continuous dependence of $h_t$ on $t$) are published in [19].

The homeomorphisms $h_t$ are constructed in [19] inductively by lifting step by step the homeomorphisms

$$h_{i,t} : U_{i,0} \to U_{i,t},$$

so that $h_{i,t}(x^{i-1}, x_i) = (h_{i-1,t}(x^{i-1}), h_{i,t,i}(x^i))$, $h_{i,t}(V_{i,0}) = V_{i,t}$, $h_{i,t}(0) = 0$. If $h_{i-1,t}$ depends continuously on $t$, then the number of roots of $F_i(h_{i-1,t}(x^{i-1}), x_i) = 0$ is independent of $t$. Therefore, if $F_n = G_1 \cdots G_k$, then the number of roots of each $G_j(h_{n-1,t}(x^{n-1}), x_n) = 0$ is independent of $t$; see Lemma 2.2 of [19]. In particular $h_t$ preserves not only $V = F_n^{-1}(0)$ but also each of $G_j^{-1}(0)$. Thus [19] implies the following.

**Theorem 3.2.** *The homeomorphisms $h_t$ of Theorem 3.1 can be chosen continuous in $t$. If $F_n = G_1 \cdots G_k$, then for each $s = 1, \ldots, k$, $h_t(G_s^{-1}(0) \cap (\{0\} \times \mathbb{C}^n)) = G_s^{-1}(0) \cap (\{t\} \times \mathbb{C}^n)$.*

## 4. Mostowski's Theorem

In this section we show Theorem 1.1.

**4.1. Generalized discriminants.** Let $f(T) = T^p + \sum_{i=1}^p a_i T^{p-i} = \prod_{i=1}^p (T - T_i)$. Then the expressions

$$\sum_{r_1, \ldots, r_{j-1}} \prod_{k<l, k, l \neq r_1, \ldots, r_{j-1}} (T_k - T_l)^2$$

are symmetric in $T_1, \ldots, T_p$ and hence polynomials in $a = (a_1, \ldots, a_p)$. We denote these polynomials by $\Delta_j(a)$. Thus $\Delta_1$ is the standard discriminant and $f$ has exactly $p - j$ distinct roots if and only if $\Delta_1 = \cdots = \Delta_j = 0$ and $\Delta_{j+1} \neq 0$.

**4.2. Construction of a normal system of equations.** Let us consider a finite set of pseudopolynomials $g_1, \ldots, g_k \in \mathbb{C}\{x\}$:

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}) x_n^{r_s - j}.$$

The coefficients $a_{n-1,s,j}$ can be arranged in a row vector $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ where $p_n := \sum_s r_s$. Let $f_n$ be the product of the $g_s$'s. The generalized discriminants $\Delta_{n,i}$ of $f_n$ are polynomials in $a_{n-1}$. Let $j_n$ be a positive integer such that

(4.1) $$\Delta_{n,i}(a_{n-1}) \equiv 0 \qquad i < j_n,$$

and $\Delta_{n,j_n}(a_{n-1}) \not\equiv 0$. Then, after a linear change of coordinates $x^{n-1}$, we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1})(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}),$$

where $u_{n-1}(0) \neq 0$ and for all $j$, $a_{n-2,j}(0) = 0$. We denote

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}$$

and the vector of its coefficients $a_{n-2,j}$ by $a_{n-2} \in \mathbb{C}\{x^{n-2}\}^{p_{n-1}}$. Let $j_{n-1}$ be the positive integer such that the first $j_{n-1} - 1$ generalized discriminants $\Delta_{n-1,i}$ of $f_{n-1}$ are identically zero and $\Delta_{n-1,j_{n-1}}$ is not. Then again we define $f_{n-2}(x^{n-2})$ as the Weierstrass polynomial associated to $\Delta_{n-1,j_{n-1}}$.

We continue this construction and define a sequence of pseudopolynomials $f_i(x^i)$, $i = 1, \ldots, n-1$, such that

$$f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$$

is the Weierstrass polynomial associated to the first non-identically zero generalized discriminant $\Delta_{i+1,j_{i+1}}(a_i)$ of $f_{i+1}$, where we denote in general $a_i = (a_{i,1}, \ldots, a_{i,p_{i+1}})$,

$$(4.2) \quad \Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i)(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}), \quad i = 0, \ldots, n-1.$$

Thus the vector of functions $a_i$ satisfies

$$(4.3) \qquad \Delta_{i+1,k}(a_i) \equiv 0, \qquad k < j_{i+1}, \quad i = 0, \ldots, n-1.$$

This means in particular that

$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \quad \text{and} \quad \Delta_{1,j_1}(a_0) \equiv u_0,$$

where $u_0$ is a non-zero constant.

