ON RANK THEOREMS AND THE NASH POINTS OF SUBANALYTIC SETS

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Abstract. We prove a generalization of Gabrielov's rank theorem for families of rings of power series which we call W-temperate. Examples include the family of complex analytic functions and of Eisenstein series. Then the rank theorem for Eisenstein series allows us to give new proofs of the following two results of W. Pawłucki:
I) The non regular locus of a complex or real analytic map is an analytic set.
II) The set of semianalytic or Nash points of a subanalytic set X is a subanalytic set, whose complement has codimension two in X.

> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
>
> Sir Michael Atiyah, (Collected works. Vol. 6. Oxford Science Publications, 2004).

1. Introduction

This article contains two sets of results concerning rank Theorems in commutative algebra and their application to analytic and subanalytic geometry.

We start by proving a rank Theorem for general families of rings which we call *W-temperate*, see Theorem 1.1, generalizing the classical Gabrielov's rank Theorem [Ga73, BCR21] (see the latter reference for a historical overview on the Theorem and its importance). These are families of Weierstrass rings $(\mathcal{K}\{\{x_1,\dots,x_r\}\})_{r\in\mathbb{N}}$, that is, families of rings of power series satisfying the Weierstrass division theorem, see Definition 2.1, where $\mathcal{K}$ is any uncountable algebraically closed field of characteristic zero, which satisfies three axioms: closure under local blowing-down, closure under restriction to generically hyperplane sections and temperateness, a closure under evaluation by algebraic elements type condition, see Definition 2.2. Examples include the family of germs of complex-analytic functions, algebraic power series, and Eisenstein power series; the latter allow us to obtain rank Theorems for families of morphisms. In particular, we obtain a new proof of Gabrielov's rank Theorem, which greatly simplifies and shortens our previous work [BCR21].

As an application of the rank Theorem for W-temperate families, we provide new proofs of two fundamental results of analytic and subanalytic geometry due to Pawłucki [Pa90, Pa92]: I) the non-regular (in the sense of Gabrielov) locus of a complex or real-analytic map $\Phi : M \to N$ is a proper analytic subset of M, see

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Theorem 1.3 and II) the set of semianalytic or Nash points of a subanalytic set X is a subanalytic set, whose complement has dimension has dimension $\leqslant \dim(X)-2$, see Theorem 1.4. In spite of being considered as fundamental results of subanalytic geometry, the original proofs of these results are considered to be very hard, as noted by Łojasiewicz: “Sans doute, parmi les faits établis en géométrie sous-analytique le théorème de Pawłucki [result II] est le plus difficile à prouver (la démonstration compte environ soixante dix pages!)", [Lo93, Page 1591].

Our first set of results concerns rank Theorems. Let $\mathcal{K}$ be an algebraically closed field. We consider families of rings of power series $(\mathcal{K}\{\{x_1,\ldots,x_r\}\})_{r\in\mathbb{N}}$ which we call Weierstrass temperate, see Definitions 2.1 and 2.2; note that the completion of $\mathcal{K}\{\{x_1,\ldots,x_r\}\}$ is $\mathcal{K}[\![x_1,\ldots,x_r]\!]$, see Proposition 2.8 i). Given a ring homeomorphism:

$$\varphi:\mathcal{K}\{\{\mathbf{x}\}\}\longrightarrow\mathcal{K}\{\{\mathbf{u}\}\}$$

where $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{u}=(u_1,\ldots,u_m)$, we say that φ is a morphism of W-temperate power series if $\varphi(f)=f(\varphi(\mathbf{x}))$ for every $f\in\mathcal{K}\{\{\mathbf{x}\}\}$, see Definition 2.3. We denote by $\widehat{\varphi}$ its extension to the ring of formal power series. We define:

$$\begin{aligned}
&\text{the Generic rank:} && \mathrm{r}(\varphi):=\operatorname{rank}_{\operatorname{Frac}(\mathcal{K}\{\{\mathbf{u}\}\})}(\operatorname{Jac}(\varphi)),\\
&\text{the Formal rank:} && \mathrm{r}^{\mathcal{F}}(\varphi):=\dim\left(\frac{\mathcal{K}[\![\mathbf{x}]\!]}{\operatorname{Ker}(\widehat{\varphi})}\right),\\
&\text{and the temperate rank:} && \mathrm{r}^{\mathcal{T}}(\varphi):=\dim\left(\frac{\mathcal{K}\{\{\mathbf{x}\}\}}{\operatorname{Ker}(\varphi)}\right),
\end{aligned}\tag{1}$$

of φ, where $\operatorname{Jac}(\varphi)$ stands for the matrix $[\partial_{u_i}\varphi(x_j)]_{i,j}$. Our first main result is:

Theorem 1.1 (W-temperate rank Theorem)**.** *Let $\varphi:\mathcal{K}\{\{\mathbf{x}\}\}\longrightarrow\mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of rings of W-temperate power series. Then*

$$\mathrm{r}(\varphi)=\mathrm{r}^{\mathcal{F}}(\varphi)\Longrightarrow\mathrm{r}(\varphi)=\mathrm{r}^{\mathcal{F}}(\varphi)=\mathrm{r}^{\mathcal{T}}(\varphi).$$

This result generalizes the original rank Theorem of Gabrielov [Ga73], which concerns the case that $\mathcal{K}\{\{\mathbf{x}\}\}$ stands for the family of complex analytic function germs. As noted before, we rely on [BCR21] for a presentation of the importance and consequences of the Theorem to local analytic geometry and commutative algebra. In spite of Gabrielov's rank Theorem being considered a fundamental result in local analytic geometry, its original proof is considered to be very difficult, cf. [Iz89, Page 1]. Recently, we have provided an alternative proof of Gabrielov's rank Theorem [BCR21], by developing geometric-formal techniques inspired by works of Gabrielov [Ga73] and Tougeron [To90]. One of the difficulties involved in the proof is the intricate interplay between algebraic geometry and complex analysis. Our new result simplifies the proof by addressing this difficulty. Indeed, the proof of Theorem 1.1 follows from algebraic geometry methods; complex analysis is only used in order to show that complex analytic functions form a W-temperate family, see §§ 3.2. As a mater of fact, we systematically generalize the arguments introduced in [BCR21] to their most general context, which demand us to introduce new commutative algebra arguments. It seems likely that the discussion of rank Theorems for non W-temperate families will demand a complete different strategy. In order to motivate this discussion, we provide a family of local rings of interest to function theory and tame geometry (that is, families of quasianalytic Denjoy-Carleman functions and families of C^∞-definable functions over an o-minimal and polynomially bounded structure) where the rank Theorem does not hold, see §§1.1.

Remark 1.2. Theorem 1.1 can be seen a "dual" of the Artin approximation Theorem. More precisely, let φ be such that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$. Then,

$$\forall F(\mathbf{x}) \in \mathcal{K}[\![\mathbf{x}]\!], \text{ such that } \ F(\varphi(\mathbf{x})) = 0,$$

$$\forall c \in \mathbb{N}, \exists F_c(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}, F_c(\varphi(\mathbf{x})) = 0 \text{ and } F(\mathbf{x}) - F_c(\mathbf{x}) \in (\mathbf{x})^c.$$

where $\varphi(\mathbf{x}) = (\varphi(x_1), \ldots, \varphi(x_n))$. Indeed, the ideals $\mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\widehat{\varphi})$ are prime ideals of $\mathcal{K}\{\{\mathbf{x}\}\}$ and $\mathcal{K}[\![\mathbf{x}]\!]$ respectively, and the equality $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{T}}(\varphi)$ is equivalent to the equality of the heights of these two ideals. Since $\mathcal{K}\{\{\mathbf{x}\}\}$ is Noetherian (see Proposition 2.8 i)), the height of $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ equals the height of $\mathrm{Ker}(\varphi)$. Now, by Artin Approximation Theorem, see Corollary 2.11, $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is again a prime ideal, so the equality $\mathrm{r}^{\mathcal{F}}(\varphi) = \mathrm{r}^{\mathcal{T}}(\varphi)$ is equivalent to the equality $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!] = \mathrm{Ker}(\widehat{\varphi})$. It is well known that, since $\mathcal{K}\{\{\mathbf{x}\}\}$ is Noetherian, $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is the closure of $\mathrm{Ker}(\varphi)$ in $\mathcal{K}[\![\mathbf{x}]\!]$ for the $(\mathbf{x})$-adic topology, and we conclude easily.

Our second set of results concerns two fundamental results of analytic and subanalytic geometry. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, and consider a $\mathbb{K}$-analytic map $\Phi : M \to N$ between $\mathbb{K}$-analytic manifolds M and N. Given $\mathfrak{a} \in M$, we denote by $\Phi_{\mathfrak{a}}$ the germ of the morphism at a point $\mathfrak{a} \in M$, and by $\Phi_{\mathfrak{a}}^* : \mathcal{O}_{\Phi(\mathfrak{a})} \to \mathcal{O}_{\mathfrak{a}}$ the associated morphism of local rings, where $\mathcal{O}_{\mathfrak{a}}$ stands for the ring of analytic function germs at $\mathfrak{a}$. For each $\mathfrak{a} \in M$, we set $\mathrm{r}_{\mathfrak{a}}(\Phi) := \mathrm{r}(\Phi_{\mathfrak{a}}^*)$ and $\mathrm{r}_{\mathfrak{a}}^{\mathcal{F}}(\Phi) := \mathrm{r}^{\mathcal{F}}(\Phi_{\mathfrak{a}}^*)$. Consider:

$$\mathcal{R}(\Phi, M) = \{\mathfrak{a} \in M;\, \mathrm{r}_{\mathfrak{a}}(\Phi) = \mathrm{r}_{\mathfrak{a}}^{\mathcal{F}}(\Phi)\},$$

which is called the set of *regular* (in the sense of Gabrielov) points of Φ. By combining Theorem 1.1 applied to Eisenstein power series, see §3.3, with the uniformization Theorem, see e.g. [BM88, Th 0.1], we prove the following result:

Theorem 1.3 (Pawłucki Theorem I, [Pa92])**.** *Let $\Phi : M \longmapsto N$ be an analytic map between connected manifolds. Then $M \setminus \mathcal{R}(\Phi, M)$ is a proper analytic subset of M.*

We now specialize our presentation to $\mathbb{K} = \mathbb{R}$, and we refer to §§5.2 and §§5.6 for all the details of the following discussion. Let $X \subset M$ be a subanalytic set. Given a point $\mathfrak{a} \in M$, we denote by $X_{\mathfrak{a}}$ the germ set of X at $\mathfrak{a}$. We say that an equidimensional subanalytic set X is a *Nash set* at $\mathfrak{a} \in M$ (which might not belong to X) if there exists a germ $Y_{\mathfrak{a}}$ of semi-analytic set at $\mathfrak{a}$ such that $X_{\mathfrak{a}} \subset Y_{\mathfrak{a}}$ and $\dim(X_{\mathfrak{a}}) = \dim(Y_{\mathfrak{a}})$. More generally, a subanalytic set $X \subset M$ of dimension d is Nash at a point $\mathfrak{a} \in M$, if X is a union of equidimensional Nash sets $\Sigma^{(k)}$, where $k = 0, \ldots, d$. We consider the sets:

$$\mathcal{N}(X) := \{\mathfrak{a} \in M \mid X_{\mathfrak{a}} \text{ is the germ of a Nash set}\}$$
$$\mathcal{SA}(X) := \{\mathfrak{a} \in M \mid X_{\mathfrak{a}} \text{ is the germ of a semianalytic set}\}.$$

It is trivially true that $M \setminus \overline{X} \subset \mathcal{SA}(X) \subset \mathcal{N}(X)$. But in general, $\mathcal{SA}(X) \neq \mathcal{N}(X)$, see example 5.25 below. Now, by combining Theorem 1.3 with the uniformization Theorem, see e.g. [BM88, Th 0.1], we prove the following result:

Theorem 1.4 (Pawłucki Theorem II, [Pa90])**.** *Let X be a subanalytic set of a real analytic manifold M. Then*

i) The sets $\mathcal{N}(X)$ and $\mathcal{SA}(X)$ are subanalytic.

ii) $\dim(M \smallsetminus \mathcal{N}(X)) \leqslant \dim(M \smallsetminus \mathcal{SA}(X)) \leqslant \dim(X) - 2$.
In particular, if $\dim(X) \leqslant 1$*, then* $\mathcal{N}(X) = \mathcal{SA}(X) = M$.

Remark 1.5. The case of $\dim(X) \leqslant 1$ was originally proved by Lojasiewicz [Lo65] and an alternative proof is given in [BM88, Theorem 6.1].

The original proof of Theorem 1.4 given in [Pa90] is an intricate construction between geometrical, algebraic, and analytic arguments, which we do not fully understand. Pawłucki then deduces Theorem 1.3 from Theorem 1.4 in [Pa92]. Our proof of these results relies heavily on algebraic arguments, namely on Theorem 1.1 and the use of Eisenstein power series, see §§3.3 instead of geometric and analytic arguments as in [Pa90]. We develop new commutative algebra methods, in particular concerning power series with coefficients in a UFD, which are of independent interest, see e.g. Theorem 3.7. Our use of geometric techniques is essentially reduced the extension Lemma 6.1 together with the use of the Uniformization Theorem of Hironaka [H73]; the former has been inspired from the work of Pawłucki [Pa90, Lemme 6.3], while the later is not used in [Pa90, Pa92].

We would like to thank Edward Bierstone for bringing the topic of this paper to our attention and for useful discussions.

1.1. **On rank Theorems for non W-temperate family.** We start by providing examples of a families of local rings where the rank Theorem does not hold. We consider an example given in [BB22, Example 1.8], which is based on a construction due to Nazarov, Sodin and Volberg [NSV04, § 5.3]. More precisely, consider quasianalytic Denjoy-Carleman classes $\mathcal{Q}_M$ which satisfies two properties:

1) There is a function $g \in \mathcal{Q}_M([0,1))$ which admits no extension to a function in $\mathcal{Q}_{M'}((-\delta,1))$, for all $\delta > 0$ and all quasianalytic Denjoy-Carleman class $\mathcal{Q}_{M'}$ (these classes exist by [NSV04, § 5.3]);
2) The shifted class $\mathcal{Q}_{M^{(p)}}$, where $M_k^{(p)} := M_{pk}$, is a quasianalytic Denjoy-Carleman class for every $p \in \mathbb{N}$.

For example, the class $\mathcal{Q}_M$ given by the sequence $M = (M_k)_{k\in\mathbb{N}}$, where $M_k = (\log(\log k))^k$, satisfies both conditions.

Let $\Phi : (-1,1) \to \mathbb{R}^2$ denote the $\mathcal{Q}_M$-morphism $\Phi(u) = (u^2, g(u^2))$, and let $\varphi = \Phi^*$ denote its pull-back at 0. Note that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$ since $G(x) = x_2 - \hat{g}(x_1)$ is a formal power series such that $\hat{\varphi}(G) = \hat{g}(u^2) - \hat{g}(u^2) \equiv 0$. Now, suppose by contradiction that there exists a function germ $h \in \mathcal{Q}_M(-\epsilon,\epsilon)$, for some $\epsilon < 1$, such that $\varphi(h) = h \circ \varphi(u) \equiv 0$. We remark that $\hat{h}(t, \hat{g}(t)) \equiv 0$ since

$$0 \equiv \hat{\varphi}(\hat{h}) = \hat{h}(u^2, \hat{g}(u^2)),$$

so we conclude that the equation $h(x_1,x_2) = 0$ admits a formal solution $x_2 = \hat{g}(x_1)$. By [BBB17, Theorem 1.1], apart from shrinking ϵ, there exists a function $f \in \mathcal{Q}_{M^{(p)}}(-\epsilon,\epsilon)$, for some $p \in \mathbb{N}$, such that $h(x_1, f(x_1)) \equiv 0$ and $\hat{f} = \hat{g}$. Since $\mathcal{Q}_{M^{(p)}}$ is quasianalytic by condition 2) and contains $\mathcal{Q}_M$, we conclude that $f_{|[0,\epsilon)} = g_{|[0,\epsilon)}$. This implies that g admits an extension in the shifted quasianalytic class $\mathcal{Q}_{M^{(p)}}$, contradicting condition 1).

We can also consider the o-minimal structure $\mathbb{R}_{\mathcal{Q}_M}$ given by expansion of the real field by restricted functions of class $\mathcal{Q}_M$ satisfying conditions 1) and 2) above, cf. [RSW03], and the quasianalytic class $\mathcal{Q}$ of $\mathcal{C}^\infty$ functions that are locally definable in $\mathbb{R}_{\mathcal{Q}_M}$. By [BBC18, Theorem 1.6], any function $h \in \mathcal{Q}((-1,1))$ belongs to a shifted

Denjoy-Carleman class $\mathcal{Q}_{M^{(p)}}$, for some positive integer p. We conclude that the morphism Φ defined above shows that the rank Theorem can not hold for $\mathcal{Q}$.

Remark 1.6. Every quasianalytic class which properly contains the analytic functions does not satisfy the Weiersstrass preparation property [PR13].

Finally, let us note that Theorem 1.1 might hold for families of rings which are not Weierstrass. For instance, if we set $\mathcal{K}\{\{x_1,\ldots,x_n\}\} = \mathcal{K}[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ for any integer n, then for any morphism $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$, we have

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = \dim\left(\mathcal{K}\{\{\mathbf{x}\}\}/\operatorname{Ker}(\varphi)\right)$$

essentially by Chevalley's constructible set Theorem. But the rings of rational functions are not Henselian local rings, so they do not form a Weierstrass family, c.f. Proposition 2.8 i) below.

2. Weierstrass Temperate families

2.1. **W-Temperate families.** Let $\mathcal{K}$ be a field of characteristic zero. For every $n \in \mathbb{N}$, we denote by $(x_1,\ldots,x_n)$ indeterminacies; we will use the compact notation $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{x}' = (x_1,\ldots,x_{n-1})$ whenever there is no risk of confusion on n. We start by recalling the notion of Weierstrass family introduced in [DL80]:

Definition 2.1. A *Weierstrass family (over $\mathcal{K}$)*, or just a W-family, of rings is a family $(\mathcal{K}\{\{x_1,\ldots,x_n\}\})_{n\in\mathbb{N}}$ of $\mathcal{K}$-algebras such that,

i) For every n,
$$\mathcal{K}[\mathbf{x}] \subset \mathcal{K}\{\{\mathbf{x}\}\} \subset \mathcal{K}[\![\mathbf{x}]\!].$$
ii) For every n and m, denoting $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_m)$:
$$\mathcal{K}\{\{\mathbf{x},\mathbf{y}\}\} \cap \mathcal{K}[\![\mathbf{x}]\!] = \mathcal{K}\{\{\mathbf{x}\}\}.$$
iii) For any permutation σ of $\{1,\ldots,n\}$, and any $f \in \mathcal{K}\{\{\mathbf{x}\}\}$,
$$f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \in \mathcal{K}\{\{\mathbf{x}\}\}.$$
iv) If $f \in \mathcal{K}\{\{\mathbf{x}\}\}$ with $f(0) \neq 0$, then f is a unit in $\mathcal{K}\{\{\mathbf{x}\}\}$.
v) The family is closed by *Weierstrass division*. More precisely, let $F \in \mathcal{K}\{\{\mathbf{x}\}\}$ be such that $F(0,x_n) = x_n^d u(x_n)$ where $u(0) \neq 0$. For every $G \in \mathcal{K}\{\{\mathbf{x}\}\}$,
$$G = FQ + R,$$
where $Q \in \mathcal{K}\{\{\mathbf{x}\}\}$ and $R \in \mathcal{K}\{\{\mathbf{x}'\}\}[x_{n+1}]$, $\deg_{x_n}(R) < d$, are unique.

A W-family satisfies several extra well-known properties which we recall in §§2.3 below; in what follows we use these properties. Let us now provide the definition of W-temperate family:

Definition 2.2. Let $\mathcal{K}$ be an uncountable algebraically closed field of characteristic zero. A *Weierstrass temperate family (over $\mathcal{K}$)*, or just a W-temperate family, of rings is a Weierstrass family $(\mathcal{K}\{\{x_1,\ldots,x_n\}\})_{n\in\mathbb{N}}$ over $\mathcal{K}$ satisfying the following three properties:

i) Closure by local blowings-down: For every $f \in \mathcal{K}[\![\mathbf{x}]\!]$, $n > 1$, we have
$$f(\mathbf{x}', x_1x_n) \in \mathcal{K}\{\{\mathbf{x}\}\} \Longrightarrow f(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}.$$

ii) Closure by generic hyperplane sections: Let $F \in \mathcal{K}[\![\mathbf{x}]\!] \smallsetminus \mathcal{K}\{\{\mathbf{x}\}\}$. Set
$$W := \{\lambda \in \mathcal{K} \mid F(\mathbf{x}', \lambda x_1) \in \mathcal{K}\{\{\mathbf{x}'\}\}\}.$$
Then the set $\mathcal{K} \smallsetminus W$ is uncountable.

iii) Temperateness: Let $\mathbf{x} = (x_1, x_2)$ and $\alpha \in \mathbb{N}^*$. Consider
$$\gamma(t) \in \mathcal{K}\{\{t\}\} \quad \text{and} \quad P(\mathbf{x}, z) = \sum_{k\in\mathbb{N}} x_1^k p_k(x_2, z) \in \mathcal{K}[\![\mathbf{x}]\!][z],$$
where $\gamma(t)$ is finite over $\mathcal{K}[t]$ and $p_k(x_2, z) \in \mathcal{K}[x_2, z]$ is such that $\deg_{x_2}(p_k) \leqslant \alpha k$ for every $k \in \mathbb{N}$. Let γ' be a conjugate of γ:
$$P(\mathbf{x}, \gamma(x_2)) \in \mathcal{K}\{\{\mathbf{x}\}\} \Longrightarrow P(\mathbf{x}, \gamma'(x_2)) \in \mathcal{K}\{\{\mathbf{x}\}\}.$$

Note that properties ii) and iii) are used only once in the paper, see §§4.2 and the proof of Theorem 4.19 respectively.

2.2. **W-Temperate morphisms and ranks.** We start by proving a detailed definition of the morphisms we consider:

Definition 2.3. Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of local rings. We call φ a *morphism of rings of W-temperate power series* if there exist W-temperate power series $\varphi_1(\mathbf{u}), \ldots, \varphi_n(\mathbf{u}) \in (\mathbf{u})\mathcal{K}\{\{\mathbf{u}\}\}$ such that
$$\forall f(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}, \quad \varphi(f) = f(\varphi_1(\mathbf{u}), \ldots, \varphi_n(\mathbf{u})).$$

For such a morphism, we have introduced in the introduction three notions of ranks: generic, formal and temperate, see (1). Note that the generic and formal ranks can be introduced, in an obvious way, for general morphisms of power series rings $\psi : \mathcal{K}[\![\mathbf{x}]\!] \longrightarrow \mathcal{K}[\![\mathbf{u}]\!]$. Let us start by showing that these ranks are well-defined:

Lemma 2.4. *Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of W-temperate power series. Then $\mathrm{r}(\varphi)$, $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{T}}(\varphi)$ are natural numbers such that:*
$$\mathrm{r}(\varphi) \leqslant \mathrm{r}^{\mathcal{F}}(\varphi) \leqslant \mathrm{r}^{\mathcal{T}}(\varphi).$$

Proof. It is straightforward that $\mathrm{r}(\varphi)$ is well-defined; $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{T}}(\varphi)$ are well-defined since $\mathcal{K}[\![\mathbf{x}]\!]/\mathrm{Ker}(\widehat{\varphi})$ and $\mathcal{K}\{\{\mathbf{x}\}\}/\mathrm{Ker}(\varphi)$ are Noetherian local rings. Next, consider a general morphism of power series ring $\psi : \mathcal{K}[\![\mathbf{x}]\!] \longrightarrow \mathcal{K}[\![\mathbf{u}]\!]$ and set $r = \mathrm{r}(\psi)$. Apart from re-ordering the coordinates, we may assume that the matrix $[\partial_{u_i}\varphi(x_j)]_{i\leqslant m, j\leqslant r}$ has rank r. Therefore, if we set $R := \mathcal{K}[\![x_1, \ldots, x_r]\!]$, $\psi_{|R}$ is injective by [Ga73, Lemma 4.2]. Thus $r = \dim(R) \leqslant \dim\left(\mathcal{K}[\![\mathbf{x}]\!]/\mathrm{Ker}(\psi)\right)$. This proves the first inequality. Finally, by Artin approximation Theorem 2.10, $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is a prime ideal, and by [Mat89, Theorem 9.4] the height of $\mathrm{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is less than or equal to the height of $\mathrm{Ker}(\widehat{\varphi})$. We conclude that $\mathrm{r}^{\mathcal{F}}(\varphi) \leqslant \mathrm{r}^{\mathcal{T}}(\varphi)$. □

Remark 2.5. The proof of the above Lemma also shows the result for a general morphism of power series rings $\psi : \mathcal{K}[\![\mathbf{x}]\!] \longrightarrow \mathcal{K}[\![\mathbf{u}]\!]$, that is, its generic and formal ranks are well defined and:
$$\mathrm{r}(\psi) \leqslant \mathrm{r}^{\mathcal{F}}(\psi).$$

Definition 2.6. Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of W-temperate power series. We say that φ is regular (in the sense of Gabrielov) if $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$.

We finish this subsection by useful results about the ranks of a morphism of W-temperate power series, which is a W-temperate version of [BCR21, Prop. 2.2]:

Proposition 2.7 (cf. [BCR21, Prop.2.2]). *Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of W-temperate power series. The ranks $\mathrm{r}(\varphi)$, $\mathrm{r}^{\mathcal{F}}(\varphi)$ and $\mathrm{r}^{\mathcal{T}}(\varphi)$ are preserved if we compose φ with:*

(1) a morphism $\sigma : \mathcal{K}\{\{u_1,\dots,u_m\}\} \longrightarrow \mathcal{K}\{\{u'_1,\dots,u'_s\}\}$ such that $\mathrm{r}(\sigma)=m$,
(2) an injective finite morphism $\tau : \mathcal{K}\{\{x'_1,\dots,x'_n\}\} \longrightarrow \mathcal{K}\{\{x_1,\dots,x_n\}\}$,
(3) an injective finite morphism $\tau : \mathcal{K}\{\{x'_1,\dots,x'_t\}\} \longrightarrow \mathcal{K}\{\{\mathbf{x}\}\}/\mathrm{Ker}(\varphi)$.

Proof. We start by proving (1). Note that it is straightforward from linear algebra that $\mathrm{r}(\sigma\circ\varphi)=\mathrm{r}(\varphi)$. In order to prove the other two equalities, it is enough to prove that σ and $\widehat{\sigma}$ are injective morphisms of local rings. This follows from Lemma 2.4, since $m=\mathrm{r}(\sigma)\leqslant \mathrm{r}^{\mathcal{F}}(\sigma)\leqslant \mathrm{r}^{\mathcal{T}}(\sigma)\leqslant \dim\mathcal{K}\{\{u_1,\dots,u_m\}\}=m$.

We now prove that $\mathrm{r}^{\mathcal{T}}(\varphi)=\mathrm{r}^{\mathcal{T}}(\varphi\circ\tau)$ under the hypothesis given in (2) and (3). Indeed, we have $\mathrm{Ker}(\varphi\circ\tau)=\mathrm{Ker}(\varphi)\cap\mathcal{K}\{\{\mathbf{x}'\}\}$ because τ is injective. Since $\mathcal{K}\{\{\mathbf{u}\}\}$ is an integral domain, $\mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\varphi\circ\tau)$ are prime ideals. Thus, by the Going-Down theorem for integral extensions [Mat89, Theorem 9.4ii], we have that $\mathrm{ht}(\mathrm{Ker}(\varphi\circ\tau))\leqslant\mathrm{ht}(\mathrm{Ker}(\varphi))$, thus $\mathrm{r}^{\mathcal{T}}(\varphi)\leqslant\mathrm{r}^{\mathcal{T}}(\varphi\circ\tau)$. On the other hand, we have the equality $\mathrm{r}^{\mathcal{T}}(\varphi)=\mathrm{r}^{\mathcal{T}}(\varphi\circ\tau)$ because $\mathrm{ht}(\mathrm{Ker}(\varphi\circ\tau))=\mathrm{ht}(\mathrm{Ker}(\varphi))$ by [Mat89, Theorem 9.3].

We now prove that $\mathrm{r}^{\mathcal{F}}(\varphi)=\mathrm{r}^{\mathcal{F}}(\varphi\circ\tau)$ under the hypothesis given in (2). Indeed, since τ is finite, $\widehat{\tau}$ is also finite by the Weierstrass division Theorem (see for instance [BCR21, Cor. 1.10] for this claim). Moreover, we have

$$\dim(\mathcal{K}[\![\mathbf{x}']\!])-\mathrm{ht}(\mathrm{Ker}\,\widehat{\tau})=\dim(\mathcal{K}[\![\mathbf{x}]\!])=n$$

since finite morphisms preserve the dimension and τ is injective. But $\mathrm{ht}(\mathrm{Ker}(\widehat{\tau}))=0$ if and only if $\mathrm{Ker}(\widehat{\tau})=(0)$ because $\mathcal{K}\{\{\mathbf{x}'\}\}$ is an integral domain. Thus, $\widehat{\tau}$ is injective and $\mathrm{r}^{\mathcal{F}}(\varphi\circ\tau)=\mathrm{r}^{\mathcal{F}}(\varphi)$.

