UNIFORM NON-AMENABILITY

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ABSTRACT. For any finitely generated group G an invariant $\text{Føl}\,G \geq 0$ is introduced which measures the "amount of non-amenability" of G. If G is amenable, then $\text{Føl}\,G = 0$. If $\text{Føl}\,G > 0$, we call G *uniformly non-amenable*. We study the basic properties of this invariant; for example, its behaviour when passing to subgroups and quotients of G. We prove that the following classes of groups are uniformly non-amenable: non-abelian free groups, non-elementary word-hyperbolic groups, large groups, free Burnside groups of large enough odd exponent, and groups acting acylindrically on a tree. Uniform non-amenability implies uniform exponential growth. We also exhibit a family of non-amenable groups (in particular including all non-solvable Baumslag-Solitar groups) which are not uniformly non-amenable, that is, they satisfy $\text{Føl}\,G = 0$. Finally, we derive a relation between our uniform Følner constant and the uniform Kazhdan constant with respect to the left regular representation of G.

INTRODUCTION

Amenability is a fundamental concept with many apparently unrelated but logically equivalent formulations in different branches of mathematics, such as measure theory, representation theory, geometry, and algebra. Following the work of Følner, the geometric notion of amenability can be paraphrased as follows:

A space is amenable if it can be exhausted by a family of sets A_n of finite volume, with boundaries $\text{Bdy}A_n$, also of finite volume, such that the ratio Volume $(\text{Bdy}A_n)$/Volume (A_n) tends to 0 as $n \to \infty$.

For a finitely generated group G the volume of a subset A of G is simply set to be its cardinality $\#A$. The boundary ∂A of A can be defined in several different manners, each of them usually dependent on the choice of a finite generating set X of G. In this paper, we use

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the *inner boundary* of A (we will simply say "boundary")

$$\partial_X A = \{a \in A \mid ax \notin A \text{ for some } x \in X^{\pm 1}\}.$$

This seems to be the most natural definition from the viewpoint of geometric group theory (as it measures nicely the isoperimetric quality of the set A), and is used in this and related contexts by many authors. For the purpose of deciding whether a group is amenable, that is, whether there exists a family of Følner sets $A_n \subset G$ with

$$\lim_{n \to \infty} \frac{\# \partial A_n}{\# A_n} = 0\,,$$

all the competing definitions for the boundary ∂A_n turn out to be equivalent: one can consider the so-called Cheeger boundary, the exterior boundary, the above defined interior boundary, etc. One of these alternative boundary notions determines the Kazhdan constant $K(\lambda_G, G, X)$ of G with respect to the left regular representation λ_G of G, which will be defined in Section 2.

This paper, however, focuses on *non-amenable groups*, and more precisely, on the amount of non-amenability a group possesses. In order to measure non-amenability, one quickly realizes that knowledge of any lower bound on $\frac{\#\partial_X A}{\# A}$ for all finite $A \subset G$ is worthless, unless it is *uniform* with respect to all finite generating systems X of G. Thus we define the *uniform Følner constant*

$$\text{Føl}\, G = \inf_X \inf_A \frac{\#\partial_X A}{\# A}\,,$$

where A ranges over all non–empty finite subsets of G and X ranges over all finite generating sets of G. Note that Føl G is, in a rather subtle way, sensitive to the choice made among the competing definitions for the boundary of A.

In this paper, we prove in particular that, if G is a group generated by n elements, then one has:

$$0 \leq \text{Føl}\, G \leq \frac{2n-2}{2n-1}.$$

The lower bound is achieved if G is amenable; correspondingly we call groups G with non-zero Følner invariant, Føl $G > 0$, *uniformly non-amenable.*

In Section 13 we show that there exist non-amenable groups which are *not* uniformly non-amenable. They deserve special interest, as, by a result of Osin [23], the much sought for examples of groups with exponential but not uniform exponential growth must be either amenable but not elementary amenable, or else non-amenable, while we show in Section 1 that the uniform growth rate $\omega(G)$ is bounded below by $\frac{1}{1-\operatorname{Føl} G}$. Together with the above mentioned Kazhdan constant, also uniformized over all finite generating systems, the situation can be summarized as follows (compare Propositions 1.4 and 2.4): Every finitely generated group G satisfies

$$0 \leq \frac{1}{2} K(\lambda_G, G)^2 \leq \operatorname{Føl} G \leq 1 - \frac{1}{\omega(G)}.$$

The maximal value, $\operatorname{Føl} G = (2n-2)/(2n-1)$, is achieved for G free of rank n, and by no other group generated by n elements, see Section 5. For surface groups S_g we give fairly close lower and upper bounds for $\operatorname{Føl} S_g$ in Section 6. A less precise lower bound is computed in Section 8 for virtually free groups. While the computation for free groups follows directly from geometric arguments in the Cayley graph, our proof for surface groups is the model for a much larger class of groups, for which uniform non-amenability is summarized in Theorem below.

It relies on two basic good properties of the uniform Følner constant (see Sections 4 and 7):

(1) $\operatorname{Føl} G \geq \operatorname{Føl} G/N$ for any normal subgroup N of G, and
(2) $\operatorname{Føl}_X G \geq \frac{1}{C} \operatorname{Føl}_Y H$ for any subgroup H of G,

where the constant C depends only on the cardinality of the generating system Y of H and the maximal length of the elements of Y with respect to the generating system $X^{\pm 1}$ of G (the terminology is given in Section 1 below).

With these tools we can establish the following theorem. Our proofs are inspired by the previously known results on uniform growth.

Theorem
The following classes of groups G are uniformly non-amenable:

(1) *non-elementary word-hyperbolic groups;*
(2) *large groups (i.e. groups containing a finite index subgroup that surjects onto a non-abelian free group);*
(3) *free Burnside groups of large enough odd exponent;*
(4) *groups which act acylindrically on a simplicial tree without global fixed points.*

Finally, we would like to direct the readers attention to recent work of [24] by Osin, which, in a late state of our work, was communicated to us by de la Harpe. The last section of our paper has been influenced by looking at this paper and at its "predecessor" by Meier [20]. We would also like to thank the referees for their helpful remarks.

1. Definitions

Let G be a group and let A be a non-empty finite subset of G.

Definition 1.1. The boundary of $A \subset G$ with respect to the finite subset $X \subset G$ is defined as:

$$\partial_X A = \{a \in A \mid ax \notin A \text{ for some } x \in X^{\pm 1}\}.$$

Usually X will be a finite generating set for G. The boundary of A is the set of group elements which are at distance one from the complement of A in the word metric relative to X. The boundary can be defined *a priori* for any subset of G, but it will be used mostly for finite sets.

Definition 1.2. We define the *Følner constant* of G with respect to the generating set X as the number

$$\mathrm{Føl}_X G = \inf_A \frac{\#\partial_X A}{\#A},$$

where A runs over all non–empty finite subsets of G. The *uniform Følner constant* for G is defined as

$$\mathrm{Føl}\, G = \inf_X \mathrm{Føl}_X G,$$

where X runs over all finite generating sets of G.