**4.3. Approximation by Nash functions.** Consider (4.2) and (4.3) as a system of polynomial equations on $a_i(x^i)$, $u_i(x^i)$. By construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables $z = (z_1, \ldots, z_s)$, an increasing function $\tau$, convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0, algebraic power series

$u_i(x^i, z) \in \mathbb{C}\langle x^i, z_1, \ldots, z_{\tau(i)}\rangle$ and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{C}\langle x^{(i)}, z_1, \ldots, z_{\tau(i)}\rangle^{p_i}$ such that the following hold:

$$z_1(x), \ldots, z_{\tau(i)}(x) \text{ depend only on } (x_1, \ldots, x_i),$$

$$a_i(x^i, z), \ u_i(x^i, z) \text{ are solutions of } (4.2), (4.3),$$

$$\text{and } a_i(x^i) = a_i(x^i, z(x^i)), \ u_i(x^i) = u_i(x^i, z(x^i)).$$

For $t \in \mathbb{C}$ we define

$$F_n(t,x) = \prod_s G_s(t,x), \quad G_s(t,x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}, tz(x^{n-1}))x_n^{r_s - j},$$

$$F_i(t,x) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}, tz(x^{i-1}))x_i^{p_i - j}, \quad i = 0, \ldots, n-1.$$

Finally we set $F_0 \equiv 1$. Because $u_i(0,0) = u_i(0, z(0)) \neq 0$, the family $F_i(t,x)$ satisfies the assumptions of Theorem 3.1 with $|t| < R$ for any $R < \infty$.

**Corollary 4.1.** *Let $(V,0) \subset (\mathbb{K}^n, 0)$ be an analytic germ defined by $g_1 = \cdots = g_k = 0$ with $g_s \in \mathbb{K}\{x\}$. Then there are algebraic power series $\hat{g}_s \in \mathbb{K}\langle x\rangle$ and a homeomorphism germ $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $h(g_s^{-1}(0)) = \hat{g}_s^{-1}(0)$ for $s = 1, \ldots, k$. In particular, $h(V)$ is the Nash set germ $\{\hat{g}_1 = \cdots = \hat{g}_k = 0\}$.*

*Proof.* For $\mathbb{K} = \mathbb{C}$ we set $\hat{g}_i(x) = G_i(0,x)$, and then the corollary follows from Theorem 3.2. The real case follows from the complex one, because if the pseudopolynomials $F_i$ of Subsection 3.1 have real coefficients, then the homeomorphisms $h_t$ constructed in [19] are conjugation invariant; cf. §6 of [19]. □

Now Theorem 1.1 follows from Corollary 4.1 and the following result.

**Theorem 4.2** ([2, Theorem 2]). *Let $(V,0) \subset (\mathbb{K}^n, 0)$ be a Nash set germ. Then there is a local Nash diffeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\sigma(V)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

## 5. Topological equivalence between analytic and algebraic function germs

In this section we show Theorem 1.2 and Theorem 1.3.

**5.1. A variant of Varchenko's method.** We replace the assumptions (2) and (3) of Subsection 3.1 by:

(2') There are $q_i \in \mathbb{N}$ such that $x_1^{q_i} F_{i-1}(t, x^{i-1}) = 0$ if and only if the equation $F_i(t, x^{i-1}, x_i) = 0$ has fewer roots than for generic $(t, x^{i-1})$.

(3') $F_1 \equiv 1$.

Then Varchenko's method gives the following result.

**Theorem 5.1.** *Under the above assumptions, $V$ is topologically equisingular along $W$ with respect to the family of sections $V_t = V \cap (\{t\} \times \mathbb{C}^n)$. Moreover all the sections $V \cap \{x_1 = const\}$ are also equisingular. This means that for all $t \in W$ there is a homeomorphism $h_t : U_{n,0} \to U_{n,t}$ such that $h_t(V_0) = V_t$, $h_t(0) = 0$, and $h_t$ preserves the levels of $x_1$*

$$(5.1) \qquad h_t(x_1, \ldots, x_n) = (x_1, \hat{h}_t(x_1, \ldots, x_n)).$$

Indeed, recall that the homeomorphisms $h_t$ are constructed inductively by lifting step by step the homeomorphisms $h_{i,t} : U_{i,0} \to U_{i,t}$, so that $h_{i,t}(x^{i-1}, x_i) = (h_{i-1,t}(x^{i-1}), h_{i,t,i}(x^i))$. At each stage such lifts $h_{i,t}$ exist and preserve the zero set of $F_i$ if $h_{i-1,t}$ depends continuously on $t$ and preserves the discriminant set of $F_i$; see [19, sections 2 and 3].