Now we prove that $\mathrm{r}^{\mathcal{F}}(\varphi)=\mathrm{r}^{\mathcal{F}}(\varphi\circ\tau)$ under the hypothesis given in (3). We denote by $\widehat{\tau}'$ the morphism

$$\mathcal{K}[\![x'_1,\dots,x'_t]\!]/\widehat{\tau}^{-1}(\mathrm{Ker}(\widehat{\varphi}))\longrightarrow\mathcal{K}[\![\mathbf{x}]\!]/\mathrm{Ker}(\widehat{\varphi}).$$

As in the previous case, since τ is finite, $\widehat{\tau}$ is also finite, therefore $\widehat{\tau}'$ is finite. Moreover, by definition, $\widehat{\tau}'$ is injective. Thus, by Theorem [Mat89, Theorem 9.3], we have $\mathrm{r}^{\mathcal{F}}(\varphi\circ\tau)=\mathrm{r}^{\mathcal{F}}(\varphi)$.

We now turn to the proof of $\mathrm{r}(\varphi)=\mathrm{r}(\varphi\circ\tau)$. We start by (2). Let J_φ and J_τ denote the Jacobian matrices of φ and τ. Then we have $J_{\varphi\circ\tau}=J_\varphi\cdot\varphi(J_\tau)$; note that it is enough to prove that the hypothesis imply that $\mathrm{r}(\tau)=n$ in order to conclude by standard linear algebra. Indeed, since τ is finite and injective, for every $i\in\{1,\dots,n\}$, there is a monic polynomial $P_i(\mathbf{x}',x_i)\in\mathcal{K}\{\{\mathbf{x}'\}\}[x_i]$ such that

$$P_i(\tau_1(\mathbf{x}'),\dots,\tau_n(\mathbf{x}'),x_i)=0 \text{ and } \frac{\partial P_i}{\partial x_i}(\tau_1(\mathbf{x}'),\dots,\tau_n(\mathbf{x}'),x_i)\neq 0.$$

Therefore, for every i and j, we have

$$\sum_{\ell=1}^{n}\frac{\partial P_i}{\partial x'_\ell}(\tau(\mathbf{x}),x_i)\frac{\partial\tau_\ell(\mathbf{x})}{\partial x_j}=\begin{cases}-\dfrac{\partial P_i}{\partial x_i} & \text{if} \quad i=j\\ 0 & \text{if} \quad i\neq j\end{cases}$$

Thus

$$\begin{bmatrix}\dfrac{\partial P_i}{\partial x'_1} & \cdots & \dfrac{\partial P_i}{\partial x'_n}\end{bmatrix}\cdot J_\tau=-\frac{\partial P_i}{\partial x_i}\cdot e_i$$

where e_i is the vector whose coordinates are zero except the i-th one which is equal to 1. In particular J_τ is generically a matrix of maximal rank, that is $\mathrm{r}(\tau) = n$, so $\mathrm{r}(\varphi) = \mathrm{r}(\varphi \circ \tau)$ by standard linear algebra. This proves (2).

Finally, let us finish the proof of (3). By adding the x'_i to the x_j, we can assume that $x'_i = x_i$ for $i \leqslant t$. By assumption, for every $i \in \{1, \ldots, n\}$, there is a monic polynomial $P_i(\mathbf{x}', x_i) \in \mathcal{K}\{\{\mathbf{x}'\}\}[x_i]$ such that

$$P_i(\tau_1(\mathbf{x}), \ldots, \tau_n(\mathbf{x}), x_i) = f_i(\mathbf{x}) \in \mathrm{Ker}(\varphi) \text{ and } \frac{\partial P_i}{\partial x_i}(\tau_1(\mathbf{x}'), \ldots, \tau_n(\mathbf{x}'), x_i) \notin \mathrm{Ker}(\varphi).$$

Thus

$$\begin{bmatrix} \frac{\partial P_i}{\partial x'_1} & \cdots & \frac{\partial P_i}{\partial x'_n} \end{bmatrix} \cdot J_\tau = -\frac{\partial P_i}{\partial x_i} \cdot e_i + \begin{bmatrix} \frac{\partial f_i}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_i}{\partial x_n}(\mathbf{x}) \end{bmatrix}$$

Since $f_i \in \mathrm{Ker}(\varphi)$, we have $f_i(\varphi(\mathbf{u})) = 0$. By differentiation we obtain

$$\forall j = 1, \ldots, m, \quad \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}(\varphi(\mathbf{u})) \frac{\partial \varphi_k(\mathbf{u})}{\partial u_j} = 0,$$

that is,

$$\begin{bmatrix} \frac{\partial f_i}{\partial x_1}(\varphi(\mathbf{u})) & \cdots & \frac{\partial f_i}{\partial x_n}(\varphi(\mathbf{u})) \end{bmatrix} \cdot J_\varphi = 0.$$

This proves that the generic rank of $J_{\varphi \circ \tau} = J_\varphi \cdot \varphi(J_\tau)$ is the rank of J_φ. □

2.3. **Properties of Weierstrass families.** We now recall several useful properties of W-families which are either proved in [DL80, Ro09] (see precise references in the proof), or which follow easily from classical results:

Proposition 2.8. *Let $(\mathcal{K}\{\{x_1, \ldots, x_n\}\})_{n \in \mathbb{N}}$ be a Weierstrass family. Then the following properties are satisfied:*

i) For every n, $\mathcal{K}\{\{\mathbf{x}\}\}$ is a Henselian, Noetherien, UFD regular local ring whose maximal ideal is generated by $(x_1, \ldots, x_n)$, and completion is $\mathcal{K}[\![\mathbf{x}]\!]$.

ii) For $f \in \mathcal{K}\{\{t, \mathbf{x}\}\}$ and any $g \in (\mathbf{x})\,\mathcal{K}\{\{\mathbf{x}\}\}$, $f(g, \mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}$.

iii) For every $f \in \mathcal{K}[\![\mathbf{x}]\!]$, and any $q \in \mathbb{N}^*$, we have

$$f(\mathbf{x}', x_n^q) \in \mathcal{K}\{\{\mathbf{x}\}\} \Longrightarrow f(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}.$$

iv) For every n and $k \leqslant n$,

$$\mathcal{K}\{\{\mathbf{x}\}\} \cap (x_k)\,\mathcal{K}[\![\mathbf{x}]\!] = (x_k)\,\mathcal{K}\{\{\mathbf{x}\}\}.$$

v) Weierstrass preparation Theorem: Let $f \in \mathcal{K}\{\{\mathbf{x}\}\}$ be such that $f(0, \ldots, 0, x_n) \neq 0$ has order d in x_n. Then there exists a unit U and a Weierstrass polynomial $P = x_n^d + a_1(\mathbf{x}')x_n^{d-1} + \cdots + a_d(\mathbf{x}')$ such that

$$f(\mathbf{x}) = U(\mathbf{x}) \cdot (x_n^d + a_1(\mathbf{x}')x_n^{d-1} + \cdots + a_d(\mathbf{x}')).$$

vi) Noether Normalization: Let $A = \mathcal{K}\{\{\mathbf{x}\}\}/I$ where I is an ideal of $\mathcal{K}\{\{\mathbf{x}\}\}$. Then, apart from a linear change of indeterminates $x_1, \ldots, x_n$, there exists an integer $r > 0$ such that the canonical morphism

$$\mathcal{K}\{\{x_1, \ldots, x_r\}\} \longrightarrow \mathcal{K}\{\{x_1, \ldots, x_n\}\}/I$$

is finite. Moreover, since the dimension does not change under finite morphisms, if $\dim\left(\mathcal{K}\{\{\mathbf{x}\}\}/I\right) = r$, then $\mathrm{ht}(I) = n - r$.

Proof. Properties i), ii), iv) are given in [DL80, Remark 1.3]; property iii) is given in [Ro09, Lemme 5.13]. To prove property v), it is enough to consider the Weierstrass division of x_n^d by $f(\mathbf{x})$. Finally, it is classical that property vi) follows from the Weierstrass division Theorem, see e.g. [dJPf00, 3.319]. ☐

We add to this list of properties the following two Theorems:

Theorem 2.9 ([PR12, Theorem 5.5])**.** *A W-temperate family of rings satisfies the Abhyankar-Jung Theorem. More precisely, let $P(\mathbf{x}, Z) \in \mathcal{K}\{\{\mathbf{x}\}\}[Z]$ be a monic polynomial in Z. Assume that*

$$\operatorname{Disc}_Z(P) = \mathbf{x}^\alpha u(\mathbf{x})$$

where $u(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}$ satisfies $u(0) \neq 0$. Then there is $q \in \mathbb{N}^*$ such that the roots of P belong to $\mathcal{K}\{\{x_1^{1/q}, \ldots, x_n^{1/q}\}\}$.

The next result motivates the introduction of the notion of W-families in [DL80].

Theorem 2.10 ([DL80, Theorem 1.1])**.** *A W-temperate family of rings satisfies the Artin Approximation Theorem: let $F = (F_1, \ldots, F_p) \in \mathcal{K}\langle\mathbf{x}\rangle[\mathbf{y}]^p$ with $\mathbf{y} = (y_1, \ldots, y_m)$, and let $\widehat{g}(\mathbf{x}) = (\widehat{g}_1(\mathbf{x}), \ldots, \widehat{g}_m(\mathbf{x})) \in \mathcal{K}[\![\mathbf{x}]\!]^m$ be a formal power series solution:*

$$F(\mathbf{x}, \widehat{g}(\mathbf{x})) = 0.$$

Let $c \in \mathbb{N}$. Then there is an algebraic solution $g^{(c)}(\mathbf{x}) = (g_1^{(c)}(\mathbf{x}), \ldots, g_m^{(c)}(\mathbf{x})) \in \mathcal{K}\langle\mathbf{x}\rangle^m$:

$$F(\mathbf{x}, g^{(c)}(\mathbf{x})) = 0$$

with $g_i^{(c)}(\mathbf{x}) - \widehat{g}_i(\mathbf{x}) \in (\mathbf{x})^c$ for every i.

In what follows, we will use the following well known corollaries of the above result (and we provide their proofs for the sake of completeness).

Corollary 2.11. *Let $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \to \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of Weierstrass power series. Then $\operatorname{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is a prime ideal.*

Proof. The following is a well-known argument. Let $\widehat{f}$, $\widehat{g} \in \mathcal{K}[\![\mathbf{x}]\!]$ be such that $\widehat{f}\widehat{g} \in \operatorname{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$. That is, there exist $f_1, \ldots, f_s \in \operatorname{Ker}(\varphi)$ and $\widehat{h}_1, \ldots, \widehat{h}_s \in \mathcal{K}[\![\mathbf{x}]\!]$ such that

$$\widehat{f}\widehat{g} - \sum_{i=1}^{s} f_i \widehat{h}_i = 0.$$

By Artin approximation Theorem applied to $y_{s+1}y_{s+2} - \sum_{i=1}^s f_i y_i$, for every $c \in \mathbb{N}^*$, there exist $f^{(c)}$, $g^{(c)}$, $h_1^{(c)}, \ldots, h_s^{(c)} \in \mathcal{K}\{\{\mathbf{x}\}\}$ such that

$$f^{(c)}g^{(c)} - \sum_{i=1}^{s} f_i h_i^{(c)} = 0$$

and $\widehat{f} - f^{(c)}$, $\widehat{g} - g^{(c)} \in (\mathbf{x})^c$. Since $\operatorname{Ker}(\varphi)$ is a prime ideal, then $f^{(c)}$ or $g^{(c)}$ is in $\operatorname{Ker}(\varphi)$. Apart from replacing f by g, we may assume that $f^{(c)} \in \operatorname{Ker}(\varphi)$ for infinitely many c. Therefore, f is the limit of elements of $\operatorname{Ker}(\varphi)$, that is, f belongs to the closure of $\operatorname{Ker}(\varphi)$ in $\mathcal{K}[\![\mathbf{x}]\!]$ for the $(\mathbf{x})$-topology. But, by [Mat89, Theorem 8.11], this closure is exactly $\operatorname{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$, so $f \in \operatorname{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$. This proves that $\operatorname{Ker}(\varphi)\mathcal{K}[\![\mathbf{x}]\!]$ is a prime ideal. ☐

Corollary 2.12. *Suppose that P and Q are monic polynomials in y, $P \in \mathcal{K}[\![\mathbf{x}]\!][y]$ and $Q \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$, that are not coprime in $\mathcal{K}[\![\mathbf{x}, y]\!]$. Then P and Q admits a common W-factor $R \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$.*

Proof. By hypothesis, there is a non unit $R \in \mathcal{K}[\![\mathbf{x}, y]\!]$ that divides P and Q in $\mathcal{K}[\![\mathbf{x}, y]\!]$. Since P is monic in y, then $R(0, y) \neq 0$, so R equals a unit times a monic polynomial by Weierstrass preparation for formal power series. By replacing R by this monic polynomial we may assume that R is monic in y. So we have $RS = Q$ where $S \in \mathcal{K}[\![\mathbf{x}]\!][y]$ is monic in y. We write

$$R = \sum_{i=0}^{d} r_i(\mathbf{x})y^i, \quad S = \sum_{i=0}^{e} s_i(\mathbf{x})y^i \quad \text{and } Q = \sum_{i=0}^{d+e} q_i(\mathbf{x})y^i.$$

The equality $RS = Q$ is equivalent to the system of equations

$$\sum_{k=\max\{0,\ell-e\}}^{\min\{\ell,d\}} r_k(\mathbf{x})s_{\ell-k}(\mathbf{x}) - q_\ell(\mathbf{x}) = 0 \text{ for } \ell = 0, \dots, d+e.$$

By Artin approximation Theorem 2.10, for any $c \in \mathbb{N}$, this system of equations has a solution $(r_i'(\mathbf{x}), s_j'(\mathbf{x})) \in \mathcal{K}\{\{\mathbf{x}\}\}^{d+e+2}$ and that coincide with $(r_i(\mathbf{x}), s_i(\mathbf{x}))$ up to $(\mathbf{x})^c$. We set $R'(\mathbf{x}, y) = \sum_{i=0}^{d} r_i'(\mathbf{x})y^i$. Since $\mathcal{K}\{\{\mathbf{x}\}\}$ is a UFD by Proposition 2.8 i), Q has finitely many monic factors of degree d in y that we denote by $R_1, \dots, R_s$. Let us choose $c \in \mathbb{N}$ large enough to insure that $R_i - R_j \notin (\mathbf{x})^c$ when $i \neq j$. Since R' equals one of the R_i, necessarily $R' = R$. This proves that $R \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$, so P has a temperate monic factor. $\square$

3. Examples of W-temperate families

3.1. **Algebraic power series.** When $\mathcal{K}$ is a field, we denote by $\mathcal{K}\langle x_1, \dots, x_n\rangle$ the subring of $\mathcal{K}[\![x_1, \dots, x_n]\!]$ of formal power series that are algebraic over $\mathcal{K}[x_1, \dots, x_n]$. We have the following proposition:

Proposition 3.1. *Let $\mathcal{K}$ be an uncountable algebraically closed field of characteristic zero. The family of algebraic power series rings $(\mathcal{K}\langle x_1, \dots, x_n\rangle)_n$ is a minimal W-temperate family, that is, it is contained in every other W-temperate family.*

Proof. Let $(\mathcal{K}\{\{x_1, \dots, x_n\}\})_n$ be a arbitrary W-temperate family. Since $\mathcal{K}\{\{\mathbf{x}\}\}$ is a Henselian local ring containing $\mathcal{K}[\mathbf{x}]_{(\mathbf{x})}$, and since $\mathcal{K}\langle\mathbf{x}\rangle$ is the Henselization of $\mathcal{K}[\mathbf{x}]_{(\mathbf{x})}$, we have $\mathcal{K}\langle\mathbf{x}\rangle \subset \mathcal{K}\{\{\mathbf{x}\}\}$ by the universal property of the Henselization.

Next, let us prove that $(\mathcal{K}\langle x_1, \dots, x_n\rangle)_n$ is a W-temperate family. The first four axioms of Definition 2.1 are classical, while the fifth axiom has been proved by Lafon in [La65], see also [Ro18b]. So, let us check that Definition 2.2 is verified. Once again, axiom i) is straightforward, and we consider:

Axiom ii) of Definition 2.2: We prove the contrapositive of the axiom, that is, let $F \in \mathcal{K}[\![\mathbf{x}]\!]$ be such that

$$W := \{\lambda \in \mathcal{K} \mid F(\mathbf{x}', \lambda x_1) \in \mathcal{K}\langle\mathbf{x}'\rangle\}$$

is uncountable and let us prove that $F \in \mathcal{K}\langle\mathbf{x}'\rangle$. Let us denote by $\mathcal{K}_0$ the algebraic closure of the field extension of $\mathbb{Q}$ generated by the coefficients of F. Since F has a countable number of coefficients, $\mathcal{K}_0$ is a countable field. Let $\lambda \in W \setminus \mathcal{K}_0$; in particular λ is transcendental over $\mathcal{K}_0$. By assumption on W, we have

$$a_0(\mathbf{x}')F(\mathbf{x}', \lambda x_1)^d + a_1(\mathbf{x}')F(\mathbf{x}', \lambda x_1)^{d-1} + \cdots + a_d(\mathbf{x}') = 0 \tag{2}$$

where the $a_i(\mathbf{x}')\in\mathcal{K}[\mathbf{x}']$. Let us denote by $(\underline{a})$ the vector whose entries are the coefficients of the $a_i(\mathbf{x}')$. Then (2) is satisfied if and only if $(\underline{a})$ satisfies a (countable) system of linear equations $(\mathcal{S})$ whose coefficients are in $\mathcal{K}_0(\lambda)$ (determined by the vanishing of the coefficients of each monomial $\mathbf{x}'^\alpha$ for $\alpha\in\mathbb{N}^{n-1}$). And $(\mathcal{S})$ is equivalent to a finite system of linear equations $(\mathcal{S}')$ with coefficients in $\mathcal{K}_0(\lambda)$. And this system has a nonzero solution in $\mathcal{K}$ if and only if it has a nonzero solution in $\mathcal{K}_0(\lambda)$, and this solution yields non trivial polynomials $\widetilde{a}_i(\mathbf{x}')\in\mathcal{K}_0(\lambda)[\mathbf{x}']$ such that

$$\widetilde{a}_0(\mathbf{x}')F(\mathbf{x}',\lambda x_1)+\widetilde{a}_1(\mathbf{x}')F(\mathbf{x}',\lambda x_1)^{d-1}+\cdots+\widetilde{a}_d(\mathbf{x}')=0.$$

By multiplying by some polynomial in $\mathcal{K}_0[\lambda]$ we may assume that the $\widetilde{a}_i(\mathbf{x}')$ belong to $\mathcal{K}_0[\mathbf{x}'][\lambda]$, thus we write $\widetilde{a}_i=\widetilde{a}_i(\mathbf{x}',\lambda)$. By dividing by a large enough power of $x_n-\lambda x_1$, we may assume that one of them is not divisible by $x_n-\lambda x_1$. Therefore not all the $\widetilde{a}_i(\mathbf{x}',x_n/x_1)$ are zero, and

$$a_0(\mathbf{x}',x_n/x_1)F(\mathbf{x}',x_n/x_1)+a_1(\mathbf{x}',x_n/x_1)F(\mathbf{x}',x_n/x_1)^{d-1}+\cdots+a_d(\mathbf{x}',x_n/x_1)=0,$$

whence $F(\mathbf{x})\in\mathcal{K}\langle\mathbf{x}\rangle$. This proves the result.

Axiom iii) of Definition 2.2: We follow the notation of axiom iii). For each $p_k(x_2,z)$, we consider its Euclidean division by the minimal polynomial Γ of γ:

$$p_k(x_2,z)=\Gamma(x_2,z)\cdot q_k(x_2,z)+r_k(x_2,z)$$

where $\deg_z(r_k)<d=\deg_z(\Gamma)$. By Lemma 3.2, there is $a\in\mathbb{N}$ such that $\deg_{x_2}(r_k(x_2,z))\leqslant ak$ for every k.

Note that $p_k(x_2,\gamma'(x_2))=r_k(x_2,\gamma'(x_2))$ for every root $\gamma'(x_2)$ of Γ, so we may consider the auxiliary function:

$$Q(x_1,x_2,z)=\sum_{k\in\mathbb{N}}x_1^kr_k(x_2,z)=\sum_{k=0}^{d-1}q_k(\mathbf{x})z^k$$

where $q_i(\mathbf{x})\in\mathbb{K}[x_2][\![x_1]\!]$, and note that $P(\mathbf{x},\gamma'(x_2))=Q(\mathbf{x},\gamma'(x_2))$ for every root $\gamma'(x_2)$ of Γ. Since $\deg_{x_2}(r_k(x_2,z))\leqslant ak$ for every k, we may write $q_k(\mathbf{x})=\widehat{q}_k(x_1,x_1x_2,\ldots,x_1x_2^a)$ for some formal power series $\widehat{q}_k(x_1,y_1,\ldots,y_a)\in\mathcal{K}[\![x_1,\mathbf{y}]\!]$. Now, there exist formal power series $\widehat{g}_i$, for $i=1,\ldots,a$, and $\widehat{k}$ such that:

$$Q(\mathbf{x},\gamma(t))=\sum_{k=0}^{d-1}\widehat{q}_k(x_1,\mathbf{y})z^k+\sum_{i=1}^{a}(y_i-x_1x_2^i)\widehat{g}_i(\mathbf{x},\mathbf{y},t,z)+(z-\gamma(t))\widehat{k}(\mathbf{x},\mathbf{y},t,z).$$

By the nested approximation Theorem for linear equations (see [CPR19, Theorem 3.1]) this equation has a non trivial nested algebraic solution

$$(\widetilde{q}_k(x_1,\mathbf{y}),\widetilde{g}_i(\mathbf{x},\mathbf{y},t,z)),\widetilde{k}(\mathbf{x},\mathbf{y},t,z))\in\mathcal{K}\langle x_1,\mathbf{y}\rangle^d\times\mathcal{K}\langle\mathbf{x},\mathbf{y},t,z\rangle^{a+1}.$$

In particular $\widetilde{Q}:=\widetilde{q}_0(x_1,x_1x_2,\ldots,x_1x_2^a)+\cdots+\widetilde{q}_{d-1}(x_1,x_1x_2,\ldots,x_1x_2^a)z^{d-1}$ is an algebraic power series satisfiying

$$\widetilde{Q}(\mathbf{x},\gamma(x_2))=Q(\mathbf{x},\gamma(x_2)).$$

Moreover we have $\widetilde{Q}=\sum_{k\in\mathbb{N}}x_1^k\widetilde{r}_k(x_2,z)$ where the $\widetilde{r}_k(x_2,z)\in\mathcal{K}[x_2,z]$ with $\deg_z(\widetilde{r}_k)\leqslant d-1$ and $\deg_{x_2}(\widetilde{r}_k)\leqslant ak$. So, since for every k, $r_k(x_2,\gamma(x_2))=\widetilde{r}_k(x_2,\gamma(x_2))$, we have $\widetilde{r}_k=r_k$ since the degree of the minimal polynomial of $\gamma(x_2)$ over $\mathcal{K}[x_2]$ is d. In particular we have

$$P(\mathbf{x},\gamma'(x_2))=Q(\mathbf{x},\gamma(x_2))=\widetilde{Q}(\mathbf{x},\gamma'(x_2))\in\mathcal{K}\langle x_1,x_2\rangle$$

for every root $\gamma'(x_2)$ of Γ. This ends the proof. $\square$

Lemma 3.2. *Let $\Gamma(\mathbf{x}, z) \in \mathcal{K}[\mathbf{x}, z]$ be a monic polynomial in z of degree e. Let $p(\mathbf{x}, z) \in \mathcal{K}[\mathbf{x}, z]$ with $\deg_z(p) \leqslant d$, where $d \geqslant e-1$. Consider the division of p by Γ:*

$$p(\mathbf{x}, z) = \Gamma(\mathbf{x}, z)q(\mathbf{x}, z) + r(\mathbf{x}, z)$$

with $\deg_z(r) < e$. Then $\deg_{\mathbf{x}}(r) \leqslant \deg_{\mathbf{x}}(p) + (d-e+1)\deg_{\mathbf{x}}(\Gamma)$.

Proof. The proof is made by induction on $d \geqslant e-1$. If $d = e-1$, it is clear. Assume that the result is proved for polynomials of degree $d-1$ where $d \geqslant e$. We can write

$$p(\mathbf{x}, z) = \Gamma(\mathbf{x}, r) \times b_e(\mathbf{x}) + \widetilde{b}(\mathbf{x}, z)$$

where $p_e(\mathbf{x})$ is the coefficient of z^e in $p(\mathbf{x}, z)$, and $\deg_z(\widetilde{p}) < \deg_z(b)$. Therefore $\deg_{\mathbf{x}}(\widetilde{p}) \leqslant \deg_{\mathbf{x}}(p) + \deg_{\mathbf{x}}(\Gamma)$. Since p and $\widetilde{p}$ have the same remainder r by the division by $\Gamma(\mathbf{x}, z)$, we apply the inductive assumption to see that

$$\deg_{\mathbf{x}}(r) \leqslant \deg_{\mathbf{x}}(\widetilde{p}) + (d-e)\deg_{\mathbf{x}}(\Gamma) \leqslant \deg_{\mathbf{x}}(p) + (d-e+1)\deg_{\mathbf{x}}(\Gamma).$$

□

3.2. **Convergent complex power series.** In our previous work [BCR21], we gave a proof of Gabrielov's rank Theorem for the family of rings of complex convergent power series $(\mathbb{C}\{x_1, \ldots, x_n\})_n$. In fact this family is also a W-temperate family. Indeed every property of Definition 2.1 is classical. Property i) and ii) of Definition 2.2 are well-known; they are respectively given in [BCR21, Lemma 2.6] and [AM70, p. 31]. Finally property 2.2 iii) has been essentially proven at the end of the proof of [BCR21, Theorem 5.18] and is based on [BCR21, Lemma 5.36]. We now recall the idea of the proof, starting by the statement of the later Lemma:

Lemma 3.3 ([BCR21, Lemma 5.36])**.** *Let $\mathcal{C} \subset \mathbb{C}^n$ be an irreducible algebraic curve, and D_1, D_2 be two compact subsets of $\mathcal{C}$, such that the interior of D_1 is nonempty. Then*

$$\exists M > 0, \forall P \in \mathbb{C}[z_1, \ldots, z_n], \quad \|P\|_{D_2} \leqslant M^{deg(P)}\|P\|_{D_1},$$

where $\|P\|_D$ denotes $\max_{z \in D}|P(z)|$.

We now follow the notation of axiom iii). Denote by $\Gamma \in \mathcal{K}[t, z]$ the minimal polynomial of γ and let $\gamma' \in \mathcal{K}[\![t]\!]$ be a conjugate root; note that γ and $\gamma' \in \mathcal{K}\{\{t\}\}$ since they are algebraic. Consider the curve $\mathcal{C} \subset \mathcal{K}^2_{t,z}$ given by $\Gamma[t, z] = 0$, and let $\pi : \mathcal{K}^2_{t,z} \to \mathbb{K}_t$ be the projection $\pi(t, z) = t$. Since γ and $\gamma' \in \mathcal{K}\{\{t\}\}$, there exists a compact disc $D_0 \subset \mathbb{K}_t$ centered at the origin such that, γ and γ' are analytic function for $t \in D_0$. Denote by D and $D' \subset \mathcal{C}$ the graphs of γ and γ' over D_0 respectively. Now, recall that $P(\mathbf{x}, \gamma(x_2)) \in \mathcal{K}\{\{x_1, x_2\}\}$, where:

$$P(\mathbf{x}, z) = \sum_{k \in \mathbb{N}} x_1^k p_k(x_2, z)$$

and $p_k(x_2, z)$ are polynomials such that $\deg_{x_2}(p_k) \leqslant \alpha k$ for some $\alpha \in \mathbb{N}$. In particular this implies that there exists A and $B > 0$ such that:

$$\|p_k(x_2, \gamma(x_2))\|_{D_0} = \|p_k(x_2, z)\|_D \leqslant A\, B^k\, k!, \quad \forall k \in \mathbb{N}$$

so that, by Lemma 3.3, we conclude that:

$$\|p_k(x_2, \gamma'(x_2))\|_{D_0} = \|p_k(x_2, z)\|_{D'} \leqslant M^{\alpha k}\|p_k(x_2, z)\|_D \leqslant A\,(M^\alpha B)^k\, k!, \quad \forall k \in \mathbb{N}$$

which implies that $P(\mathbf{x}, \gamma'(x_2)) \in \mathcal{K}\{\{x_1, x_2\}\}$ as we wanted to prove.

3.3. **Eisenstein power series.** Let $\mathcal{O}$ be a UFD, and let $\mathcal{K}$ be an algebraic closure of its fraction field. The ring of Eisenstein series over $\mathcal{O}$ is the filtered limit of rings:

$$\bigcup_{\mathfrak{c}\in\mathcal{K}}\bigcup_{f\in\mathcal{O}\smallsetminus\{0\}}\mathcal{O}_f[\![x_1,\ldots,x_n]\!][\mathfrak{c}]$$

where $\mathcal{O}_f$ denotes the localization of $\mathcal{O}$ with respect to the multiplicative family $\{1,f,f^2,\ldots,\}$.

