

ON STABLE NORM IN WORD HYPERBOLIC GROUPS

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ABSTRACT. This work is concerned with the stable norm in word hyperbolic groups, as defined in [Gr]. We give a short elementary proof of one of its basic property, that is existence of a computable non null uniform lower bound for stable norm in a word hyperbolic group.

INTRODUCTION

Though in word hyperbolic groups (cf. [Gr], [CDP], [GdlH] or [GHVS]) the stable norm has appeared to be a useful tool, some of its basic properties have only been sketched in papers like [Gr] and [De], at least as far as we're concerned. We are interested in the existence of a computable uniform lower bound for the stable norm in a word hyperbolic group and give a short elementary complete proof.

1. DEFINITION OF STABLE NORM

Given a group G on a finite generating set S , a word on $S \cup S^{-1}$ naturally represents an element of G . We denote by $\text{lgr}(\cdot)$ its length as a word and :

$$|g| = \inf \{ \text{lgr}(\omega) \mid \omega \text{ is a word on } S \cup S^{-1} \text{ representing } g \}$$

which allows to define the word metric d_S on G by $d_S(g_1, g_2) = |g^{-1}g_2|$.

Definition. Let G be a group and $g \in G$ an element with infinite order, we denote by :

$$[[g]] = \lim_{n \rightarrow +\infty} \frac{|g^n|}{n}$$

the **stable norm** of the element g .

The limit of $|g^n|/n$ exists : with the triangle inequalities, for any $n, p \in \mathbb{N}$, $|g^{n+p}| \leq |g^n| + |g^p|$, that is the sequence $(|g^n|)_n$ is sub-additive.

Lemma 1. Let $(u_n)_n$ be a sub-additive sequence of positive real numbers, then the sequence $(u_n/n)_n$ converges.

Proof. Let $n \geq m > 0$; consider the euclidian division of n by m : $n = pm + r$ with $p, r \in \mathbb{N}$, $0 \leq r < m$. Since $(u_n)_n$ is sub-additive, $0 \leq u_n \leq u_{pm} + u_r \leq pu_m + \max_{0 \leq r < m} u_r$. Then :

$$0 \leq \frac{u_n}{n} \leq \frac{pm}{n} \frac{u_m}{m} + \frac{1}{n} \max_{0 \leq r < m} u_r \leq \frac{u_m}{m} + \frac{1}{n} \max_{0 \leq r < m} u_r$$

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Making n tend to ∞ one obtains for any $m \in \mathbb{N}^*$: $0 \leq \limsup u_n/n \leq u_m/m$, so that $\limsup u_n/n$ is finite and moreover : $\limsup u_n/n \leq \inf_{m>0} u_m/m \leq \liminf u_m/m$, so that : $\limsup u_n/n \leq \liminf u_n/n$, which proves the assumption. \square

2. BASIC PROPERTIES

The two following are basic properties of stable norm :

Property 1. *The stable norm is an invariant of conjugacy classes.*

Proof. Suppose u and v are infinite order elements of a group G , and $u = ava^{-1}$ for some $a \in G$. Then for any $n \in \mathbb{N}^*$, $u^n = av^n a^{-1}$ which implies that $||u^n| - |v^n|| \leq 2|a|$. Dividing by n and making n tend to ∞ one obtains $[[u]] = [[v]]$. \square

Property 2. *For any $n \in \mathbb{Z}^*$, $[[g^n]] = |n| \cdot [[g]]$.*

Proof. Given $n \in \mathbb{Z}^*$,

$$[[g^n]] = \lim_{t \rightarrow +\infty} \frac{|g^{nt}|}{t} = \lim_{t \rightarrow +\infty} |n| \frac{|g^{|n|t}|}{|n|t} = |n| \lim_{t \rightarrow +\infty} \frac{|g^t|}{t} = |n| \cdot [[g]]$$

\square

3. UNIFORM LOWER BOUND FOR STABLE NORM

We now give the proof of the property we are interested in.