Because $F_1 \equiv 1$, by (2'), the discriminant set of $F_2$ is either empty or given by $x_1 = 0$. Therefore we may take $h_{1,t}(x_1) = x_1$. Then we show by induction on $i$ that each $h_{i,t}$ can be lifted so that the lift $h_{i+1,t}$ preserves the zero set of $F_{i+1}$ and the values of $x_1$. The former condition follows by inductive assumption; $h_{i,t}$ preserves the discriminant set of $F_{i+1}$. The latter condition is satisfied trivially since $h_{i+1,t}$ is a lift of $h_{i,t}$.

**5.2. Equisingularity of functions.** We apply Theorem 5.1 to study the equisingularity of analytic function germs as follows. Let $G(t, y) : (\mathbb{C}^l \times \mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ be analytic, $y = (y_1, \ldots, y_{n-1})$. We consider $G$ as an analytic family of analytic function germs $G_t : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ parametrized by $t \in W$, where $W$ is a neighborhood of the origin in $\mathbb{C}^l$. We associate to $G$ its graph

$$V = \{(t, x_1, x_2, \ldots, x_n); x_1 = G_t(x_2, \ldots, x_n)\},$$

thus fixing the following notation:

$$(5.2) \qquad x = (x_1, x_2, \ldots, x_n) = (x_1, y).$$

**Theorem 5.2.** *Suppose that $V$ and $W$ satisfy the assumptions of Theorem 5.1. Then the family of analytic function germs $G_t$ is topologically equisingular. This means that there is a family of local homeomorphisms $\sigma_t : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^{n-1}, 0)$ such that*

$$G_0 = G_t \circ \sigma_t.$$

*Proof.* It follows from (5.1) by setting $\sigma_t(y) = \hat{h}_t(G_0(y), y)$. Indeed, since $h_t$ preserves $V$ we have

$$h_t(G_0(y), y) = (G_t(\hat{h}_t(G_0(y), y)), \hat{h}_t(G_0(y), y)),$$

and since it preserves the levels of $x_1$,

$$G_0(y) = G_t(\hat{h}_t(G_0(y), y)). \qquad \square$$

**5.3. Construction of a normal system of equations for a finite family of function germs.** Let $g_m : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$, $m = 1, \ldots, p$, be a finite family of analytic function germs that we assume are not identically equal to zero. After a linear change of coordinates $(x_2, \ldots, x_n)$, we find that $\prod_{m=1}^p (x_1 - g_m(x_2, \ldots, x_n))$ is equivalent to a pseudopolynomial; that is, we may write

$$\prod_{m=1}^p (x_1 - g_m(x_2, \ldots, x_n)) = u_n(x)(x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j}(x^{n-1}) x_n^{p_n - j}),$$

where $u_n(0) \neq 0$ and $a_{n-1,j}(0) = 0$. We denote

$$f_n(x) = x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j}(x^{n-1}) x_n^{p_n - j}$$

so that

$$(5.3) \qquad u_n(x) f_n(x) = \prod_{m=1}^p (x_1 - \sum_{k=2}^n x_k b_{m,k}(x_2, \ldots, x_n))$$

with $g_m = \sum_{k=2}^n x_k b_{m,k}$. We denote by $b \in \mathbb{C}\{x\}^{p(n-1)}$ the vector of the coefficients $b_{m,k}$ and by $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ the one of the coefficients $a_{n-1,j}$.

The generalized discriminants $\Delta_{n,i}$ of $f_n$ are polynomials in $a_{n-1}$. Let $j_n$ be a positive integer such that

$$\Delta_{n,i}(a_{n-1}) \equiv 0, \qquad i < j_n,$$

and $\Delta_{n,j_n}(a_{n-1}) \not\equiv 0$. After a change of coordinates $(x_2, \ldots, x_{n-1})$ we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1}) x_1^{q_{n-1}} (x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2}) x_{n-1}^{p_{n-1} - j}),$$

where $u_{n-1}(0) \neq 0$ and $a_{n-2,j}(0) = 0$. We denote

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2}) x_{n-1}^{p_{n-1} - j}$$

and the vector of its coefficients $a_{n-2,j}$ by $a_{n-2} \in \mathbb{C}\{x^{n-2}\}^{p_{n-1}}$. Let $j_{n-1}$ be the positive integer such that the first $j_{n-1} - 1$ generalized discriminants $\Delta_{n-1,i}$ of $f_{n-1}$ are identically zero and $\Delta_{n-1,j_{n-1}}$ is not. Then again we divide $\Delta_{n-1,j_{n-1}}$ by the maximal power of $x_1$ and, after a change of coordinates $(x_2, \ldots, x_{n-2})$, denote the associated Weierstrass polynomial by $f_{n-2}(x^{n-2})$.