Remark 3.4. In our proofs of Theorems 1.3 and 1.4, we will use of Eisenstein power series in the proof of Lemma 5.18 below, in the following way. Given a closed polydisc $D\subset\mathbb{C}^n$, denote by $\mathcal{O}(D)$ the ring of analytic functions defined in a neighborhood of D, and note that it is an UFD by [Da74]. We then consider the Eisenstein power series given by $\mathcal{O}=\mathcal{O}(D)$.

The main result of this subsection is the following:

Proposition 3.5. *If $\mathcal{O}$ is a UFD containing an uncountable characteristic zero field $\Bbbk$, the ring of Eisenstein series is a W-temperate family over $\mathcal{K}$.*

Proof. Axioms i) ii) iii) iv) of Definition 2.1 are easily verified.

Axiom v) of Definition 2.1: consider F and $G\in\mathcal{O}_f[\![\mathbf{x}]\!][\mathfrak{c}]$ as in the statement of Axiom v). We have $F(0,x_n)=x_n^du(x_n)$. If we multiply f by $u(0)$, we may assume that $u(x_n)$ is a unit in $\mathcal{O}_f[\mathfrak{c}][\![x_n]\!]$. Let $\mathbb{L}$ be the fraction field of $\mathcal{O}_f[\mathfrak{c}]$. By the Weierstrass division theorem for power series in $\mathbb{L}[\![\mathbf{x}]\!]$, $G=QF+R$ where $Q\in\mathbb{L}[\![\mathbf{x}]\!]$ and $R\in\mathbb{L}[\![\mathbf{x}']\!][x_n]$ and $\deg_{x_n}(R)<d$. We claim that the coefficients of Q and R are also in $\mathcal{O}_f[\mathfrak{c}]$. Indeed, fix the following order on the monomials: We have $\mathbf{x}^\alpha<\mathbf{x}^\beta$ if

$$(\alpha_1+\cdots+\alpha_{n-1}+(d+1)\alpha_n,\alpha_1,\ldots,\alpha_n)<_{\text{lex}}(\beta_1+\cdots+\beta_{n-1}+(d+1)\beta_n,\beta_1,\ldots,\beta_n)$$

where $<_{\text{lex}}$ denotes the lexicographic order. In particular the nonzero monomial of least order in the expansion of F is Cx_n^d where C is a unit in $\mathcal{O}_f[\mathfrak{c}]$. For a series $H\in\mathbb{L}[\![\mathbf{x}]\!]$ we denote by $\operatorname{in}(H)$ the monomial of least weight in the expansion of H.

We now consider an inductive way to construct the unique coefficients Q and R. We start by setting $G^{(0)}=G$, $Q^{(0)}=0$ and $R^{(0)}=0$. Fix $k\geqslant 0$, and assume that $Q^{(\ell)}$ and $R^{(\ell)}$ have been constructed for every $\ell\leqslant k$ in such a way that $G^{(\ell)}=G-FQ^{(\ell)}-R^{(\ell)}$ satisfies $\operatorname{ord}(G_{\ell+1})\geqslant\operatorname{ord}(G_\ell)$. We consider the two following cases:

i) If $\operatorname{in}(G^{(k)})$ is divisible by x_n^d, we set $R^{(k+1)}:=R^{(k)}$ and $Q^{(k+1)}:=Q^{(k)}+{}^{\operatorname{in}(G^{(k)})}\!/_{\operatorname{in}(F)}$.

ii) If $\operatorname{in}(G^{(k)})$ is not divisible by x_n^d, we set $R^{(k+1)}:=R^{(k)}+\operatorname{in}(G^{(k)})$ and $Q^{(k+1)}:=Q^{(k)}$.

By the formal Weierstrass division Theorem, this process converges as $G^{(k)}\longrightarrow 0$, $Q^{(k)}\longrightarrow Q$ and $R^{(k)}\longrightarrow R$ when $k\longrightarrow\infty$. But we see that we do not need to introduce elements of $\mathbb{L}$ that does not belong to $\mathcal{O}_f[\mathfrak{c}]$ because the coefficient of initial term of F is a unit in $\mathcal{O}_f[\mathfrak{c}]$. This proves the claim.

Axiom i) of Definition 2.2: This property is easily verified.

Axiom ii) of Definition 2.2: The property follows from the following Lemma, which is stronger version of the axiom valid for Eisenstein power series:

Lemma 3.6. *In the conditions of the statement of Proposition 3.5, consider $F \in \mathcal{K}[\![\mathbf{x}]\!]$ and assume that $F \notin \mathcal{K}\{\{\mathbf{x}\}\}$. Then the following set is countable*

$$W := \{\lambda \in \Bbbk \mid F(\mathbf{x}', \lambda x_1) \in \mathcal{K}\{\{\mathbf{x}'\}\}\}.$$

Proof. We start by a general claim. Let $P(x_1, x_2) \in \mathcal{O}[x_1, x_2]$ be a homogeneous polynomial. Write

$$P = \sum_{k=0}^{d} p_k x_1^{d-k} x_2^k,$$

so that $P(x_1, \lambda x_1) = \left(\sum_k p_k \lambda^k\right) x_1^d$. Let $g \in \mathcal{O}$, $g \neq 0$; we claim that if $\gcd(p_k, k = 0, \ldots, d) = 1$, then $\gcd\left(\sum_k p_k \lambda^k, g\right) \neq 1$ for at most finitely many $\lambda \in \Bbbk$. Indeed, assume that $\gcd\left(\sum_k p_k \lambda^k, g\right) \neq 1$ for infinitely many $\lambda \in \Bbbk$. Since g has finitely many factors, this implies that g has an irreducible factor h such that, for infinitely many $\lambda \in \Bbbk$, h divides $\sum_k p_k \lambda^k$. Hence the polynomial $Q(T) := \sum_k p_k T^k \in \mathrm{Frac}(\mathcal{O}/(h))[T]$ has infinitely many roots in $\Bbbk$, which is possible only if h divides all the p_k since $\mathrm{Frac}(\mathcal{O}/(h))$ is an infinite field (it is a field containing $\Bbbk$). This contradicts the hypothesis, proving the Claim.

Now, we prove the contrapositive of the Lemma, that is, consider an element $F \in \mathcal{K}[\![\mathbf{x}]\!]$ such that

$$W := \{\lambda \in \Bbbk \mid F(\mathbf{x}', \lambda x_1) \in \mathcal{K}\{\{\mathbf{x}'\}\}\}$$

is uncountable, and let us prove that $F \in \mathcal{K}\{\{\mathbf{x}\}\}$. Let $\mathbb{L}$ be the fraction field of $\mathcal{O}$. Since F has countably many coefficients, the field extension of $\mathbb{L}$ generated by the coefficients of F is a $\mathbb{L}$-vector space of countable dimension. Let $(\mathfrak{c}_k)_{k\in\mathbb{N}}$ be a $\mathbb{L}$-basis of this vector space, so

$$F(\mathbf{x}) = \sum_{k\in\mathbb{N}} \mathfrak{c}_k F_k(\mathbf{x})$$

where the $F_k(\mathbf{x})$ are in $\mathbb{L}[\![\mathbf{x}]\!]$ and, for each $\alpha \in \mathbb{N}^n$, the coefficient of $\mathbf{x}^\alpha$ is zero in all but finitely many $F_k(\mathbf{x})$. Moreover, we can write

$$F_k(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^{n-2}} \left(\sum_{d\in\mathbb{N}} \frac{P_{k,\alpha,d}(x_1, x_n)}{g_{k,a,d}}\right) x_2^{a_2} \cdots x_{n-1}^{\alpha_{n-2}}$$

where the $P_{k,\alpha,d} \in \mathcal{O}[x_1, x_n]$ are homogeneous polynomials of degree d, and $g_{k,\alpha,d}$ is coprime with the gcd of the coefficients of $P_{k,\alpha,d}$. Now $\gcd(P_{k,\alpha,d}(1, \lambda), g_{k,\alpha,d}) = 1$ for $\lambda \in E_{k,\alpha,d}$ where $E_{k,\alpha,d} \subset W$ is cofinite by the Claim. Thus the complement of the set $E := \cap_{k,\alpha,d} E_{k,\alpha,d}$ in $\Bbbk$ is at most countable. Therefore $E \cap W \neq 0$ if $\Bbbk$ is uncountable. Hence, by choosing $\lambda \in E \cap W$, there is $f \in \mathcal{O}$ such that $F_k(\mathbf{x}', \lambda x_1) \in \mathcal{O}_f[\![\mathbf{x}']\!]$ for every k. Then we see that for every k, α and d, $g_{k,\alpha,d}$ divides a power of f, whence $F(\mathbf{x}) \in \mathcal{O}_f[\![\mathbf{x}]\!]$. $\square$

Axiom iii) of Definition 2.2: The proof of this result is based on the following Galois-type result whose proof we postpone to §§§3.3.1

Theorem 3.7. *Let A be a UFD, $\mathbb{L}$ be its fraction field and $\mathfrak{c}$ be in an algebraic closure of $\mathbb{L}$ and be separable over $\mathbb{L}$. Let Γ in $A[t, z]$ be irreducible. Assume that Γ splits as a polynomial with coefficients in $A[\mathfrak{c}][\![t]\!]$ and let the $\gamma_i(t) \in A[\mathfrak{c}][\![t]\!]$ denote the roots of Γ. Then there is $f \in A$ such that, for every $Q \in \mathbb{L}[t, z]$:*

$$Q(t, \gamma_1) \in A[\mathfrak{c}][\![t]\!] \Longrightarrow Q(t, \gamma_2) \in A_f[\mathfrak{c}][\![t]\!].$$

We now follow the notation of axiom iii). By the definition of Eisenstein power series and the assumption, we know that $P(\mathbf{x},\gamma(x_2)) \in \mathcal{O}_g[\mathfrak{c}][\![x_1,x_2]\!]$ for some $g \in \mathcal{O}$ and some $\mathfrak{c} \in \mathcal{K}$; note that $\mathfrak{c}$ is separable over $\mathcal{K}$ since $\mathcal{K}$ is of characteristic zero. Moreover, by assumption:

$$P(\mathbf{x},z) = \sum_{k\in\mathbb{N}} x_1^k p_k(x_2,z)$$

where $p_k(x_2,z)$ are polynomials such that $\deg_{x_2}$ is bounded by a liner function in k. Since $P(\mathbf{x},\gamma(x_2)) \in \mathcal{O}_g[\mathfrak{c}][\![x_1,x_2]\!]$, we conclude that $p_k(x_2,\gamma(x_2)) \in \mathcal{O}_g[\mathfrak{c}][\![x_2]\!]$. Let γ' be a conjugate root of γ. By Theorem 3.7 applied to $A = \mathcal{O}_g$, there is $f \in \mathcal{O}$ such that, for every $k \in \mathbb{N}$, $p_k(x_2,\gamma'(x_2)) \in O_{fg}[\mathfrak{c}][\![x_2]\!]$. Thus $P(\mathbf{x},\gamma'(x_2)) \in \mathcal{O}_{fg}[\mathfrak{c}][\![\mathbf{x}]\!]$, proving that the axiom is verified. □

Remark 3.8. The following example shows that we really need f in the statement of Theorem 3.7. Let $f \in A$ irreducible and let $\Gamma(t,z) = z^2 - (1+ft)$. So $\gamma_1 = \sqrt{1+ft}$ and $\gamma_2 = -\sqrt{1+ft}$. For $Q = \frac{1}{f}(1-z)$, we have

$$Q(\gamma_1) = \frac{1}{f}\left(1-\sqrt{1+ft}\right) \in A[\![t]\!], \quad Q(\gamma_2) = \frac{1}{f}\left(1+\sqrt{1+ft}\right) \in A_f[\![t]\!] \setminus A[\![t]\!].$$

3.3.1. *Algebraic power series with coefficients in a UFD and proof of Theorem 3.7.* In this subsubsection, we provide a proof of Theorem 3.7, and we collect results concerning algebraic power series which are of independent interest. We start with a simple Lemma:

Lemma 3.9. *Let $\Gamma(t,z) \in A[t,z]$ be a polynomial with coefficients in an integral domain A. Let us write $\Gamma = a_0(t)z^d + \cdots + a_d(t)$. Assume that $\Gamma(t,z)$ has a root in $A[t]$ of degree D. Then $D \leqslant \max_i\{\deg_t(a_i(t))\}$.*

Proof. After changing the indices we may assume that $a_0(t) \neq 0$. Let $F(t) \in A[t]$ with $\deg_t(F(t)) = D$ and assume that $D > \max_i\{\deg_t(a_i(t))\}$. Then for $i > 0$:

$$\deg_t(a_i(t)F(t)^{d-i}) \leqslant \max_j\{\deg_t(a_j(t))\} + (d-i)D < dD \leqslant \deg_t(a_0(t)F(t)^d).$$

Therefore $\Gamma(t,F(t)) \neq 0$. □

The next Lemma shows that the coefficients of an algebraic infinite series over an UFD satisfies strong relations:

Lemma 3.10. *Let A be a UFD, $\mathfrak{c}$ in an algebraic closure of $\mathrm{Frac}(A)$ be finite and separable over A. Let $\mathcal{P}$ be a representative family of primes of A (i.e. each principal prime ideal of A is generated by a unique element of $\mathcal{P}$) and $F(t) \in A[\mathfrak{c}][\![t]\!] \setminus A[\mathfrak{c},t]$ be algebraic over $A[t]$. Then the following set is finite*

$$\{g \in \mathcal{P} \mid F(t) \in A[\mathfrak{c},t] + (g)A[\mathfrak{c}][\![t]\!]\}.$$

Proof. We start by showing that we can reduce the Lemma to the case that $F(t) \in A[\![t]\!] \setminus A[t]$, that is, F is independent of $\mathfrak{c}$. Indeed let e denote the degree of $\mathfrak{c}$ over A. Then $F(t)$ can be written in a unique way as

$$F(t) = F_0(t) + F_1(t)\mathfrak{c} + \cdots + F_{e-1}(t)\mathfrak{c}^{e-1}$$

where the $F_i(t)$ belong to $A[\![t]\!]$. We denote by $\mathfrak{c}_2, \ldots, \mathfrak{c}_e$ the distinct conjugates of $\mathfrak{c}$ over A. The power series $F^{(j)}(t) = F_0(t) + F_1(t)\mathfrak{c}_j + \cdots + F_{e-1}(t)\mathfrak{c}_j^{e-1}$ for $j = 2$,

..., e are the conjugates of $F(t)$ over $A[\![t]\!]$, therefore they are also algebraic over $A[t]$. We have

$$\begin{bmatrix} F(t) \\ F^{(2)}(t) \\ \vdots \\ F^{(e)}(t) \end{bmatrix} = \begin{bmatrix} 1 & \mathfrak{c} & \mathfrak{c}^2 & \cdots & \mathfrak{c}^{e-1} \\ 1 & \mathfrak{c}_2 & \mathfrak{c}_2^2 & \cdots & \mathfrak{c}_2^{e-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathfrak{c}_e & \mathfrak{c}_e^2 & \cdots & \mathfrak{c}_e^{e-1} \end{bmatrix} \cdot \begin{bmatrix} F_0(t) \\ F_1(t) \\ \vdots \\ F_{e-1}(t) \end{bmatrix}$$

The Vandermonde matrix is invertible, its entries are algebraic over A, thus the entries of its inverse are algebraic over A. Therefore the $F_i(t) \in A[\![t]\!]$ are algebraic over $A[t]$. Thus, it is enough to prove the lemma for the $F_i(t)$; we may therefore assume that $F(t) \in A[\![t]\!]$.

Write $F(t) = \sum_{k\in\mathbb{N}} F_k t^k$ with $F_k \in A$ for every k. Let $g \in A$ and $N \in \mathbb{N}^*$. We have that $F(t)$ is equal to a polynomial of degree $\leqslant N$ in t modulo $gA[\![t]\!]$ if and only if

$$\forall k > N, \quad F_k \in (g).$$

Therefore for distinct $g_1, \ldots, g_s \in \mathcal{P}$ and because A is a UFD, $F(t)$ is equal to a polynomial of degree $\leqslant N$ in t modulo every $g_i A[\![t]\!]$ if and only if

$$\forall k > N, \quad F_k \in (g_1 \cdots g_s).$$

Since $F(t) \notin A[t]$, we conclude that there does not exist an infinite subset $\mathcal{G}_N \subset \mathcal{P}$ such that for every $g \in \mathcal{G}_N$, $F(t)$ is equal to a polynomial of degree $\leqslant N$ in t modulo $gA[\![t]\!]$. In particular, if we assume by contradiction that the set $\{g \in \mathcal{P} \mid F(t) \in A[t] + (g)A[\![t]\!]\}$ is not finite, then there exists a sequence (g_n) of distinct primes in $\mathcal{P}$, such that $F(t)$ is equal to a polynomial of degree N_n modulo $g_n A[\![t]\!]$ where the sequence $(N_n)_n$ is increasing and tends to infinity. In what follows, we show that the existence of this sequence would contradict Lemma 3.9.

Indeed, since $F(t)$ is algebraic, we may consider $\Gamma(t,z) := a_0(t)z^d + \cdots + a_d(t) \in A[t,z]$ a polynomial such that $\Gamma(t, F(t)) = 0$ and $a_0(t) \neq 0$. Denote by $F_n(t)$ (resp. $\Gamma_n(t,z)$) the image of $F(t)$ in $A/(g_n)[\![t]\!]$ (resp. of $\Gamma(t,z)$ in $A/(g_n)[t,z]$). We have $\deg_t(F_n(t)) = N_n$ and $\Gamma_n(t, F_n(t)) = 0$. For $n \in \mathbb{N}$ large enough we have that $\Gamma_n(t,z) \neq 0$ and $\deg_z(\Gamma_n(t,z)) = d$, because any given $a \in A$ has finitely many prime divisors. We conclude from Lemma 3.9 that $N_n \leqslant \max_i\{\deg_t(a_i(t))\}$ for every n sufficiently big, yielding a contradiction. □

Remark 3.11. Recall that, in general, an irreducible polynomial $\Gamma(z)$ with coefficients in a UFD may be reducible modulo infinitely many primes of A. One classical example is given by $\Gamma(z) = z^4 + 1$ that is irreducible over $\mathbb{Z}[z]$ but reducible modulo every prime number p. In contrast, Lemma 3.10 guarantees that for an irreducible polynomial $\Gamma(t,z) \in A[t,z]$, the set

$$\{g \in \mathcal{P} \mid \Gamma(t,z) \text{ is reducible modulo } (g)\}$$

is finite, provided that Γ has a root in $A[\mathfrak{c}][\![t]\!] \smallsetminus A[\mathfrak{c}][t]$.

Before proving Theorem 3.7, recall that given a UFD A and $f \in A$, $f \neq 0$, the the localization A_f is also a UFD; we will use this observation implicitly below. We recall that this claim follows from the fact that a UFD is a Krull domain in which every prime ideal of height 1 is principal. Since A is a UFD, it is a Krull domain so A_f is also a Krull domain. Because the localization morphism $A \longrightarrow A_f$ induces an isomorphism between the primes of A_f and the primes of A avoiding f, every prime ideal of A_f of height 1 is necessarily principal. We are now ready to prove our main result about algebraic power series with coefficients in a UFD:

Proof of Theorem 3.7. We start by showing that we may suppose that Γ is monic. We write

$$\Gamma = p_0 z^d + p_1 z^{d-1} + \cdots + p_d.$$

We have $p_0^{d-1}\Gamma(t,z) = R(t, p_0 z)$ where

$$R(t,z) = T^d + p_1 T^{d-1} + p_2 p_0 T^{d-2} + \cdots + p_d p_0^{d-1}.$$

We have $R(t, p_0\gamma_i) = 0$ for $i = 1$, 2. If we set $\gamma_i' = p_0\gamma_i$, and we prove the statement of the Theorem for the γ_i' then we also deduce the statement of the Theorem for the γ_i, since $Q(t, p_0^{-1}z) \in \mathbb{L}[t,z]$ if and only if $Q \in \mathbb{L}[t,z]$; therefore, we suppose that Γ is monic in z.

Let us first treat the case that $\mathfrak{c} \in \mathbb{L}$. By replacing A by A_g for some well chosen $g \in A$, we can assume that $\mathfrak{c} \in A$. We claim that there exists $f \in A$ such that

$$\forall P \in A[t,z], \forall g \in A,\ P(t,\gamma_1) \in gA[\![t]\!] \Longrightarrow P(t,\gamma_2) \in gA_f[\![t]\!]. \tag{3}$$

Note that the Theorem then follows from the Claim. Indeed, if $Q \in \mathbb{L}[t,z]$ then there exists $g \in A$ such that $P = g\,Q \in A[t,z]$. In particular, $Q(t,\gamma_1(t)) \in A[\![t]\!]$ implies that $P(t,\gamma_1(t)) \in gA[\![t]\!]$, so the Claim implies that $P(t,z) \in gA_f[\![t]\!]$ and, therefore, $Q \in A_f[\![t]\!]$. In order to prove the Claim, we start by noting that, since A is a UFD, it is enough to prove the Claim for every irreducible element g of A. By replacing P by its remainder under its Euclidean division by Γ, furthermore, we may assume that $\deg_z(P) < d$. So let's consider the set

$$\mathcal{G} := \{g \in A \text{ prime } \mid \exists P,\ P(t,\gamma_1) \in gA[\![t]\!], P(t,z) \notin gA[t,z] \text{ and } \deg_z(P) < d\},$$

and let's prove that it is finite (up to multiplication by a unit). Indeed, note that if $P(t,\gamma_1) \in gA[\![t]\!]$ and $P(t,z) \notin gA[t,z]$, we have that $\overline{\Gamma}$ is not irreducible $A/g[t,z]$, where $\overline{R}$ denote the image of a polynomial $R \in A[\![t]\!][z]$ in $A/g[\![t]\!][z]$. Thus

$$\prod_{i \in E_g} (z - \overline{\gamma}_i) \in A/g[t,z]$$

for some $E_g \subsetneq \{1,\dots,d\}$. For $E \subsetneq \{1,\dots,d\}$, we set $\Gamma_E := \prod_{i\in E}(z-\gamma_i)$.

Now assume by contradiction that $\mathcal{G}$ is infinite. In this case, there is $E \subsetneq \{1,\dots,d\}$ such that $\Gamma_E \in A/g[t,z]$ for infinitely many primes g. But the coefficients of Γ_E are in $A[\![t]\!]$, and at least one of them is not in $A[t]$ because Γ is irreducible in $A[t,z]$. Since the γ_i are algebraic over $A[t]$, the coefficients of Q_E are also algebraic over $A[t]$. We therefore obtain a contradiction with Lemma 3.10, and conclude that $\mathcal{G}$ is finite. We may therefore define

$$f = \prod_{g\in\mathcal{G}} g.$$

Note that the Claim is verified with this choice of f for every irreducible g by construction. Thus the Claim is proved, finishing the case that $\mathfrak{c} \in \mathbb{L}$.

Now we assume that $\mathfrak{c} \notin \mathbb{L}$. Since $\mathfrak{c}$ is algebraic, we may write $a_0\mathfrak{c}^e + a_1\mathfrak{c}^{e-1} + \cdots + a_e = 0$ where $a_i \in A$ for every $i = 0,\dots,e$. By replacing A by A_{a_0} we may assume that $\mathfrak{c}$ is finite over A (of degree e). For every i we can write in a unique way

$$\gamma_i = \gamma_{i,0} + \gamma_{i,1}\mathfrak{c} + \cdots + \gamma_{i,e-1}\mathfrak{c}^{e-1} \tag{4}$$

where the $\gamma_{i,j}$ belong to $A[\![t]\!]$ and are algebraic over $A[t]$. For $Q \in \mathbb{L}[t,z]$ we can expand in a unique way

$$Q(t, z_0 + z_1\mathfrak{c} + \cdots + z_{e-1}\mathfrak{c}^{e-1}) = \sum_{k=0}^{e-1} Q_k(t, z_0, \ldots, z_{e-1})\mathfrak{c}^k$$

where the Q_k belong to $\mathbb{L}[t, z_0, \ldots, z_{e-1}]$. For $i \neq 1$, γ_i is obtained from γ_1 expanded as in (4) by replacing the $\gamma_{1,j}$ by its conjugates $\gamma_{i,j}$. Following the same logic as of the first case, we are reduced to proving the claim that there is $f \in A$ such that for every $P \in A[t, z_0, \ldots, z_{e-1}]$ and every $g \in A$,

$$P(t, \gamma_{1,0}, \ldots, \gamma_{1,e-1}) \in gA[\![t]\!] \Longrightarrow P(t, \gamma_{2,0}, \ldots, \gamma_{2,e-1}) \in gA_f[\![t]\!].$$

By the primitive element Theorem (that we can apply since $\mathbb{L}[t] \longrightarrow \mathbb{L}\langle t\rangle$ is separable), we have that

$$\mathbb{L}(t, \gamma_{i,0}, \ldots, \gamma_{i,e-1}) = \mathbb{L}\left(t, \sum_{k=0}^{e-1} \lambda_k \gamma_{i,k}\right)$$

for every $(\lambda_k)_k$ in a Zariski open dense subset V_i of $\mathbb{L}^e$. Therefore we may choose $(\lambda_k)_k \in \cap_{i=1}^d V_i$ and assume that for every $i = 1, \ldots, d$

$$\mathbb{L}(t, \gamma_{i,0}, \ldots, \gamma_{i,e-1}) = \mathbb{L}\left(t, \sum_{k=0}^{e-1} \lambda_k \gamma_{i,k}\right).$$

Thus there is $\Gamma_{i,k} \in \mathbb{L}(t)[U]$ such that

$$\gamma_{i,k} = \Gamma_{i,k}\left(t, \sum_{k=0}^{e-1} \lambda_k \gamma_{i,k}\right). \tag{5}$$

By replacing the $\gamma_{i,k}$ by their conjugates $\gamma_{i',k}$ in (5) we see that we can choose the $\Gamma_{i,k}$ to be independent of i. From now we denote $\Gamma_{i,k}$ by Γ_k, and $\sum_{k=0}^{e-1} \lambda_k \gamma_{i,k}$ by δ_i. By the claim made in the first case where we assumed that $\mathfrak{c} \in \mathbb{L}$, there exists $f \in A$ such that

$$\forall P' \in A[t,z], \forall g \in A,\ P'(t, \delta_1) \in gA[\![t]\!] \Longrightarrow P'(t, \delta_2) \in gA_f[\![t]\!].$$

Now, let $P \in A[t, z_0, \ldots, z_{e-1}]$ and $g \in A$ such that

$$P(t, \gamma_{1,0}, \ldots, \gamma_{1,e-1}) \in gA_f[\![t]\!].$$

Let $D(t) \in A[t]$ be a common denominator of the Γ_k, that is, a polynomial such that $D(t)\Gamma_k \in A[t,U]$ for every k. Then there is an integer ℓ, depending on P, such that

$$R(t,z) := D(t)^\ell P(t, \Gamma_0(t,z), \ldots, \Gamma_{e-1}(t,z)) \in A[t,z].$$

By assumption $R(t, \delta_1) \in gA[\![t]\!]$, whence $R(t, \delta_2) \in gA_f[\![t]\!]$. We can write $D(t) = f_0 t^{d_0} \times u(t)$ where $f_0 \in A$ and $u(t) \in A_{f_0}[t]$ satisfies $u(0) = 1$. This shows that $P(t, \gamma_{2,0}, \ldots, \gamma_{2,e-1}) \in gA_{ff_0}[\![t]\!]$ proving the Claim, and the Theorem is proven. □

4. Proof of W-temperate rank Theorem

4.1. **Reduction of Theorem 1.1 to Theorem 4.1.** We start by proving by contradiction that the following result implies Theorem 1.1:

Theorem 4.1. *Let $\varphi : \mathcal{K}\{\{\mathbf{x}, y\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ be a morphism of rings of W-temperate power series such that*

i) The kernel of φ is generated by one Weierstrass polynomial $P \in \mathcal{K}[\![\mathbf{x}]\!][y]$.
ii) $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = n$.

Then $P \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$.

Reduction of Theorem 1.1 to Theorem 4.1. We follow closely [BCR21, page 1347]. Assume that Theorem 1.1 does not hold, that is, there exists a morphism of rings of W-temperate power series $\varphi : \mathcal{K}\{\{\mathbf{x}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_m)$, such that $\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) \geqslant 1$, but $\mathrm{r}^{\mathcal{F}}(\varphi) < \mathrm{r}^{\mathcal{T}}(\varphi)$. Consider the induced injective morphism $\mathcal{K}\{\{\mathbf{x}\}\}/\mathrm{Ker}(\varphi) \longrightarrow \mathcal{K}\{\{\mathbf{u}\}\}$ and, by the Noether normalization given in Proposition 5.16 vi), there exists a finite injective morphism $\tau : \mathcal{K}\ \{\{\tilde{\mathbf{x}}\}\} \longrightarrow \mathcal{K}\{\{\mathbf{x}\}\}/\mathrm{Ker}(\varphi)$. By Proposition 2.7, we can replace φ by $\varphi \circ \tau$, that is, we may assume that φ is injective.