Our definitions are motivated by the well-known *Følner condition* on a group which is equivalent to the amenability of the group. Using the above introduced notation this characterisation can be stated as follows: a finitely generated group G is *amenable* if and only if $\mathrm{Føl}_X G = 0$ for some (and hence for every) generating set X, see for instance [30, 13]. Clearly then, every amenable group has uniform Følner constant zero. Following [14] (for instance), we will use the following term:

Definition 1.3. A finitely generated group G is said to be *uniformly non-amenable* if $\mathrm{Føl}\, G > 0$.

Amenability originates from a more general context which will be indicated in the next section. It is worth noticing that in Section 13 we give examples of *non-amenable* groups with uniform Følner constant zero.

Recall that associated naturally to a group G and a finite generating set X there is a locally finite connected graph, the *Cayley graph*, which can be considered as a metric space by associating length 1 to every edge. It realizes on its vertex set G the word metric relative to X. A family of Følner sets can be viewed as an analogue of the sequence of balls $B_X(n)$ of radius n around the origin $1 \in G$, and $\text{Føl}_X G$ measures in some sense the growth of these "generalized balls".

On the other hand, the *growth rate* of G with respect to X is defined to be

$$\omega_X(G) = \lim_{n\to\infty} \sqrt[n]{\#B_X(n)}$$

(the existence of this limit follows from the submultiplicativity property of the function $\#B_X(n)$: $\#B_X(m+n) \leq \#B_X(m)\#B_X(n)$ for $n, m \geq 0$, see for example [11]). The *uniform growth rate* of G is defined as

$$\omega(G) = \inf_X \omega_X(G).$$

where the infimum is taken over all finite generating sets X for G.

It is not hard to show that the growth rate of a free group F_k of rank k, with respect to a free basis X_k, is $\omega_{X_k}(F_k) = 2k-1$. In fact this is the uniform growth rate of the free group of rank k, see for instance [11].

An important open problem on uniform growth rates, posed by Gromov in [12, remarque 5.12], is the question whether there exist groups of exponential growth (i.e. $\omega_X(G) > 1$ for all generating sets X) but with uniform growth rate equal to $\omega(G) = 1$. (Wilson [31] has recently produced a non–amenable example of such a group.) In view of the following result such groups must have uniform Følner constant zero.

Proposition 1.4. *Let G be a finitely generated group, and let X be a finite generating set. Then*

$$\text{Føl}_X G \leq 1 - \frac{1}{\omega_X(G)},$$

and hence,

$$\text{Føl}\, G \leq 1 - \frac{1}{\omega(G)}.$$

Proof. The Følner constant is an infimum taken over all non–empty finite subsets of G. Since $\partial_X B_X(n) \subseteq B_X(n) - B_X(n-1)$, we have

$$\text{Føl}_X G \leq \frac{\#\partial_X B_X(n)}{\#B_X(n)} \leq 1 - \frac{\#B_X(n-1)}{\#B_X(n)}$$

for all $n \geq 1$. And, since

$$\liminf_{n\to\infty} \frac{\#B_X(n)}{\#B_X(n-1)} \leq \omega_X(G),$$

we deduce

$$\text{Føl}_X G \leq 1 - \frac{1}{\omega_X(G)}.$$

□

Note that the inequalities in Proposition 1.4 may very well be strict. However, in a preprint [2] which has appeared while this paper was in final revision, Arzhantseva, Guba and Guyot have shown that there are k generator amenable groups with growth rates arbitrarily close to $\frac{1}{2k-1}$, and thus with uniform Følner constant 0, but such that the upper bound of the inequality is arbitrarily close to $\frac{2k-2}{2k-1}$. The Baumslag-Solitar group $BS(1,2)$ (see Section 13 for the definition) provides an example of an amenable group, hence with the Følner constant zero, whose the uniform growth rate is different from 1 [4]. In general, for amenable groups with exponential growth, the inequalities involving a fixed generating set are always strict. We also see that $\omega(G) > 1$ whenever $\text{Føl}\, G > 0$:

Corollary 1.5. *Uniformly non-amenable groups have uniform exponential growth different from one.*

2. Amenability and Kazhdan's constants

A locally compact group Γ is called *amenable* if there exists a left-invariant, finitely additive measure μ defined on all Borel subsets of Γ and satisfying $\mu(\Gamma) = 1$. For more information and background on amenability, see for instance [9, 13, 30] and the references therein. The characterization of amenability which we shall use throughout this paper is the existence of a family of Følner sets, that is, a sequence $\{A_n\}$ of subsets of Γ of finite Haar measure such that for all $g \in \Gamma$ we have

$$\lim_{n\to\infty} \frac{\#(A_n \triangle g A_n)}{\# A_n} = 0. \tag{2.1}$$

(We use $\#$ to denote Haar measure; later we shall simply consider cardinality of finite sets.)

Amenability can also be characterized from the point of view of representations. Let λ_Γ be the left regular representation of Γ on the Hilbert space $\mathcal{H} = L^2(\Gamma)$, that is, $\lambda_\Gamma(g)u(f) = u(g^{-1}f)$ for $u \in L^2(\Gamma)$ and $f, g \in \Gamma$.

Definition 2.1. Let Γ be a locally compact group, and let λ_Γ be the left regular representation of Γ. We say that the trivial representation is *weakly contained* in λ_Γ if, for any $\varepsilon > 0$ and any compact subset $S \subset \Gamma$, there exists $u \in L^2(\Gamma)$ with $\|u\| = 1$ such that

$$|\langle u, \lambda_\Gamma(s)u\rangle - 1| < \varepsilon \tag{2.2}$$

for any $s \in S$.

Theorem 2.2 ([16]). *A group Γ is amenable if and only if the left regular representation of Γ weakly contains the trivial representation.*

Related to Følner constants are the well-known Kazhdan constants, which are defined in terms of unitary representations.

Let Γ be a locally compact group, $\mathcal{H}$ a separable Hilbert space, and $S \subset \Gamma$ a compact set. For a unitary representation $(\pi, \mathcal{H})$ of Γ we define the number

$$K(\pi, \Gamma, S) = \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|}.$$

Then the *Kazhdan constant* with respect to the set S is defined as

$$K(\Gamma, S) = \inf_{\pi} K(\pi, \Gamma, S),$$

where the infimum is taken over unitary representations π having no invariant vectors. We also define the *uniform* Kazhdan constant (with respect to π) as

$$K(\pi, \Gamma) = \inf_{S} K(\pi, \Gamma, S)$$

where the infimum is taken over all generating sets S.

A group Γ is said to have *Kazhdan property* (T) (or to be a Kazhdan group) if there exists a compact set $S \subset \Gamma$ with $K(\Gamma, S) > 0$. There are explicit computations or estimates of Kazhdan constants in the literature [3, 5, 6, 8, 21, 27, 28, 32].