We continue this construction and define a sequence of pseudopolynomials $f_i(x^i)$, $i = 1, \ldots, n-1$, such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}) x_i^{p_i - j}$ is the Weierstrass polynomial associated to the first non-identically zero generalized

discriminant $\Delta_{i,j_i}(a_{i+1})$ of $f_{i+1}$, divided by the maximal power of $x_1$, where we denote in general $a_i = (a_{i,1}, \ldots, a_{i,p_i})$,

(5.4)
$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i)x_1^{q_i}(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}), \quad i = 0, \ldots, n-1.$$

Thus the vector of functions $a_i$ satisfies

(5.5)
$$\Delta_{i+1,k}(a_{i-1}) \equiv 0, \qquad k < j_{i+1}, \quad i = 0, \ldots, n-1.$$

These equations mean in particular that

(5.6)
$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \quad \text{and} \quad \Delta_{1,j_1}(a_0) \equiv u_0 x_1^{q_0},$$

where $u_0$ is a non-zero constant. Hence $f_1 \equiv 1$.

**5.4. Approximation by Nash functions.** Consider (5.3), (5.4), (5.5) as a system of polynomial equations on $a_i(x^i)$, $u_i(x^i)$, and $b(x)$. By construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables $z = (z_1, \ldots, z_s)$, an increasing function $\tau$, convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0, algebraic power series $u_i(x^i, z) \in \mathbb{C}\langle x^i, z_1, \ldots, z_{\tau(i)}\rangle$, and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{C}\langle x^{(i)}, z_1, \ldots, z_{\tau(i)}\rangle^{p_i}$, $b(x, z) \in \mathbb{C}\langle x, z\rangle^{n-1}$, such that the following hold:

$z_1(x), \ldots, z_{\tau(i)}(x)$ depend only on $(x_1, \ldots, x_i)$,

$a_i(x^i, z), u_i(x^i, z), b(x, z)$ are solutions of (5.3), (5.4), (5.5),

and $a_i(x^i) = a_i(x^i, z(x^i)), u_i(x^i) = u_i(x^i, z(x^i)), b(x) = b(x, z(x))$.

For $t \in \mathbb{C}$ we define

$$F_i(t, x) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}, tz(x^{i-1}))x_i^{p_i-j}.$$

In particular, by (5.6), $F_1 \equiv 1$. Since

$$u_n(x, tz(x))F_n(t, x) = \prod_{m=1}^{p} (x_1 - \sum_{k=2}^{n} x_k b_{m,k}(x, tz(x))),$$

by the Implicit Function Theorem there are algebraic power series

$$G_m \in \mathbb{C}\langle t, y_1 \ldots, y_{n-1}\rangle$$

such that

$$F_n^{-1}(0) = \bigcup_m \{(t, x); x_1 = G_m(t, x_2, \ldots, x_n)\}$$

as germs at the origin. Then $g_m(y) = G_m(1, y)$ and $G_m(0, y) \in \mathbb{C}\langle y\rangle$. We denote $\hat{g}_m(y) = G_m(0, y)$.

Because $u_i(0,0) = u_i(0, z(0)) \neq 0$, the family $F_i(t,x)$ satisfies the assumptions of Theorem 5.1 with $|t| < R$ for arbitrary $R < \infty$. By Theorem 5.1 there is a continuous family of homeomorphism germs $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $h_t(x) = (x_1, \hat{h}_t(x_1, x_2, \ldots, x_n))$, such that

$$h_t(g_m(y), y) = (G_m(t, \hat{h}_t(g_m(y), y)), \hat{h}_t(g_m(y), y)).$$

Fix one $m$, for instance $m = 1$, and set

$$\sigma_t(y) = \hat{h}_t(g_1(y), y)$$

as in the proof of Theorem 5.2 (we use here the notation (5.2)). Then $g_1(y) = G_1(t, \sigma_t(y))$ and in particular

(5.7) $$g_1(y) = \hat{g}_1(\sigma_0(y)).$$

It is not true in general that $g_m(y) = \hat{g}_m(\sigma_0(y))$ since the homeomorphism $\sigma_t$ is defined by restricting $h_t$ to the graph of $G_1$. If we define

$$\sigma_{m,t}(y) = \hat{h}_t(g_m(y), y),$$

then we have

$$g_m(y) = \hat{g}_m(\sigma_{m,0}(y)).$$

Both homeomorphisms coincide on $X_m = \{y \in (\mathbb{C}^{n-1}, 0); (g_m - g_1)(y) = 0\}$. Therefore if we define $\hat{X}_m = \{y \in (\mathbb{C}^{n-1}, 0); (\hat{g}_m - \hat{g}_1)(y) = 0\}$, then

(5.8) $$\sigma_0(\hat{X}_m) = X_m.$$

Therefore we have the following result.

