PŁOSKI APPROXIMATION THEOREM

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Abstract. The aim of this paper is to review how some approximation results in commutative algebra are being used to construct equisingular deformations of singularities. The first example of such an approximation result appeared for the first time in A. Płoski's PhD thesis.

1. Introduction

The Artin's seminal approximation Theorem [1], which established the density of convergent solutions in the space of formal solutions for systems of analytic equations, has been an important result having a lot of applications in different fields, in particular in singularity theory [22]. This theorem lead to new problems of approximations in local algebra and local analytic geometry [3].

In his PhD thesis, A. Płoski made a significant contribution to this area by proposing a strengthened version of Artin's theorem, now known as the Płoski's Approximation Theorem. His result not only extended Artin's theorem but also gave hints for generalizations in commutative algebra and provided applications to singularity theory. In particular it has implications for the study of equisingular deformations of singularities, which play a key role in understanding the topological and analytical properties of singular spaces.

This paper reviews Płoski's Approximation Theorem, examining its implications and generalizations, particularly in the context of equisingular deformations and algebrization of analytic sets and functions. By integrating recent advancements, we aim to shed light on the broader relevance of Płoski's Theorem. Furthermore, we explore their connections to the algebraization of function germs and meromorphic functions, with a focus on two variables cases. These results demonstrate the far-reaching impact of Płoski's work on contemporary mathematical research.

2. Approximation Theorems

2.1. **Artin approximation Theorem.** In 1968 M. Artin proved the following result:

Theorem 2.1 (Artin approximation Theorem)**.** [1] *Let* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_p)$ *be an* n*-tuple and a* p*-tuple of indeterminates. Let* $f = (f_1, \ldots, f_m) \in$

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$\mathbb{C}\{x,y\}^m$ be an m-tuple of convergent power series. Assume given a formal power series solution vector $\widehat{y}(x) \in \mathbb{C}[\![x]\!]^p$ vanishing at 0:

$$f(x, \widehat{y}(x)) = 0.$$

Then, for every $c \in \mathbb{N}$, there is a convergent power series solution vector $y(x) \in \mathbb{C}\{x\}^p$ vanishing at 0:

$$f(x, y(x)) = 0$$

such that

$$\forall i, \qquad y_i(x) - \widehat{y}_i(x) \in (x)^c.$$

This result means that the set of convergent solutions of $f(x, y) = 0$ is dense in the set of formal solutions for the (x)-adic topology. One year later, M. Artin proved several extensions of this theorem. In particular he proved that this result remains true if we replace the rings of convergent power series $\mathbb{C}\{x\}$ and $\mathbb{C}\{x,y\}$ by the rings of algebraic power series $\mathbb{k}\langle x\rangle$ and $\mathbb{k}\langle x, y\rangle$ for a characteristic zero field $\mathbb{k}$. Let us recall that the elements of $\mathbb{k}\langle x\rangle$ are the formal power series $f(x)$ such that $P(x, f(x)) = 0$ for some nonzero polynomial $P(x,t) \in \mathbb{k}[x,t]$. Moreover this set of algebraic power series is a Noetherian local ring satisfying the Weierstrass division and preparation theorems.

2.2. **Płoski approximation Theorem.** In his PhD thesis [15], A. Płoski made a deep study of the method of M. Artin to prove a strengthened version of his theorem:

Theorem 2.2 (Płoski approximation Theorem)**.** *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_p)$ be an n-tuple and a p-tuple of indeterminates. Let $f = (f_1, \ldots, f_m) \in \mathbb{C}\{x,y\}^m$ be an m-tuple of convergent power series. Assume given a formal power series solution vector $\widehat{y}(x) \in \mathbb{C}[\![x]\!]^p$ vanishing at 0:*

$$f(x, \widehat{y}(x)) = 0.$$

Then there is a convergent power series solution $y(x,z) \in \mathbb{C}\{x,z\}^p$, where $z = (z_1, \ldots, z_s)$ is a new s-tuple of indeterminates:

$$f(x, y(x,z)) = 0$$

and a vector of formal power series $\widehat{z}(x) \in \mathbb{C}[\![x]\!]^s$ such that

$$y(x, \widehat{z}(x)) = \widehat{y}(x).$$

This result trivially implies Artin approximation Theorem. Indeed, with the notations of Theorem 2.1 and Theorem 2.2, it is enough to replace $\widehat{z}(x)$, by any vector of convergent power series $z(x)$ such that $\widehat{z}(x) - z(x) \in (x)^c$, and set $y(x) := y(x, z(x))$ to obtain the conclusion of Theorem 2.1.

A. Płoski published this theorem in [16] without the details of the proof. He did not published any complete proof (except of his PhD thesis [15] written in Polish) before 2017. At a conference at Lille in 1999, he gave a course presenting this result, and wrote notes with a complete proof [17] (written in French). These notes are available on the web, but have not been formally published. Therefore in 2017, A. Płoski published a complete proof in [18].

Remark 2.3. Płoski's result can be roughly rephrased as follows: any formal power series solution of $f(x,y) = 0$ is a formal point in an analytically parametrized family of solutions. This result can be thought as a kind of uniformization theorem.