Observe that according to the Definition 2.1 and Theorem 2.2, the group Γ is amenable if and only if $K(\lambda_\Gamma, \Gamma, S) = 0$ for all compact $S \subset \Gamma$.

Let us now return to finitely generated groups, i.e. Γ is replaced by G, S by a finite generating system X, and $L^2(\Gamma)$ by $\ell^2(G)$. We can rewrite condition (2.1) in terms of the boundaries of finite sets $\{A_n\}$ forming a Følner family, see for instance [9]. Thus, using the notations above and results already mentioned, we can summarize several equivalent characterizations of amenability as follows:

Proposition 2.3. *Let G be a finitely generated group. The following conditions on G are equivalent.*

(i) *G is amenable;*

(ii) *There exist a finite generating set X of G and a sequence of non–empty finite subsets $\{A_n\}$ of G satisfying*

$$\lim_{n\to\infty} \frac{\#\partial_X A_n}{\#A_n} = 0;$$

(iii) $\text{Føl}_X G = 0$ *for every finite generating set X;*

(iv) $K(\lambda_G, G, X) = 0$ *for every finite generating set* X.

The uniform Kazhdan constant of the left regular representation is related to the Følner constant as follows:

Proposition 2.4. *Let G be a finitely generated group. Then one has*

$$\mathrm{Føl}\, G \geq \frac{1}{2} K(\lambda_G, G)^2.$$

In particular, if $\mathrm{Føl}\, G = 0$ *then* $K(\lambda_G, G) = 0$.

Proof. For any $\varepsilon > \mathrm{Føl}\, G$ there exist a finite generating set X for G and a non–empty finite subset $A \subseteq G$ so that $\dfrac{\#\partial_X A}{\#A} < \varepsilon$. Let $\chi_{A^{-1}}$ be the characteristic function of A^{-1}. Then for $x \in X$ and $u = \dfrac{\chi_{A^{-1}}}{\sqrt{\#A}}$ of norm $\|u\| = 1$ we obtain

$$\|\lambda_G(x)u - u\|^2 = \frac{1}{\#A} \sum_{g \in G} \left(\chi_{A^{-1}}(x^{-1}g) - \chi_{A^{-1}}(g)\right)^2 \leq 2\frac{\#\partial_X A}{\#A} \leq 2\varepsilon\,.$$

This implies $K(\lambda_G, G, X) \leq \sqrt{2\varepsilon}$, so $K(\lambda_G, G) \leq \sqrt{2\mathrm{Føl}\, G}$. □

It is a subtle question whether the implication in the last sentence of Proposition 2.4 can be reversed. In particular, it would be interesting to know whether the 2-generator infinite periodic groups G with $K(\lambda_G, G) = 0$ exhibited in [24], which are not amenable, satisfy $\mathrm{Føl}\, G = 0$: If so, they would be examples of non-amenable but not uniformly non-amenable groups which do not contain non-abelian free subgroups. [1]

It follows from Proposition 2.4 and Proposition 1.4 that $K(\lambda_G, G) > 0$ implies that G has uniform exponential growth. In [28], Shalom shows that $K(\lambda_G, G) > 0$ for non-elementary residually finite word hyperbolic groups. In [17], Koubi proves that non-elementary word hyperbolic groups have uniform exponential growth, a result that we will use later.

[1]The referee has pointed out that the 2-generator infinite periodic group Q constructed in [24] actually does have zero Følner constant. This is because Q is a quotient of all non-elementary word hyperbolic groups, and the infinum of their uniform Følner constants is zero since the closure of the set of non-elementary word hyperbolic groups in the space of marked groups contains an amenable group [25]. Applying Lemma 13.2 and Theorem 4.1, it follows that $\mathrm{Føl}\, Q = 0$.

3. Subgroups

Before beginning with the computation of the Følner constants of free groups, we establish some results which relate the Følner constants of a group to those of its subgroups and quotients.

Lemma 3.1. *Let G be a finitely generated group, X a finite generating system, and $g \in G$. Let $Y = X \cup \{g\}$. Then*

$$\mathrm{Føl}_X G \leq \mathrm{Føl}_Y G.$$

Proof. The Cayley graph of G with respect to Y is the same as the one with respect to X, but at each vertex v there is an extra edge labelled g leaving v and an extra edge labelled g arriving at v. Consider a non–empty finite subset A. Obviously, adding edges to a Cayley graph cannot move a boundary point of A to the interior. The only thing that can happen is that an interior point now becomes a boundary point if its corresponding edge g or g^{-1} has its other endpoint outside A. So the boundary with respect to Y is at least as large as the boundary with respect to X. □

The following result gives lower bounds on the Følner constants of a group in terms of those of certain subgroups:

Theorem 3.2 (First Subgroup Theorem). *Let G be a group, and let $X = \{x_1, \ldots, x_n\}$ be a finite generating set of G. Let $m < n$, and let H be the subgroup of G generated by the set $Y = \{x_1, \ldots, x_m\}$. Then,*

$$\mathrm{Føl}_X G \geq \mathrm{Føl}_Y H.$$

Proof. Let A be a non–empty finite subset of G, and choose $y_1, \ldots, y_k$ elements of G in such a way that $y_i H \cap y_j H = \varnothing$ if $i \neq j$, and $A \cap y_i H \neq \varnothing$. Namely, the y_i are representatives of the cosets of H which intersect A. Let $A_i = A \cap y_i H$.

The Cayley graph of H with respect to Y sits inside the Cayley graph of G with respect to X. Considering only the edges labelled in Y, the cosets for H form disjoint "parallel" copies of the Cayley graph of H. Note that A_i is a finite subgraph of the component corresponding to the coset $y_i H$. Clearly, by the definition of the Følner constant, we have

$$\frac{\#\partial_Y A_i}{\#A_i} = \frac{\#\partial_Y (y_i^{-1} A_i)}{\#(y_i^{-1} A_i)} \geq \mathrm{Føl}_Y H.$$

Now, using the argument of the lemma above, it is clear that the boundary for A using only elements of Y is smaller than the X-boundary of A. Then,

$$\frac{\#\partial_X A}{\#A} \geq \frac{\#\partial_Y A}{\#A} = \frac{\sum_i \#\partial_Y A_i}{\sum_i \#A_i} \geq \mathrm{Føl}_Y H,$$

which concludes the proof. □

This result has many interesting applications. For instance, it allows us to prove the following proposition, which is a special case of the main result of the next section.