Next, since $\mathrm{r}^{\mathcal{F}}(\varphi) < \mathrm{r}^{\mathcal{T}}(\varphi) = m$, we know that $\mathrm{Ker}(\widehat{\varphi}) \neq (0)$. Now, suppose that $\mathrm{Ker}(\widehat{\varphi})$ is not principal or, equivalently, that its height is at least 2. By the normalization theorem for formal power series, after a linear change of coordinates, the canonical morphism

$$\pi : \mathcal{K}[\![x_1, \ldots, x_{\mathrm{r}(\varphi)}]\!] \longrightarrow \frac{\mathcal{K}[\![\mathbf{x}]\!]}{\mathrm{Ker}(\widehat{\varphi})}$$

is finite and injective. Thus, the ideal $\mathfrak{p} := \mathrm{Ker}(\widehat{\varphi}) \cap \mathcal{K}[\![x_1, \ldots, x_{\mathrm{r}(\varphi)+1}]\!]$ is a nonzero height one prime ideal. Because $\mathcal{K}[\![x_1, \ldots, x_{\mathrm{r}(\varphi)+1}]\!]$ is a unique factorization domain, $\mathfrak{p}$ is a principal ideal (see [Mat89, Theorem 20.1] for example). After a linear change of coordinates, we may assume that $\mathfrak{p}$ is generated by a Weierstrass polynomial $P \in \mathcal{K}[\![x_1, \ldots, x_{\mathrm{r}(\varphi)}]\!][x_{\mathrm{r}(\varphi)+1}]$.

Now, denote by φ' the restriction of φ to $\mathcal{K}\{\{x_1, \ldots, x_{\mathrm{r}(\varphi)+1}\}\}$. By definition P is a generator of $\mathrm{Ker}(\widehat{\varphi}')$, thus $\mathrm{r}^{\mathcal{F}}(\varphi') = \mathrm{r}(\varphi) + 1 - 1 = \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi)$. Since φ is injective, φ' is injective and P does not belong to $\mathcal{K}\{\{x_1, \ldots, x_{\mathrm{r}(\varphi)}\}\}[x_{\mathrm{r}(\varphi)+1}]$. Moreover, since π is finite, we can use again Proposition 2.7, to see that

$$\mathrm{r}(\varphi') = \mathrm{r}(\widehat{\varphi'}) = \mathrm{r}(\widehat{\varphi}) = \mathrm{r}(\varphi).$$

Therefore we have $\mathrm{r}(\varphi') = \mathrm{r}^{\mathcal{F}}(\varphi') = m - 1$, contradicting Theorem 4.1. □

4.2. **Reduction to the low-dimensional case.** We now prove by contradiction that the following result implies Theorem 4.1:

Theorem 4.2. *Let $\varphi : \mathcal{K}\{\{x_1, x_2, y\}\} \longrightarrow \mathcal{K}\{\{u_1, u_2\}\}$ be a morphism of rings of W-temperate power series such that*

i) $\varphi(x_1) = u_1$ and $\varphi(x_2) = u_1 u_2$,
ii) $\mathrm{Ker}(\widehat{\varphi})$ is generated by one reduced Weierstrass polynomial $P \in \mathcal{K}[\![\mathbf{x}]\!][y]$.

Then $P \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$.

Reduction of Theorem 4.1 to Theorem 4.2. We follow closely [BCR21, 3rd Reduction]. Assume that there is a morphism φ satisfying the hypothesis of Theorem 4.1 but where $P \notin \mathcal{K}\{\{\mathbf{x}\}\}[y]$.

(1) First, after a linear change of coordinates in $\mathbf{u}$ we may assume that $\varphi(x_1)(u_1, 0, \ldots, 0) \neq 0$. Thus, the morphism $\sigma \circ \varphi$, where σ is given by

$$\sigma(u_1) = u_1 \text{ and } \sigma(u_i) = u_1 u_i \quad \forall i > 1$$

satisfies the hypotheses of Theorem 4.1 and its kernel is generated by P, by Proposition 2.7. Thus we may assume that $\varphi(x_1) = u_1^e \times U(\mathbf{u})$ where $U(\mathbf{u})$ is a unit. And by replacing x_1 by $\frac{1}{U(0)}x_1$ we may further assume that $U(0) = 1$.

(2) We define the morphism τ by

$$\tau(x_1) = x_1^e \text{ and } \tau(x_i) = x_i \quad \forall i > 1.$$

Let $V(\mathbf{u}) \in \mathcal{K}[\![\mathbf{u}]\!]$ be a power series such that $V(\mathbf{u})^e = U(\mathbf{u})$. Such a power series exists since $U(0) = 1$ and, by the Implicit Function Theorem cf. Proposition 2.8 i), $V(\mathbf{u}) \in \mathcal{K}\{\{\mathbf{u}\}\}$. We define the morphism ψ by

$$\psi(x_1) = u_1 V(\mathbf{u}) \text{ and } \psi(x_i) = \varphi(x_i) \quad \forall i > 1.$$

Then $\psi \circ \tau = \varphi$ and $P(x_1^e, x_2, \ldots, x_n, y) \in \operatorname{Ker}(\widehat{\psi})$. Since $P(x_1^e, x_2, \ldots, x_n, y)$ is a Weierstrass polynomial in y, $\operatorname{Ker}(\widehat{\psi})$ is generated by a Weierstrass polynomial Q that divides P. Thus P is the product of Q with the distinct polynomials $Q(\xi x_1, x_2, \ldots, x_n, y)$ where ξ runs over the e-th roots of unity. Therefore, if $Q \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$, $P \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$ which contradicts the hypothesis. Thus, ψ satisfies the hypothesis of Theorem 4.1 but $\operatorname{Ker}(\widehat{\psi})$ is generated by a Weierstrass polynomial that is not in $\mathcal{K}\{\{\mathbf{x}\}\}[y]$. By Proposition 2.7, we may replace φ by ψ and assume that $\psi(x_1) = x_1$ by composing ψ by the inverse of the temperate automorphism that sends u_1 onto $u_1 V(\mathbf{u})$.

(3) Now we have $\varphi(x_1) = u_1$ and we perform "Gabrielov's trick", cf. [BCR21, Example 3.5]. We denote by $\varphi_i(\mathbf{u})$ the image of x_i by φ. We consider the temperate automorphism χ defined by

$$\chi(x_1) = x_1 \text{ and } \chi(x_i) = x_i - \varphi_i(x_1, 0, \ldots, 0) \quad \forall i > 1.$$

If we replace φ by $\varphi \circ \chi$ we may assume that every nonzero monomial of $\varphi(x_i)$ is divisible by one of the u_i for $i > 1$. Then by replacing φ by $\sigma \circ \varphi$, where σ is defined above, we may assume that $\varphi(x_i) = u_1^{a_i} g_i(\mathbf{u})$ where $a_i > 0$, $g_i(0) = 0$ and $g_i(0, x_2, \ldots, x_n) \neq 0$, for $i > 1$. Moreover, by composing with the morphism

$$x_i \longmapsto x_i^{\prod_{k \neq i} a_k}$$

for $i > 1$, we may assume that $a_i = a$ is independent of i. Finally, by replacing x_1 by x_1^{a+1} we may assume that $\varphi(x_1) = u_1^{a+1}$. Composing φ with these morphisms does not change the ranks, by Proposition 2.7.

(4) Now we set, for $\lambda = (\lambda_2, \ldots, \lambda_n) \in \mathcal{K}^{n-1} \setminus \{0\}$, $h_\lambda = x_1 - \sum_{i=2}^n \lambda_i x_i$. We have

$$\varphi(h_\lambda) = u_1^a g_\lambda(\mathbf{u})$$

where $g_\lambda(\mathbf{u}) = u_1 - \sum_{i=2}^n \lambda_i g_i(\mathbf{u}) \in \mathcal{K}\{\{\mathbf{u}\}\}$. By the implicit function Theorem, there exists a unique nonzero $\xi_\lambda(u_2, \ldots, u_n) \in \mathcal{K}\{\{\mathbf{u}\}\}$ such that $\xi_\lambda(0) = 0$ and

$$g_\lambda(\xi_\lambda(x_2, \ldots, x_n), x_2, \ldots, x_n) = 0.$$

Let $M(\mathbf{u})$ be a nonzero minor of the Jacobian matrix of φ that is of maximal rank. Then assume that $M(\mathbf{u})$ is divisible by $h_\lambda(\mathbf{u})$ for every $\lambda \in \Lambda$, where Λ cannot be written as a finite union of sets included in proper affine subsets of $\mathcal{K}^{n-1}$. Thus, because $\mathcal{K}[\![\mathbf{u}]\!]$ is a UFD, there is a finite number of subsets $\Lambda_k \subset \mathcal{K}^{n-1}$, $k = 1, \dots,$ N, whose union equals Λ, and such that for every λ, $\lambda' \in \Lambda_k$, h_λ and $h_{\lambda'}$ are equal up to multiplication by a unit. Thus, by the assumption on Λ, there is k such that Λ_k contains n vectors of $\mathcal{K}^{n-1}$, denoted by $\lambda^{(1)}, \dots, \lambda^{(n)}$ such that the vectors $\lambda^{(i)} - \lambda^{(1)}$ are $\mathcal{K}$-linearly independent. Therefore, there are units $U_i(\mathbf{u})$, for $i = 2,$ $\dots, n$, such that

$$u_1 - (g)\cdot\lambda^{(i)} = U_i(\mathbf{u})(u_1 - (g)\cdot\lambda^{(1)}) \ \ \forall i = 2, \dots, n$$

where (g) denote the vector whose entries are the $g_i(\mathbf{u})$. This implies that the $(g)\cdot(\lambda^{(i)} - \lambda^{(1)})$ are divisible by $h_{\lambda^{(1)}}$. But the $\lambda^{(i)} - \lambda^{(1)}$ being $\mathcal{K}$-linearly independent, every $g_i(\lambda)$ is divisible by $h_{\lambda^{(1)}}$, thus u_1 is divisible by $h_{\lambda^{(1)}}$, which implies that $\xi_{\lambda^{(1)}} = 0$ contradicting the assumption. Therefore, there is a finite union of proper affine subspaces of $\mathcal{K}^{n-1}$, denoted by Λ, such that $M(\mathbf{u})$ is not divisible by h_λ for every $\lambda \in \mathcal{K}^{n-1} \setminus \Lambda$. In particular $\mathcal{K}^{n-1} \setminus \Lambda$ is uncountable.

After a linear change of coordinates, we may assume that $\mathcal{K} \times \{0\}^{n-2}$ is not included in Λ, in particular $(\mathcal{K} \times \{0\}^{n-2}) \cap \Lambda$ is finite. For any $(\lambda, 0, \dots, 0) \in (\mathcal{K} \times \{0\}^{n-2}) \setminus \Lambda$ the morphism

$$\psi_\lambda : \frac{\mathcal{K}[\![x_1, \dots, x_n]\!][y]}{(x_1 - \lambda x_2)} \longrightarrow \frac{\mathcal{K}[\![u_1, \dots, u_n]\!][y]}{(u_1 - \lambda g_2(\mathbf{u}))}$$

is of rank $\mathrm{r}(\varphi_\lambda) = n - 1$. Then if $n > 3$, by Bertini's Theorem [BCR21, Theorem 3.4], and by Definition 2.1 ii) (note that this is the only point of the paper where we use Definition 2.1 ii)), the polynomial P remains irreducible and not in $\mathcal{K}\{\{\mathbf{x}\}\}[y]$ in

$$\frac{\mathcal{K}[\![x_1, \dots, x_n]\!][y]}{(x_1 - \lambda x_2)} \simeq \mathcal{K}[\![x_2, \dots, x_n]\!][y]$$

when λ belongs to $W \subset \mathcal{K}$ that is uncountable. Therefore we can choose $(\lambda, 0, \dots, 0) \in (W \times \{0\}^{n-2}) \setminus \Lambda$, and this allows us replace n by $n-1$ in Theorem 4.1. By repeating this process, we construct an example of a morphism φ with $n = 2$ satisfying Theorem 4.2 (i) such that $\mathrm{Ker}(\widehat{\varphi})$ is generated by a Weierstrass polynomial that is not $\mathcal{K}\{\{\mathbf{x}\}\}[y]$; note that we must stop the reduction at $n = 2$, because Bertini's Theorem does not hold for $n < 3$, cf. [BCR21, Remark 3.6(3)]. Moreover, by repeating the argument given in part (2) if necessary, we may assume that $\varphi(x_1) = u_1$, and $\varphi(x_2)$ has the form $u_1^a g(\mathbf{u})$ with $g(0) = 0$ and $g(0, u_2) \neq 0$.

By composing φ with the morphism σ defined in (1), we can assume that $g(\mathbf{u}) = u_2^b U(\mathbf{u})$ for some unit $U(\mathbf{u})$. Now let σ' be the morphism defined by $\sigma'(u_1) = u_1^b$ and $\sigma'(u_2) = u_1 u_2^{a+1}$. Then, we have

$$\sigma' \circ \varphi(x_1) = u_1^b \text{ and } \sigma' \circ \varphi(x_2) = (u_1 u_2)^{b(a+1)} V(\mathbf{u})$$

for some unit $V(\mathbf{u})$. Therefore, as done in (2), we can assume that

$$\varphi(x_1) = u_1 \text{ and } \varphi(x_2) = u_1 u_2.$$

Hence, we have constructed a morphism φ that satisfies the hypothesis of Theorem 4.2, but $\mathrm{Ker}(\widehat{\varphi})$ is generated by Weierstrass polynomial in y that is not in $\mathcal{K}\{\{\mathbf{x}\}\}[y]$, contradicting Theorem 4.2. □

The rest of this section is devoted to the proof of Theorem 4.2, given in §§4.7.

4.3. **Newton-Puiseux-Eisenstein Theorem.** In [BCR21, Section 5], we provided a framework allowing us to obtain a good factorization of a polynomial in $\mathbb{C}[\![\mathbf{x}]\!][y]$. We recall here the main definitions and adapt the main results to the more general context of polynomials in $\mathcal{K}[\![\mathbf{x}]\!][y]$.

Consider the ring of power series $\mathcal{K}[\![\mathbf{x}]\!]$ where $\mathbf{x} = (x_1, \dots, x_n)$ and denote by $\mathcal{K}(\!(\mathbf{x})\!)$ its field of fractions. We denote by ν the $(\mathbf{x})$-adic valuation on $\mathcal{K}[\![\mathbf{x}]\!]$. The valuation ν extends to $\mathcal{K}(\!(\mathbf{x})\!)$ by defining $\nu(f/g) = \nu(f) - \nu(g)$ for every f, $g \in \mathcal{K}[\![\mathbf{x}]\!]$, $g \neq 0$. Denote V_ν the valuation ring of ν in $\mathcal{K}(\!(\mathbf{x})\!)$, and $\widehat{V}_\nu$ its completion with respect to ν. Let us now recall the notion of homogeneous element:

Definition 4.3 (Homogeneous elements)**.** Let $\omega = p/e \in \mathbb{Q}_{>0}$. We say that $\Gamma \in \mathcal{K}[\mathbf{x}, z]$ is *ω-weighted homogeneous* if $\Gamma(x_1^e, \cdots, x_n^e, z^p)$ is homogeneous.

A *homogeneous element* γ is an element of an algebraic closure of $\mathcal{K}(\mathbf{x})$, satisfying a relation of the form $\Gamma(\mathbf{x}, \gamma) = 0$ for some ω-weighted homogeneous polynomial $\Gamma(\mathbf{x}, z)$, where $\omega \in \mathbb{Q}_{>0}$. Furthermore, if $\Gamma(\mathbf{x}, z)$ is monic in z, we say that γ is an *integral homogeneous element*. In this case, ω is called the *degree* of γ.

Given an integral homogeneous element γ of degree ω, there exists an extension of the valuation ν, still denoted by ν, to the field $\mathcal{K}(\mathbf{x})[\gamma]$, defined by

$$\nu\left(\sum_{k=0}^{d-1} a_k(\mathbf{x})\gamma^k\right) = \min\{\nu(a_k) + k\omega\}.$$

where d is the degree of the field extension $\mathcal{K}(\mathbf{x}) \longrightarrow \mathcal{K}(\mathbf{x})[\gamma]$. We denote $V_{\nu,\gamma}$ the valuation ring of ν in $\mathcal{K}(\mathbf{x})[\gamma]$, and $\widehat{V}_{\nu,\gamma}$ its completion with respect to ν.

Definition 4.4 (Projective rings and temperate projective rings)**.** Let $h \in \mathcal{K}[\mathbf{x}]$ be a homogeneous polynomial. Denote by $\mathbb{P}_h(\!(x)\!)$ the ring of elements A for which there is $k_0 \in \mathbb{Z}$, α, $\beta \in \mathbb{N}$ and $a_k(\mathbf{x})$ homogeneous polynomials in $\mathcal{K}[x]$ for $k \geqslant k_0$ such that:

$$A = \sum_{k \geqslant k_0} \frac{a_k(\mathbf{x})}{h^{\alpha k + \beta}}, \qquad \text{where } \nu(a_k) - (\alpha k + \beta)\nu(h) = k, \ \forall\, k \geqslant k_0$$

We denote by $\mathbb{P}_h[\![\mathbf{x}]\!]$ the subring of $\mathbb{P}_h(\!(\mathbf{x})\!)$ of elements A such that k_0 belongs to $\mathbb{Z}_{\geqslant 0}$, and we denote by $\mathbb{P}_h\{\{\mathbf{x}\}\}$ the subring of $\mathbb{P}_h[\![\mathbf{x}]\!]$ of elements A such that

$$\sum_{k \geqslant k_0} a_k(\mathbf{x}) \in \mathcal{K}\{\{\mathbf{x}\}\}.$$

When γ is an integral homogeneous element, we denote by $\mathbb{P}_h[\![\mathbf{x}, \gamma]\!]$ the subring of $\widehat{V}_{\nu,\gamma}$, whose elements ξ are of the form:

$$\xi = \sum_{k=0}^{d-1} A_k(\mathbf{x})\gamma^k, \qquad \text{where } A_k \in \mathbb{P}_h(\!(\mathbf{x})\!) \text{ and } \nu(A_k(\mathbf{x})\gamma^k) \geqslant 0, \ \ k = 0, \dots, d-1.$$

Remark 4.5. Lemma 4.15 below shows that if $A \in \mathbb{P}_h[\![\mathbf{x}]\!]$, the fact that $A \in \mathbb{P}_h\{\{\mathbf{x}\}\}$ is independent of the presentation of A, that is, $\mathbb{P}_h\{\{\mathbf{x}\}\}$ is well-defined. This result replaces and greatly simplifies [BCR21, Prop 5.13], which relied in complex analysis.

The next two results have been proven (in greater generality) in [Ro17], but we refer the reader to [BCR21] where the statement is given when $\mathcal{K} = \mathbb{C}$, but whose proof remains valid in the case of a general characteristic zero field.

Theorem 4.6 (Newton-Puiseux-Eisenstein, cf. [BCR21, Th 5.8])**.** *Let $\mathcal{K}$ be a characteristic zero field and let $P(\mathbf{x}, y) \in \mathcal{K}[\![\mathbf{x}]\!][y]$ be a monic polynomial. There exists an integral homogeneous element γ, and a homogeneous polynomial $h(\mathbf{x})$, such that $P(\mathbf{x}, y)$ factors as a product of degree 1 monic polynomials in y with coefficients in $\mathbb{P}_h[\![\mathbf{x}, \gamma]\!]$.*

The following result is a convenient reformulation of Theorem 4.6:

Corollary 4.7 (Newton-Puiseux-Eisenstein factorization, cf. [BCR21, Cor 5.9])**.** *Let $\mathcal{K}$ be a characteristic zero field and let $P \in \mathcal{K}[\![\mathbf{x}]\!][y]$ be a monic polynomial. Then, there is a homogenous polynomial h and integral homogenous elements $\gamma_{i,j}$, such that P can be written as*

$$P(\mathbf{x}, y) = \prod_{i=1}^{s} Q_i, \quad \text{and} \quad Q_i = \prod_{j=1}^{r_i} (y - \xi_i(\mathbf{x}, \gamma_{i,j})) \tag{6}$$

where

(i) the $Q_i \in \mathbb{P}_h[\![\mathbf{x}]\!][y]$ are irreducible in $\widehat{V}_\nu[y]$,

(ii) for every i, there are $A_{i,k}(\mathbf{x}) \in \mathbb{P}_h(\!(\mathbf{x})\!)$, for $0 \leqslant k \leqslant k_i$ such that

$$\xi_i(\mathbf{x}, \gamma_{i,j}) = \sum_{k=0}^{k_i} A_{i,k}(\mathbf{x}) \gamma_{i,j}^k \in \mathbb{P}_h[\![\mathbf{x}, \gamma_{i,j}]\!]$$

(iii) for every i, the $\gamma_{i,j}$ are distinct conjugates of an homogeneous element γ_i, that is, roots of its minimal polynomial Γ_i over $\mathcal{K}(\mathbf{x})$.

4.4. **Blowups and the geometric setting.** In what follows, we use algebraic-geometry methods concerning blowings-up $\sigma : N' \to N$", where N will stand for some affine space over $\mathcal{K}$ (the precise meaning of this statement will be clarified in this subsection). Nevertheless, and in contrast to usual algebraic and analytic geometry, we do not have access as far as we know to a theory of varieties and sheaves valid for W-temperate families. We do not have the ambition to develop such a general theory in here, but rather to introduce the minimal set of definitions which are necessary for this work. In particular, we will greatly exploit the fact that we only need to work over $\mathcal{K}\{\{\mathbf{x}\}\}$, where $\mathbf{x} = (x_1, x_2)$ stands for *two indeterminate*, in order to avoid a more technical discussion.

Let us start by fixing a set of indeterminate $\mathbf{x} = (x_1, \ldots, x_n)$, and a W-temperate ring $\mathcal{K}\{\{\mathbf{x}\}\}$. In what follows, we will often need to change indeterminate:

Definition 4.8 (Temperate automorphism)**.** Let φ be a $\mathcal{K}$-automorphism of the ring of power series $\mathcal{K}[\![\mathbf{x}]\!]$. We say that φ is *temperate* if $\varphi(\mathcal{K}\{\{\mathbf{x}\}\}) \subset \mathcal{K}\{\{\mathbf{x}\}\}$.

Lemma 4.9. *A $\mathcal{K}$-automorphism φ given by series $\varphi(x_i) \in \mathcal{K}[\![\mathbf{x}]\!]$ is temperate if and only if for every i we have $\varphi(x_i) \in \mathcal{K}\{\{\mathbf{x}\}\}$. In this case φ^{-1} is also temperate.*

Proof. The condition is necessary by definition, and sufficient from the fact that $\mathcal{K}\{\{\mathbf{x}\}\}$ is stable by composition. Now φ^{-1} is also temperate since $\mathcal{K}\{\{\mathbf{x}\}\}$ satisfies the implicit function Theorem, cf. Proposition 2.8 i). $\square$

It follows directly from the Lemma that any $\mathcal{K}$-linear automorphism in $\mathbf{x}$ is a temperate automorphism. We are ready to introduce the notion of temperate coordinate systems:

Definition 4.10 (Temperate coordinates)**.** Let $\mathcal{K}\{\{\mathbf{x}\}\}$ be a temperate ring. A system of parameters $\widetilde{\mathbf{x}}$ of the ring $\mathcal{K}[\![\mathbf{x}]\!]$ is said to be *temperate* if $\widetilde{\mathbf{x}}$ is obtained from $\mathbf{x}$ by a temperate $\mathcal{K}$-automorphism. A system of parameters $\widehat{\mathbf{x}}$ of $\mathcal{K}[\![\mathbf{x}]\!]$ which is not temperate will be called formal.

We will denote by $\mathcal{O}$ the intrinsic ring of temperate power series associated to $\mathcal{K}\{\{\mathbf{x}\}\}$, that is, $\mathcal{O}$ denotes a ring which is isomorphic to $\mathcal{K}\{\{\widetilde{\mathbf{x}}\}\}$ for any temperate coordinate system $\widetilde{\mathbf{x}}$.

We now specialize to the case that $n = 2$. Let $\mathcal{K}\{\{x_1, x_2\}\}$ be a temperate ring and $N_0 = \widehat{\mathbb{A}}^2_{\mathcal{K}}$ be the formal scheme associated to the complete local ring $\mathcal{K}[\![x_1, x_2]\!]$. We denote by $\mathcal{O}_0$ and $\widehat{\mathcal{O}}_0$ the rings of temperate and formal power series at 0. We consider the formal blowing-up of the origin:

$$\sigma : (N_1, E_1) \to (N_0, 0)$$

where $\sigma^{-1}(0) = E$ is called the exceptional divisor. Given any closed point $\mathfrak{b} \in E$, we can localize σ to $\mathfrak{b}$ in order to obtain a morphism between local rings $\sigma_{\mathfrak{b}} : \widehat{\mathcal{O}}_0 \to \widehat{\mathcal{O}}_{\mathfrak{b}}$, where $\widehat{\mathcal{O}}_{\mathfrak{b}}$ stands for the local ring of formal power series at $\mathfrak{b}$. Now, apart from a $\mathcal{K}$-linear change of indeterminacy in $\mathbf{x}$ (which is a temperate change of coordinates), we may suppose that $\mathfrak{b}$ is in the origin of the x_1-chart of the blowing-up, that is, there exists a system of parameters $\mathbf{v} = (v_1, v_2)$ of $\widehat{\mathcal{O}}_{\mathfrak{b}}$ such that $\sigma_{\mathfrak{b}} : \mathcal{K}[\![\mathbf{x}]\!] \to \mathcal{K}[\![\mathbf{v}]\!]$ is given by

$$(x_1, x_2) \longmapsto (v_1, v_1 v_2).$$

We note that the ideal of E is the ideal generated by v_1 in this chart.

Definition 4.11. Following the above construction, we say that $\mathbf{v} = (v_1, v_2)$ is a set of temperate coordinates at $\mathfrak{b}$. In particular $\sigma_{\mathfrak{b}}$ induces a morphism:

$$\sigma_{\mathfrak{b}} : \mathcal{K}\{\{\mathbf{x}\}\} \to \mathcal{K}\{\{\mathbf{v}\}\}.$$

The next Lemma shows that this definition is consistent with temperate changes of coordinates, allowing us to write:

$$\sigma^*_{\mathfrak{b}} : \mathcal{O}_0 \to \mathcal{O}_{\mathfrak{b}}.$$

Lemma 4.12. *Let $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2)$ be a different set of temperate coordinates at 0, that is, there exists a temperate authomorphism $\varphi : \mathcal{K}\{\{\widetilde{\mathbf{x}}\}\} \to \mathcal{K}\{\{\mathbf{x}\}\}$. Suppose that there exists a system of parameters $\widetilde{\mathbf{v}} = (\widetilde{v}_1, \widetilde{v}_2)$ of $\widehat{\mathcal{O}}_{\mathfrak{b}}$ such that:*

$$(\widetilde{x}_1, \widetilde{x}_2) \longmapsto (\widetilde{v}_1, \widetilde{v}_1 \widetilde{v}_2).$$

Then $\widetilde{\mathbf{v}}$ is a temperate coordinate, that is, there exists a temperate automorphism $\psi : \mathcal{K}\{\{\widetilde{\mathbf{v}}\}\} \to \mathcal{K}\{\{\mathbf{v}\}\}$.

Proof. Let $\varphi(\widetilde{x}_1) = \varphi_1(x_1, x_2)$ and $\varphi(\widetilde{x}_2) = \varphi_2(x_1, x_2)$. From the assumption

$$\widetilde{v}_1 = \varphi_1(v_1, v_1 v_2), \quad \widetilde{v}_2 = \frac{\varphi_2(v_1, v_1 v_2)}{\varphi_1(v_1, v_1 v_2)}, \tag{7}$$

and from usual formal algebraic geometry, we know that (7) defines an authomorphism of $\widehat{\mathcal{O}}_{\mathfrak{b}}$. Let us show that this automorphism is temperate. We consider the Taylor expansion of φ_1 and φ_2 in order to get:

$$\varphi_1(v_1, v_1 v_2) = v_1 \left(a_{1,1} + a_{1,2} v_2 + v_1 \Phi_1\right), \quad \varphi_2(v_1, v_1 v_2) = v_1 \left(a_{2,1} + a_{2,2} v_2 + v_1 \Phi_2\right)$$

where the $\mathcal{K}$-matrix $A = [a_{i,j}]$ is invertible and Φ_1 and Φ_2 are temperate functions by Proposition 2.8 iv). Therefore:

$$\widetilde{v}_2 = \frac{a_{2,1} + a_{2,2}v_2 + v_1\Phi_2}{a_{1,1} + a_{1,2}v_2 + v_1\Phi_1}$$

and we conclude that $a_{1,1} \neq 0$ and $a_{2,1} = 0$. The result is now immediate from the implicit function Theorem, cf. Proposition 2.8 i). $\square$

In what follows, we will consider sequences of point blowings-up

$$(\widehat{\mathbb{A}}^2_{\mathcal{K}}, 0) = (N_0, 0) \xleftarrow[\sigma_1]{} (N_1, F_1) \xleftarrow[\sigma_2]{} \cdots \xleftarrow[\sigma_r]{} (N_r, F_r)$$

and it will be convenient to fix notation. We set $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$ and, for $j \in \{1, \ldots, r\}$, F_j is a simple normal crossing divisor that can be decomposed as

$$F_j = F_j^{(1)} \cup F_j^{(2)} \cup \cdots \cup F_j^{(j)}$$

where $F_j^{(k)}$ is the strict transform of $F_{j-1}^{(k)}$ (when $k < j$) and $F_j^{(j)}$ is the exceptional divisor of σ_j. Now, fixed a temperate ring $\mathcal{O} = \mathcal{K}\{\{x_1, x_2\}\}$ at 0, the formal morphism σ can be localized at every point $\mathfrak{b} \in F_r$ in order to generate a morphism between temperate rings, that is, there are system of parameters $\mathbf{v} = (v_1, v_2)$ of $\widehat{\mathcal{O}}_{\mathfrak{b}}$ such that $\sigma_{\mathfrak{b}}^* : \mathcal{K}\{\{\mathbf{x}\}\} \to \mathcal{K}\{\{\mathbf{v}\}\}$ is well-defined and can be written $\sigma_{\mathfrak{b}}^* : \mathcal{O}_0 \to \mathcal{O}_{\mathfrak{b}}$.