Remark 2.4. Ploski's proof works in exactly the same way when replacing rings of convergent series with rings of algebraic series over a characteristic zero field. Indeed, the essential arguments of the proof are the Jacobian criterion and the Weierstrass division theorem. In fact, this theorem holds in the more general framework of Weierstrass systems in any characteristic [21].

Example 2.5. Assume that $f(x,y)$ satisfies the assumptions of the Implicit function Theorem : $f(0,0)=0$ and $\frac{\partial f}{\partial y}(0,0)$ is an invertible matrix (here we assume $m=p$). Therefore, the equation $f(x,y)=0$ has only one solution $y^0(x)\in\mathbb{C}\{x\}^p$ which is convergent. Therefore Theorem 2.2 is obviously true with $s=0$ and $y(x)=y^0(x)$.

Example 2.6. Consider one equation $f(x,y):=y_1^2-y_2^3=0$. Then, since $\mathbb{C}\{x\}$ are $\mathbb{C}[\![x]\!]$ are unique factorization domains, the set of convergent (resp. formal) solutions is

$$\{(z(x)^3,z(x)^2)\mid z(x)\in\mathbb{C}\{x\}\ (resp.\mathbb{C}[\![x]\!])\}.$$

Therefore Theorem 2.2 is true with $s=1$. Moreover, in this case the analytically parametrized family of solutions is the whole set of solutions of $f(x,y)=0$.

Example 2.7. Consider the same $f(x,y)$ as before and assume that x is a single indeterminate. Let $(\widehat{y}_1(x),\widehat{y}_2(x))$ be a nonzero formal power series solution. Let $d:=\mathrm{ord}(\widehat{y}_1(x))>0$. Since $\widehat{y}_1(x)^2=\widehat{y}_2(x)^3$, we have that $d\in3\mathbb{Z}$. Let $d=3(e+1)$ where $e\in\mathbb{N}$. Then we have $\widehat{y}_1(x)=x^{3e}\widehat{z}(x)^3$ and $\widehat{y}_2(x)=x^{2e}\widehat{z}(x)^2$ for some formal power series $\widehat{z}(x)$ vanishing at 0. Then in Theorem 2.2 we can choose $(y_1(x,z),y_2(x,z)):=(x^{3e}z^3,x^{2e}z^2)$. The analytically parametrized family of solutions we obtain in this way does not cover the whole set of solutions of $f(x,y)=0$.
The proof of Płoski is completely effective, and if we follows his proof for this particular example, this is the analytically parametrized family of solutions that we obtain. Therefore, the proof of Płoski does not provide the whole set of solutions of $f(x,y)=0$ in general.

Remark 2.8. Giving a formal power series solution $\widehat{y}(x)$ of $f(x,y)=0$ is equivalent to the data of a morphism of $\mathbb{C}\{x\}$-algebra:

$$\varphi:\frac{\mathbb{C}\{x\}[y]}{(f(x,y))}\longrightarrow\mathbb{C}[\![x]\!]$$

Theorem 2.2 asserts that this morphism factors as

$$\begin{tikzcd} & \mathbb{C}[\![x]\!] \\ \dfrac{\mathbb{C}\{x,y\}}{(f(x,y))} \arrow[ur, "\varphi"] \arrow[r] & \mathbb{C}\{x,z\} \arrow[u] \end{tikzcd}$$

Having taken note of Płoski's Theorem, D. Popescu conjectured that this result was more general: given a Henselian local ring A, $\widehat{A}$ denoting its completion, for every finitely generated A-algebra B and any A-morphism $\varphi:B\longrightarrow\widehat{A}$, there exists a *smooth* A-algebra C such that φ factors through C:

$$\begin{tikzcd} & \widehat{A} \\ B \arrow[ur, "\varphi"] \arrow[r] & C \arrow[u] \end{tikzcd}$$

Then D. Popescu proved his conjecture in [19] (see also [23], [25], [20], [26] for subsequent proofs). Moreover, in [25], M. Spivakovsky proved a nested version of Popescu's Theorem. We will not state this statement, which is a bit technical, but from Spivakovsky's Theorem we can deduce the following nested version of Płoski result:

Theorem 2.9 (Nested Artin-Płoski-Popescu Approximation Theorem, [4])**.** *Let $f(x,y) \in \Bbbk\langle x\rangle[y]^m$ and let us consider a solution $y(x) \in \Bbbk[\![x]\!]^p$ of*

$$f(x, y(x)) = 0.$$

Let us assume that $y_i(x)$ depends only on $(x_1, \ldots, x_{\sigma(i)})$ where $i \longmapsto \sigma(i)$ is an increasing function. Then there exist a new set of indeterminates $z = (z_1, \ldots, z_s)$, an increasing function τ, convergent power series $z_i(x) \in \Bbbk[\![x]\!]$ vanishing at 0 such that $z_1(x), \ldots, z_{\tau(i)}(x)$ depend only on $(x_1, \ldots, x_{\sigma(i)})$, and an algebraic power series vector solution $y(x,z) \in \Bbbk\langle x, z\rangle^p$ of

$$f(x, y(x,z)) = 0,$$

such that for every i,

$$y_i(x,z) \in \Bbbk\langle x_1, \ldots, x_{\sigma(i)}, z_1, \ldots, z_{\tau(i)}\rangle \text{ and } y(x) = y(x, z(x)).$$

Remark 2.10. Let us mention that this nested version of Płoski's Theorem is no longer true for rings of convergent power series by an example of A. Gabrielov [8].