Proposition 3.3. *Suppose that G is a finitely generated group, and that there is a surjective homomorphism $\phi : G \to F_2$. Then*

$$\text{Føl}\, G \geq \text{Føl}\, F_2.$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of generators for G. For $1 \leq i < j \leq n$, the subgroup $\langle \phi(x_i), \phi(x_j) \rangle$ of F_2 generated by the images $\phi(x_i), \phi(x_j)$ is either free of rank 2, or cyclic. As $\phi(G) = F_2$, there are generators x_i, x_j such that $\langle \phi(x_i), \phi(x_j) \rangle$ is free of rank 2, and hence $\langle x_i, x_j \rangle$ is a free non-abelian subgroup of G. The result now follows from the First Subgroup Theorem. □

4. Quotients

The Følner constant of a group is bounded below by the constants of its quotients as follows:

Theorem 4.1. *Let G be a finitely generated group and let X be a finite generating system for G. Let N be a normal subgroup of G, π the canonical homomorphism of G onto G/N and $X' = \pi(X)$. Then,*

$$\text{Føl}_X G \geq \text{Føl}_{X'} G/N,$$

and hence,

$$\text{Føl}\, G \geq \text{Føl}\, G/N.$$

Proof. It is clear that the second inequality follows from the first, because the first inequality is valid for any finite generating system of G. We have

$$\text{Føl}_X G \geq \text{Føl}_{X'} G/N \geq \text{Føl}\, G/N,$$

so the infimum of $\text{Føl}_X G$ has the same lower bound.

To prove the theorem, consider a finite set $A \subset G$, and let $B = \pi(A)$. By definition,

$$\frac{\#\partial_{X'} B}{\# B} \geq \text{Føl}_{X'} G/N.$$

For all $i \geq 1$, we define the i-level subset of B as

$$B_i = \{ b \in G/N \mid \# \left(\pi^{-1}(b) \cap A \right) \geq i \} \subseteq B.$$

Note that $B_1 = B$ and that $B_i = \emptyset$ whenever $i > \# A$.

We need the following lemma:

Lemma 4.2. *Let $b, c \in G/N$ such that there exists $x \in X$ with image $x' = \pi(x)$ satisfying $bx' = c$ (i.e. b and c are at distance at most one in G/N). Suppose $b \in B_i$ and $c \in B_j - B_{j+1}$, for some $i > j \geq 0$. Then, there exist at least $i - j$ points in the set $\partial_X A \cap \pi^{-1}(b)$.*

Proof. Let $a_1, \ldots, a_i$ be i different points in $\pi^{-1}(b) \cap A$ and consider the elements $a_1 x, \ldots, a_i x$. These points are all in $\pi^{-1}(c)$, but at most j of them are in A, since $\pi^{-1}(c) \cap A$ has exactly j elements. Hence, there are at least $i - j$ points in $\pi^{-1}(b) \cap A$ which are in the boundary of A. This completes the proof of the lemma. □

We have the inequality

$$\text{Føl}_{X'} G/N \leq \frac{\#\partial_{X'} B_i}{\#B_i}$$

for all non-empty B_i, so

$$\text{Føl}_{X'} G/N \leq \frac{\sum_{i\geq 1} \#\partial_{X'} B_i}{\sum_{i\geq 1} \#B_i} = \frac{\sum_{i\geq 1} \#\partial_{X'} B_i}{\#A} .$$

The proof will conclude when we prove that

$$\sum_{i\geq 1} \#\partial_{X'} B_i \leq \#\partial_X A .$$

So let now b be a point in some $\partial_{X'} B_i$, and let $i_0 = \#(\pi^{-1}(b) \cap A) \geq i$. We want to consider the neighbour of b which has the least number of preimages: let

$$i_1 = \min_{x' \in X'} \{j \mid bx' \in B_j - B_{j+1}\}$$

Then, since b has i_0 preimages in A, and it has a neighbour with exactly i_1 preimages, the contribution of b to

$$\sum_{i\geq 1} \#\partial_{X'} B_i$$

is exactly zero if $i_0 \leq i_1$ and $i_0 - i_1$ otherwise, since b appears in all the sets $\partial_{X'} B_j$ for all $j = i_1 + 1, \ldots, i_0$. But now, according to Lemma 4.2, in the preimage of b there must be at least $i_0 - i_1$ points in the boundary $\partial_X A$. So the conclusion is that

$$\sum_{i\geq 1} \#\partial_{X'} B_i \leq \#\partial_X A,$$

as desired. □

5. Free groups

The quintessential examples of nonamenable groups are free non-abelian groups. So it is reasonable to start our computations by calculating their Følner constants. However, all computations below depend only on the combinatorial properties of Cayley graphs of free groups with respect to free bases. Such Cayley graphs are always regular trees. It is remarkable to observe that the computations agree with the situation for a free group of rank one, that is, the group of integers. All results will be stated including this special case.

Let F_k be the free group on $k \geq 1$ generators, and let $X_k = \{x_1, \ldots, x_k\}$ be a basis for F_k. The growth rate of a free group F_k with respect to a free basis is $2k - 1$. From Proposition 1.4 we immediately obtain:

Lemma 5.1. *Let F_k be a free group of rank k. Then*

$$\text{Føl}\, F_k \leq \text{Føl}_{X_k} F_k \leq \frac{2k-2}{2k-1}.$$

We will devote the rest of this section to proving that these two inequalities are indeed equalities, i.e. the uniform Følner constant for free groups is achieved by considering balls and free bases.

Recall that the Cayley graph of the free group of rank k with respect to a free basis is a $2k$-regular tree. A subset $A \subset F_k$ can also be considered as a subgraph of this Cayley graph; in this way, A is a forest.

Proposition 5.2. *Let A be any non–empty finite subset of F_k, with $k \geq 1$. Then,*

$$\frac{\#\partial_{X_k} A}{\# A} > \frac{2k-2}{2k-1} \ .$$

Proof. We can assume that the graph A (i.e. the vertices in A together with all edges with both ends in A) is connected, because if the result is satisfied by all the connected components, then it is clearly satisfied by their union.

Let V_i denote the number of vertices of valence i in A, $i = 1, \ldots, 2k$. Since A is a tree its Euler characteristic must be zero,

$$1 - (V_1 + \cdots + V_{2k}) + \frac{V_1 + 2V_2 + \cdots + 2kV_{2k}}{2} = 0.$$

So, $V_1 = 2 + \sum_{i=3}^{2k} (i-2)V_i$. Now,

$$\begin{aligned}\frac{\#\partial_{X_k}A}{\#A} &= \frac{V_1+\cdots+V_{2k-1}}{V_1+\cdots+V_{2k}}\\ &= 1-\frac{V_{2k}}{V_1+\cdots+V_{2k}}\\ &= 1-\frac{V_{2k}}{2+V_2+\sum_{i=3}^{2k}(i-1)V_i}\\ &> 1-\frac{1}{2k-1}\\ &= \frac{2k-2}{2k-1}.\end{aligned}$$

□

This last result, together with the upper bound obtained from the growth rate, completes the proof of the calculation of the Følner constant for free groups with respect to free bases:

Proposition 5.3. *If X_k is a basis for the free group F_k, then*

$$\text{Føl}_{X_k} F_k = \frac{2k-2}{2k-1}\ .$$

□

To conclude the computation of the uniform Følner constant of the free groups, we only need to use the first subgroup theorem.