Remark 4.13. If $\mathfrak{b} \in F_r^{(1)}$ then, from usual combinatorial considerations about blowings-up, we may further suppose that $\sigma_{\mathfrak{b}}^* : \mathcal{K}\{\{\mathbf{x}\}\} \to \mathcal{K}\{\{\mathbf{v}\}\}$ is given by:

$$(x_1, x_2) \mapsto (v_1 v_2^c, v_1 v_2^{c+1})$$

for some natural number $0 \leqslant c \leqslant r$.

4.5. **Blowups and Projective rings.** We present in this subsection different results about the behavior of projective series and temperate projective series under blowing-up, which will be most useful in the sequel.

Definition 4.14. Let $A \in \mathbb{P}_h[\![\mathbf{x}]\!]$ and $\sigma : (N_r, F_r) \to (N_0, 0)$ a sequence of point blowings-up. We say that A *extends at a point* $\mathfrak{b} \in F_r$ if $A_{\mathfrak{b}} := \sigma_{\mathfrak{b}}^*(A)$ belongs to $\widehat{\mathcal{O}}_{\mathfrak{b}}$. Furthermore, we say that A *extends temperately* if $A_{\mathfrak{b}} \in \mathcal{O}_{\mathfrak{b}}$, where we recall that $\mathcal{O}_{\mathfrak{b}}$ stands for the ring of W-temperate functions at $\mathfrak{b}$.

Lemma 4.15 (Characterization for temperate extension)**.** *Let $A \in \mathbb{P}_h[\![\mathbf{x}]\!]$ and let $\sigma : (N_r, F_r) \to (N_0, 0)$ be a sequence of point blowings-up. Let $\mathfrak{b} \in F_r^{(1)}$ be such that A extends at $\mathfrak{b}$ (that is the case for instance when $\mathfrak{b}$ does not belong to the strict transform of $h = 0$ or the in the intersection with $F_r^{(j)}$ for some $j > 1$). Then $A \in \mathbb{P}_h\{\{\mathbf{x}\}\}$ if and only if $A_{\mathfrak{b}} \in \mathcal{O}_{\mathfrak{b}}$.*

Proof. Let $A \in \mathbb{P}_h[\![\mathbf{x}]\!]$ and fix a point $\mathfrak{b} \in F_r^{(1)}$. By definition 4.4 and the local expressions of blowings-up given in Remark 4.13, we can write:

$$A = \sum_{k \geqslant k_0} \frac{a_k(\mathbf{x})}{h(\mathbf{x})^{\alpha k + \beta}}, \quad \text{and} \quad A_{\mathfrak{b}} = \sum_{k \geqslant k_0} (v_1 v_2^c)^k \frac{a_k(1, v_2)}{h(1, v_2)^{\alpha k + \beta}}.$$

Denote by d the degree of h, and consider:

$$\widetilde{A} = \sum_{k \geqslant k_0} a_k(\mathbf{x}), \quad \text{and} \quad \widetilde{A}_{\mathfrak{b}} = \sigma_{\mathfrak{b}}^*(\widetilde{A}) = \sum_{k \geqslant k_0} (v_1 v_2^c)^{(\alpha k + \beta)d + k} a_k(1, v_2).$$

Let us define the following auxiliary function:

$$B(\mathbf{w}) := \sum_{k\geqslant k_0} (w_1 w_2^c)^k a_k(1, w_2) \in \mathcal{K}[\![w]\!].$$

Now, writing $h(1, v_2) = v_2^m g(v_2)$, where g is a unit and $m \in \mathbb{N}$, we have:

$$\widetilde{A}_{\mathfrak{b}}(\mathbf{v}) = (v_1 v_2^c)^{\beta d} B(v_1^{\alpha+1} v_2^{c\alpha d}, v_2)$$
$$B(\mathbf{w}) = w_2^{m\beta} g(w_2)^{\beta} A_{\mathfrak{b}}(w_1 w_2^{m\alpha} g(w_2)^{\alpha}, w_2).$$

Since being temperate is closed by division by a coordinate, ramification and local blowing-down (see Proposition 2.8 iii), iv) and Definition 2.2), and $A_{\mathfrak{b}} \in \mathcal{K}[\![\mathbf{v}]\!]$ by hypothesis, we conclude that

$$\widetilde{A}_{\mathfrak{b}}(\mathbf{v}) \in \mathcal{K}\{\{\mathbf{v}\}\} \iff B(\mathbf{w}) \in \mathcal{K}\{\{\mathbf{w}\}\} \iff A_{\mathfrak{b}}(\mathbf{v}) \in \mathcal{K}\{\{\mathbf{v}\}\},$$

finishing the proof. $\square$

We conclude this subsection with a useful characterization of projective series which are not formal power series:

Lemma 4.16. *Let $A \in \mathbb{P}_h[\![\mathbf{x}]\!] \setminus \mathcal{K}[\![\mathbf{x}]\!]$ and consider a point blowing-up σ centered at the origin. There exists a point $\mathfrak{b} \in \sigma^{-1}(0)$ such that $A_{\mathfrak{b}} = \sigma_{\mathfrak{b}}^*(A)$ is not a power series, that is, $\sigma_{\mathfrak{b}}^*(A) \notin \widehat{\mathcal{O}}_{\mathfrak{b}}$.*

Proof. Let $A \in \mathbb{P}_h[\![\mathbf{x}]\!] \setminus \mathcal{K}[\![\mathbf{x}]\!]$; from definition 4.4 we may write

$$A = \sum_{k\in\mathbb{N}} \frac{a_k(\mathbf{x})}{b_k(\mathbf{x})}$$

where the a_k and b_k are homogeneous polynomials in $\mathcal{K}[\mathbf{x}]$ such that $\deg(a_k) - \deg(b_k) = k$ and $\gcd(a_k, b_k) = 1$. By hypothesis, there exists k_0 such that $b_{k_0}(\mathbf{x})$ is not a constant polynomial. Apart from a $\mathcal{K}$-linear change of coordinates in $\mathbf{x}$, we may furthermore suppose that $b_{k_0}(1, 0) = 0$. It follows that after the local blowing-up $\sigma : (x_1, x_2) \longmapsto (v_2, v_1 v_2)$ we obtain

$$\sigma_{\mathfrak{b}}^*(A) = \sum_{k\in\mathbb{N}} v_1^k \frac{a_k(1, v_2)}{b_k(1, v_2)},$$

this expression has a pole in the term k_0, and we conclude easily. $\square$

4.6. **Extension along the exceptional divisor.** We introduce the notion of Laurent series with support in a strongly convex cone, and we refer the reader to [AI09] for extra details.

Definition 4.17. Let $\omega \in (\mathbb{R}_{>0})^n$ be a vector whose coordinates are $\mathbb{Q}$-linearly independent. This vector defines a total order on the set of monomials by setting

$$\mathbf{x}^{\alpha} \preceq \mathbf{x}^{\beta} \text{ if } \alpha\cdot\omega \leqslant \beta\cdot\omega.$$

Let Σ be a strongly rational cone. We say that Σ is ω-positive if $s \cdot \omega > 0$ for every $s \in \Sigma \setminus \{0\}$; under this hypothesis, $\Sigma \cap \mathbb{Z}^n$ and $\Sigma \cap \frac{1}{q}\mathbb{Z}^n$ for $q \in \mathbb{N}^*$ are well-ordered for $\preceq$, and $(\mathbb{R}_{\geqslant 0})^{\subset}\Sigma$ because $\omega \in (\mathbb{R}_{>0})^n$.

Assume that Σ is ω-positive strongly rational cone. We denote by $\mathcal{K}[\![\Sigma]\!]$ (resp. $\mathcal{K}[\![\Sigma \cap \frac{1}{q}\mathbb{Z}^n]\!]$ for $q \in \mathbb{N}^*$) the set of Laurent series with support in $\Sigma \cap \mathbb{Z}^n$ (resp. with support in $\Sigma \cap \frac{1}{q}\mathbb{Z}^n$). Since $\Sigma \cap \mathbb{Z}^n$ and $\Sigma \cap \frac{1}{q}\mathbb{Z}^n$ are well-ordered for $\preceq$, they are rings containing respectively $\mathcal{K}[\![\mathbf{x}]\!]$ and $\mathcal{K}[\![\mathbf{x}^{1/q}]\!]$. These rings are commutative integral domains, and we denote by $\mathcal{K}(\!(\Sigma)\!)$ and $\mathcal{K}(\!(\Sigma\cap\frac{1}{q}\mathbb{Z}^n)\!)$ their respective fraction fields.

Theorem 4.18. *Let $P\in\mathcal{K}[\![\mathbf{x}]\!][y]$ be a monic reduced polynomial, and let Q be an irreducible factor of P in some $\mathbb{P}_h[\![\mathbf{x}]\!][y]$ for a convenient $h\in\mathcal{K}[\mathbf{x}]$ as in Corollary 4.7. Let $\sigma:(N_r,F_r)\to(N_0,0)$ be a sequence of point blowings-up such that $\sigma^*(\Delta_P)$ is everywhere monomial, that is, at any point $\mathfrak{b}$ there exist (non necessarily temperate) coordinates $\widehat{\mathbf{v}}$ such that*

$$\sigma^*(\Delta_P)=\widehat{\mathbf{v}}^\alpha\times\text{unit}\,.$$

Then Q extends at every point $\mathfrak{b}'\in F_r^{(1)}$.

Proof. Let $\mathfrak{b}\in F_r^{(1)}$. From Remark 4.13, there are coordinates $\widehat{\mathbf{v}}=(\widehat{v}_1,\widehat{v}_2)$ centered at $\mathfrak{b}'$ and $c\in\mathbb{N}$ such that

$$(x_1,x_2)=(\widehat{v}_1\widehat{v}_2^c,\widehat{v}_1\widehat{v}_2^{c+1}).$$

Let A be a coefficient of Q. By definition 4.4, and by writing $h(1,\widehat{v}_2)=\widehat{v}_2^m g(\widehat{v}_2)$ where g is a unit and $m\in\mathbb{N}$, we have

$$A=\sum_{k\in\mathbb{N}}\frac{a_k(\mathbf{x})}{h(\mathbf{x})^{\alpha k+\beta}}\quad\text{so that}\quad A_{\mathfrak{b}}=\widehat{v}_2^{-m\beta}\sum_{k\in\mathbb{N}}\widehat{v}_1^k\widehat{v}_2^{k(c-m\alpha)}\frac{a_k(1,\widehat{v}_2)}{g(1,\widehat{v}_2)^{\alpha k+\beta}}. \tag{8}$$

Note that the series $\widehat{v}_2^{m\beta}A_{\mathfrak{b}}$ has support in a translation of the strongly convex cone Σ generated by the vectors $(0,1)$ and $(1,\min\{0,c-m\alpha\})$, thus $A_{\mathfrak{b}}$ belongs to $\mathcal{K}(\!(\Sigma)\!)$. We conclude that $Q_{\mathfrak{b}}=\sigma_{\mathfrak{b}}^*(Q)$ is a factor of $P_{\mathfrak{b}}=\sigma_{\mathfrak{b}}^*(P)$ in $\mathcal{K}(\!(\Sigma)\!)[y]$.

Now, by the Abhyankar-Jung Theorem for formal power series, the roots of $P_{\mathfrak{b}}$ can be written as Puiseux power series in $\mathcal{K}[\![\widehat{v}_1^{1/q},\widehat{v}_2^{1/q}]\!]\subset\mathcal{K}[\![\Sigma\cap\frac{1}{q}\mathbb{Z}^2]\!]$ for some $q\in\mathbb{N}^*$. Since $\mathcal{K}(\!(\Sigma\cap\frac{1}{q}\mathbb{Z}^n)\!)$ is a field, we conclude that $Q_{\mathfrak{b}}$ splits in $\mathcal{K}(\!(\Sigma\cap\frac{1}{q}\mathbb{Z}^n)\!)[y]$ and its roots are in $\mathcal{K}[\![\widehat{v}_1^{1/q},\widehat{v}_2^{1/q}]\!]$. By (8), we conclude that $Q_{\mathfrak{b}}\in\mathcal{K}[\![\widehat{\mathbf{v}}]\!]$. $\square$

We are ready to prove the main result of this subsection:

Theorem 4.19. *Let $P\in\mathcal{K}[\![x_1,x_2]\!][y]$ be a monic reduced polynomial, and h be a homogeneous polynomial for which Theorem 4.6 is satisfied. Let $\sigma:(N_r,F_r)\to(N_0,0)$ be a sequence of point blowings-up. Suppose:*

- *At every point $\mathfrak{b}\in F_r^{(1)}$, the pulled-back discriminant $\sigma_{\mathfrak{b}}^*(\Delta_P)$ is monomial;*
- *There exists $\mathfrak{b}_0\in F_r^{(1)}$ such that $P_{\mathfrak{b}_0}=\sigma_{\mathfrak{b}_0}^*(P)$ admits a factor in $\mathcal{O}_{\mathfrak{b}_0}$.*

Then P admits a non-constant factor $Q\in\mathbb{P}_h\{\!\{\mathbf{x}\}\!\}[y]$, such that either P/Q is constant, or $\sigma_{\mathfrak{b}}^*(P/Q)$ admits no non-constant temperate factor for all $\mathfrak{b}\in F_r^{(1)}$.

Proof. Consider the factorization $P=\prod_{i=1}^s Q_i$ given in Corollary 4.7, where the Q_i belong to some $\mathbb{P}_h[\![\mathbf{x}]\!][y]$. It follows from Remark 4.13 that there exists temperate coordinates $\mathbf{v}=(v_1,v_2)$ centered at $\mathfrak{b}_0$ such that $\sigma_{\mathfrak{b}_0}^*$ is locally given by

$$(x_1,x_2)\mapsto(v_1v_2^c,v_1v_2^{c+1}),$$

so that we get:

$$P_{\mathfrak{b}_0}=\prod_{i=1}^s\sigma_{\mathfrak{b}_0}^*(Q_i)$$

where the $\sigma_{\mathfrak{b}_0}^*(Q_i)\in\mathcal{K}[\![\mathbf{v}]\!][y]$ have formal power series coefficients according to Theorem 4.18. Moreover, since $\mathcal{K}[\![\mathbf{x}]\!]$ is a UFD, for some i_0, $\sigma_{\mathfrak{b}_0}^*(Q_{i_0})$ has a common factor with a polynomial in $\mathcal{K}\{\{\mathbf{v}\}\}[y]$. Therefore, by Corollary 2.12, $\sigma^*_{\mathfrak{b}_0}(Q_{i_0})$ has a non trivial divisor $R \in \mathcal{K}\{\{\mathbf{v}\}\}[y]$ that is monic in y.

We claim that $\sigma^*_{\mathfrak{b}_0}(Q_{i_0})$ has its coefficients in $\mathcal{K}\{\{\mathbf{v}\}\}$. Note that the Theorem immediately follows from the Claim applied to every polynomial Q_i having a temperate factor at some point of $F_r^{(1)}$. Let us prove the Claim. For simplicity we denote Q_{i_0} by Q, and $\sigma^*_{\mathfrak{b}_0}(Q_{i_0})$ by $Q_{\mathfrak{b}_0}$. Now, we may suppose without loss of generality that the discriminant of P is monomial in respect to the temperate coordinate system (v_1, v_2). Indeed, up to making a blowing-up with center $\mathfrak{b}_0$, we may suppose that the discriminant of P is monomial in respect to the temperate coordinate system (v_1, v_2) by considering, for example, the point $\mathfrak{c}_0 = F_{r+1}^{(1)} \cap F_{r+1}^{(r+1)}$; we note that if we show that $\sigma^*_{\mathfrak{c}_0}(Q)$ is W-temperate, then so is $Q_{\mathfrak{b}_0}$ by Def. 2.2 i).

At the one hand, we may apply Abhyankhar-Jung Theorem for formal power series in order to show that $Q_{\mathfrak{b}_0}$ splits in $\mathcal{K}[\![\mathbf{v}^{1/q}]\!]$ for some $q \in \mathbb{N}$, that is $Q_{\mathfrak{b}_0} = \prod(y - \psi_j)$ where $\psi_j \in \mathcal{K}[\![\mathbf{v}^{1/q}]\!]$. Furthermore, we may apply the temperate Abhyankhar-Jung Theorem 2.9 to the temperate factor of $Q_{\mathfrak{b}_0}$, in order to conclude that one of these roots is temperate, say, $\psi_1 \in \mathcal{K}\{\{\mathbf{v}^{1/q}\}\}$. At the other hand, by Theorem 4.6, the roots of Q belong to a ring $\mathbb{P}_h[\![\mathbf{x}, \gamma_1]\!]$, where γ_1 is an integral homogeneous element. Let $\Gamma(\mathbf{x}, z) \in \mathcal{K}[\mathbf{x}, z]$ be the irreducible ω-weighted homogeneous polynomial having γ_1 as a root (see Definition 4.3) and that is monic in z. By Corollary 4.7, the roots of Q are given by $\xi(\mathbf{x}, \gamma)$ where γ runs over the roots of $\Gamma(\mathbf{x}, z)$, that is, $Q = \prod(y - \xi(\mathbf{x}, \gamma'))$ where the product is taken over every root γ' of Γ. In what follows, we perform a detailed study on how roots of Q in $\mathbb{P}_h[\![\mathbf{x}, \gamma_1]\!][y]$ transform by the blowing-up in order to compare them to the temperate root of $Q_{\mathfrak{b}_0}$ in $\mathcal{K}[\![\mathbf{v}^{1/q}]\!]$.

We start by describing how the roots γ' of Γ transform by σ. Consider

$$\Gamma(\mathbf{x}, z) = z^d + \sum_{i=1}^{d} f_i(\mathbf{x}) z^{d-i}$$

where the $f_i(\mathbf{x})$ are homogeneous polynomials of degree ωi. Since $\mathcal{K}$ is algebraically closed, we may suppose that $\omega > 0$ (otherwise Q is a degree one polynomial, and the Claim is trivial), that is, $\Gamma(\mathbf{x}, z)$ is a Weierstrass polynomial in z. We write $\omega = p/e$ with $\gcd(p, e) = 1$, and we note that $f_i = 0$ if e does not divide i. Furthermore, because Γ is irreducible, $f_d \neq 0$, hence, e divides d. We have

$$\begin{aligned}\Gamma(v_1 v_2^c, v_1 v_2^{c+1}, v_1^\omega z) &= z^d + \sum_{i=1}^{d} f_i(v_1 v_2^c, v_1 v_2^{c+1})(v_1^\omega z)^{d-i} \\ &= v_1^{d\omega}\left(z^d + \sum_{j=1}^{d/e} v_2^{cpj} f_{ej}(1, v_2) z^{d-ej}\right)\end{aligned}$$

and we set

$$\overline{\Gamma}(v_2, z) = z^d + \sum_{j=1}^{d/e} v_2^{cpj} f_{ej}(1, v_2) z^{d-ej} \in \mathcal{K}[v_2, z^e] \subset \mathcal{K}[v_2, z].$$

Note that $\widetilde{\gamma}$ is a root of $\sigma^*(\Gamma) = \Gamma(v_1 v_2^c, v_1 v_2^{c+1}, z)$ if and only if $\widetilde{\gamma} = v_1^\omega \overline{\gamma}$ where $\overline{\gamma}$ is a root of $\overline{\Gamma}(v_2, z)$. Now, let us remark that $\overline{\Gamma}$ is irreducible in $\mathcal{K}[v_2, z^e]$. Indeed, if $\overline{\Gamma} = \overline{\Gamma}_1 \overline{\Gamma}_2$ where $\overline{\Gamma}_i \in \mathcal{K}[v_2, z^e]$ have positive degree ℓ_i in z^e, we set

$\Gamma'_i(v_1, v_2, z^e) := v_1^{\ell_i e\omega}\overline{\Gamma}_i$ for $i = 1, 2$. Then we would have

$$\Gamma(\mathbf{x}, z) = \overline{\Gamma}_1\left(\frac{x_1^{c+1}}{y^c}, \frac{x_2}{x_1}, \frac{x_2^{cp}}{x_1^{(c+1)p}}z^e\right)\overline{\Gamma}_2\left(\frac{x_1^{c+1}}{y^c}, \frac{x_2}{x_1}, \frac{x_2^{cp}}{x_1^{(c+1)p}}z^e\right)$$

contradicting the irreducibility of $\Gamma(\mathbf{x}, z)$. In particular, this implies that the irreducible factors of $\overline{\Gamma}(v_2, z)$ are conjugates up to multiplication of z by a e-th root of unity. This means that we may write

$$\overline{\Gamma}(v_2, z) = \prod \overline{\Gamma}_\eta(v_2, z)$$

where η runs through a subgroup H of the group of the e-th root of unity and the $\overline{\Gamma}_\eta(v_2, z)$ are irreducible (monic in z) polynomials, such that

$$\overline{\Gamma}_\eta(v_2, z) = \overline{\Gamma}_1(v_2, \eta z).$$

It follows that we may parametrize all roots of $\overline{\Gamma}$ by $\overline{\gamma}_{i,\eta}$ for $1 = 1, \ldots, d/e'$ and $\eta \in H$, where $e' = |H|$ and $\overline{\gamma}_{i,\eta} = \eta \cdot \overline{\gamma}_{i,1}$. We may index the roots of Γ, therefore, by $\gamma_{i,\eta}$ in such a way that $\sigma^*_{\mathfrak{b}_0}(\gamma_{i,\eta}) = \widetilde{\gamma}_{i,\eta} = v_1^\omega\overline{\gamma}_{i,\eta}$ are the roots of $\sigma^*(\Gamma) = \Gamma(v_1 v_2^c, v_1 v_2^{c+1}, z)$. We fix the convention that $\sigma^*_{\mathfrak{b}_0}(\gamma_1)/v_1^\omega = \overline{\gamma}_{1,1}$ and, more generally, that $\sigma^*_{\mathfrak{b}_0}(\gamma_i)/v_1^\omega = \overline{\gamma}_{i,1}$ are all the roots of $\overline{\Gamma}_1$. Next, by Newton-Puiseux Theorem, we can write the roots of $\overline{\Gamma}(v_2, z)$ as Puiseux series in $\mathcal{K}\langle v_2^{1/q}\rangle$, even if it means replacing q by a larger integer.

Now, we use the normal form given by Definition 4.4, in order to write

$$\xi(\mathbf{x}, \gamma_{i,\eta}) = \sum_{j=0}^{d-1} A_j(\mathbf{x})\gamma_{i,\eta}^j \text{ with } A_j(\mathbf{x}) = \sum_{k \geqslant k_j} \frac{a_{k,j}(\mathbf{x})}{h^{\alpha_j k + \beta_j}(\mathbf{x})}$$

where the $a_{k,j}(\mathbf{x})$ are homogeneous polynomials. Since there are only finitely many j, apart from multiplying the numerators and the denominators of the coefficients of $A_j(\mathbf{x})$ by a power of $h(\mathbf{x})$, we may assume that the α_j (resp. the β_j) are all independent of j and equal to some integer α (resp. β). Note that

$$\begin{aligned}\sigma^*_{\mathfrak{b}_0}(\xi(\mathbf{x}, \gamma_{i,\eta})) = \sum_{j=0}^{d-1}\sigma^*_{\mathfrak{b}_0}(A_j(\mathbf{x}))\widetilde{\gamma}_{i,\eta}^j &= \sum_{j=0}^{d-1}\overline{\gamma}_{i,\eta}^j \sum_{k \geqslant k_j} v_1^{k+\omega j} v_2^{ck}\frac{a_{k,j}(1, v_2)}{h^{\alpha k + \beta}(1, v_2)} \\ &= \sum_{k \in \frac{1}{e}\mathbb{N}} \frac{v_1^k}{h(1, v_2)^{\alpha k + \beta}} b_k(v_2, \overline{\gamma}_{i,\eta})\end{aligned} \tag{9}$$

where $b_k \in \mathcal{K}[v_2, z]$ with $\deg_z(b_k) \leqslant d - 1$. We remark that $\deg_{v_2}(b_k)$ is bounded by a linear function in k because, for each j, $\deg_{\mathbf{x}}(a_{k,j}(\mathbf{x}))$ is bounded by a linear function in k.

We note that we can write $h(1, v_2) = v_2^m g(v_2)$ for some unit $g(v_2)$, and some $m \in \mathbb{N}$. Therefore, as already shown in the proof of Theorem 4.18, the series $v_2^{m\beta}(A_j)_{\mathfrak{b}}$ are Laurent series with support in the strongly convex cone Σ generated by the vectors $(0, 1)$ and $(1, \min\{0, c - m\alpha\}$. Therefore, if we identify the $\overline{\gamma}_{i,\eta}$ with their expansions as Puiseux series of $\mathcal{K}\langle v_1^{1/q}\rangle$, we have that

$$v_2^{m\beta}\sigma_{\mathfrak{b}_0}(\xi(\mathbf{x}, \gamma_{i,\eta})) \in \mathcal{K}\left[\!\left[\Sigma \cap \frac{1}{eq}\mathbb{Z}^2\right]\!\right].$$

Since $\mathcal{K}\{\{\mathbf{v}\}\} \subset \mathcal{K}[\![\Sigma \cap \frac{1}{eq}\mathbb{Z}^2]\!] \subset \mathcal{K}(\!(\Sigma \cap \frac{1}{eq}\mathbb{Z}^2)\!)$, and $\mathcal{K}(\!(\Sigma \cap \frac{1}{eq}\mathbb{Z}^2)\!)[y]$ is a UFD, we conclude that the set of roots $\sigma_{\mathfrak{b}_0}^*(\xi(\mathbf{x}, \gamma_{i,\eta}))$ and ψ_j of $Q_{\mathfrak{b}_0}$ must coincide when we expand the $\overline{\gamma}_{i,\eta}$ as Puiseux series. From now, the $\overline{\gamma}_{i,\eta} \in \mathcal{K}\langle v_2^{1/q}\rangle$. We set $\psi_{i,\eta} = \sigma_{\mathfrak{b}_0}^*(\xi(\mathbf{x}, \gamma_{i,\eta}))$; note that $\psi_{i,\eta} \in \mathcal{K}[\![v_1^{1/e}, v_2^{1/q}]\!]$ for every i and η and, apart from re-indexing, we have $\psi_{1,1} \in \mathcal{K}\{\{v_1^{1/e}, v_2^{1/q}\}\}$.