3. Algebrization of the germ of an analytic set or of an analytic function

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $x = (x_1, \ldots, x_n)$. If $f \in \mathbb{K}\{x\}$ is an isolated singularity then, by a result of Samuel [24], f is *finitely determined*, that is, there exists an integer k such that for every $g \in \mathbb{K}\{x\}$ with $f - g \in (x)^k$, there exists an analytic diffeomorphism $h : (\mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^n, 0)$ such that

$$f \circ h(x) = g(x).$$

In particular, if we choose g to be the truncation of f at an order $\geq k$, we obtain that f can be transformed into a polynomial after a local analytic change of coordinates (one says that f is analytically equivalent to a polynomial function germ). Several authors generalized Samuel's result, and eventually Kucharz [10] proved that every (non necessarily reduced) analytic function in two variables is finitely determined (he also proved that any analytic function in n variables is equivalent to a polynomial in two variables whose coefficients are analytic functions in $n-2$ variables).

Thus in the case of $n = 2$ any reduced analytic singularity can be transformed by an analytic change of coordinates to a polynomial one. When f is a (reduced) convergent power series in three or more variables, this is no longer true: H. Whitney [30] gave an example of a three variable reduced convergent power series which is not equivalent to a polynomial function neither to an algebraic one.

We explain below how to construct a deformation of a given analytic function germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ that is topologically trivial (with repect to the right equivalence, i.e. by a homeomorphism in the source $(\mathbb{K}^n, 0)$), and such that one fibre of this deformation is an algebraic function germ. This construction is based on Theorem 2.9 and the notion of Zariski equisingularity.

3.1. **Zariski equisingularity.** Zariski equisingularity of families of singular varieties was introduced by Zariski in [31] in the context of equisingularity of a hypersurface along a smooth subvariety. It can be formulated over any field of characteristic zero in the algebroid set-up (varieties defined by the formal power series) and over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ in the analytic case. For a survey on Zariski equisingularity see [12], it also contains an appendix on the higher order (also called generalized) discriminants that are used in the construction below.

Definition 3.1. Let $V = F^{-1}(0)$, $F \in \mathbb{K}\{t, x\}$, be an analytic hypersurface in a neighborhood of the origin in $\mathbb{K}^\ell \times \mathbb{K}^n$. We say that V is *Zariski equisingular with respect to the parameter $t \in \mathbb{K}^\ell$ (and a local system of coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$)* if (here $\pi : (\mathbb{K}^\ell \times \mathbb{K}^n, 0) \longrightarrow (\mathbb{K}^\ell \times \mathbb{K}^n, 0)$ denotes the canonical projection):

(1) $(\mathbb{K}^\ell, 0) \times (\{0\}) \subset (V, 0)$.
(2) The restriction $\pi_{|_{(V,0)}}$ is (algebraically) finite.
(3) The branch locus of $\pi_{|_{(V,0)}}$ is itself Zariski equisingular with respect to the parameter t.
(4) When $\ell = \dim(V, 0)$, then $(V, 0)$ is Zariski equisingular with respect to the parameter t if $(V, 0) = (\mathbb{C}^\ell, 0)$ (or at some stage the branch locus is empty).

We say that V is *Zariski equisingular with respect to the parameter $t \in \mathbb{K}^\ell$* if it is so after a local change of coordinates $(\mathbb{K}^\ell \times \mathbb{K}^n, 0) \to (\mathbb{K}^\ell \times \mathbb{K}^n, 0)$ preserving the parameter t.

Remark 3.2. Note that this notion depends heavily on the local choice of coordinates. Therefore often we consider Zariski equisingularity with respect to generic or generic linear x coordinates, see [32]. Whether a generic linear choice is generic in the sense of [32] is an open problem for singularities of codimension ≥ 3. For singularities of codimension 1, i.e. equisingular families of plane curves the positive answer follows from Zariski's theory of equisingularity of families of plane curves. For singularities in codimension 2, i.e. equisingular families of surfaces singularities in $\mathbb{C}^3$, the positive answer was given in [14].