Theorem 1. *Let G be a word hyperbolic group ; there exists a computable constant $K > 0$ such that for any infinite order element g of G , $[[g]] \geq K$.*

Proof. Denote by $\Gamma(G)$ the Cayley graph of G respectively to a given finite generating set; G is given the word metric which makes it isometric to the vertex set of $\Gamma(G)$. A word on the generators of G is said *cyclically reduced* if all its cyclic conjugates are geodesic words. Obviously for any element $g \in G$ there exists a cyclically reduced word ω representing an element in the conjugacy class of g . Whenever ω is a word representing an infinite order element of G , one can consider a segment in $\Gamma(G)$ with label ω and the infinite path, that we denote $[\omega^-, \omega^+]$, which is the orbit of this segment under the action of the cyclic subgroup generated by ω . If ω is a cyclically reduced word of length $k > 0$, then $[\omega^-, \omega^+]$ is a k -local geodesic, that is, any of its subpaths of length at most k is a geodesic.

Since the word problem is solvable in G one can algorithmically transform a given word w into a cyclically reduced one lying in the same conjugacy class. We will make use of this algorithm together with the fact (theorem 3.1.4, [CDP]) that there exists computable constant numbers $k > 0$, $l \geq 1$ and $\varepsilon \geq 0$ such that each k -local geodesic in the Cayley graph of G is a (l, ε) -quasigeodesic. In the following we fix the constants k, l, ε .

Denote by \mathcal{B} the ball in G with center 1 and radius $k - 1$. Apply the following process for each element g in \mathcal{B} : Initially $i = 2$; change g^i into a cyclically reduced word in the same conjugacy class. If its length is at least k then stop with $n_g = i$ else if it is non null restart with $i + 1$ instead of i , else stop. The process will terminate since either g has finite order or g^p cannot be conjugate with g^q for $0 < p < q$ (hyperbolic groups do not contain any Baumslag-Solitar group). Among all the

integers n_g obtained for (infinite order) elements g , denote by n_{max} the supremum ; clearly $1 < n_{max} < \infty$.

Let g be an infinite order element of G and ω a cyclically reduced word in its conjugacy class. If ω has length at least k , the path $[\omega^-, \omega^+]$ is a k -local geodesic, and then a (l, ε) -quasigeodesic. In particular, for any $n \in \mathbb{N}$, one has $n|\omega| \leq l|\omega^n| + \varepsilon$, and hence :

$$\frac{|\omega|}{l} - \frac{\varepsilon}{ln} \leq \frac{|\omega^n|}{n}$$

Making n tend to ∞ one obtains $[[\omega]] \geq |\omega|/l \geq k/l$. Together with property 1 one has :

$$[[g]] \geq \frac{k}{l}$$

If ω has length less than k , ω lies in \mathcal{B} . Consider the integer n_ω as defined above, then following the same argument :

$$[[g^{n_\omega}]] \geq \frac{k}{l} \text{ which implies } n_\omega [[g]] \geq \frac{k}{l} \text{ with property 2}$$

and finally for any infinite order element of G :

$$[[g]] \geq K = \frac{k}{ln_{max}}$$

which proves the assumption. □

REFERENCES

- [CDP] M.COORNAERT, T.DELZANT, and A.PAPADOPOULOS, *Géométrie et théorie des groupes, les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics **1441**, Springer-Verlag, 1991.
- [De] T.DELZANT, *Sous-groupes distingués et quotients des groupes hyperboliques*, Duke Math. J. **83** (1996), n° 3, 661-682.
- [GHVS] E.GHYS, A.HAEFLIGER, A.VERJOVSKY, and H.SHORT, *Notes on word hyperbolic groups in "Group theory from a geometrical viewpoint"*, (H.SHORT ed.), World Scientific, 1991.
- [GdlH] E.GHYS and P.DE LA HARPE (ed.), *Les groupes hyperboliques d'après Gromov*, Progress in Mathematics, Birkhäuser, 1990.
- [Gr] M.GROMOV, *Hyperbolic groups in "Essays in group theory"*, MSRI Publications, Springer (1987), 75-263.