Proposition 5.4. *One has*

$$\text{Føl}\, F_k = \frac{2k-2}{2k-1}.$$

Proof. Let Y be any finite generating system for F_k. Let π be the abelianization map

$$\pi : F_k \longrightarrow \mathbb{Z}^k,$$

and let $Y^{\text{ab}} = \pi(Y)$. Since Y^{ab} generates $\mathbb{Z}^k$, we can find $y_1, \ldots, y_k \in Y$ such that $\pi(y_1), \ldots, \pi(y_k) \in Y^{\text{ab}}$ are linearly independent, and hence, they generate a subgroup of $\mathbb{Z}^k$ which is isomorphic to $\mathbb{Z}^k$. Then, $H = \langle y_1, \ldots, y_k \rangle$ is a subgroup of F_k which is also isomorphic to a free group of rank k since it maps onto $\mathbb{Z}^k$ which cannot be generated by less than k elements (note that, in general, H is not necessarily equal to the original F_k). Thus, using the First Subgroup Theorem, we can deduce that

$$\text{Føl}_Y F_k \geq \text{Føl}_{\{y_1,\ldots,y_k\}} H = \frac{2k-2}{2k-1},$$

and since Y is any finite generating system of F_k, we conclude the desired result, by using Proposition 5.3. $\square$

The above result can be improved to say that the free group of rank k is the unique group with Følner constant $\frac{2k-2}{2k-1}$, among groups admitting a system of generators with k elements. This follows from the analogous result about the growth rate proved by Koubi in [18].

Proposition 5.5. *Let G be a group generated by a set X such that $|X| = k \geq 2$. Then $\text{Føl}_X G \leq \frac{2k-2}{2k-1}$, and the equality holds if and only if G is free with basis X.*

Proof. As G has k generators, it is clear that $\omega_X(G) \leq 2k-1$. From Proposition 1.4, we have the upper bound

$$\text{Føl}_X G \leq 1 - 1/\omega_X(G) \leq \frac{2k-2}{2k-1}.$$

If $\text{Føl}_X G = \frac{2k-2}{2k-1}$, we see that $\omega_X(G) \geq 2k-1$, and thus $\omega_X(G) = 2k-1$. Koubi's Proposition 1.2 in [18] states that $\omega_X(G) \leq 2k-1$ with equality if and only if G is free with basis X. In our case $\omega_X(G) = 2k-1$ and so, by Koubi's result, G is free with basis X.

Conversely, if G is free on X, the equality holds by Proposition 5.3. $\square$

Analogously, the same can be said with the uniform Følner constant.

Theorem 5.6. *Let G be a k-generated group, $k \geq 2$. Then, $\text{Føl}\, G \leq \frac{2k-2}{2k-1}$, and the equality holds if and only if G is free of rank k.* $\square$

Now that Følner constants for free groups have been calculated exactly, using the quotient theorem, we see that knowledge of the Følner constant of a group gives information about its rank and about the rank of its free quotients.

Corollary 5.7. *Let G be a finitely generated group. If k is a positive integer such that*

$$\frac{2k-2}{2k-1} \leq \text{Føl}\, G$$

then the rank of G is at least k. $\square$

Corollary 5.8. *Let G be a finitely generated group. If k is a positive integer such that*

$$\text{Føl}\, G \leq \frac{2k-2}{2k-1}$$

and G admits a free quotient of rank ℓ, then $\ell \leq k$. $\square$

6. Surface groups

Let S_g be the fundamental group of a closed orientable surface of genus g. The rank of S_g is $2g$, so that $(4g-2)/(4g-1)$ is an upper bound for its uniform Følner constant.

A lower bound for the Følner constants for S_g can be obtained as follows. Let X be the usual set of $2g$ generators for S_g. By Magnus' Freiheitssatz, any subset of $2g-1$ elements of X generates a free group of rank $2g-1$, and the first subgroup theorem applies to conclude that the Følner constant for S_g with respect to X has a lower bound given by the constant for F_{2g-1}. In fact this bound is uniform.

Theorem 6.1. *Let S_g be the fundamental group of a closed orientable surface of genus g. Then*

$$\frac{4g-4}{4g-3} \leq \mathrm{Føl}\, S_g < \frac{4g-2}{4g-1}$$

Proof. Let Y be any finite set of generators for S_g. Since the abelianization of S_g is $\mathbb{Z}^{2g}$, there is a subset Y' consisting of $2g$ elements of Y whose images in the abelianization are linearly independent. Let Y'' be a subset of Y' with $2g-1$ elements. Since every subgroup of a surface group is either free or else a surface group of higher rank, $\langle Y''\rangle$ is necessarily free. And since $\langle Y''\rangle$ maps onto $\mathbb{Z}^{2g-1}$, which cannot be generated by less than $2g-1$ elements, we see that $\langle Y''\rangle$ has rank $2g-1$. Now, the First Subgroup Theorem and Proposition 5.3, tell us that

$$\frac{4g-4}{4g-3} = \mathrm{Føl}_{Y''}(\langle Y''\rangle) \leq \mathrm{Føl}_Y S_g,$$

which is valid for every Y.

Finally, the second inequality is consequence of the fact that S_g has rank $2g$, and it is strict because S_g is not free (and equality would therefore contradict Theorem 5.6). □

Notice that the subset Y' in the above proof generates a subgroup which is either free of rank $2g$, or S_g itself. The First Subgroup Theorem implies that either $\mathrm{Føl}_Y S_g \geq \frac{4g-2}{4g-1}$ or Y' is a minimal set of generators for S_g. In the first case, $\mathrm{Føl}_Y S_g$ is bounded away from the uniform Følner constant. Thus to obtain a Følner constant close to the uniform one, it suffices to consider minimal sets of generators. The definition of the Følner constants and the First Subgroup Theorem already suggest that this should be true in general, as larger sets of generators will apparently provide larger boundaries. The examples of the free groups, together with this behaviour for surface groups, appear to confirm this.

The calculation of the exact uniform Følner constant as well as the exact uniform growth rate of surface groups is an open problem.

7. Subgroups revisited

The subgroup theorem obtained in Section 3 only applies to those subgroups which were generated by a subsystem of the system of generators for the group. In this section we will state a general result which can be used for any subgroup and any system of generators, but which will give worse bounds for the Følner constants.

Theorem 7.1 (Second Subgroup Theorem). *Let G be a finitely generated group and let $X = \{x_1, \dots, x_n\}$ be a system of generators for G. Let $H \leq G$ be a subgroup, generated itself by a system $Y = \{y_1, \dots, y_m\}$. Choose expressions w_j for the y_j as words on X, and let L be the maximum among their lengths. Then,*

$$\mathrm{Føl}_X G \geq \frac{1}{1+mL}\,\mathrm{Føl}_Y H \ .$$

Proof. Let A be a non–empty finite subset of G. As in Theorem 3.2, we consider A as a finite union of intersections A_i of A with right cosets of H, and we write $\partial_Y A = \cup_i \partial_Y A_i$, viewing each A_i as existing inside a copy of the Cayley graph of H with respect to Y. With the same argument as in Theorem 3.2, we have $\mathrm{Føl}_Y H \leq \frac{\#\partial_Y A}{\#A}$, even if A is not a subset of H.