In practice one may argue as follows. Let F be an analytic function defining $(V, 0)$ and suppose that $F(0, x) \not\equiv 0$. Then after a linear change of coordinates x, we may assume that F is a pseudopolynomial F_n times a unit $u_n(t, x)$. This means that F_n is a polynomial in x_n with coefficients that are analytic in ~~t~~ (t, x^{n-1}) (Here $x^{n-1} = (x_1, \dots, x_{n-1})$):

$$F_n(t, x) = x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j}(t, x^{n-1}) x_i^{p_i - j}$$

where $a_{n-1,j}(0) = 0$ for every j. Then, we denote by $\Delta_n(t, x^{n-1})$ the discriminant of F_n seen as a polynomial in x_n. If F is reduced, Δ_n is not identically zero. We assume again that $\Delta_n(0, x) \not\equiv 0$. In general, if F is not reduced we replace it by $(F)_{red}$, or equivalently we consider the higher order discriminants of F_n. Thus by induction we define a sequence of pseudopolynomials

$$F_i(t,x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t,x^{i-1})x_i^{p_i-j}, \qquad i = 0,\dots,n,$$

$t \in \mathbb{K}^\ell$, $x^i := (x_1,\dots,x_i) \in \mathbb{K}^i$, with analytic coefficients $a_{i-1,j}$, that satisfy

(1) $F_{i-1}(t,x^{i-1}) = 0$ if and only if $F_i(t,x^{i-1},x_i) = 0$ considered as an equation in x_i with (t,x^{i-1}) fixed, has fewer complex (!) roots than for generic (t,x^{i-1}).
(2) $F_0 \equiv 1$.
(3) There are positive reals $\delta_k > 0$, $k = 1,\dots,\ell$, and $\varepsilon_j > 0$, $j = 1,\dots,n$, such that F_i are defined on the polydiscs $U_i := \{|t_k| < \delta_k, |x_j| < \varepsilon_j, k = 1,\dots,\ell, j = 1,\dots,i\}$.
(4) All roots of $F_i(t,x^{i-1},x_i) = 0$, for $(t,x^{i-1}) \in U_{i-1}$, lie inside the circle of radius ε_i.
(5) Either $F_i(t,0) \equiv 0$ or $F_i \equiv 1$ (and in the latter case we define $F_k \equiv 1$ for all $k \le i$).

In practice, to define F_{i-1} in term of F_i, we do the following: if F_i is not reduced then we denote by Δ_i the first nonzero generalized discriminant of F_i as a polynomial in x_i. Then, after a linear change of coordinates in (t,x^{i-1}), we may assume that

$$\Delta_i(t,x^{i-1}) = u_{i-1}(t,x^{i-1})F_{i-1}(t,x^{i-2},x_{i-1}) \tag{1}$$

where F_{i-1} is a pseudopolynomial in x_{i-1}. After this linear change of coordinates, F_j, for $j \ge i$, is transformed into a new pseudopolynomial in x_j of degree p_j satisfying again Properties (1), (3) and (4), so it does not affect the form of the previous pseudopoynomials.

An important result due to Varchenko is the following one:

Theorem 3.3 ([27], ([28], [29]). *A Zariski equisingular family of singularities is topologically trivial.*

3.2. **Construction of a deformation of the germ of an analytic set.** Suppose now that $(V,0)$ is the germ at the origin of an analytic subset of $\mathbb{K}^n$ given by one equation $f(x) = 0$. We do not assume f reduced but we assume $f \not\equiv 0$. We define a local system of coordinates $x = (x_1,\dots,x_n)$ and a sequence of distinguished pseudopolynomials

$$f_i(x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}, \quad i = 1,\dots,n,$$

as follows.

Let, after a local change of coordinates x, f_n be the Weierstrass polynomial associated to f. Then we consider the generalized discriminants $\Delta_{n,i}$ of f_n that are polynomials in the entries of $a_{n-1} := (a_{n-1,1},\dots,a_{n-1,p_n})$. Let l_n be a positive integer such that

$$\Delta_{n,l}(a_{n-1}) \equiv 0 \qquad l < l_n, \tag{2}$$

and $\Delta_{n,l_n}(a_{n-1}) \not\equiv 0$. Then, after a local change of coordinates x^{n-1}, by the Weierstrass Preparation Theorem, we may write

$$\Delta_{n,l_n}(a_{n-1}) = u_{n-1}(x^{n-1})\Big(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}\Big),$$

where $u_{n-1}(0) \neq 0$ and for all j, $a_{n-2,j}(0) = 0$. Note that if f is reduced then $l_n = 1$ and the only generalized discriminant we consider is the standard one. Then we set

$$f_{n-1} := x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j},$$

We continue this construction and define a sequence of pseudopolynomials $f_i(x^i)$, $i = 1, \dots, n-1$, such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first non-identically zero generalized discriminant $\Delta_{i+1,l_{i+1}}(a_i)$ of f_{i+1}, where we denote in general $a_i = (a_{i,1}, \dots, a_{i,p_{i+1}})$,

$$\Delta_{i+1,l_{i+1}}(a_i) = u_i(x^i)\Big(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}\Big), \quad i = 0, \dots, n-1, \tag{3}$$

and $a_{i-1,j}(0) = 0$. Thus, for $i = 0, \dots, n-1$, the vector of functions a_i satisfies

$$\Delta_{i+1,l}(a_i) \equiv 0 \text{ for } l < l_{i+1}, \quad \Delta_{i+1,l_{i+1}}(a_i) \not\equiv 0. \tag{4}$$

This means in particular that

$$\Delta_{1,l}(a_0) \equiv 0 \quad \text{for } l < l_1 \text{ and } \Delta_{1,l_1}(a_0) \equiv u_0,$$

where u_0 is a nonzero constant.