Next, note that for every e-th root of unity η, there exists a e-th root of unity $\widetilde{\eta}$ such that $\widetilde{\eta}^p = \eta$ since $\gcd(e,p) = 1$, so that:

$$\begin{aligned}\psi_{1,\eta}(v_1^{1/e}, v_2) = \widehat{\sigma}_{\mathfrak{b}_0}^*(\xi(\mathbf{x}, \eta\,\gamma_1)) &= \sum_{j=0}^{d-1} (\eta\,\overline{\gamma}_1)^j \sum_{k \geqslant k_j} v_1^{k+\omega j} v_2^{ck} \frac{a_{k,j}(1, v_2)}{h^{\alpha k+\beta}(1, v_2)} \\ &= \sum_{j=0}^{d-1} \overline{\gamma}_1^j \sum_{k \geqslant k_j} (\widetilde{\eta}\, v_1^{1/e})^{ek+pj} v_2^{ck} \frac{a_{k,j}(1, v_2)}{h^{\alpha k+\beta}(1, v_2)} = \psi_{1,1}(\widetilde{\eta}\, v_1^{1/e}, v_2),\end{aligned}$$

so that $\psi_{1,\eta} \in \mathcal{K}\{\{v_1^{1/e}, v_2\}\}$ for every e-th root of unity η. More generally, this argument shows that:

$$\forall i, \qquad \psi_{i,1} \in \mathcal{K}\{\{v_1^{1/e}, v_2\}\} \implies \psi_{i,\eta} \in \mathcal{K}\{\{v_1^{1/e}, v_2\}\},$$

for every $\eta \in H$. We are, therefore, reduced to show that $\psi_{i,1} = \sigma_{\mathfrak{b}_0}^*(\xi(\mathbf{x}, \gamma_i)) \in \mathcal{K}\{\{v_1^{1/e}, v_2^{1/q}\}\}$ for all $i = 1, \ldots, d/e'$, where we recall that the $\overline{\gamma}_i$ are the roots of the irreducible polynomial $\overline{\Gamma}_1$. Now, we introduce the auxiliary function

$$B(\mathbf{w}, z) := \sum_{k \in \frac{1}{e}\mathbb{N}} w_1^{ek} b_k(w_2^q, z) \in \mathcal{K}[\![w_1, w_2^q]\!][z]. \tag{10}$$

where $\deg_{w_2}(b_k)$ is bounded by a linear function in k. Since $\psi_{1,1}(\mathbf{v}) \in \mathcal{K}\{\{v_1^{1/e}, v_2\}\}$, (9) and (10) imply that

$$B(\mathbf{w}, \overline{\gamma}_1(w_2)) = w_2^{\beta m q} g(w_2^q)^\beta \cdot \psi_{1,1}(w_1^e w_2^{qm\alpha} g(w_2^q)^\alpha, w_2^q) \in \mathcal{K}\{\{w_1, w_2^q\}\}.$$

Moreover, because $B(\mathbf{w}, z) \in \mathcal{K}[\![w_1, w_2^q]\!][z]$ and $B(w_1, \zeta w_2, \overline{\gamma}_1(\zeta w_2)) \in \mathcal{K}\{\{w_1, w_2^q\}\}$, we have that $B(\mathbf{w}, \overline{\gamma}_1(\zeta w_2)) \in \mathcal{K}\{\{w_1, w_2^q\}\}$ for every q-th root of unity. We remark that $\overline{\Gamma}_1(w_2^q, z)$ may factor as a product of monic polynomials that are conjugated under the action of a subgroup G of the q-th roots of unity. Thus, the set $\{\overline{\gamma}_1(\zeta w_2) \mid \zeta \in G\}$ contains exactly one root of every factor of $\overline{\Gamma}_1(w_2^q, z)$. Therefore, by definition 2.2 iii) (and we highlight that this is the only point of the paper where Definition 2.2 iii) intervenes), we conclude that:

$$B(\mathbf{w}, \overline{\gamma}_i(w_2)) \in \mathcal{K}\{\{w_1, w_2\}\}$$

for every $\overline{\gamma}_i$ which is a root of $\overline{\Gamma}_1$. Now, note that:

$$B(\mathbf{w}, \overline{\gamma}_i(w_2)) = w_2^{\beta m q} g(w_2^q)^\beta \cdot \psi_{i,1}(w_1^e w_2^{qm\alpha} g(w_2^q)^\alpha, w_2^q) \in \mathcal{K}\{\{w_1, w_2\}\}$$

for every $i = 1, \ldots, d/e'$. Since we also know that $\psi_{i,1} \in \mathcal{K}[\![v_1^{1/e}, v_2^{1/q}]\!]$, we conclude from the fact that being temperate is closed under division, ramification and local blowings-up, see Proposition 2.8 iii), iv) and Definition 2.2 i), that $\psi_{i,1}(\mathbf{v}) \in \mathcal{K}\{\{v_1^{1/e}, v_2\}\}$, finishing the proof. □

4.7. **Proof of Theorem 4.2.** Let P be a Weierstrass polynomial in y as in the statement of Theorem 4.2. Since P is reduced,the discriminant of P is a formal curve $\Delta(P)$. By resolution of singularities, there exists a sequences of point blowings-up

$$(\widehat{\mathbb{A}}^2_{\mathcal{K}},0) = (N_0,0) \xleftarrow[\sigma_1]{} (N_1,F_1) \xleftarrow[\sigma_2]{} \cdots \xleftarrow[\sigma_r]{} (N_r,F_r)$$

such that the discriminant of $P_{\mathfrak{b}} = \sigma^*_{\mathfrak{b}}(P)$ is everywhere monomial; we denote by $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$. Apart from blowing-up the origin once, we can always suppose that the sequence of blowings-up has at least one blowing-up, that is, $r \geqslant 1$. In particular, the blowings-up $\sigma_1 : (N_1,F_1) \to (N_0,0)$ is always defined. Now, there exists a point $\mathfrak{b} \in F_1$ and coordinate systems $\mathbf{v} = (v_1,v_2)$ where $(\sigma_1)^*_{\mathfrak{b}}$ is given by $(x_1,x_2 = (v_1,v_1v_2)$. It follows from the expressions of φ and P given in the statement of Theorem 4.2, that:

$$(\sigma_{\mathfrak{b}})^*_1(P) = P(v_1,v_1v_2,y), \quad \widetilde{\varphi}(v_1,v_2) = (v_1,v_2,\psi(\mathbf{v})),$$

are such that $\widetilde{\varphi} \circ \sigma = \varphi$; moreover, since $P \in \mathrm{Ker}(\varphi)$, $(\sigma_{\mathfrak{b}})^*_1(P)$ is divisible by $y - \psi(v_1,v_2) \in \mathcal{K}\{\{\mathbf{v}\}\}[y]$ and, therefore, admits a temperate factor. We conclude that there exists a point $\mathfrak{b}_0 \in F_r$ where $P_{\mathfrak{b}_0} = \sigma^*_{\mathfrak{b}_0}(P)$ admits a temperate factor. In order to finish the proof, it is enough to prove the following result:

Proposition 4.20. *Let $P \in \mathcal{K}[\![\mathbf{x}]\!][y]$, and let $\sigma : (N_r,F_r) \to (N_0,0)$ be a sequence of point blowings-up such that the discriminant of $P \circ \sigma$ is everywhere monomial. Let $\mathfrak{b} \in F_r^{(k)}$ be such that $P_{\mathfrak{b}} = \sigma^*_{\mathfrak{b}}(P)$ has a temperate factor. Then P has a non-constant temperate factor.*

Proof. We prove this result by induction on the lexicographical order on (r,k). First, suppose that $(r,k) = (r,1)$ with $r \geqslant 1$. By Theorem 4.19, there is a non-constant factor $Q \in \mathbb{P}_h\{\{\mathbf{x}\}\}[y]$ of P. Without loss of generality, we may suppose that Q is the monic factor of P in $\mathbb{P}_h\{\{\mathbf{x}\}\}[y]$ of maximal degree. Note that Q extends at every point of $F_1^{(r)}$ by Theorem 4.18, and furthermore, Q extends temperately by Lemma 4.15. If $r = 1$, then we conclude from Lemma 4.16 that $Q \in \mathcal{K}[\![\mathbf{x}]\!][y]$, so that $Q \in \mathcal{K}\{\{\mathbf{x}\}\}[y]$ since $\mathcal{K}[\![\mathbf{x}]\!] \cap \mathbb{P}_h\{\{\mathbf{x}\}\} = \mathcal{K}\{\{\mathbf{x}\}\}$ by Lemma 4.15.

If $r > 1$, let $\mathfrak{a}_1,\ldots,\mathfrak{a}_j$ be the points of $F_1^{(1)}$ that are centres of subsequent blowups, and denote $P_i := (\sigma_1)^*_{\mathfrak{a}_i}(P)$. Denote $\sigma' := \sigma_2 \circ \cdots \circ \sigma_r$. By the induction hypothesis, for every i, P_i has a temperate factor. Indeed, denoting by $\mathfrak{b}_i$ the point of $F_r^{(1)}$ which is sent to $\mathfrak{a}_i$ by σ', we get that $\sigma^*_{\mathfrak{b}_i}(Q)$ is a temperate factor of ${\sigma'}^*_{\mathfrak{b}_i}(P_i) = \sigma^*_{\mathfrak{b}_i}(P)$, obtained after only $r-1$ blowups.

Now, denote by q_i the monic temperate factor of P_i of maximal degree. Then ${\sigma'}^*_{\mathfrak{b}_i}(q_i)$ is a temperate factor of $(\sigma')^*_{\mathfrak{b}_i}(P_i)$ such that $(\sigma')^*_{\mathfrak{b}_i}(P_i/q_i)$ has no non-constant temperate factor, otherwise by induction hypothesis, P_i/q_i would have a non-constant temperate factor. Next, note that, by Theorem 4.19, $\sigma^*_{\mathfrak{b}_i}(Q)$ is also a temperate factor of $\sigma^*_{\mathfrak{b}_i}(P)$ such that $\sigma^*_{\mathfrak{b}_i}(P/Q)$ has no non-constant temperate factor. We conclude that ${\sigma'}^*_{\mathfrak{b}_i}(q_i) = \sigma^*_{\mathfrak{b}_i}(Q)$, hence q_i coincides with $(\sigma_1)^*_{\mathfrak{a}_i}(Q)$, which therefore admits a temperate extension at $\mathfrak{a}_i$. Moreover at every point $\mathfrak{b}'$ of $F_1^{(1)} \setminus \{\mathfrak{a}_1,\ldots,\mathfrak{a}_j\}$, $(\sigma_1)^*_{\mathfrak{b}'}(Q)$ is temperate, since it coincides with $\sigma^*_{\mathfrak{b}'}(Q)$. We conclude now, exactly as in the case $r = 1$, that lemma 4.16 implies that Q is a temperate factor of P.

Finally, suppose that (r,k) is such that $k > 1$, and denote by $\mathfrak{a} \in F_{k-1}^{(j)}$ the center of σ_k, for some $j \leqslant k-1$. Denote $P_{\mathfrak{a}} := (\sigma_1 \circ \cdots \circ \sigma_{k-1})^*_{\mathfrak{a}}(P)$. Then by the induction hypothesis, $P_{\mathfrak{a}}$ has a non-constant temperate factor. Therefore, by denoting $\sigma' := \sigma_k \circ \cdots \circ \sigma_r$, at every point $\mathfrak{b}' \in (\sigma')^{-1}(\mathfrak{a})$, the polynomial $(\sigma')^*_{\mathfrak{b}'}(P_{\mathfrak{a}})$ has a temperate factor. In particular, if $\mathfrak{b}' \in (\sigma')^{-1}(\mathfrak{a}) \cap F_r^{(j)}$, we get that $\sigma^*_{\mathfrak{b}'}(P)$ has a temperate factor at a point of $F_r^{(j)}$ with $j < k$, and we conclude by induction. □

5. Application: Regularity of analytic maps and Nash points

5.1. Analytic set and spaces.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and fix an analytic manifold M.

Definition 5.1 (Real-analytic set)**.** A subset X of M is analytic if each point of M admits a neighborhood U and an analytic function $f \in \mathcal{O}(U)$ such that:

$$X \cap U = \{\mathfrak{a} \in U;\ f(\mathfrak{a}) = 0\}.$$

We say that X is an analytic set generated by global sections in $\mathcal{O}(M)$ if we can take $U = M$.

Definition 5.2 (cf. [GuRo65, Ch. V, Def 6])**.** A (coherent) $\mathbb{K}$-analytic space is a locally ringed space $(X, \mathcal{O}_X)$, where:

(1) X is a Hausdorff topological space and $\mathcal{O}_X$ is a coherent sheaf of functions,
(2) at each point $\mathfrak{a}$ of X there is a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to a ringed space $(Y, \mathcal{O}_Y)$ where Y is an analytic subset of an open set $V \subset \mathbb{K}^n$ and $\mathcal{O}_Y$ is its sheaf of analytic functions. That is, there exist $\mathbb{K}$-analytic functions $(f_1, \ldots, f_d) \in \mathcal{O}(V)$ such that:

$$Y = \{\mathfrak{a} \in V;\ f_k(\mathfrak{a}) = 0,\ k = 1 \ldots, d\} \quad \text{and} \quad \mathcal{O}_Y = \mathcal{O}_V/(f_1, \ldots, f_d).$$

A subspace of $(X, \mathcal{O}_X)$ is an analytic space $(Z, \mathcal{O}_Z)$ such that $Z \subset X$ and the inclusion $i : Z \to X$ is an injection that is, an injective map such that $i^* : \mathcal{O}_X \to \mathcal{O}_Z$ is surjective.

If $\mathbb{K} = \mathbb{R}$, then it is not true that every $\mathbb{R}$-analytic set X admits the structure of a $\mathbb{R}$-analytic space, as illustrated by examples of Cartan, see e.g. [Na66, Ch. V, §3]. In contrast, if $\mathbb{K} = \mathbb{C}$, then every $\mathbb{C}$-analytic set X admits the structure of $\mathbb{C}$-analytic space, essentially by a Theorem of Oka, see e.g. [Ho88, Ch. VII, Th 7.1.5]. We refer to [GuRo65, page 155] for a definition of *irreducible* complex analytic subspace $X \subset M$, and we recall that if X is irreducible then it is not the union of two proper complex analytic sets $Y, Z \subset M$, that is, if $X = Y \cup Z$ then either $Y = X$ or $Z = X$.

Remark 5.3. Note that if $X \subset \Omega \subset \mathbb{C}^n$ is an irreducible complex analytic set generated by global sections in a connected open set Ω, then the ring $\mathcal{O}(X)$ is an integral domain.

5.2. Semianalytic and Subanalytic sets.

We follow the presentation of [BM88]. Fix a real analytic manifold M.

Definition 5.4 (Semianalytic set)**.** A subset X of M is semianalytic if each point of M admits a neighborhood U and analytic functions $f_i \in \mathcal{O}(U)$ and $g_{i,j} \in \mathcal{O}(U)$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$ such that:

$$X \cap U = \bigcup_{i=1}^{p} \{\mathfrak{a} \in U;\ f_i(\mathfrak{a}) = 0,\ g_{i,j}(\mathfrak{a}) > 0,\ 1, \ldots, q\}.$$

Definition 5.5 (Subanalytic set)**.** A subset X of M is subanalytic if each point of M admits a neighborhood U such that $X \cap U$ is the projection of a relatively compact semi-analytic set.

The following is an important general example of subanalytic set:

Example 5.6. Let $\varphi : N \to M$ be a proper analytic map. The image $X = \varphi(N)$ is a subanalytic set of M. Indeed, note that the graph $\Gamma(\varphi) \subset M \times N$ is a closed analytic set and that the set X is the projection of $\Gamma(\varphi)$ onto M, that is, the image of $\Gamma(\varphi)$ by the projection $\pi : M \times N \to M$. It is now enough to remark that since φ is proper, given a relatively compact set $U \subset M$, the intersection $\pi^{-1}(U) \cap \Gamma(\varphi)$ is relatively compact.

Definition 5.7. A subset X of $\mathbb{R}^n$ is *finitely subanalytic* if its image under the map

$$\pi_n : \mathbf{x} \in \mathbb{R}^n \longmapsto \left(\frac{x_1}{\sqrt{1 + \|x\|^2}}, \ldots, \frac{x_n}{\sqrt{1 + \|x\|^2}} \right) \in \mathbb{R}^n$$

is subanalytic.

Remark 5.8. Because π_n is a semialgebraic diffeomorphism, every finitely subanalytic subset of $\mathbb{R}^n$ is subanalytic, but the converse is not true in general: for instance

$$X = \{(t, \sin(t)) \mid t \in \mathbb{R}\}$$

is subanalytic but not finitely subanalytic.

Let X be a subanalytic set. We say that X is smooth (of dimension d) at a point $\mathfrak{a} \in X$ if there exists a neighborhood U of $\mathfrak{a}$ where $X \cap U$ is an analytic submanifold (of dimension d). The dimension of X is defined as the highest dimension of its smooth points, c.f. [BM88, Remark 3.5]. Given a subanalytic (respectively, semianalytic) set X and a number $k \in \mathbb{N}$, the set of all smooth points of X of dimension k, which we denote by $X^{(k)}$, is subanalytic [Ta81], [BM88, Theorem 7.2] (respectively, semianalytic [BM88, Remark 7.3]). The set of pure dimension k of X is the set $\Sigma^{(k)} = \overline{X^{(k)}} \cap X$, which is subanalytic. If there exists $d \in \mathbb{N}$ such that $X = \Sigma^{(d)}$, we say that X has pure dimension d. Note that $X = \cup_{k=0}^{d} \Sigma^{(k)}$, where d is the dimension of X.

Example 5.9. Let $M = \mathbb{R}^3$ endowed with coordinate system (x, y, z), and consider the Whitney umbrella $X = \{x^2 - zy^2 = 0\} \subset \mathbb{R}^3$. Then:

$$\Sigma^{(2)} = \{x^2 - zy^2 = 0 \text{ and } z \geqslant 0\}, \quad \Sigma^{(1)} = \{x = y = 0, \text{ and } z \leqslant 0\}.$$

Note that their intersection is non-empty.

We now recall a classical result about subanalytic sets due to Hironaka [H73]; we follow the presentation of [BM88, Theorem 0.1]:

Theorem 5.10 (Uniformization Theorem I)**.** *Let $X \subset M$ be a closed subanalytic set of dimension d. There exists an analytic manifold N of dimension d and a proper analytic map $\varphi : N \to M$ such that $\varphi(N) = X$.*

In what follows, we use the following variant of the above result:

Theorem 5.11 (Uniformization Theorem II)**.** *Let $X \subset M$ be a closed subanalytic set of dimension d. There exists $d+1$ analytic manifolds N_k, where $k = 0, \ldots, d$, where the dimension of N_k is equal to k, and $d+1$ proper and generically immersive analytic maps $\pi_k : N_k \to M$ such that $\pi_k(N_k) = \Sigma^{(k)}$.*

Proof. It is enough to prove the result when X is an equidimensional subanalytic set, that is, when $X = \Sigma^{(d)}$. Let $\varphi : N \to M$ be the proper analytic map given by Theorem 5.10 such that $\varphi(N) = X$. We note that $N = \cup_{\iota \in I} N_\iota$ where each N_ι is a connected manifold and I is an index set. Denote by $\varphi_\iota := \varphi|_{N_\iota} : N_\iota \to M$. Note that the generic rank of φ is constant along connected components of N, and denote by r_ι the generic rank associated to each φ_ι. Let $J \subset I$ be the subindex set of $\iota \in I$ such that $r_\iota = d$; since $\varphi(N) = X$ is of dimension d, we conclude that $J \neq \emptyset$ and that $r_\iota < d$ for every $\iota \in I \setminus J$. We consider the manifold $N_d = \cup_{\iota \in J} N_\iota$ and the associated proper analytic morphism $\varphi_d : N_d \to M$, which we claim to satisfy all properties of the Theorem.

Indeed, we start by noting that $X \setminus \varphi_d(N_d)$ is a subanalytic set of dimension smaller than d and, therefore, the closure of $\varphi_d(N_d)$ is equal to X. Since φ is proper and continuous, we conclude that $\varphi_d(N_d) = X$. It is now enough to prove that the mapping is generically immersive. This easily follows from the fact that φ is generically of the same rank as the dimension of N_d. □

We finish this section with a sufficient condition for a subanalytic to be analytic:

Lemma 5.12 ([Pa92, Lemma 3])**.** *Let $X \subset M$ be a subanalytic set which is a union of countably many analytic subsets. Then X is an analytic set.*

Proof. We claim that if X is a subanalytic set contained in a union of countably many analytic subsets $(Y_k)_{k\in\mathbb{N}}$, then it is locally contained in a union of a finite number of the analytic sets $(Y_k)_{k\in\mathbb{N}}$. Note that the lemma easily follows from the claim. Since $X = \cup\Sigma^{(k)}$, where $\Sigma^{(k)}$ is a subanalytic equidimensional set, it is enough to prove the claim in the case that X is an equidimensional set. By the uniformization Theorem 5.11 there exists a proper analytic map $\varphi : N \to M$ such that $\varphi(N) = \overline{X}$ and φ is generically of rank $d = \dim(X)$; the later condition implies that $\varphi^{-1}(X)$ is subanalytic set of N whose interior is dense in N. Let us fix $\mathfrak{a} \in \overline{X}$; since φ is proper, the fiber $\varphi^{-1}(\mathfrak{a})$ has a finite number of connected components $T_1, \ldots, T_r$; denote by $U_1, \ldots, U_r$ connected open neighborhoods of the T_k. Now, given an analytic subset $Y \subset M$, its pre-image $Z = \varphi^{-1}(Y)$ is analytic in N. It follows that for each $k = 1, \ldots, r$, either $Z \cap U_k = U_k$, or $Z \cap U_k$ is a closed set with empty interior in U_k. Since X is contained in countable many analytic sets, and the union of countable many closed sets with empty interior has empty interior by Baire's Theorem, we conclude that for each $k = 1, \ldots, r$, there is an analytic set $Y_k \subset X$ such that $\varphi^{-1}(Y_k) \cap U_k = U_k$. We conclude easily. □

5.3. **Regular locus of analytic maps.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider an analytic map $\Phi : \Omega \subset \mathbb{K}^m \longrightarrow \mathbb{K}^n$ where Ω is an open set. The set of regular points of Φ is given by:
$$\mathcal{R}(\Omega) = \{\mathfrak{a} \in \Omega;\, \mathrm{r}_\mathfrak{a}(\Phi) = \mathrm{r}^{\mathcal{F}}_\mathfrak{a}(\Phi)\}.$$
We recall that Gabrielov's rank Theorem [Ga71, BCR21] states that:
$$\mathrm{r}(\Phi_\mathfrak{a}) = \mathrm{r}^{\mathcal{F}}(\Phi_\mathfrak{a}) \implies \mathrm{r}(\Phi_\mathfrak{a}) = \mathrm{r}^{\mathcal{F}}(\Phi_\mathfrak{a}) = \mathrm{r}^{\mathcal{A}}(\Phi_\mathfrak{a}).$$
In particular, the set $\mathcal{R}(\Omega)$ is open. As a matter of fact it also contains a non-empty analytic-Zariski set:

Lemma 5.13. *Let $\Phi : \Omega \subset \mathbb{K}^m \longrightarrow \mathbb{K}^n$ be an analytic map. Then the set*
$$\mathcal{R}(\Phi) := \{\mathfrak{a} \in \Omega \mid \Phi_\mathfrak{a} \text{ is regular }\}$$

contains a set of the form $\Omega \setminus Z$ where Z is a proper analytic set of Ω generated by global equations in $\mathcal{O}(\Omega)$.

Proof. It is enough to prove the Lemma in the case that Ω is connected. Let r be the generic rank of Φ and denote by Z the set of points $\mathfrak{a} \in \Omega$ where the rank of Φ is smaller than r. Note that F is a proper analytic subset generated by global equations in $\mathcal{O}(\Omega)$; indeed, it is the zero set of the r-minors of the Jacobian of Φ. It is now enough to note that Φ is regular at every point of $\Omega \setminus Z$ by the constant rank Theorem. □

We now recall a result that relates the regular locus of complex and real analytic morphisms due to Milman [Mi78], but which we state as in [Pa92]:

Lemma 5.14 ([Pa92, Lemma 4])**.** *Let $\Phi : \Omega \subset \mathbb{C}^m \to \mathbb{C}^n$ be a complex analytic map and denote by $\Phi^{\mathbb{R}}$ its real-analytic counterpart. Then $\mathcal{R}(\Phi, \Omega) = \mathcal{R}(\Phi^{\mathbb{R}}, \Omega)$.*

Proof. The inclusion $\mathcal{R}(\Phi, \Omega) \subset \mathcal{R}(\Phi^{\mathbb{R}}, \Omega)$ is immediate. In order to prove the other inclusion, suppose that $\Phi^{\mathbb{R}}$ is regular at $\mathfrak{a}$ and denote by $r = \mathrm{r}_{\mathfrak{a}}(\Phi^{\mathbb{R}})$. Since $\Phi^{\mathbb{R}}$ is the real-analytic counterpart of Φ, $r = 2s$ where $s = \mathrm{r}_{\mathfrak{a}}(\Phi)$. The result is now immediate from [Mi78, Theorem 2]. □

5.4. **Family of morphisms.**

Definition 5.15. Consider two analytic maps $\Phi : \Omega \subset \mathbb{K}^m \longrightarrow \mathbb{K}^n$ and $\varphi : \Lambda \subset \mathbb{K}^l \to \Omega$, where Ω is a connected open set and one of the following holds:

(1) $\Lambda = \Omega$ and φ is the identity;
(2) Λ is a connected open set and φ is an analytic map;
(3) $\Lambda \subset \Omega$ is an analytic subspace of Ω such that $\mathcal{O}(\Lambda)$ is an integral domain, and φ is its inclusion.

An *admissible family of analytic germs* (associated to Φ and φ) is the analytic map

$$\Psi : \Lambda \times (\mathbb{K}^m, 0) \quad \longrightarrow \quad (\mathbb{K}^n, 0)$$

given by $\Psi(\mathfrak{a}, \mathbf{u}) = \Phi(\varphi(\mathfrak{a}) + \mathbf{u}) - \Phi(\varphi(\mathfrak{a}))$. We denote by $\Psi_{\mathfrak{a}} : (\mathbb{K}^m, 0) \to (\mathbb{K}^n, 0)$ the associated germ at $\mathfrak{a}$; in particular $\Psi_{\mathfrak{a}} = \Phi_{\mathfrak{a}} - \Phi(\varphi(\mathfrak{a}))$.

Lemma 5.16. *Given an admissible family of analytic germs:*

(1) The generic rank is constant along Λ, that is,

$$\forall \mathfrak{a}, \mathfrak{b} \in \Lambda, \ \mathrm{r}(\Psi_{\mathfrak{a}}) = \mathrm{r}(\Psi_{\mathfrak{b}}).$$

(2) The map $\mathfrak{a} \in \Lambda \longmapsto \mathrm{r}^{\mathcal{A}}(\Psi^*_{\mathfrak{a}}) \in \mathbb{N}$ is upper semi-continuous for the Euclidean topology.
(3) The ring of global sections $\mathcal{O}(\Lambda)$ is an integral domain.

Proof. Condition (1) and (3) are straightforward. In order to prove (2), let $f_1, \ldots, f_s$ be generators of $\mathrm{Ker}(\Phi^*_{\varphi(\mathfrak{a})})$ and U be an open neighborhood of $\varphi(\mathfrak{a})$ such that the f_i are well defined on U. Let V be a connected neighborhood of $\mathfrak{a}$ contained in $\varphi^{-1}(U)$. Since Φ is analytic, apart from shrinking U and V, we have that $f_i \circ \Phi \circ \varphi_{\mathfrak{b}} \equiv 0$, for all $\mathfrak{b} \in V$. We conclude easily. □

Now fix an admissible family of analytic map germs

$$\Psi : \Lambda \times (\mathbb{K}^m, 0) \longrightarrow (\mathbb{K}^n, 0).$$

and let $\mathcal{L}$ denote the fractions field of the ring $\mathcal{O}(\Lambda)$ of analytic functions on Λ. Note that Ψ induces a morphism of power series rings:

$$\Psi^*_{\mathcal{L}} : \mathcal{L}[\![\mathbf{x}]\!] \longrightarrow \mathcal{L}[\![\mathbf{u}]\!]$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$ and

$$\Psi^*_{\mathcal{L}}(x_i) = \sum_{\gamma \in \mathbb{N}^m \smallsetminus 0} F_{i,\gamma} \mathbf{u}^\gamma, \quad F_{i,\gamma} = \frac{1}{\gamma!} \frac{\partial^{|\gamma|}}{\partial \mathbf{w}^\gamma}(x_i \circ \Phi) \circ \varphi \in \mathcal{O}(\Lambda).$$

where $\mathbf{w} = (w_1, \dots, w_m)$ are globally defined coordinate systems over Ω. Note that $F_{i,0} = 0$ for every $i = 1, \dots, n$, which guarantees that $\Psi^*_{\mathcal{L}}$ is well-defined.

Now let $r = \mathrm{r}(\Psi^*_{\mathcal{L}})$. Thus any $(r+1) \times (r+1)$ minor of the Jacobian matrix of $\Phi^*_{\mathcal{L}}$ is zero, therefore $\mathrm{r}(\Psi_{\mathfrak{a}}) \leqslant r$ for every $\mathfrak{a} \in \Lambda$. On the other hand, there is a $r \times r$ minor of the Jacobian matrix of $\Phi^*_{\mathcal{L}}$, denoted by M, that is not identicaly zero. So, for a generic $\mathfrak{a} \in \Lambda$, we have $M(\mathfrak{a}) \neq 0$ and $\mathrm{r}(\Psi_{\mathfrak{a}}) = r$. Therefore, by Lemma 5.16(1), we have that:

$$\mathrm{r}(\Psi^*_{\mathcal{L}}) = \mathrm{r}(\Psi_{\mathfrak{a}}), \quad \forall \mathfrak{a} \in \Lambda.$$

We now turn to the problem of relating the formal rank of Ψ at a point $\mathfrak{a} \in \Lambda$ with the formal rank of $\Psi^*_{\mathcal{L}}$:

Proposition 5.17. *Let $\Psi : \Lambda \times (\mathbb{K}^m, 0) \longrightarrow (\mathbb{K}^n, 0)$ be an admissible family of analytic map germs. If there is $\mathfrak{a} \in \Lambda$ such that $\mathrm{r}(\Psi_{\mathfrak{a}}) = \mathrm{r}^{\mathcal{F}}(\Psi_{\mathfrak{a}})$, then:*

$$\mathrm{r}(\Psi^*_{\mathcal{L}}) = \mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathcal{L}}). \tag{11}$$

In particular, the set

$$\mathcal{R}(\Psi, \Lambda) := \{\mathfrak{a} \in \Lambda \mid \Psi_{\mathfrak{a}} \text{ is regular } \}$$

is either empty or contains a set of the form $\Lambda \smallsetminus W$ where W is a countable union of proper analytic subsets of Λ generated by global equations in $\mathcal{O}(\Lambda)$.