By definition, every element $\zeta \in \partial_Y A_i$ can be joined with a point outside A_i, and so outside A, by multiplication by some y_j, which we think of as a path labelled w_j in the generators X. If $\zeta \in \partial_X A$, then this path begins at ζ. If $\zeta \notin \partial_X A$, then the path must necessarily pass through a vertex in $\partial_X A$, which is not the final vertex of the path, just before leaving A. Consider a vertex $z \in \partial_X A$; it may be that $z \in \partial_Y A$. Otherwise, there are at most $\ell(w_j) - 1 \leq L - 1$ ways in which a path labelled w_j may pass through z in such a way that z is neither the initial nor the final vertex.

Thus a vertex $z \in \partial_X A$ corresponds to at most $1 + \sum_{j=1}^{m}(\ell(w_j) - 1) \leq 1 + mL$ different vertices in $\partial_Y A$ (and each vertex in $\partial_Y A$ has at least one corresponding vertex in $\partial_X A$). It follows that

$$\#\partial_Y A \leq (1 + \sum_{j=1}^{m}(\ell(w_j) - 1))\#\partial_X A \leq (1 + mL)\#\partial_X A \ .$$

Since the previous inequality is valid for any non–empty finite subset of G, we deduce the result. □

Remark 7.2. Observe that the only obstacle to the Second Subgroup Theorem providing a lower bound for the uniform Følner constant for G is the fact that, for different generating systems of G, the lengths of the generators of H, and their number, can grow arbitrarily. So, for all those examples where one can find bounds on these numbers, the Second Subgroup Theorem can be used to estimate the uniform Følner constant, as we shall see in the next three sections, where we find such bounds for virtually free groups, large groups and hyperbolic groups.

8. Virtually free groups

Shalen and Wagreich [26] showed that, for a subgroup H of finite index k in a group G, the uniform growth rates are related by $\omega(G) \geq \omega(H)^{1/(2k-1)}$. This bound can be improved to $\omega(G) \geq \omega(H)^{1/(k+1)}$ using Lemma 8.1 below.

The Second Subgroup Theorem 7.1 can be used to give a lower bound for the uniform Følner constant for virtually free groups. This is a special case of Theorem 10.1, but is treated here separately: the proof below is much more direct, the method employed is interesting in itself, and it is also used again in Section 9.

Let G be a finitely generated group, and let H be a finite index subgroup, of index $k = [G : H]$. The length of the generators is controlled by the following lemma, which is a direct consequence of the fact that a subgroup of index k is $\frac{k}{2}$-quasiconvex:

Lemma 8.1. *Let G be a group and H be a subgroup of index k. Given a generating set X for G, there exists a generating system Y for H where all the generators have length at most $k+1$ with respect to X.* □

Theorem 8.2. *Every virtually free group G is uniformly non-amenable. More precisely, if G contains a non–abelian free subgroup H of index k, then*

$$\text{Føl}\, G \geq \frac{1}{1+2(k+1)}\frac{2}{3} > \frac{1}{3(k+2)}\ .$$

Proof. Let X be a generating system for G and let H be a free subgroup of rank p and index k. Using the lemma above, we know that there exists a system of generators Y for H whose elements have length at most $k+1$ with respect to X.

As H is non–abelian and free, there is a pair of generators $Y' \subset Y$ which freely generate a free subgroup H' of rank two. The Second Subgroup Theorem applies to H' and we have

$$\text{Føl}_X G \geq \frac{1}{1+2(k+1)}\text{Føl}_{Y'} H' \geq \frac{1}{1+2(k+1)}\frac{2}{3} > \frac{1}{3(k+2)}\ .$$

□

9. Large groups

We can now use our results concerning subgroups to prove that certain classes of well-known groups are uniformly non-amenable.

Definition 9.1. A group is said to be *large* (or as large as F_2) if it contains a finite index subgroup which has a quotient isomorphic to F_2, the free group of rank two.

Many classes of groups are known to be large: groups with deficiency 2 presentations, not virtually abelian Coxeter groups [19], torsion-free one-relator groups, deficiency 1 presentations where no relator is a proper power and most generalized triangle groups, see [15] for bibliography and more such classes.

Proposition 9.2. *Large groups are uniformly non-amenable.*

Proof. Let G be a large group and H a subgroup of index k which admits a quotient isomorphic to F_2. Let X be a generating system for G. From Lemma 8.1, we can construct a set of generators Y for H whose elements all have length at most $k+1$. We have no control on the size of Y but, because of the existence of a quotient of H isomorphic to F_2, we can choose two elements of Y which are a basis of a free subgroup of H. Since the length of these two elements is bounded by $k+1$, the Second Subgroup Theorem can be used to obtain a uniform lower bound for $\operatorname{Føl}_X G$. This implies $\operatorname{Føl} G \geq \frac{1}{1+2(k+1)}\frac{2}{3} > \frac{1}{3(k+2)} > 0$.
□

10. Hyperbolic groups

A result of Koubi [17] proves that a non-elementary hyperbolic group has uniform exponential growth. The precise statement used here is as follows:

Theorem 10.1 ([17])**.** *Let G be a non-elementary word hyperbolic group. Then there exists a constant $N(G) > 0$ such that for any generating set X of G there are two elements f and g of lengths $|f|_X, |g|_X \leq N(G)$ freely generating a nonabelian free subgroup.*

This result establishes the necessary upper bound on the number of generators (two) and their lengths, so that Remark 7.2 enables to use the Second Subgroup Theorem 7.1 to give a lower bound on Følner constants: $\operatorname{Føl} G \geq \frac{1}{1+2N(G)}\frac{2}{3} > \frac{1}{3(1+N(G))}$.

Corollary 10.2. *Let G be a non-elementary word hyperbolic group. Then G is uniformly non-amenable.* □

11. Burnside groups

In [1] Adian proved that for any $m \geq 2$ and $n \geq 665$ odd the free Burnside group $B(m,n) = F_m/F_m^n$ is non-amenable. The following results follow from Theorem 41 (page 303) of [22]:

Given integers $n > 0$ odd and large enough and $m \geq 2$, there exist words $u(x,y), v(x,y)$ in the alphabet $\{x,y\}$ such that, if a,b are two non–commuting elements of $B(m,n)$, then the subgroup H generated by $Y = \{u(a,b), v(a,b)\}$ is isomorphic to the free Burnside group $B(2,n)$.