Next we apply the algebraic power series version of Theorem 2.9 to the system of equations given by (3) and (4). By construction, this system admits convergent solutions. Therefore, by Theorem 2.9, there exist a new set of variables $z = (z_1, \dots, z_s)$, an increasing function τ, convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0, algebraic power series $u_i(x^i, z) \in \mathbb{C}\langle x^i, z_1, \dots, z_{\tau(i)}\rangle$, and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{C}\langle x^{(i)}, z_1, \dots, z_{\tau(i)}\rangle^{p_i}$, $b(x, z) \in \mathbb{C}\langle x, z\rangle^{n-1}$, such that the following holds:

$$\begin{aligned}
&z_1(x), \dots,\ z_{\tau(i)}(x) \text{ depend only on } (x_1, \dots, x_i),\\
&a_i(x^i, z),\ u_i(x^i, z), \text{ are solutions of (3) and (4)}\\
&\text{and } a_i(x^i) = a_i(x^i, z(x^i)), u_i(x^i) = u_i(x^i, z(x^i)),\ b(x) = b(x, z(x)).
\end{aligned}$$

It is essential that the new solutions $u_i(t, x^i), a_{i,j}(t, x^i)$ depend only on the first i variables in x. Thanks to this property it is easy to check that the one parameter deformation $t \to \{((t, x); F(t, x) = 0\}$ of

$$F(t, x) = x_n^{p_i} + \sum_{j=1}^{p_n} a_{n-1,j}(x^{n-1}, tz(x^{n-1}))x_n^{p_n-j}$$

is Zariski equisingular. Because $F(1, x) = f(x)$ and $F(0, x)$ is algebraic we have shown

Theorem 3.4 ([11], [4]). *Every analytic set germ given by one equation $f = 0$, $f \in \mathbb{K}\{x\}$, is Zariski equisingular to a germ defined by an algebraic power series.*

Remark 3.5. The first proof of algebraicity of analytic set germs was given by Mostowski in [11]. At that time Popescu's Theorem was not yet available. Instead, Mostowski proposes an ingenious recursive construction of the system of equations (3) and (4) giving Zariski equisingularity conditions by local linear changes of coordinates and, at the same time, step by step, provides the deformation to an algebraic power series solution following the recipe given by Płoski in [16].

3.3. **Algebraization. Final steps.** Using Varchenko's theorem, Theorem 3.3, Theorem 3.4 shows that every analytic set germ is homeomorphic to a one defined by the algebraic power series, moreover by an ambient homeomorphism. This holds not only for hypersurfaces. If $(V, 0)$ is given by a finite system of equations $g_s = 0, s = 1, \ldots, k$, $g_s \in \mathbb{K}\{x\}$, then we proceed as follows. In a system of local coordinates we replace $g_s = 0$ by the associated pseudopolynomials:

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}) x_n^{r_s - j},$$

and arrange the coefficients $a_{n-1,s,j}$ in a row vector $a_{n-1} \in \mathbb{K}\{x^{n-1}\}^{p_n}$ where $p_n := \sum_s r_s$. Then apply the previous construction to f_n being the product of the g_s's. After solving the system of equations for $t \in \mathbb{K}$ we define

$$F_n(t, x) = \prod_s G_s(t, x), \quad G_s(t, x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}, tz(x^{n-1})) x_n^{r_s - j}.$$

Then Varchenko's proof of Theorem 3.3 provides a topological trivialization that preserves the zero sets of $G_s(x, t) = 0, s = 1, \ldots, k$, and thus the ambient homeomorphism between $(V, 0)$ and $\{G(x, 0)_s = 0, s = 1, \ldots, k\}$.

Finally, the following theorem due to Bochnak and Kucharz [6], based on Artin-Mazur Theorem, gives an equivalence, up to a Nash diffeomorphism, between the zeros of algebraic power series (or equivalently germs of Nash functions) and the local zeros of polynomial functions.

Theorem 3.6. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $g_s : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, be a finite family of Nash function germs. Then there is a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and analytic (even Nash) units $u_s : (\mathbb{K}^n, 0) \to \mathbb{K}$, $u_s(0) \neq 0$, such that for all s, $u_s(x) g_s(h(x))$ are germs of polynomials.*

3.4. **Algebraization of function germs.** The algebraization of analytic set germs can be extended to analytic function germs, that is the mappings with values in $\mathbb{K}$. The following theorem was proven in [4].