The proof of this Proposition is based on an extension result, namely Lemma 6.1 below, whose proof is strongly inspired by an argument of Pawłucki cf. [Pa90, Lemme 6.3]. We postpone the proof to §6. Condition (11) is the deepest statement of the above Proposition which, together with Theorem 1.1, allows us to prove the following crucial technical result:

Lemma 5.18. *Let $\Psi : \Lambda \times (\mathbb{K}^m, 0) \to (\mathbb{K}^n, 0)$ be an admissible family of analytic germs where Λ is a connected open set of $\mathbb{K}^l$ (that is, we consider cases (1) and (2) of Definition 5.15). Then either $\mathcal{R}(\Psi, \Lambda) = \emptyset$ or, for every $\mathfrak{a} \in \Lambda$, there exists an open neighborhood $U_{\mathfrak{a}} \subset \Lambda_{\mathfrak{a}}$ and a proper analytic set $Z \subset U_{\mathfrak{a}}$ such that $\mathcal{R}(\Psi, U_{\mathfrak{a}}) \supset U_{\mathfrak{a}} \smallsetminus Z$.*

Proof. Let $\mathcal{L}$ denote the fraction field of $\mathcal{O}(\Lambda)$. Note that Ψ yields a morphism $\Psi^*_{\mathcal{L}} : \mathcal{L}[\![\mathbf{x}]\!] \to \mathcal{L}[\![\mathbf{u}]\!]$ and that $\mathrm{r}(\Psi^*_{\mathcal{L}}) = \mathrm{r}(\Psi_{\mathfrak{a}})$ for any $\mathfrak{a} \in \Lambda$. Now, suppose that; $\mathcal{R}(\Psi, \Lambda) \neq \emptyset$ so that Proposition 5.17 yields:

$$\mathrm{r}(\Psi^*_{\mathcal{L}}) = \mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathcal{L}}).$$

We now first prove the Lemma in the case that $\mathbb{K} = \mathbb{C}$. Let $\mathfrak{a} \in \Lambda$ be fixed and consider a sufficiently small closed polydisc $D_{\mathfrak{a}} \subset \Lambda$ centered at $\mathfrak{a}$. Let $\mathcal{O}(D_{\mathfrak{a}})$ denote the ring of analytic functions defined in a neighborhood of $D_{\mathfrak{a}}$; note that this ring is a UFD by [Da74]. Let $\mathcal{K}$ denote the algebraic closure of the fraction field of $\mathcal{O}(D_{\mathfrak{a}})$ and recall that the family of Eisenstein rings $(\mathcal{K}\{\{v_1, \dots, v_n\}\})_{n \in \mathbb{N}}$, defined in §3.3, is temperate by Proposition 3.5. We note that the restriction of Ψ to $D_{\mathfrak{a}}$, yields a temperate morphism $\Psi^*_{\mathcal{K}} : \mathcal{K}\{\{\mathbf{x}\}\} \to \mathcal{K}\{\{\mathbf{u}\}\}$. It is clear that $\mathrm{r}(\Psi^*_{\mathcal{K}}) = \mathrm{r}(\Psi^*_{\mathcal{L}})$, and since the restriction from Λ to $D_{\mathfrak{a}}$ yields an injective morphism from $\mathcal{O}(\Lambda)$ into $\mathcal{O}(D_{\mathfrak{a}})$, we conclude that:

$$\mathrm{r}(\Psi^*_{\mathcal{K}}) = \mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathcal{K}}).$$

so that we may apply Theorem 1.1 in order to get

$$\mathrm{r}(\Psi^*_{\mathcal{K}}) = \mathrm{r}^{\mathcal{T}}(\Psi^*_{\mathcal{K}}) =: r.$$

Now, up to a $\mathcal{K}$-linear change of coordinates, applying Remark 2.8 vi), the morphism $\mathcal{K}\{\{x_1, \ldots, x_r\}\} \to \mathcal{K}\{\{\mathbf{x}\}\}/\mathrm{Ker}(\Psi^*_{\mathcal{K}})$ is finite, which means that there are non-zero Weierstrass polynomials

$$Q_i(x_1, \ldots, x_r, x_{r+i}) \in \mathcal{K}\{\{x_1, \ldots, x_r\}\}[x_{r+i}] \text{ for } i = 1, \ldots, n-r,$$

such that $\Psi^*_{\mathcal{K}}(Q_i) \equiv 0$. By the definition of $\mathcal{K}\{\{\mathbf{x}\}\}$ and the primitive element theorem, there exists $f \in \mathcal{O}(D_{\mathfrak{a}})$ and $\mathfrak{c} \in \mathcal{K}$ of degree d such that $Q_i \in \mathcal{O}(D_{\mathfrak{a}})_f[\![\mathbf{x}]\!][\mathfrak{c}]$, that is

$$Q_i = \sum_{j=0}^{d-1} Q_{i,j}\mathfrak{c}^j, \quad Q_{i,j} \in \mathcal{O}(D_{\mathfrak{a}})_f[\![\mathbf{x}]\!].$$

Note that $\Psi^*_{\mathcal{K}}(\mathfrak{c}) = \mathfrak{c}$ and $\{1, \mathfrak{c}, \ldots, \mathfrak{c}^{d-1}\}$ are linearly independent over $\mathcal{O}(D_{\mathfrak{a}})$. Hence, up to replacing Q_i by $Q_{i,0}$, which is monic, we can choose the Q_i in $\mathcal{O}(D_{\mathfrak{a}})_f[\![\mathbf{x}]\!]$.

Let $U_{\mathfrak{a}} \subset D_{\mathfrak{a}}$ be any open neighborhood of $\mathfrak{a}$. We set $Z = \{\mathfrak{b} \in U_{\mathfrak{a}};\, f(\mathfrak{b}) = 0\}$. Note that Q_i yields a power series $Q_{i,\mathfrak{b}} \in \mathbb{C}[\![\mathbf{x}]\!]$ at each $\mathfrak{b} \in U_{\mathfrak{a}} \setminus Z$ and that $\Psi^*_{\mathfrak{b}}(Q_{i,\mathfrak{b}}) \equiv 0$, for every $i = 1, \ldots, n-r$. We conclude that $\mathrm{r}(\Psi^*_{\mathfrak{b}}) = \mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathfrak{b}})$ for every $\mathfrak{b} \in U_{\mathfrak{a}} \setminus Z$ as we wanted to prove.

Now let us consider the case that $\mathbb{K} = \mathbb{R}$. Denote by $\Lambda^{\mathbb{C}}$ a complex open neighborhood of Λ such that $\Lambda^{\mathbb{C}} \cap \mathbb{R}^l = \Lambda$, over which Ψ admits an holomorphic extension:

$$\Psi^{\mathbb{C}} : \Lambda^{\mathbb{C}} \times (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0).$$

By the first part of the proof, for each $\mathfrak{a} \in \Lambda^{\mathbb{C}}$, there exists a neighborhood $U^{\mathbb{C}}_{\mathfrak{a}}$ and a complex analytic set $Z^{\mathbb{C}} \subset U^{\mathbb{C}}_{\mathfrak{a}}$ such that $\mathcal{R}(\Psi, U_{\mathfrak{a}}) \supset U_{\mathfrak{a}} \setminus Z$. We fix a point $\mathfrak{a} \in \Lambda$ and we consider the neighborhood $U_{\mathfrak{a}} = U^{\mathbb{C}}_{\mathfrak{a}} \cap \mathbb{R}^l$ and the intersection $Z := Z^{\mathbb{C}} \cap U_{\mathfrak{a}}$. It is now enough to note that Z is a proper real-analytic subset of $U_{\mathfrak{a}}$, finishing the proof. □

5.5. **Proof of Theorem 1.3.** We start by a well-known result, which follows from the geometrical statement of Proposition 5.17:

Proposition 5.19 (cf. [Pa92, Prop. 1])**.** *Let $\Phi : \Omega \subset \mathbb{K}^m \longrightarrow \mathbb{K}^n$ be an analytic map where Ω is open. Then $\Omega \setminus \mathcal{R}(\Phi, \Omega)$ is a union of countably many analytic subsets.*

Proof. Let us first argue the case that $\mathbb{K} = \mathbb{C}$, in which case every complex analytic set is a complex analytic space. By Proposition 5.17 applied to each connected component of Ω, $X := \Omega \setminus \mathcal{R}(\Phi, \Omega)$ is included in the union of countably many analytic subsets $\bigcup_{i=0}^{\infty} Y_i$ of Ω. We may assume that the Y_i are irreducible (in Ω) by replacing each Y_i by its irreducible components, and we change the family $\{Y_i\}_i$ according to the following rule:

(R) For a given i_0, if there is countably many irreducible analytic subspaces $Y_{i_0,k}$ of Ω of dimension $< \dim(Y_{i_0})$ such that $X \cap Y_{i_0} \subset \bigcup_{k=0}^{\infty} Y_{i_0,k}$, we replace the family $\{Y_i\}_{i\in\mathbb{N}}$ by $\{Y_i\}_{i\neq i_0} \cup \{Y_{i_0,k}\}_{k\in\mathbb{N}}$.

By repeating this rule countably many times, we can assume that the family $\{Y_i\}_{i\in\mathbb{N}}$ is minimal in respect to (R) and contains X. Now assume by contradiction that $X \neq \bigcup_{i\in\mathbb{N}} Y_i$. This means that there is $i_0 \in \mathbb{N}$ such that $Y_{i_0} \not\subset X$ but $Y_{i_0} \cap X \neq \emptyset$. By Proposition 5.17 applied to Y_{i_0} (cf. Remark 5.3 and Definition 5.15(3)) we have that $Y_{i_0} \cap X$ is included in a countable number of proper analytic subsets $\{Y_{i_0,k}\}_{k\in\mathbb{N}}$ of Y_{i_0} that are of dimension $< \dim(Y_{i_0})$. Since Y_{i_0} is an analytic subspace of Ω, we conclude that each $Y_{i_0,k}$ is analytic subspace of Ω, which contradicts the minimality of the family $\{Y_i\}_{i\in\mathbb{N}}$ in respect to (R).

If $\mathbb{K} = \mathbb{R}$, the result follows from considering a complexification of Φ, and noting that the set of regular points is non-empty by Lemma 5.13. □

We are now ready to prove the following result:

Theorem 5.20 (Pawłucki Theorem I [Pa92])**.** *Let $\Phi : \Omega \subset \mathbb{K}^m \longmapsto \mathbb{K}^n$ be an analytic map where Ω is open. Then $\Omega \setminus \mathcal{R}(\Phi, \Omega)$ is a proper analytic subset of Ω.*

Proof. By Lemma 5.14 and Corollary 5.19, it is enough to consider the case where $\mathbb{K} = \mathbb{R}$. Furthermore, from Lemma 5.12 and Proposition 5.19, it is enough to show that $\mathcal{R}(\Phi, \Omega)$ is a subanalytic set of Ω. Note that being subanalytic is a local property, so we may suppose that Ω is a subanalytic open set.

We claim that for every closed subanalytic set $X \subset \Omega$, the intersection $X \cap \mathcal{R}(\Phi, \Omega)$ is subanalytic in Ω. The result then follows from the Claim applied to $X = \Omega$. We prove the claim by induction on the dimension of X.

When $\dim(X) = 0$, the result immediate. Assume the Claim is proved for $d - 1 \geqslant 0$ and let X be a subanalytic subset of Ω of dimension d. Consider its equidimensional part $\Sigma^{(d)}$ and let $E = \overline{X \setminus \Sigma^{(d)}}$, which is a closed subanalytic set of dimension $< d$. By induction $E \cap \mathcal{R}(\Phi, \Omega)$ is a subanalytic subset of Ω. It is, therefore, enough to prove the claim when $X = \Sigma^{(d)}$ is an equidimensional set.

By Corollary 5.11, there exists a proper and generically immersive analytic morphism $\varphi : N \to X$ such that $\varphi(N) = X$. Now fix a point $\mathfrak{a} \in \Lambda$ and a connected open neighborhood $\Lambda_{\mathfrak{a}}$ of $\mathfrak{a}$. We consider the family of admissible morphism:

$$\begin{array}{rccc}\Psi : & \Lambda_{\mathfrak{a}} \times (\mathbb{R}^m, 0) & \longrightarrow & (\mathbb{R}^n, 0) \\ & (\mathfrak{a}', \mathbf{u}) & \longmapsto & \Phi(\varphi(\mathfrak{a}') + \mathbf{u}) - \Phi(\varphi(\mathfrak{a}'))\end{array}$$

By Lemma 5.18, apart from shrinking $\Lambda_{\mathfrak{a}}$, we conclude that either $\varphi(\Lambda_{\mathfrak{a}}) \subset \Omega \setminus \mathcal{R}(\Phi, \Omega)$ or there exists a analytic proper set $Z_{\mathfrak{a}} \subset \Lambda_{\mathfrak{a}}$ such that $\varphi(\Lambda_{\mathfrak{a}} \setminus Z_{\mathfrak{a}}) \subset \mathcal{R}(\Phi, \Omega)$. Note that, since $\mathfrak{a} \in N$ was arbitrary and both of these properties are open, they hold globally over each different connected component of N. We conclude that there exist two closed subanalytic subsets Y and Z of X, such that: Y is of dimension d and $Y \subset \Omega \setminus \mathcal{R}(\Phi, \Omega)$; and Z is of dimension $< d$ and $X \setminus (Y \cup Z) \subset \mathcal{R}(\Phi, \Omega)$. The result now follows from induction applied over Z. □

Remark 5.21. Theorem 5.20 is a local version of Theorem 1.3, which easily follows from it.

5.6. **The Nash and the Semianalytic locus.** Given a subanalytic set $X \subset M$ and a point $\mathfrak{a} \in M$, we will denote by $X_{\mathfrak{a}}$ the germ set of X at $\mathfrak{a}$, that is, the equivalence relation induced by considering the intersection $U \cap X$ for every neighborhood U of $\mathfrak{a}$.

Definition 5.22 (Nash points)**.** Let $X \subset M$ be a subanalytic set of pure dimension d. We say that X is a *Nash set* at $\mathfrak{a} \in M$ (which might not belong to X) if there exists a germ $Y_{\mathfrak{a}}$ of semi-analytic set at $\mathfrak{a}$ such that $X_{\mathfrak{a}} \subset Y_{\mathfrak{a}}$ and $\dim(X_{\mathfrak{a}}) = \dim(Y_{\mathfrak{a}})$. More generally, a subanalytic set $X \subset M$ of dimension d is Nash at a point $\mathfrak{a} \in M$, if $\Sigma_{\mathfrak{a}}^{(k)}$ is Nash for each $k = 0, \ldots, d$. We consider the set:

$$\mathcal{N}(X) := \{\mathfrak{a} \in M \mid X_{\mathfrak{a}} \text{ is the germ of a Nash set}\}$$

We say that X is a Nash set if it is Nash at every point, that is, if $\mathcal{N}(X) = M$.

It is clear that every semi-analytic set is Nash subanalytic. A more general example is given by the following Lemma:

Lemma 5.23. *Let $\varphi : N \to M$ be a proper and regular analytic map, that is, at every point $\mathfrak{a} \in N$, $\mathrm{r}_{\mathfrak{a}}(\varphi) = \mathrm{r}_{\mathfrak{a}}^{\mathcal{F}}(\varphi) = \mathrm{r}_{\mathfrak{a}}^{\mathcal{A}}(\varphi)$. Suppose that $X = \varphi(N)$ is equidimensional of dimension d. Then X is Nash subanalytic.*

Proof. Indeed, fix a point $\mathfrak{b} \in X$. Consider a relatively compact neighborhood V of $\mathfrak{b}$, and note that $\varphi^{-1}(V) = U$ is a relatively compact open set of N. Now, for each point $\mathfrak{a} \in \overline{U}$, it follows from the regularity of the mapping that there exists an open neighborhood $U_{\mathfrak{a}}$ of $\mathfrak{a}$ and a semi-analytic set $Y_{\mathfrak{a}} \subset M$ of dimension at most d such that $\varphi(U_{\mathfrak{a}}) \subset Y_{\mathfrak{a}}$. From the relative compactness of U, it follows that there exists a semi-analytic set Y of dimension at most d (given as the union of a finite number of sets $Y_{\mathfrak{a}}$) such that $\varphi(U) \subset Y$, finishing the proof. $\square$

Indeed, we may generalize the above idea to provide a description of the Nash locus in terms of the regular points of a morphism:

Lemma 5.24. *Let $\varphi : N \to M$ be a proper generically immersive analytic morphism such that $\varphi(N) = X$ is a closed equidimensional set. Then*

$$X \smallsetminus \mathcal{N}(X) = \varphi(N \smallsetminus \mathcal{R}(\varphi, N)).$$

Proof. First, let us show that $X \smallsetminus \mathcal{N}(X) \subset \varphi(N \smallsetminus \mathcal{R}(\varphi, N))$ by proving the associated inclusion of their complements. Fix a point $\mathfrak{b} \in X \smallsetminus \varphi(N \smallsetminus \mathcal{R}(\varphi, N))$. This means that φ is regular on the pre-image of $\varphi^{-1}(\mathfrak{b})$. Since being regular is an open property, there exists a neighborhood U of $\varphi^{-1}(\mathfrak{b})$ such that $\varphi|_U$ is everywhere regular. Moreover, since φ is proper and continuous, there exists a neighborhood V of $\mathfrak{b}$ such that $\varphi^{-1}(V) \subset U$. By Lemma 5.23 applied to $X \cap V$, we conclude that $\mathfrak{b} \in \mathcal{N}(X)$ as desired.

Now, let us prove that $\varphi(N \smallsetminus \mathcal{R}(\varphi, N)) \subset X \smallsetminus \mathcal{N}(X)$ by proving the associated inclusion of their complements. Fix a point $\mathfrak{b} \in \mathcal{N}(X)$ and let $Y_{\mathfrak{b}}$ be the germ of a semi-analytic set of dimension d which contains $X_{\mathfrak{b}}$; let V be a subanalytic and relatively compact neighborhood of $\mathfrak{b}$ where $Y_{\mathfrak{b}}$ admits a representative Y defined in V such that $X \cap V \subset Y$. Let $U = \varphi^{-1}(V)$, which is a relatively compact neighborhood of $\varphi^{-1}(\mathfrak{b})$. It follows that $\varphi(U) \subset Y$, which implies that φ is regular at every point $\mathfrak{a} \in U$; in particular, at every point $\mathfrak{a} \in \varphi^{-1}(\mathfrak{b})$. We conclude that $\mathfrak{b} \notin \varphi(N \smallsetminus \mathcal{R}(\varphi, N))$, finishing the proof. $\square$

We now consider the following set:

$$\mathcal{SA}(X) := \{\mathfrak{a} \in M \mid X_{\mathfrak{a}} \text{ is the germ of a semianalytic set}\}.$$

It is trivially true that $M \setminus \overline{X} \subset \mathcal{SA}(X)$ and $\mathcal{SA}(X) \subset \mathcal{N}(X)$. But in general, $\mathcal{SA}(X) \neq \mathcal{N}(X)$ as is illustrated by the following examples:

Example 5.25.

i) Consider a subanalytic two dimensional set S in $\mathbb{R}^3$ such that the germ at the origin S_0 is not semianalytic (for instance, the image of a compact set through the Osgood mapping [Os1916] provides such a surface). We consider $X := \mathbb{R}^3 \setminus S$; X is subanalytic and of pure dimension 3, thus it is Nash subanalytic since $X \subset \mathbb{R}^3$. But the germ X_0 is not semianalytic. Note that $0 \notin X$.

ii) We may modify the example as follows: we set
$$X := \mathbb{R}^4 \setminus (\mathbb{R}^3 \times \{0\}) \cup (S \times \{0\} \times \{0\}).$$
Then X is equidimensional of dimension 4, and $\mathcal{N}(X) = \mathbb{R}^4$, but X_0 is not semianalytic. Note that $0 \in X$.

Remark 5.26. We recall that the closure of a semianalytic (respectively, a subanalytic) set is semianalytic (respectively, subanalytic) set of the same dimension. It follows that $\mathcal{N}(X) = \mathcal{N}(\overline{X})$ for every subanalytic set $X \subset M$. In contrast, we can only conclude from this argument that $\mathcal{SA}(X) \subset \mathcal{SA}(\overline{X})$, c.f. example 5.25(i).

5.7. **Proof of Theorem 1.4.** We start by proving the following Corollary of the uniformization Theorem 5.11 and Theorem 1.3. This result is the difficult case of the original proof from [Pa90]:

Proposition 5.27. *Let X be a subanalytic set of a real analytic manifold M. Then*

i) The set $\mathcal{N}(X)$ is subanalytic.

ii) $\dim(M \setminus \mathcal{N}(X)) \leqslant \dim(X) - 2$.

In particular, if $\dim(X) \leqslant 1$*, then* $\mathcal{N}(X) = M$.

Proof. By remark 5.26, we may suppose without loss of generality that X is a closed subanalytic set. First consider the equidimensional case $X = \Sigma^{(d)}$. Denote by $\varphi : N \to M$ the proper generically immersive analytic morphism given by Corollary 5.11, where N is of dimension d and $\varphi(N) = X$. In particular $\mathrm{r}(\varphi) = d$. By Theorem 5.20, $N \setminus \mathcal{R}(\varphi, N)$ is a proper analytic subset of N. It follows from Lemma 5.24 that $X \setminus \mathcal{N}(X)$ is a subanalytic set of codimension at least 1. It remains to prove that it has codimension 2.

Denote by F the set of points in N where φ does not have maximal rank. Note that F is analytic (it is given by the zero locus of the Jacobean ideal of φ) so, apart from applying resolution of singularities, we may suppose that F is a simple normal crossing divisor in N. Now, note that $N \setminus F \subset \mathcal{R}(\varphi, N)$ since $\varphi|_{N \setminus F}$ is a local submersion. It follows that $N \setminus \mathcal{R}(\varphi, N) \subset F$. So, it is enough to prove that the image $\varphi(E \setminus \mathcal{R}(\varphi, N))$ has dimension at most $d-2$ for every irreducible (in particular connected) component $E \subset F$. Fix such an E and consider the morphism $\varphi_E = \varphi|_E : E \to M$. Let r denote the generic rank of φ_E and note that $r \leqslant d-1$ since E has dimension $d-1$. If $r < d-1$, then $\varphi(E)$ is a subanalytic set of dimension at most $d-2$ and the result is clear. So we may suppose that $r = d-1$.

Fix a point $\mathfrak{a} \in E$ and consider a local coordinate system $(u, v) = (u, v_1, \ldots, v_{d-1})$ of N centered at $\mathfrak{a}$ and defined in an open neighborhood U of $\mathfrak{a}$, such that $E \cap U = (u = 0)$. From the rank condition over φ_E, and the inverse function Theorem, there exists a coordinate system $(x,y,z) = (x_1,\ldots,x_{d-1},y,z_{d+1},\ldots,z_n)$ centered at $\varphi(\mathfrak{a}) = \mathfrak{b}$ such that:

$$\varphi^*(x_i) = v_i, \quad i = 1,\ldots,d-1.$$

Now, apart from an analytic change of coordinates in the target and a permutation of y and the z_k, we may further suppose that there exists a positive integer a such that:

$$\begin{aligned}\varphi^*(y) &= u^a g_d(u,v)\\ \varphi^*(z_k) &= u^a g_k(u,v), \qquad k = d+1,\ldots,n\end{aligned}$$

where $g_d(0,v) \not\equiv 0$. In particular, the set of points of $E \cap U$ where $g_d(0,v) \neq 0$ is an open dense set E' of $E \cap U$. We claim that at every point of E', φ is a regular mapping; this claim implies that $\mathcal{R}(\varphi,N) \cap E \cap U$ is a proper analytic set of E and, therefore, $\varphi(E \setminus \mathcal{R}(\varphi,N))$ has dimension at most $d-2$. We turn to the proof of the Claim: suppose that $\mathfrak{a}$ is a point in E'. Apart from shrinking U and making a change of coordinates in the source and target, we may further suppose that:

$$\varphi^*(y) = u^a,$$

and we consider the following functions defined in the target:

$$P_k(x,y,z) = \prod_{i=1}^{a}(z_k - y g_k(x, \xi^i y^{1/a})), \quad k = d+1,\ldots,n$$

where ξ is a primitive a-root of the unit. By construction, it is clear that $P_k \circ \varphi|_U \equiv 0$ for every $k = d+1,\ldots,d$. We conclude that $\mathrm{r}_{\mathfrak{a}}(\varphi) = \mathrm{r}^{\mathcal{A}}_{\mathfrak{a}}(\varphi) = d$ proving the claim and finishing the proof of the Theorem in the case of an equidimensional subanalytic set X.

We now consider a general closed subanalytic set X. Consider the morphisms from Corollary 5.11 $\varphi_k : N_k \to M$, for $k = 0,\ldots,d-1$. From the previous argument applied to each set $\Sigma^{(k)}$, we conclude that $M \setminus \mathcal{N}(\Sigma^{(k)})$ is a subanalytic set of dimension at most $k-2$. It follows from the definition of $\mathcal{N}(X)$ that:

$$\mathcal{N}(X) = \cap_{k=0}^{d} \mathcal{N}(\Sigma^{(k)})$$

which is a subanalytic set. Furthermore, its complement is equal to the union of the complements of $\mathcal{N}(\Sigma^{(k)})$, and therefore is a subanalytic set of dimension at most $d-2$, finishing the proof. □

We are now ready to complete the proof of Theorem 1.4, following an argument from [BM87]. We start with two Lemmas:

Lemma 5.28. *Let X be a subanalytic set of dimension d. Then*

$$\mathcal{SA}(X) = \mathcal{SA}(X \setminus X^{(d)}) \cap \mathcal{SA}(X^{(d)}).$$

Proof. Note that $\mathcal{SA}(X \setminus X^{(d)}) \cap \mathcal{SA}(X^{(d)}) \subset \mathcal{SA}(X)$ is trivial. In order to prove the other inclusion, let $\mathfrak{a} \in \mathcal{SA}(X)$; in particular $X_{\mathfrak{a}}$ is a semi-analytic germ. Let U be a sufficiently small neighborhood of $\mathfrak{a}$ where $X_{\mathfrak{a}}$ is realizable by $X \cap U$, which is semi-analytic. We recall that if Y is semi-analytic, the n $Y^{(d)}$ is a semi-analytic set, see e.g. [BM88, Remark 7.3], so we conclude that $X^{(d)} \cap U$ is semi-analytic and $\mathfrak{a} \in \mathcal{SA}(X^{(d)})$. Since $(X \setminus X^{(d)}) \cap U = X \cap U \setminus (X^{(d)} \cap U)$, we conclude easily. □

Lemma 5.29 (c.f. [BM87, p. 200]). *Let X be a closed subanalytic set of equidimension d and let $Y = X \setminus X^{(d)}$. Then:*

$$\mathcal{SA}(X^{(d)}) = \mathcal{SA}(Y) \cap \mathcal{N}(X^{(d)}).$$

Proof. Clearly we have $\mathcal{SA}(X^{(d)}) \subset \mathcal{N}(X^{(d)})$. Moreover, if $\mathfrak{a} \in \mathcal{SA}(X^{(d)})$, then $X_{\mathfrak{a}}^{(d)}$ is semianalytic, so its closure, which is $X_{\mathfrak{a}}$, is semianalytic and $X_{\mathfrak{a}} \setminus X_{\mathfrak{a}}^{(d)}$ is semianalytic. Thus $\mathcal{SA}(X^{(d)}) \subset \mathcal{SA}(Y) \cap \mathcal{N}(X^{(d)})$.

In order to prove the other inclusion, let $\mathfrak{a} \in \mathcal{SA}(Y) \cap \mathcal{N}(X^{(d)})$. Since the result is local, apart from replacing M by a sufficiently small neighborhood of $\mathfrak{a}$, we may suppose that Y is semianalytic and that there exists a closed analytic set Z of dimension d such that $X^{(d)} \subset Z$; we conclude that $X \subset Z$. Let $\mathrm{Sing}(Z)$ denote the singular points of Z. It follows that $X \setminus (Y \cup \mathrm{Sing}(Z))$ is open and closed in $Z \setminus (Y \cup \mathrm{Sing}(Z))$ and, thus, $X \setminus (Y \cup \mathrm{Sing}(Z))$ is semi-analytic. Since the closure of this set is equal to X, we conclude that X is semianalytic, and we conclude by Lemma 5.28. □

Proof of Theorem 1.4. Because of Proposition 5.27, it only remains to show that $\mathcal{SA}(X)$ is a subanalytic set whose complement is of dimension at most $d-2$. We prove this result by induction on the dimension of X; the case that $d = 0$ being trivial. So, fix a subanalytic set X of dimension d and consider the set $E = X \setminus X^{(d)}$, which is a subanalytic set of dimension at most $d-1$. By Lemmas 5.28 and 5.29 we get:

$$\mathcal{SA}(X) = \mathcal{SA}(E) \cap \mathcal{SA}(X^{(d)}) = \mathcal{SA}(E) \cap \mathcal{SA}(\overline{X^{(d)}} \setminus X^{(d)}) \cap \mathcal{N}(X^{(d)}).$$

By induction applied to E and $\overline{X^{(d)}} \setminus X^{(d)}$, and by Proposition 5.27 applied to $X^{(d)}$, we conclude that $\mathcal{SA}(X)$ is a subanalytic set whose complement has dimension smaller or equal to $d-2$. □

We finish this section by proving the following corollary:

Corollary 5.30. *Let $X \subset \mathbb{R}^n$ be a finitely subanalytic set. Then $\mathcal{N}(X)$ and $\mathcal{SA}(X)$ are finitely subanalytic.*

Proof. Let us denote by π the map

$$\mathbf{x} \in \mathbb{R}^n \longmapsto \left(\frac{x_1}{\sqrt{1+\|x\|^2}}, \dots, \frac{x_n}{\sqrt{1+\|x\|^2}} \right) \in \mathbb{R}^n.$$

By hypothesis the image $Y = \pi(X)$ is a subanalytic set. By Theorem 1.4 $\mathcal{N}(Y)$ is subanalytic. Furthermore, since π is a semialgebraic diffeomorphism, we conclude that $\pi(\mathcal{N}(X)) = \mathcal{N}(Y) \cap \pi(\mathbb{R}^n)$, which proves that $\pi(\mathcal{N}(X))$ is finitely subanalytic.