In particular one has $\text{Føl}_Y H > 0$. As any generating system X of $B(m,n)$ must contain at least two non-commuting elements x_i, x_j, we consider H for $a = x_i$, $b = x_j$ and apply Remark 7.2 to obtain directly from the Second Subgroup Theorem 7.1 the following:

Corollary 11.1. *For $m \geq 2$ and n odd large enough the free Burnside group $B(m,n)$ is uniformly non-amenable.* □

12. Groups acting on trees

Recall that in [4], Bucher and de la Harpe studied uniform exponential growth of HNN-extensions and amalgamated products of groups. The next result provides, in some sense, a generalization of their work by asserting that a group which acts in a proper way on a tree is uniformly non-amenable. Let us point out, however, that some attention should be given to the hypotheses: Among the groups acting on trees for which uniform exponential growth was shown in [4] there are in particular non–amenable Baumslag-Solitar groups. Those, however, do not satisfy the acylindricity condition in the proposition below. The fact, proved in Section 13, that they give examples of non-amenable groups G that satisfy $\text{Føl}\, G = 0$, shows that our conditions on the tree action are sharp, although they are more restrictive than the ones in [4].

Proposition 12.1. *Let G be a finitely generated group which is not virtually cyclic, and assume that G acts on a simplicial tree T without a global fixed point, and such that, for some $k \geq 0$, the action is k-acylindrical in the sense of Sela (i.e. for any $g \in G - \{1\}$ the set of fixed points $\text{Fix}(g) \subset T$ has diameter $\leq k$, with respect to the simplicial metric).*

Then, for every finite generating system X of G, there exist two elements $a, b \in G$ which generate a free subgroup of rank 2 and satisfy $|a|_X, |b|_X \leq \max(8k, 16)$. In particular, the group G is uniformly non-amenable.

Proof. Notice first that, given any two elements $a, b \in G$ which act hyperbolically on T, such that their axes are disjoint or intersect in a segment J which is shorter than the translation length of both, then it follows from a standard ping-pong argument that the subgroup generated by a and b is isomorphic to F_2. Hence our goal is to produce such elements which are of bounded length with respect to an arbitrary fixed generating system $X = \{x_1, \ldots, x_q\}$ of G.

If one of the x_i defines a hyperbolic action on T, we set $g = x_i$. If all of the x_i act as elliptic elements on T, and any two of them have a common fixed point, then, as X is finite (and T a tree), there would be a common fixed point for all the x_i, which contradicts the hypothesis that G has no global fixed point on T. Thus we can assume that $\mathrm{Fix}(x_1) \cap \mathrm{Fix}(x_2)$ is empty, which implies that $g = x_1 x_2$ is a hyperbolic element, whose axis will be denoted by $\mathrm{ax}(g)$.

If one of the x_j fixes one end of $\mathrm{ax}(g)$, then the commutator $x_j g x_j^{-1} g^{-1}$ fixes pointwise an infinite subarc of $\mathrm{ax}(g)$ which defines that end. Hence from our hypothesis of k-acylindricity of the action, this commutator must be trivial, and in particular x_j leaves $\mathrm{ax}(g)$ invariant.

Thus, if each of the x_j fixes an end of $\mathrm{ax}(g)$ or interchanges its ends, then all of G acts on $\mathrm{ax}(g)$, which means that either G is virtually cyclic, or else the commutator subgroup contains elements of infinite order which fix all of $\mathrm{ax}(g)$, so that the action of G would again not be k-acylindrical.

Thus we find an element $h = x_j g x_j^{-1}$ which is also hyperbolic and has the property that $J = \mathrm{ax}(g) \cap \mathrm{ax}(h)$ is empty or has finite diameter $d \geq 0$. Without loss of generality we can assume that, if J is non-empty, then g and h shift their axes along J in the same direction. If $\frac{d}{4}$ is greater than or equal to the translation length of both g and h, then $g^{-1}h^{-1}gh$ fixes pointwise a final segment of J of length $\geq \frac{d}{2}$, and the acylindricity hypothesis implies that $2k \geq d$. It follows that the elements $a = g^{2k}$ and $b = h^{2k}$ have the desired properties, since every edge in T has length 1.

If $\frac{d}{4}$ is smaller than the translation length of g (or of h), then we can define $a = g^4$ and $b = hg^4h^{-1}$ to find the desired elements.

If J is empty, then one can take $a = g$ and $b = h$. □

13. Non-amenable groups with Følner constant zero

Let $\mathcal{Q}_m$ denote the set of marked m-generated groups, that is, the set of all quotients of the free group $F(X)$ where X is a fixed free generating set containing $m \geq 2$ elements. The set $\mathcal{Q}_m$ can also be considered as the set of all normal subgroups of $F(X)$ or, geometrically, as the set of all corresponding Cayley graphs $\mathcal{C}(F(X)/N, X)$ (where, abusing notation, X denotes also the generating set of the quotient $F(X)/N$).

We define a metric (hence a topology, known as the Cayley topology) on $\mathcal{Q}_m$ as follows. Given two normal subgroups $N_1, N_2 \le F(X)$, let $\mathcal{C}_i$, for $i = 1, 2$, be the Cayley graph for $F(X)/N_i$. Then, the distance between N_1 and N_2 is defined as

$$D(N_1, N_2) = \inf\left\{\frac{1}{n+1}; B_{\mathcal{C}_1}(n) \text{ is isometric to } B_{\mathcal{C}_2}(n)\right\}$$

where $B_{\mathcal{C}}(n)$ is the ball of radius n in the Cayley graph $\mathcal{C}$ centered at the identity, and the isometry preserves the edge labels. This topology was introduced in [10], see also [29, 7] for background and interesting applications.

For a sequence of normal subgroups $\{N_k\}_{k=1}^{\infty}$, we say that N is the limit normal subgroup of the sequence if

$$\lim_{k\to\infty} D(N, N_k) = 0.$$

The corresponding quotient $G = F(X)/N$ is called the limit group of the sequence $\{G_k\}_{k=1}^{\infty}$ with $G_k = F(X)/N_k$. In particular, if $N_1 \ge N_2 \ge N_3 \ge \dots$ form a chain of normal subgroups, then the limit normal subgroup is $N = \bigcap_{k=1}^{\infty} N_k$. Similarly, if $N_1 \le N_2 \le N_3 \le \dots$ is now an ascending chain, then the limit is the union $N = \bigcup_{k=1}^{\infty} N_k$.