Theorem 3.7. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ \sigma$ is the germ of a polynomial.*

The idea how to adapt Zariski equisingularity to the function case comes from [27]. Given a family $g_t(y) = g(t, y_1, \ldots, y_{n-1})$ of such germs parameterized by $t \in \mathbb{K}^\ell$, the idea is to consider the associated family of set germs defined by the graph of g, the zero set of $F(t, x_1, \ldots, x_n) := x_1 - g(t, x_2, \ldots, x_n)$. If the family $V = V(F)$ is Zariski equisingular, with respect to the system of coordinates $x_1, \ldots, x_n$, then the trivialization constructed in [27] does not move the variable x_1

$$h_t(x_1, \ldots x_n) = (x_1, \hat{h}_t(x_1, x_2, \ldots, x_n)). \tag{5}$$

Set $\sigma_t(y) := \hat{h}_t(g(y), y)$. Then

$$g_t \circ \sigma_t = g_{t_0},$$

To complete the passage from algebraic power series to polynomials one uses again Theorem 3.6. See [4] for details.

4. Algebraization of the germ of a meromorphic function in dimension 2

As said before, every two variables analytic function germ is finitely determined, so it is analytically equivalent to a polynomial function germ. The case of meromorphic function germs has been studied in [7]. The author shows that, on the contrary of analytic function germs, a two variables meromorphic germ is not always finitely determined. But the question whether a two variables meromorphic function germ is analytically equivalent to a rational function germ remains open. The following result has been proved recently:

Theorem 4.1. [9] *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let φ be the germ of a meromorphic function at $(\mathbb{K}^2, 0)$. Then there is the germ of an analytic diffeomorphism $h : (\mathbb{K}^2, 0) \longrightarrow (\mathbb{K}^2, 0)$ such that $\varphi \circ h$ is the germ of an algebraic meromorphic function, that is, $\varphi \circ h = \frac{f}{g}$ where f, $g \in \mathbb{C}\langle x_1, x_2\rangle$.*
Moreover, for any $k \in \mathbb{N}$, we may assume that $\varphi(x) - x \in (x)^k$.

Sketch of proof. The main case is when $\mathbb{K} = \mathbb{C}$. We consider this case now. The idea is to associate to φ a germ of analytic 1-form ω, and then to use Płoski Theorem to construct an analytic deformation of ω that has one algebraic fibre. Then we use a result of D. Cerveau and J.-F. Mattei to prove that this deformation is analytically trivial. Let us give more details.

Step 1. Let us write $\varphi = \frac{f}{g}$ where f and g are coprime convergent power series. Let us write

$$f = f_1^{\ell_1} \cdots f_p^{\ell_p} \text{ and } g = g_1^{k_1} \cdots g_q^{k_q}$$

where the f_i and g_j are irreducible convergent power series and the ℓ_i and k_j are positive integers. We set

$$\theta := \frac{f_1 \cdots f_p g_1 \cdots g_q}{fg}(gdf - fdg).$$

Then θ is a holomorphic 1-form and each of its analytic divisors $h \in \mathbb{C}\{x\}$ is coprime with fg (here $x = (x_1, x_2)$). Let us recall that a *divisor of* θ is a common divisor of the coefficients of dx_1 and dx_2 in the expansion of θ. Then we can prove the following lemma:

Lemma 4.2. [9, Prop. 3.2] *Let $h \in \mathbb{C}\{x\}$ be an irreducible convergent power series. Then h divides θ if and only if there is $c \in \mathbb{C}$ such that h divide $f - cg$. In this case, if μ is the largest power of h dividing θ, $\mu+1$ is the largest power of h dividing $f - cg$.*

Denote by $h_1, \ldots, h_e$ the irreducible divisors of θ, and by $\mu_1, \ldots, \mu_e \in \mathbb{N}^*$ the maximal exponents such that $h_1^{\mu_1} \cdots h_e^{\mu_e}$ divides θ. We define

$$\omega := \frac{1}{h_1^{\mu_1} \cdots h_e^{\mu_e}}\theta.$$

This is a holomorphic 1-form with (at most) an isolated singularity (let us recall that the singular locus of ω is the zero locus of the coefficients of dx_1 and dx_2 in ω). We recall that a *first integral of* ω is a meromorphic function R such that $\omega \wedge dR = 0$; so φ is a first integral of ω.

We denote by $c_1, \ldots, c_e \in \mathbb{C}$ the complex numbers corresponding to the h_i in Lemma 4.2, that is, $f - c_i g = h_i^{\mu_i+1}\rho_i$ for some convergent power series ρ_i.

Step 2. We consider the following system of equations:

$$(S) \qquad \begin{cases} y_{1,1}^{\ell_1}\cdots y_{1,p}^{\ell_p} - c_1 y_{2,1}^{k_1}\cdots y_{2,q}^{k_q} & = & y_{3,1}^{\mu_1+1}y_{4,1} \\ y_{1,1}^{\ell_1}\cdots y_{1,p}^{\ell_p} - c_2 y_{2,1}^{k_1}\cdots y_{2,q}^{k_q} & = & y_{3,2}^{\mu_2+1}y_{4,2} \\ \vdots & & \\ y_{1,1}^{\ell_1}\cdots y_{1,p}^{\ell_p} - c_e y_{2,1}^{k_1}\cdots y_{2,q}^{k_q} & = & y_{3,e}^{\mu_e+1}y_{4,e} \end{cases}$$