**Proposition 5.3.** *Let $(V_i, 0) \subset (\mathbb{K}^n, 0)$ be a finite family of analytic set germs and let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there are Nash set germs $(\hat{V}_i, 0) \subset (\mathbb{K}^n, 0)$, an algebraic power series $\hat{g} \in \mathbb{K}\langle x \rangle$, and a homeomorphism germ $\hat{\sigma} : (\mathbb{K}^{n-1}, 0) \to (\mathbb{K}^{n-1}, 0)$ such that $\sigma(\hat{V}_i) = V_i$ and $g \circ \hat{\sigma} = \hat{g}$.*

*Proof.* Let $\mathbb{K} = \mathbb{C}$. Choose a finite family $g_m : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$, $m = 1, \ldots, p$, of analytic function such that $g_1 = g$ and for every $i$, the ideal of $V_i$ is generated by some of the differences $g_m - g_1$. We apply to the family $g_m$ the procedure of Subsections 5.3 and 5.4 and set $\hat{\sigma} = \sigma_0$. The claim now follows from (5.7) and (5.8).

The real case follows from the complex one because if the pseudopolynomials $F_i$ of Subsection 3.1 have real coefficients, then the homeomorphisms $h_t$ constructed in [19] are conjugation invariant; cf. §6 of [19].                $\square$

**5.5. Proof of Theorem 1.2 and Theorem 1.3.** It suffices to show Theorem 1.3. It will follow from Proposition 5.3 and the next two results.

**Theorem 5.4.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a finite family of Nash function germs. Then there are a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and analytic (even Nash) units $u_i : (\mathbb{K}^n, 0) \to \mathbb{K}$, $u_i(0) \neq 0$, such that for all $i$, $u_i(x) f_i(h(x))$ are germs of polynomials.*

*Proof.* For $\mathbb{K} = \mathbb{C}$ Theorem 5.4 follows from Theorem 5 of [2]. Indeed, the implication "(i) $\implies$ (ii)" of this theorem gives:

*If $(V, 0) \subset (\mathbb{K}^n, 0)$ is a Nash set germ, then there is a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that for any analytic irreducible component $W$ of $(V, 0)$ the ideal of functions vanishing on $h(W)$ is generated by polynomials.*

Now, if $\mathbb{K} = \mathbb{C}$, it suffices to apply the above result to $(V, 0)$ defined as the zero set of the product of $f_i$'s.

If $\mathbb{K} = \mathbb{R}$ such a set theoretic statement is not sufficient, but in this case Theorem 5.4 follows from the proof of Theorem 5 of [2]. We sketch this argument below.

First we consider $\mathbb{K} = \mathbb{C}$. Choose representatives $f_i : U \to \mathbb{C}$ of the germs $f_i$, $i = 1, \ldots, m$, and let $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$. By the Artin-Mazur Theorem, [1, Theorem 8.4.4], [2, Proposition 2], there is an algebraic set $X \subset \mathbb{C}^n \times \mathbb{C}^N$ of dimension $n$, a polynomial map $\Phi : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{C}^m$, and a Nash map $s : U \to X$ such that $f = \Phi \circ s$, $s : U \to s(U)$ is a Nash diffeomorphism and $s(U) \cap Sing(X) = \emptyset$. ($X$ is the normalization of the Zariski closure of the graph of $f$.) We may assume that $p = s(0)$ is the origin in $\mathbb{C}^n \times \mathbb{C}^N$.

Let $\pi : X \to \mathbb{C}^n$ be a generic linear projection. Then the germ $h$ of $(\pi \circ s)^{-1}$ satisfies the claim. Indeed, denote $X_i = X \cap \Phi_i^{-1}(0)$. Then for each $i = 1, \ldots, m$, $Z_i = \pi(X_i)$ is an algebraic subset of $\mathbb{C}^n$ and, moreover, $\pi$ induces a local isomorphism $(X_i, 0) \to (Z_i, 0)$. We fix a reduced polynomial $P_i$ that defines $Z_i$. Then $f_i \circ h$, as a germ at the origin, vanishes exactly on $Z_i$ and hence equals a power of $P_i$ times an analytic unit.

If $\mathbb{K} = \mathbb{R}$, then we apply the complex case to the complexifications of the $f_i$'s keeping the construction conjugation invariant. In particular the linear projection can be chosen real (that is conjugation invariant). Indeed, this projection is from the Zariski open dense subset $\mathcal{U}$ of the set of linear projections $L(\mathbb{C}^n \times \mathbb{C}^N, \mathbb{C}^n)$. Then $\mathcal{U} \cap L(\mathbb{R}^n \times \mathbb{R}^N, \mathbb{R}^n)$ is non-empty. (A complex polynomial of $M$ variables that vanishes on $\mathbb{R}^M$ is identically equal to zero.) This ends the proof. $\square$

**Theorem 5.5.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ and let $u : (\mathbb{K}^n, 0) \to \mathbb{K}$ be an analytic unit, $u(0) \neq 0$ ($u(0) > 0$ if $\mathbb{K} = \mathbb{R}$). Let $(V_i, 0) \subset (\mathbb{K}^n, 0)$ be a finite family of analytic set germs. Then*

there is a homeomorphism germ $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $(\sigma(V_i), 0) = (V_i, 0)$ for each $i$ and $uf = f \circ \sigma$.