The proof that $\mathcal{SA}(X)$ is finitely subanalytic is identical. □

6. Proof of Proposition 5.17

6.1. **Extension Lemma.** The goal of this subsection is to prove the following:

Lemma 6.1 (Extension Lemma). *Let $\Psi : \Lambda \times (\mathbb{K}^m, 0) \longrightarrow (\mathbb{K}^n, 0)$ be an admissible family of analytic map germs (see Definition 5.15) and let $\mathcal{L}$ be the field of fractions of $\mathcal{O}(\Lambda)$. Let $(\mathbf{x}, y)$ be a coordinate system of $(\mathbb{K}^n, 0)$ where y is a distinguished variable. Let U be an open and connected subset of Λ and suppose that there exists a polynomial in y*

$$f(\mathbf{x}, y) = y^d + a_1(\mathfrak{a}, \mathbf{x}) y^{d-1} + \cdots + a_d(\mathfrak{a}, \mathbf{x})$$

such that

i) $a_i(\mathfrak{a},\mathbf{x})\in\mathcal{O}(U)[\![\mathbf{x}]\!]$, $i=1,\ldots,d$;

ii) $a_i(\cdot,0)\equiv 0$ *on* U, $i=1,\ldots,d$;

iii) for all $\mathfrak{a}\in U$, $f(\mathfrak{a},\mathbf{x},y)$ *is a generator of* $\operatorname{Ker}(\widehat{\Psi}^*_{\mathfrak{a}})$.

Let us write $a_i(\mathfrak{a},\mathbf{x})=\sum_{\beta\in\mathbb{N}^{n-1}}a_{i,\beta}(\mathfrak{a})\mathbf{x}^\beta$. *Then, for every* i *and* β, *there is a proper global analytic subset* $Z_{i,\beta}\subsetneq\Lambda$ *such that* $a_{i,\beta}$ *extends on* $\Lambda\smallsetminus Z_{i,\beta}$ *as an analytic function* $\overline{a}_{i,\beta}\in\mathcal{L}$. *Moreover if we set*

$$\overline{f}:=y^d+\sum_{\beta\in\mathbb{N}^{n-1}}\overline{a}_{1,\beta}(\mathfrak{a})\mathbf{x}^\beta y^{d-1}+\cdots+\sum_{\beta\in\mathbb{N}^{n-1}}\overline{a}_{d,\beta}(\mathfrak{a})\mathbf{x}^\beta\in\mathcal{L}[\![\mathbf{x}]\!][y]$$

then $f(\mathbf{x},y)\in\operatorname{Ker}(\Psi^*_{\mathcal{L}})$.

The proof of this result is strongly inspired by the proof of [Pa90, Lemme 6.3], and is based on Chevalley's Lemma:

Proposition 6.2 (Chevalley's Lemma)**.** [Ch43, Lemma 7] *Let* $\mathbb{k}$ *be a field. Let* $\varphi:\mathbb{k}[\![\mathbf{x}]\!]\longrightarrow\mathbb{k}[\![\mathbf{u}]\!]$ *be a morphism of formal power series rings. Then there exists a function* $\lambda:\mathbb{N}\longrightarrow\mathbb{N}$ *such that*

$$\forall k\in\mathbb{N},\ \varphi^{-1}((\mathbf{u})^{\lambda(k)})\subset(\mathbf{x})^k+\operatorname{Ker}(\varphi).$$

The smallest function satisfying this property is called the Chevalley's function *of* φ, *and is denoted by* λ_φ.

We start by fixing notation and by proving a Corollary of Chevalley's Lemma. Let $\mathbb{k}$ be a field and $\varphi:\mathbb{k}[\![\mathbf{x}]\!]\longrightarrow\mathbb{k}[\![\mathbf{u}]\!]$ be a morphism of formal power series rings. We set $\mathbf{x}':=(x_1,\ldots,x_{n-1})$. Let us consider the images of the x_i by φ:

$$\varphi_i=\sum_{\alpha\in\mathbb{N}^m}\varphi_{i,\alpha}\mathbf{u}^\alpha$$

where the $\varphi_{i,\alpha}\in\mathbb{k}$. Let

$$F(x):=x_n^d+A_1(\mathbf{x}')x_n^{d-1}+\cdots+A_d(\mathbf{x}') \tag{12}$$

where the A_i are universal power series

$$A_i:=\sum_{\beta\in\mathbb{N}^{n-1}}A_{i,\beta}\mathbf{x}'^\beta \tag{13}$$

and the $A_{i,\beta}$ are new indeterminates. Then we can expand

$$F(\varphi_1,\ldots,\varphi_m)=\sum_{\gamma\in\mathbb{N}^m}F_\gamma\mathbf{u}^\gamma$$

where

$$F_\gamma=\sum_{i,\beta}M_{\gamma,i,\beta}A_{i,\beta}+B_\gamma$$

with $M_{\gamma,i,\beta}$ and B_γ polynomials in the $\varphi_{j,\alpha}$.

Let R be a ring. Then the system of linear equations

$$\forall\gamma\in\mathbb{N}^m,\quad F_\gamma(A_{i,\beta})=0 \tag{S_∞}$$

has a solution $(a_{i,\beta})\in R^{\mathbb{N}}$ if and only if $\operatorname{Ker}(\varphi)$ contains a non zero Weierstrass polynomial

$$f=x_n^d+a_1(\mathbf{x}')x_n^{d-1}+\cdots+a_d(\mathbf{x}'),\ \text{where } a_i(\mathbf{x}')=\sum_{\beta\in\mathbb{N}^{n-1}}a_{i,\beta}{\mathbf{x}'}^\beta. \tag{14}$$

Let us consider the systems of linear equations

$$(S_k) \qquad \forall \gamma \in \mathbb{N}^m, |\gamma| < k, \quad F_\gamma(A_{i,\beta}) = 0$$

where k runs over $\mathbb{N}$. We have

Corollary 6.3 (Approximation)**.** *Let $\Bbbk$ be a field. Assume that f, given as in (14), is a generator of $\mathrm{Ker}(\varphi)$. Then $(a_{i,\beta})$ is the unique solution of (S_∞) in $\Bbbk^{\mathbb{N}}$. Moreover, there is a function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $k \in \mathbb{N}$, all solutions $(\widetilde{a}_{i,\beta}) \in \Bbbk^{\mathbb{N}}$ of $(S_{\mu(k)})$ satisfies*

$$\forall \beta \in \mathbb{N}^n,\ |\beta| \leqslant k \Longrightarrow \widetilde{a}_{i,\beta} = a_{i,\beta}.$$

Proof. Let $(\widetilde{a}_{i,\beta})$ be a solution of (S_∞). Then

$$\widetilde{f} := x_n^d + \sum_{\beta \in \mathbb{N}^{n-1}} \widetilde{a}_{1,\beta} \mathbf{x}^\beta x_n^{d-1} + \cdots + \sum_{\beta \in \mathbb{N}^{n-1}} \widetilde{a}_{d,\beta} \mathbf{x}^\beta \in \mathrm{Ker}(\varphi).$$

Since f is a generator of $\mathrm{Ker}(\varphi)$, there is $g \in \Bbbk[\![\mathbf{x}]\!]$ such that $\widetilde{f} = fg$. Since f and $\widetilde{f}$ are Weierstrass polynomials, by the uniqueness of the decomposition of a series as a product of a Weierstrass polynomials with a unit, we have that $g = 1$ and $\widetilde{f} = f$. This shows that $(a_{i,\beta})$ is the unique solution of (S_∞). Next, for $k \in \mathbb{N}$ we set

$$\mu(k) = \lambda\left((d+1)^d (k+1)\right)$$

where λ is given in Proposition 6.2. Consider a solution $(\widetilde{a}_{i,\beta}) \in \Bbbk^{\mathbb{N}}$ of $(S_{\mu(k)})$. Set

$$\widetilde{f} := x_n^d + \widetilde{a}_1(\mathbf{x}') x_n^{d-1} + \cdots + \widetilde{a}_d(\mathbf{x}'), \quad \text{where} \quad \widetilde{a}_i := \sum_{\beta \in \mathbb{N}^{n-1}} \widetilde{a}_{i,\beta} \mathbf{x}^\beta,\ i = 1, \ldots, d.$$

Since $\varphi(\widetilde{f}) \in (\mathbf{u})^{\mu(k)}$, by Proposition 6.2, $\widetilde{f} \in (x)^{(d+1)^d(k+d+1)} + \mathrm{Ker}(\varphi)$. Therefore

$$\widetilde{f} = fg + \sum_{i=1}^d (\widetilde{a}_i - a_i) x_n^{d-i}$$

for some g, where

$$\sum_{i=1}^d (\widetilde{a}_i - a_i) x_n^{d-i} \in \mathrm{Ker}(\varphi) + (\mathbf{x})^{(d+1)^d(k+d+1)}.$$

Thus we can write

$$\sum_{i=1}^d (\widetilde{a}_i - a_i) x_n^{d-i} = fh + \varepsilon \tag{15}$$

where $\varepsilon \in (\mathbf{x})^{(d+1)^d(k+d+1)}$. We denote by ν the monomial valuation defined by

$$\nu\left(\sum_{\alpha \in \mathbb{N}^n} g_\alpha x^\alpha\right) := \min\{(d+1)(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n \mid g_\alpha \neq 0\}.$$

For a power series g, we denote by $\mathrm{in}(g)$ its initial term in respect to this monomial valuation. We remark that, for any g, $(d+1)\,\mathrm{ord}(g) \geqslant \nu(g) \geqslant \mathrm{ord}(g)$.
Note that $\mathrm{in}(f) = x_n^d$. But, in (15), we see that the initial term of the left hand side is not divisible by x_n^d. Therefore $\nu\left(\sum_{i=1}^d (\widetilde{a}_i - a_i) x_n^{d-i}\right) \geqslant \nu(\varepsilon)$. Therefore

$$(d+1)\,\mathrm{ord}\left(\sum_{i=1}^d (\widetilde{a}_i - a_i) x_n^{d-i}\right) \geqslant \mathrm{ord}(\varepsilon).$$

Thus, there is a i_0 such that

$$\operatorname{ord}((\widetilde{a}_{i_0} - a_{i_0})x_n^{d-i_0}) \geqslant (d+1)^{d-1}(k+d+1).$$

In particular $\widetilde{a}_{i_0} - a_{i_0} \in (\mathbf{x})^{(d+1)^{d-1}(k+d+1)-(d-i_0)} \subset (\mathbf{x})^{k+1}$. On the other hand we have that $\sum_{i\neq i_0}(\widetilde{a}_i - a_i)x_n^{d-i} \in \operatorname{Ker}(\varphi) + (\mathbf{x})^{(d+1)^{d-1}(k+d+1)}$. The result is proved by induction on the number of terms in the sum. □

We are now ready to turn to the proof of the main result of this subsection:

Proof of the Extension Lemma 6.1. We consider, for each $\mathfrak{a} \in U$, the following system of linear equations

$$(S_\infty(\mathfrak{a})) \qquad \forall \gamma \in \mathbb{N}^m, \quad F_\gamma(\mathfrak{a})(A_{i,\beta}) = 0$$

where F, A_i are as in equations (12) and (13), respectively. Set $\Psi_k = \pi_k \circ \Psi$ where $\pi_k : \mathbb{K}^n \to \mathbb{K}$ is the projection to the k-entry, and note that all of its derivatives $\frac{\partial^{|\gamma|}}{\partial \mathbf{u}^\gamma}\Psi_k(\cdot, 0)$ are globally defined morphisms over Λ. Now consider:

$$F(\Psi^*_{1,\mathfrak{a}}, \ldots, \Psi^*_{n,\mathfrak{a}}) = \sum_{\gamma\in\mathbb{N}^m} F_\gamma(\mathfrak{a})\,\mathbf{u}^\gamma$$

where $\mathfrak{a} \in \Lambda$, and

$$F_\gamma(\mathfrak{a}) = \sum_{i=1}^{d} \sum_{\beta\in\mathbb{N}^{n-1},\, \beta\leqslant\gamma} M_{\gamma,i,\beta}(\mathfrak{a})A_{i,\beta} + B_\gamma(\mathfrak{a})$$

with $M_{\gamma,i,\beta}(\mathfrak{a})$ and $B_\gamma(\mathfrak{a})$ polynomials in the derivatives of $\Psi^*_{\mathfrak{a}}$. In particular, note that $M_{\gamma,i,\beta}(\mathfrak{a})$ and $B_\gamma(\mathfrak{a})$ belong to $\mathcal{O}(\Lambda)$.

As before, for any $k \in \mathbb{N}$, we consider the finite system of linear equations:

$$(S_k(\mathfrak{a})) \qquad \forall \gamma \in \mathbb{N}^m, |\gamma| < k, \quad F_\gamma(\mathfrak{a})(A_{i,\beta}) = 0.$$

Let s_k denote the number of indexes γ such that $|\gamma| < k$. The system $(S_k(\mathfrak{a}))$ can be written as

$$M^{(k)}(\mathfrak{a}) \cdot A^{(k)} + B^{(k)}(\mathfrak{a}) = 0$$

where $M^{(k)}(\mathfrak{a})$ is the $(s_k \times ds_k)$-matrix with entries $M_{\gamma,i,\beta}(\mathfrak{a})$, $A^{(k)}$ is the $(ds_k \times 1)$-column with entries $A_{i,\beta}$, and $B^{(k)}(\mathfrak{a})$ is the $(s_k \times 1)$-column with entries $B_\gamma(\mathfrak{a})$. We denote by $M^{(k)}_{i,\beta}(\mathfrak{a})$ the column of $M^{(k)}(\mathfrak{a})$ corresponding to $A_{i,\beta}$, that is:

$$M^{(k)}(\mathfrak{a}) \cdot A^{(k)} = \sum_{i=1}^{d} \sum_{|\beta|<k} M^{(k)}_{i,\beta}(\mathfrak{a})A_{i,\beta}$$

Let us fix $i_0 \in \{1, \ldots, d\}$ and $\beta_0 \in \mathbb{N}^{n-1}$ and let us prove that there exists $\overline{a}_{i_0,\beta_0} \in \mathcal{L}$ whose restriction to U is equal to a_{i_0,β_0}. For every $k \in \mathbb{N}$ with $k > |\beta_0|$, let us denote by $t_0^{(k)}(\mathfrak{a})$ the dimension of the $\mathbb{K}$-vector space $T_0^{(k)}(\mathfrak{a})$ generated by the $M^{(k)}_{i,\beta}(\mathfrak{a})$ for $(i, \beta) \neq (i_0, \beta_0)$. There is an analytic proper subset $D^{(k)}$ of Λ such that for every $\mathfrak{a} \in \Lambda \setminus D_0^{(k)}$, $t_0^{(k)}(\mathfrak{a})$ is maximal; denote by $t_0^{(k)}$ this maximal value.

We now fix $\mathfrak{a} \in U \setminus \bigcup_{k>|\beta|} D_0^{(k)}$ and consider $\mu_{\mathfrak{a}}$ the Chevalley function of Corollary 6.3 associated to $f(\mathfrak{a}, x)$. We now fix $k = |\beta_0| + 1$ and we set $\ell = \mu_{\mathfrak{a}}(k)$. To simplify the notation, set $t_0^{(\ell)} = t_0$, and consider $\mathbb{K}$-linearly independent vectors $M^{(\ell)}_{i_1,\beta_1}(\mathfrak{a})$, $M^{(\ell)}_{i_2,\beta_2}(\mathfrak{a})$, ..., $M^{(\ell)}_{i_{t_0},\beta_{t_0}}(\mathfrak{a})$ which generate $T_0^{(\ell)}(\mathfrak{a})$.

Claim 6.4. *There exists a neighborhood $U_{\mathfrak{a}}$ of $\mathfrak{a}$ such that $M^{(\ell)}_{i_0,\beta_0}(\mathfrak{b})$ does not belong to the vector space generated by $T_0^{(\ell)}(\mathfrak{b})$ for every $\mathfrak{b} \in U_{\mathfrak{a}}$.*

Proof. Indeed, from the definition of $T_0^{(\ell)}(\mathfrak{a})$ the equality $M^{(\ell)}(\mathfrak{a}) \cdot A^{(\ell)} + B^{(\ell)}(\mathfrak{a}) = 0$ can be re-written as:

$$A_{i_0,\beta_0} M^{(\ell)}_{i_0,\beta_0}(\mathfrak{a}) + \sum_{j=1}^{t_0} (A_{i_j,\beta_j} + L_j) M^{(\ell)}_{i_j,\beta_j}(\mathfrak{a}) + B^{(\ell)}(\mathfrak{a}) = 0.$$

where the L_j are $\mathbb{K}$-linear combinations of the terms $A_{i,\beta}$ with $(i,\beta) \neq (i_j,\beta_j)$ for $j = 0,\dots,t_0$. We recall that, by Corollary 6.3, there exists a unique entry $a_{i_0,\beta_0} = A_{i_0,\beta_0}$ for which the above system admits a solution. It is now immediate that $M^{(\ell)}_{i_0,\beta_0}(\mathfrak{a}) \notin T_0^{(\ell)}(\mathfrak{a})$ (otherwise, for each choice of A_{i_0,β_0}, it would be possible to compensate the terms $A_{i,\beta}$ with $(i,\beta) \neq (i_0,\beta_0)$ in order to get a different solution). We conclude easily from the analyticity of the vectors $M^{(\ell)}_{i,\beta}$. □

Now, by analyticity of the entries $M^{(\ell)}_{i,\beta}$, there is a proper analytic subset E_0 of Λ such that, for every $\mathfrak{b} \in \Lambda \setminus E_0$, the vectors $M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})$, for $0 \leqslant j \leqslant t_0$, are $\mathbb{K}$-linearly independent. Moreover, since $t_0 = \max_{\mathfrak{c}}\{t_0^{(\ell)}(\mathfrak{c})\}$, these vectors form a basis of the vector space generated by all the $M^{(\ell)}_{i,\beta}(\mathfrak{b})$. Therefore, for a given $(i,\beta) \neq (i_j,\beta_j)$ for $j = 0,\dots,t_0$ and for a given $\mathfrak{b} \in \Lambda \setminus E_0$, the equation $\sum_{j=0}^{t_0} M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b}) X_j = M_{i,\beta}(\mathfrak{b})$ has a unique solution $X = (X_0,\dots,X_{t_0}) \in \mathbb{K}$. Let us denote by $M_0(\mathfrak{b})$ the $s_\ell \times (t_0+1)$-matrix with columns $M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})$ for $j = 0,\dots,t_0$. By Cramer's rule, the X_i have the form $g_i(\mathfrak{b})/\Delta_0(\mathfrak{b})$ where $g_i(\mathfrak{b})$ is a minor of a matrix whose entries are some of the entries of the $M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})$ and of $M_{i,\beta}(\mathfrak{b})$, and $\Delta_0(\mathfrak{b})$ is the determinant of a (t_0+1)-square sub-matrix $N_0(\mathfrak{b})$ of $M_0(\mathfrak{b})$. Therefore, there is a proper analytic subset E_1 of Λ, such that for every $\mathfrak{b}' \in \Lambda \setminus E_1$, $\Delta_0(\mathfrak{b}') \neq 0$. In particular the system $(S_\ell(\mathfrak{b}))$, for $\mathfrak{b} \in \Lambda \setminus (E_0 \cup E_1)$, can be rewritten as

$$\sum_{j=0}^{t_0} M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})(A_{i_j,\beta_j} + L_{i_j,\beta_j}(\mathfrak{b})) + B^{(\ell)}(\mathfrak{b}) = 0$$

where the $L_{i_j,\beta_j}(\mathfrak{b})$ are linear forms in the $A_{i,\beta}$ for $(i,\beta) \neq (i_j,\beta_j)$ for $j = 0,\dots,t_0$, with analytic coefficients. We claim that $L_{i_0,\beta_0}(\mathfrak{b}) \equiv 0$. Indeed, by Claim 6.4, note that for every $\mathfrak{c} \in U_{\mathfrak{a}} \setminus \bigcup_{k>|\beta|} D_0^{(k)}$ we have that $M^{(\ell)}_{i_0,\beta_0}(\mathfrak{c})$ does not belong to the t_0-vector space $T_0^{(\ell)}(\mathfrak{c})$, implying that $L_{i_0,\beta_0}(\mathfrak{c})$ is equal to zero in an open set; by analyticity $L_{i_0,\beta_0} \equiv 0$. In particular the system $(S_\ell(\mathfrak{b}))$, for $\mathfrak{b} \in \Lambda \setminus (E_0 \cup E_1)$, can be rewritten as:

$$M^{(\ell)}_{i_0,\beta_0}(\mathfrak{b}) A_{i_0,\beta_0} + \sum_{j=1}^{t_0} M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})(A_{i_j,\beta_j} + L_{i_j,\beta_j}(\mathfrak{b})) + B^{(\ell)}(\mathfrak{b}) = 0.$$

It now follows from Cramer's rule that there exists a solution $\overline{a}_{i_0,\beta_0}(\mathfrak{b})$ of the truncated system which can be expressed as a division $Q_0(\mathfrak{b})/\Delta_0(\mathfrak{b})$, where $Q_0(\mathfrak{b})$ depends on the entries of $M^{(\ell)}_{i_j,\beta_j}(\mathfrak{b})$ for $j = 1,\dots,t_0\}$ and $B^{(\ell)}(\mathfrak{b})$. We now remark that Claim 6.4 implies that $\overline{a}_{i_0,\beta_0}(\mathfrak{b}) = a_{i_0,\beta_0}(\mathfrak{b})$ for every $\mathfrak{b} \in U_{\mathfrak{a}} \setminus (D_0 \cup Z_0)$, which implies that they are equal over $U \setminus Z_0$. We conclude that a_{i_0,β_0} can be extended as a holomorphic function on $\Lambda \setminus Z_0$ that belongs to $\mathcal{L}$. Since the choice of (i_0, β_0) was arbitrary, this proves the Lemma. $\square$

6.2. **Proof of Proposition 5.17.** Let $\Phi : \Omega \to \mathbb{K}^n$ and $\varphi : \Lambda \to \Omega$ be the two morphisms from the definition of admissible family 5.15, and recall that $\Psi(\mathfrak{a}, \mathbf{u}) = \Phi(\varphi(\mathfrak{a}) + \mathbf{u}) - \Phi(\varphi(\mathfrak{a}))$. Let $\mathfrak{a} \in \mathcal{R}(\Psi, \Lambda)$ and set $r := \mathrm{r}(\Psi_{\mathfrak{a}}) = \mathrm{r}^{\mathcal{F}}(\Psi_{\mathfrak{a}})$; in particular, $r = \mathrm{r}(\Phi_{\varphi(\mathfrak{a})}) = \mathrm{r}^{\mathcal{F}}(\Phi_{\varphi(\mathfrak{a})})$. It follows from Gabrielov's rank Theorem (or the rank Theorem 1.1) that $\mathrm{r}^{\mathcal{A}}(\Phi_{\varphi(\mathfrak{a})}) = r$.

Apart from a translation in $\mathbf{x}$, we may suppose that $\Phi(\varphi(\mathfrak{a})) = 0$. Let $(Z, 0)$ be the germ of analytic set defined by $\mathrm{Ker}(\Phi^*_{\varphi(\mathfrak{a})})$ and note that $r = \dim(Z, 0)$. Apart from a linear change of coordinates in $\mathbf{x}$, we may assume that the projection $\pi : (Z, 0) \longrightarrow (\mathbb{K}^r, 0)$ on the first r coordinates is finite. In particular, each function x_i, for $i > r$, is finite over the ring of convergent power series $\mathbb{K}\{x_1, \ldots, x_r\}$. That is, by the Weierstrass preparation theorem, there exist non zero Weierstrass polynomials

$$P_i(x_1, \ldots, x_r, x_{r+i}) \in \mathbb{K}\{x_1, \ldots, x_r\}[x_{r+i}], \quad \text{for } i = 1, \ldots, n - r,$$

belonging to $\mathrm{Ker}(\Phi^*_{\varphi(\mathfrak{a})})$. By replacing each P_i by one of its irreducible factors we may assume that the P_i are irreducible Weierstrass polynomials at 0.

We claim that, apart from changing the choice of point $\mathfrak{a} \in \mathcal{R}(\Psi, \Lambda)$ and recentering the coordinate system $\mathbf{x}$ accordingly, there exists a neighborhood U of $\mathfrak{a}$ such that P_i are well-defined and irreducible at every point in $\Phi(\varphi(U))$. Indeed, let V be an open neighborhood of 0 in $\mathbb{K}^n$ on which the P_i are well-defined, and U be an open connected neighborhood of $\mathfrak{a}$ such that $\Phi(\varphi(U)) \subset V$. Apart from shrinking U and V, we may suppose that $P_i \in \mathrm{Ker}(\Phi^*_{\varphi(\mathfrak{b})})$ for every $\mathfrak{b} \in U$; in particular, $U \subset \mathcal{R}(\Psi, \Lambda)$. Now, recall that being not irreducible is an open property for the Euclidean topology, thus the property of being irreducible is a closed property. If one of P_i is not irreducible at a point $\Phi(\varphi(\mathfrak{b}))$, for some $\mathfrak{b} \in U$, we may replace $\mathfrak{a}$ by $\mathfrak{b}$, P_i by one of its irreducible factors at this point, and we shrink U and V accordingly. Since the degree of the P_i is a positive integer, this process should end in a finite number of steps, proving the claim.

Fix $s = 1, \ldots, n - r$, set $\mathbf{x}^{(s)} = (x_1, \ldots, x_r, x_{r+s})$, $\Phi^{(s)} := (\Phi_1, \ldots, \Phi_r, \Phi_{r+s})$, and denote by $\Psi^{(s)} = (\Psi_1, \ldots, \Psi_r, \Psi_{r+s})$ the family associated to $\Phi^{(s)}$ and φ. Note that $U \subset \mathcal{R}(\Psi^{(s)}, \Lambda)$ by construction. Moreover $\mathrm{Ker}({\Psi^{(s)}_{\mathfrak{a}}}^*)$ is generated by P_s since P_s is irreducible and $\mathrm{Ker}({\Psi^{(s)}_{\mathfrak{a}}}^*)$ is a height one prime ideal of $\mathbb{K}\{\mathbf{x}^{(s)}\}$. We set:

$$f_s(\mathfrak{b}, \mathbf{x}^{(s)}) := P_i\left(\Phi^{(s)}(\varphi(\mathfrak{b})) + \mathbf{x}^{(s)}\right)$$

for every $\mathfrak{b} \in U$, which can be written as:

$$f_s(\mathfrak{b}, \mathbf{x}^{(s)}) = y^d + a_1(\mathfrak{b}, \mathbf{x}')y^{d-1} + \cdot + a_d(\mathfrak{b}, \mathbf{x}')$$

where $y = x_{r+s}$ and $\mathbf{x}' = (x_1, \ldots, x_r)$. First, note that $a_i(\mathfrak{b}, \mathbf{x}') \in \mathcal{O}(U)[\![\mathbf{x}']\!]$ since $\Phi \circ \varphi$ is an analytic map defined on U and P_i is well defined in $\Phi(\varphi(U))$. Second, note that $f_s(\mathfrak{b}, \mathbf{x}^{(s)}) \in \mathrm{Ker}({\Psi^{(s)}_{\mathfrak{b}}}^*)$ for every $\mathfrak{b} \in U$ since ${\Psi^{(s)}_{\mathfrak{b}}}^*(f_i) = P_s \circ \Phi^{(s)}_{\varphi(\mathfrak{b})} \equiv 0$, and that $f_s(\mathfrak{b}, \mathbf{x}^{(s)})$ generates $\mathrm{Ker}({\Psi^{(s)}_{\mathfrak{b}}}^*)$ since P_i is irreducible. Third, note that $f_i(\mathfrak{b}, 0, y) = y^{k(\mathfrak{b})}U(\mathfrak{b}, y)$ for some $1 \leqslant k(\mathfrak{b}) \leqslant d$ and $U(\mathfrak{b}, y)$ is a monic polynomial in y coprime with y. By Hensel Lemma (see [Gro67, 18.5.13]), this implies that $f_i(\mathfrak{b}, \mathbf{x}', y)$ is the product of two monic polynomials of degree $k(\mathfrak{b})$ and $d - k(\mathfrak{b})$ respectively. From the fact that P_i is irreducible and $k(\mathfrak{b}) > 0$ at every point $\mathfrak{b} \in U$, we conclude that $k(\mathfrak{b}) = d$, that is, $f_s(\mathfrak{b}, 0, y) = y^d$. These three observations show that f_s satisfies all hypothesis of Lemma 6.1, so that it can be extended as a power series $\overline{f}_s(\mathbf{x}^{(s)})$ of $\mathcal{L}[\![\mathbf{x}]\!]$, where $\mathcal{L}$ is the fraction field of $\mathcal{O}(\Lambda)$, such that $\Psi^*_{\mathcal{L}}(\overline{f}_s) = 0$. We conclude that $\mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathcal{L}}) \leqslant r$, and since $\mathrm{r}(\Psi^*_{\mathcal{L}}) = r$, we get that $\mathrm{r}(\Psi) = \mathrm{r}^{\mathcal{F}}(\Psi^*_{\mathcal{L}})$, finishing the proof.

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