By assumption

$$y(x) := (f_1(x), \ldots, f_p(x), g_1(x), \ldots, g_q(x), h_1(x), \ldots, h_e(x), \rho_1(x), \ldots, \rho_e(x))$$

is a solution of (S). By Theorem 2.2, there exists a vector of algebraic power series $y(x,z) \in \mathbb{C}\langle x,z\rangle$ solution of (S), and convergent power series $z_1(x), \ldots, z_s(x) \in \mathbb{C}\{x\}$ such that

$$y(x) = y(x, z(x)).$$

We denote by

$$f_1(x,z), \ldots, f_p(x,z), f_1(x,z), \ldots, g_q(x,z), h_1(x,z), \ldots, h_e(x,z), \rho_1(x,z), \ldots, \rho_e(x,z)$$

the components of $y(x,z)$. Let $k_0 \in \mathbb{N}$. For $t \in [0,1]$, we set

$$z(x,t) = z_{k_0}(x) + (1-t)r_{k_0}(x)$$

where $z_{k_0}(x)$ is the truncation of $z(x)$ at order k_0 and $r_{k_0}(x) = z(x) - z_{k_0}(x)$. So $z(x,0) = z(x)$ and $z(x,1)$ is a polynomial. We set

$$F_i(x,t) := f_i(x, z(x,t)),\ G_j(x,t) := g_j(x, z(x,t)),\ H_k(x,t) := h_k(x, z(x,t))$$

and

$$F(x,t) := F_1(x,t)^{\ell_1}\cdots F_p(x,t)^{\ell_p},\ G(x,t) := G_1(x,t)^{k_1}\cdots G_q(x,t)^{k_q},$$
$$H(x,t) = H_1(x,t)^{\mu_1}\cdots H_e(x,t)^{\mu_e}.$$

We set

$$\Phi(x,t) := \frac{F(x,t)}{G(x,t)}.$$

We have that $\Phi(x,0) = \varphi(x)$, and $\Phi(x,1)$ is an algebraic meromorphic function germ. We set

$$\Theta := \frac{F_1\cdots F_p G_1\cdots G_q}{FG}(GdF - FdG) \ \text{ and } \ \Omega := \frac{\Theta}{H_1^{\mu_1}\cdots H_e^{\mu_e}}.$$

Step 3. We denote by Ω_τ the restriction of Ω to the plane of equation $t = \tau$. Then $\Omega_0 = \omega$ has an isolated singularity at 0 and Ω_1 is a 1-form having $\Phi(x,1)$ as a first integral.
In fact one can prove that Ω_t has an isolated singularity for any $t \in [0,1]$ if k_0 is chosen large enough (See [9] for more details). This comes from the fact that the coefficients of dx_1 and dx_2 in Ω_t are tangent to those of $\Omega_0 = \omega$ at order $\geq k_0$. Then we use the following lemma:

Lemma 4.3. [7, Lemme 2.1, p. 149] *Let ω be an analytic 1-form germ with an isolated singularity at $0 \in \mathbb{C}^n$. Then there is an integer N such that, for every integrable analytic 1-form germ Ω at $(\mathbb{C}^{n+m}, 0)$ satisfying*

(1) $\Omega_0 = \omega$,
(2) $\Omega - \omega \in (x)^N$,

there is a germ of biholomorphism Ψ, of the form $\Psi(x,t) = (\Psi_1(x,t), t)$, and a unit $u \in \mathbb{C}\{x,t\}$, such that

$$\Psi_* \Omega = u\omega.$$

Therefore, if $k_0 \geq N$, we have that $\Psi_* \Omega_1 = u\omega$ and $\varphi \circ \Psi_1(x,1)$ is a first integral of Ω_1. Now we use [7, Thm 1.1, p. 137] that asserts that the set of first integrals of a meromorphic 1-form having a meromorphic first integral has the form

$$\mathbb{C}(\kappa) = \{\gamma \circ \kappa \mid \gamma \in \mathbb{C}(T)\}.$$

We can apply this to Ω_1: let κ such a meromorphic function, so $\Phi(x,1) = R(\kappa)$ for some rational function $R \in \mathbb{C}(T)$. Since $\Phi(x,1)$ is algebraic, it satisfies $P(x, \Phi(x,1)) = 0$ for some nonzero polynomial $P(x,y)$. So κ annihilates $P(x, R(z)) = 0$. But $P(x, R(z))$ is a rational function, so κ is a root of its denominator, thus $\kappa \in \mathbb{C}\langle x \rangle$. In particular all the first integrals of Ω_1 are algebraic. But $\varphi \circ \Psi_1(x,1)$ is also a first integral of Ω_1, so $\varphi \circ \Psi_1(x,1)$ is algebraic. This proves the theorem for $\mathbb{K} = \mathbb{C}$ with $h(x) := \Psi(x,1)$.