*Proof.* If $\mathbb{K} = \mathbb{C}$ we suppose additionally that the segment that joins $u(0)$ and 1 does not contain 0. The general case can be reduced to this one.

Fix a small neighborhood $U$ of the origin in $\mathbb{K}^n$ so that the representatives $V_i \subset U$, $f : U \to \mathbb{K}$, and $u : U \to \mathbb{K}$ are well-defined. In the proof we often shrink $U$ when necessary. Let $I$ denote a small neighborhood of $[0, 1]$ in $\mathbb{R}$. We construct a Thom stratification of the deformation $\Psi(x, t) = (F(x, t), t) : U \times I \to \mathbb{K} \times I$, where

$$F(x, t) = f(x)(1 - t + tu(x)) = f(x)(1 + t(u(x) - 1)), \quad (x, t) \in U \times I,$$

which connects $f(x) = F(x, 0)$ and $u(x)f(x) = F(x, 1)$. Then we conclude by the second Thom-Mather Isotopy Lemma. For the Thom stratification we refer the reader to [4, Chapter 1], and for the Thom-Mather Isotopy Lemmas to [4, Chapter 2].

Fix a Thom stratification $\mathcal{S}' = \{S'_j\}$ of $f : U \to \mathbb{K}$ such that each $V_i$ is a union of strata. That means that $\mathcal{S}'$ is a Whitney stratification of $U$, compatible with $f^{-1}(0)$ and $f^{-1}(\mathbb{K} \setminus \{0\})$, that satisfies Thom's $a_f$ condition. (It is well-known that such a stratification exists; the existence of $a_f$ regular stratifications was first proved in the complex analytic case by Hironaka in [5], using resolution of singularities, under the assumption "sans éclatement", which is always satisfied for functions. In the real subanalytic case it was first shown in [9].)

First we show that $\mathcal{S} = \{S_j = S'_j \times I\}$ as a stratification of $U \times I$ satisfies $a_F$ condition

($a_F$) for every stratum $S \subset F^{-1}(\mathbb{K} \setminus \{0\})$ and every sequence of points $p_i = (x_i, t_i) \in S$ that converges to a point $p_0 = (x_0, t_0) \in S_0 \subset F^{-1}(0)$, such that $\ker d_{p_i} F|_{T_{p_i} S} \to T$, we have $T \supset T_{p_0} S_0$.

By the curve selection lemma it suffices to check this condition on every real analytic curve $p(s) = (x(s), t(s)) : [0, \varepsilon) \to S \cup S_0$, $p(0) \in S_0$ and $p(s) \in S$ for $s > 0$. Since $\mathcal{S}'$ satisfies $a_f$, the condition $a_F$ for $\mathcal{S}$ follows from the following lemma.

**Lemma 5.6.** *Let $S = S' \times I \subset F^{-1}(\mathbb{K} \setminus \{0\})$ and $S_0 \subset F^{-1}(0)$ be two strata of $\mathcal{S}$ and let $p(s) = (x(s), t(s)) : [0, \varepsilon) \to U \times I$ be a real analytic curve such that $p_0 = p(0) \in S_0$ and $p(s) \in S$ for $s > 0$. Then for $s > 0$ and small, $\mathrm{grad}_x F|_S(p(s)) \neq 0$ and*

$$(5.9) \qquad \lim_{s \to 0} \frac{\mathrm{grad}\, F|_S(p(s))}{\| \mathrm{grad}\, F|_S(p(s)) \|} = \lim_{s \to 0} \frac{(\mathrm{grad}\, f|_{S'}(x(s)), 0)}{\| \mathrm{grad}\, f|_{S'}(x(s)) \|}.$$

*Proof.* By assumption $f(x(s)) = \sum_{i=i_0}^{\infty} a_i s^i$, with $i_0 > 0$ and $a_{i_0} \neq 0$. By differentiating we obtain

$$|\frac{df}{ds}| = |\langle \operatorname{grad} f|_{S'}, x'(s)\rangle| \leq \|\operatorname{grad} f|_{S'}\|\|x'(s)\|.$$

Hence there exists $C > 0$ such that for small $s > 0$,

(5.10) $$|f(x(s))| \leq sC\|\operatorname{grad} f|_{S'}\|.$$

Moreover

(5.11)
$$\operatorname{grad} F|_S(p(s)) = (\operatorname{grad} f|_{S'}(x(s)), 0)(1 + t(s)(u(x(s)) - 1))$$
$$+ f(x(s))(t(s) \operatorname{grad} u|_{S'}(x(s)), u(x) - 1).$$