Proposition 13.1. *Let $G = F(X)/N$ be the limit group of a sequence $\{G_k\}_{k=1}^{\infty}$ with $G_k = F(X)/N_k$, for $k \ge 1$. Then*

$$\mathrm{F\o l}_X G \ge \limsup_{k\to\infty} \mathrm{F\o l}_X G_k.$$

Proof. We denote by $\mathcal{C}$ and $\mathcal{C}_k$ the Cayley graph relative to X of G and G_k respectively. For an arbitrary $\varepsilon > 0$, by definition of $\mathrm{F\o l}_X G$, there exists a finite set $A \subset G$ satisfying $\mathrm{F\o l}_X G \le \frac{\#\partial_X A}{\#A} < \mathrm{F\o l}_X G + \varepsilon$. Since A is finite, it is contained in some ball $B_{\mathcal{C}}(n)$ of radius $n = n(A)$ in $\mathcal{C}$. By definition of the limit normal subgroup, there exists $K = K(n) > 0$ such that for any $k > K(n)$ we have $D(N_k, N) < \frac{1}{n+2}$. That is, for such indices k, the balls $B_{\mathcal{C}}(n+1)$ and $B_{\mathcal{C}_k}(n+1)$ are isometric via an isometry φ_k. Putting $A_k = \varphi_k(A) \subset G_k$ we obtain $\mathrm{F\o l}_X G \le \frac{\#\partial_X A_k}{\#A_k} < \mathrm{F\o l}_X G + \varepsilon$. It implies that $\mathrm{F\o l}_X G_k < \mathrm{F\o l}_X G + \varepsilon$ for any $k > K(n)$. Hence $\limsup_{k\to\infty} \mathrm{F\o l}_X G_k \le \mathrm{F\o l}_X G$. □

Corollary 13.2. (cf. [25]) *If the limit group $G = F(X)/N$ is amenable then*

$$\lim_{k\to\infty} \mathrm{F\o l}_X F(X)/N_k = 0.$$

We recall that the Baumslag-Solitar groups are given by the presentations

$$BS(p,q) = \langle a, t, \mid t^{-1}a^p t = a^q \rangle, \quad p, q \in \mathbb{N}.$$

Using Britton's lemma for HNN-extensions, it's easy to see that, if $p, q > 1$, then the elements t and $a^{-1}ta$ generate a free subgroup of rank 2 in $BS(p,q)$. It follows that the group is non-amenable.

Proposition 13.3. *For relatively prime $p \neq 1, q \neq 1$ the group $BS(p,q)$ is a non-amenable group with* $\text{Føl}\, BS(p,q) = 0$.

Proof. We have to show that $\text{Føl}\, G = 0$ for $G = BS(p,q)$ with relatively prime $p \neq 1, q \neq 1$. We define a homomorphism $\phi : G \to G$ as follows:

$$a \mapsto a^p,$$
$$t \mapsto t.$$

Since p and q are relatively prime, we have that ϕ is surjective. We denote by N_i its iterated kernel, i.e. $N_i = \ker \phi^i, i \geq 1$. Note that $N_1 \leq N_2 \leq \ldots$. Let L denote the corresponding limit group, that is, $L = G/\bigcup_{k=1}^{\infty} N_k$. Then this group is amenable. Indeed, the kernel of the homomorphism $L \to \mathbb{Z}$ defined by $a \mapsto 1$ and $t \mapsto t$ is abelian, since it is generated by $\{t^{-n}at^n,\ n \in \mathbb{Z}\}$. Any two such generators are conjugate to elements $t^{-\ell}at^{\ell}$ and a, for some $\ell \in \mathbb{N}$. These elements commute in $\phi^{\ell}(G)$ and hence in L. Thus L is an extension of an abelian group by a cyclic one, so it is solvable and hence amenable. By the previous corollary, $\lim_{k\to\infty} \text{Føl}_X F_2/N_k = 0$ where $X = \{a, t\}$. But for all $k \geq 1$ there is an isomorphism between the quotient F_2/N_k and the group G. It follows that $\text{Føl}\, G = 0$, which completes the proof. □

The previous result can be extended to a more general class of groups.

Theorem 13.4. *Let $A = \langle x_1, \ldots, x_m \mid \mathcal{R} \rangle$ be an amenable group with a set of defining relations $\mathcal{R}$. Let $\mu, \nu : A \to A$ be injective homomorphisms satisfying*

(i) $\mu \circ \nu = \nu \circ \mu$;
(ii) $\mu(A) \cup \nu(A)$ *generate* A.

Then for the group $G = \langle t, A \mid t^{-1}\mu(x_i)t = \nu(x_i) \rangle$ we have $\text{Føl}\, G = 0$. *If in addition,*

(iii) $\mu(A) \cup \nu(A) \neq A$,

then G is a non-amenable group with $\text{Føl}\, G = 0$.

Proof. Let $\Phi : G \to G$ be defined by $\Phi(t) = t$, $\Phi(x_i) = \mu(x_i), i = 1, \ldots, m$. Condition (i) implies that Φ is homomorphism, (ii) that Φ is surjective. Considering the iterated kernels of Φ, we obtain the limit group $L = G/\bigcup_{k=1}^{\infty} \operatorname{Ker} \Phi^k$ which is amenable by the hypothesis on A. Indeed, as above, we consider the kernel K of the homomorphism $L \to \mathbb{Z}$ defined by $x_i \mapsto 1, i = 1, \ldots, m$, and $t \mapsto t$. We claim that it is amenable. Then L is amenable as an extension of an amenable group by a cyclic group. Hence, as in Proposition 13.3, we have $\operatorname{Føl} G = 0$.

In order to prove our claim we recall that a countable group is amenable if and only if every finitely generated subgroup of this group is amenable. Every finitely generated subgroup H of K is generated by finitely many products of finitely many conjugates $t^{-n}x_it^n$ with $i = 1, \ldots, m$ and $n \in \mathbb{Z}$. Taking a conjugate of H (if necessary) by an appropriate power of t, we assume that $n \in \mathbb{N}$ in these conjugates. Then using the defining relation $t^{-1}\mu(x_i)t = \nu(x_i)$ of G we find a number $\ell \in \mathbb{N}$ such that $\Phi^\ell(h) \in A$ for each generator h of H. Hence $\Phi^\ell(H) \leq A$. This image is amenable because it is a finitely generated subgroup of the amenable group A. Since Φ is an isomorphism of L we obtain the amenability of H and hence that of K. This proves the claim.

Condition (iii) provides a free subgroup of rank 2 in G and hence non-amenability of G. Namely, using Britton's lemma for HNN-extensions, one can check that it is a subgroup freely generated by b and $b^{-1}tb$, for any $b \in A - (\mu(A) \cup \nu(A))$. □

Remark 13.5. Groups satisfying conditions (i), (ii), and (almost) (iii) above with A being *abelian or a direct product* were introduced by Meier in order to construct non-hopfian HNN-extensions [20]. In particular, such groups are non-hopfian.

As an immediate corollary of Theorem 2.4 and Theorem 13.4 we see that $K(\lambda_G, G) = 0$ for a group G satisfying (i) and (ii) above (i.e. the left regular representation is not uniformly isolated from the trivial representation for these groups). If in addition (iii) holds we obtain such a non-amenable group. This provides further examples of a negative answer to a question of Shalom [28], solved first by Osin [24], where groups with similar conditions and an *abelian* group A were considered.

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