The real case proceeds essentially in the same way, but we have to prove that at each step of the proof, the objects that intervene are real. □

References

[1] M. Artin, On the solutions of analytic equations, *Invent. math.*, **5**, (1968), 277-291.
[2] M. Artin, Algebraic approximation of structures over complete local rings, *Publ. I.H.E.S.*, **36**, (1969), 23-58.
[3] M. Artin, Construction techniques for algebraic spaces, *Actes Congres Intern. Math.*, **1**, (1970), 419-423.
[4] M. Bilski, A. Parusiński, G. Rond, Local topological algebraicity of analytic function germs, *J. Algebraic Geometry*, **26**, (2017), 177-197.
[5] M. Bilski, K. Kurdyka, A. Parusiński, G. Rond, Higher order approximation of analytic sets by topologically equivalent algebraic sets, *Math. Z.*, **288**, (2018), no. 3-4, 1361-1375.
[6] J. Bochnak, W. Kucharz, Local algebraicity of analytic sets, *J. Reine Angew. Math.*, **352**, (1984), 1–14.
[7] D. Cerveau, J.-F. Mattei, Formes intégrables holomorphes singulières, *Astérisque*, **97**, Société Mathématique de France, Paris, 1982.
[8] A. Gabrielov, A. M. Gabrielov, The formal relations between analytic functions, *Funkcional. Anal. i Prilovzen*, **5**, (1971), 64-65.
[9] Y. Genzmer, R. Mol, On the algebraicity of germs of meromorphic functions, preprint Arxiv:2302.12860.
[10] W. Kucharz, Power series and smooth functions equivalent to polynomials, *Proc. Amer. Math. Soc.*, **98:3**, (1986), 527-533.
[11] T. Mostowski, Topological equivalence between analytic and algebraic sets, *Bull. Polish Acad. Sci. Math.*, **32**, (1984), no. 7–8,
[12] A. Parusiński, Algebro-geometric equisingularity of Zariski, Cisneros-Molina, José Luis (ed.) et al., *Handbook of geometry and topology of singularities II.* Cham: Springer. 177-222 (2021).
[13] A. Parusiński, L. Păunescu, Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity, *Adv. Math.*, **309**, 254-305 (2017).

[14] A. Parusiński, L. Păunescu, *Zariski's dimensionality type of singularities. Case of dimensionality type 2*, In: J. Algebraic Geom. 33.1 (2024), pp. 117–142. issn: 1056–3911,1534–7486., https://doi.org/10.1090/jag/815

[15] A. Płoski, Rozwiązania formalne i zbieżne równań analitycznych, PhD thesis, Institute of Mathematics, Polish Academy of Sciences (1973).

[16] A. Płoski, Note on a theorem of M. Artin, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **22**, (1974), 1107-1109.

[17] A. Płoski, Les solutions formelles et convergentes des équations analytiques, Semestre de Singularités à Lille, 1999, `https://gasiull.webs.ull.es/Lille.pdf`.

[18] A. Płoski, Formal and convergent solutions of analytic equations, *Analytic and algebraic geometry.* **2**, 161-173. Łódź University Press, Łódź, 2017.

[19] D. Popescu, General Néron desingularization, *Nagoya Math. J.*, **100**, (1985), 97-126.

[20] R. Quarez, The Artin conjecture for $\mathbb{Q}$-algebras, *Rev. Mat. Univ. Complut. Madrid*, **10**, (1997), no. 2, 229-263.

[21] G. Rond, Sur la lissification de type Płoski-Popescu, *C. R. Math. Acad. Sci. Paris*, **348**, no. 13-14, (2010), 727-729.

[22] G. Rond, Artin Approximation, *J. Singul.*, *17* (2018), 108-192.

[23] C. Rotthaus, On the approximation property of excellent rings, *Invent. Math.*, **88**, (1987), no. 1, 39-63.

[24] P. Samuel, Algébricité de certains points singuliers algébroïdes, *J. Math. Pures Appl.*, **35**, (1956), 1-6.

[25] M. Spivakovsky, A new proof of D. Popescu's theorem on smoothing of ring homomorphisms, *J. Amer. Math. Soc.*, **12**, (1999), no. 2, 381-444.

[26] R. Swan, Néron-Popescu desingularization, Algebra and geometry (Taipei, 1995), 135-192, Lect. Algebra Geom., 2, Internat. Press, Cambridge, MA, 1998.

[27] A. N. Varchenko, Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial mappings, *Math. USSR Izviestija*, **6**, (1972), 949–1008.

[28] A. N. Varchenko, The relation between topological and algebro-geometric equisingularities according to Zariski, *Funkcional. Anal. Appl.*, **7**, (1973), 87–90.

[29] A. N. Varchenko, Algebro-geometrical equisingularity and local topological classification of smooth mappings, *Proceedings of the International Congress of Mathematicians* (Vancouver, B.C., 1974), **1**, pp. 427–431. Canad. Math. Congress, Montreal, Que., 1975.

[30] H. Whitney, Local properties of analytic varieties, *1965 Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, 205–244 Princeton Univ. Press, Princeton, N. J.

[31] O. Zariski, Some open questions in the theory of singularities, Bull. Amer. Math. Soc., 77 (1971), pp. 481–491.

[32] O. Zariski, Foundations of a General Theory of Equisingularity on r-Dimensional Algebroid and Algebraic Varieties, of Embedding Dimension $r+1$, *Amer. J. Math.* **101**, 2, (1979) 453–514; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 573–635.

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