Now (5.9) follows easily from (5.10) and (5.11).  $\square$

Finally $\mathcal{S}$ as a stratification of $U \times I$ together with $((\mathbb{K} \setminus \{0\}) \times I, \{0\} \times I)$ as a stratification of $\mathbb{K} \times I$ is a Thom stratification of $\Psi$. Indeed, $\mathcal{S}$ is a Whitney stratification as the product of a Whitney stratification of $U$ times $I$. Then for any pair of strata $S = S' \times I \subset F^{-1}(\mathbb{K} \setminus \{0\})$ and $S_0 = S_0' \times I \subset F^{-1}(0)$ it satisfies $a_F$ and hence also $a_\Psi$ condition. Therefore Theorem 5.5 follows from the second Thom-Mather Isotopy Lemma [4, Chapter 2 (5.8)].  $\square$

Now we may conclude the proof of Theorem 1.3. By Proposition 5.3 we may assume that $g$ is a Nash function germ and the $V_i$'s are Nash set germs. Moreover by Theorem 5.4, after composing with the Nash diffeomorphism $h$, we may assume that $g$ equals a polynomial times an analytic unit and that the ideal of analytic function germs defining each $V_i$ is generated by polynomials. In particular each $V_i$ is algebraic. Finally we apply Theorem 5.5 to show that, after composing with a homeomorphism preserving each $V_i$, $g$ becomes a polynomial.

## 6. Examples

**Example 6.1.** We give here an example showing that the $C^1$ analog of Theorems 1.1 or 1.2 is false in the real case (this example is well-known; see [2] for example). The germ $(V, 0) \subset (\mathbb{R}^3, 0)$, defined by the vanishing of

$$f(t, x, y) = xy(y - x)(y - (3 + t)x)(y - \gamma(t)x)$$

where $\gamma(t) \in \mathbb{R}\{t\}$ is transcendental and $\gamma(0) = 4$, is not $C^1$-diffeomorphic to the germ of an algebraic set as follows from the argument of Whitney; cf. Section 14 of [22]. Indeed $V$ is the union of five smooth surfaces intersecting along the $t$-axis and its tangent cone at the point $(t, 0, 0)$ is the union of five planes intersecting along a line. The cross-ratio of the first four planes is $3 + t$, and the cross-ratio of the first three and the last plane is $\gamma(t)$. Since the

cross-ratio is preserved by linear maps, these two cross-ratios are preserved by $C^1$-diffeomorphisms. But these two cross-ratios are algebraically independent; thus the image of $V$ under a $C^1$-diffeomorphism cannot be algebraic.

**Example 6.2.** The previous example also shows that the $C^1$ analogs of Theorem 1.1 or 1.2 are false in the complex case. Define $V$ in a neighborhood of 0 in $\mathbb{C}^3$ by the vanishing of the polynomial of Example 6.1. Modifying Whitney's argument (cf. [22, pp. 240, 241]) we will show that the germ of $V$ at 0 is not $C^1$-equivalent to any Nash germ in $\mathbb{C}^3$.

For any $(t,0,0) \in \mathbb{C}^3$ with $|t|$ small, the tangent cone to $V$ at $(t,0,0)$ is the union of five two-dimensional $\mathbb{C}$-linear spaces $L_{1,t},\dots,L_{5,t}$, where $L_{j,t}$ corresponds to the $j$'th factor of $f$. Suppose that there is a $C^1$-diffeomorphism $\Phi : (\mathbb{C}^3,0) \to (\mathbb{C}^3,0)$ such that $\Phi(V)$ is a germ of a Nash set in $\mathbb{C}^3$. Then the tangent cone to $\Phi(V)$ at $\Phi(t,0,0)$ is the union of $d_{(t,0,0)}\Phi(L_{j,t})$, for $j = 1,\dots,$ 5. In particular, every $d_{(t,0,0)}\Phi(L_{j,t})$ is a $\mathbb{C}$-linear subspace of $\mathbb{C}^3$ of dimension 2.

Let us check that for every $t \in \mathbb{R}$ with $|t|$ small and for every pairwise distinct $k_1,\dots,k_4 \in \{1,\dots,5\}$, the cross-ratio of $L_{k_1,t},\dots,L_{k_4,t}$ is equal to the cross-ratio of $d_{(t,0,0)}\Phi(L_{k_1,t}),\dots,d_{(t,0,0)}\Phi(L_{k_4,t})$. By a real line in $\mathbb{C}^3$ we mean a set of the form $\{a + tb : t \in \mathbb{R}\}$ where $a,b \in \mathbb{C}^3$. For $t \in \mathbb{R}$, $L_{k_1,t},\dots,L_{k_4,t}$ are defined by real equations so there is a real line $l_t \subset \mathbb{C}^3$ intersecting $L_{k_1,t} \cup \dots \cup L_{k_4,t}$ at exactly four points, say $a_{1,t},\dots,a_{4,t}$. Then the cross-ratio of $L_{k_1,t},\dots,L_{k_4,t}$ equals the cross-ratio of $a_{1,t},\dots,a_{4,t}$. Moreover, $d_{(t,0,0)}\Phi(l_t)$ is also a real line and it intersects $d_{(t,0,0)}\Phi(L_{k_1,t}) \cup \dots \cup d_{(t,0,0)}\Phi(L_{k_4,t})$ at $d_{(t,0,0)}\Phi(a_{1,t}),\dots,d_{(t,0,0)}\Phi(a_{4,t})$. The cross-ratio of the last four points equals that of $a_{1,t},\dots,a_{4,t}$ because $d_{(t,0,0)}\Phi$ is $\mathbb{R}$-linear and $a_{1,t},\dots,a_{4,t} \in l_t$. Since the cross-ratio of $d_{(t,0,0)}\Phi(a_{1,t}),\dots,d_{(t,0,0)}\Phi(a_{4,t})$ equals the cross-ratio of $d_{(t,0,0)}\Phi(L_{k_1,t}),\dots,d_{(t,0,0)}\Phi(L_{k_4,t})$, we obtain our claim.

Now observe that the complex $t$-axis is the singular locus of $V$ and its image $S$ by $\Phi$ is the singular locus of $\Phi(V)$. Clearly, $S$ is a smooth complex Nash curve. Moreover, the cross-ratios $h_1, h_2$ of $d_{(t,0,0)}\Phi(L_{1,t}),\dots,d_{(t,0,0)}\Phi(L_{4,t})$ and $d_{(t,0,0)}\Phi(L_{1,t}), d_{(t,0,0)}\Phi(L_{2,t}), d_{(t,0,0)}\Phi(L_{3,t}), d_{(t,0,0)}\Phi(L_{5,t})$, respectively, depend algebraically on $s = \Phi(t,0,0) \in S$ (cf. [22, p. 241]); i.e. $h_1, h_2 : S \to \mathbb{C}$ are complex Nash functions. On the other hand, the cross-ratios of $L_{1,t},\dots,L_{4,t}$ and of $L_{1,t}, L_{2,t}, L_{3,t}, L_{5,t}$ equal $3 + t$ and $\gamma(t)$, respectively.

The last two paragraphs imply that for $t \in \mathbb{R}$ with $|t|$ small, we have

$$h_1(\Phi(t,0,0)) = 3 + t \quad \text{and} \quad h_2(\Phi(t,0,0)) = \gamma(t).$$

Since $S$ is a smooth complex Nash curve, we may assume that $h_1, h_2$ are defined in some neighborhood of $0 \in \mathbb{C}$ and that $\Psi(t) = \Phi(t,0,0)$ is a map into $\mathbb{C}$. We have $\Psi(t) = \Psi_1(t) + i\Psi_2(t)$ where $\Psi_1, \Psi_2$ are real valued continuous

functions and $h_1(s) = u_1(s) + iv_1(s)$, where $u_1, v_1$ are real Nash functions, and $u_1(\Psi_1(t), \Psi_2(t)) = 3 + t$, and $v_1(\Psi_1(t), \Psi_2(t)) = 0$ for $t \in \mathbb{R}$ with $|t|$ small. Since $u_1, v_1$ satisfy the Cauchy-Riemann equations and $h_1$ is not constant, neither of $u_1, v_1$ is constant. Consequently, $\Psi_1|_{\mathbb{R}}$, $\Psi_2|_{\mathbb{R}}$ are semi-algebraic functions, which contradicts the fact that $h_2(\Psi(t)) = \gamma(t)$ for real $t$.

**Example 6.3.** Theorem 1.2 cannot be extended to many functions or to maps to $\mathbb{K}^m$, $m > 1$. For example the one variable analytic germs $x$ and $e^x - 1$ cannot be made polynomial (or Nash) simultaneously by composing with the same homeomorphism.

**Example 6.4.** The key point in the previous examples is the fact that two one variable functions which are algebraically independent remain algebraically independent after composition with a homeomorphism. Theorem 1.2 also cannot be extended to many functions, even if we assume them to be algebraically dependent. For instance, one variable Nash germs $x$ and $y(x) = \sqrt{\varphi(x)} - 2$, with $\varphi(x) = (x-1)(x+2)(x-2)$, cannot be made polynomial simultaneously since the cubic $y^2 = \varphi(x)$ is not rational.

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