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Report on the thesis of Jean-Philippe Préaux[1]

This thesis is dedicated entirely to proving the following theorem: if M is a compact, orientable 3-manifold that satisfies Thurston's geometrization conjecture, then the fundamental group of M has a solvable conjugacy problem.

In order to solve the conjugacy problem for a finitely presented group one must show that there exists an algorithm that, given two finite words in the given generators of the group, determines if these words represent conjugate elements of the group or not. Max Dehn pointed out in 1912 that the conjugacy problem, the word problem and the isomorphism problem are the most fundamental of all problems in combinatorial group theory. The study of these problems formed much of the core of combinatorial group theory throughout the last century. Moreover, Dehn discovered their importance in the course of his work on low-dimensional manifolds and immediately upon stating the problems pointed out that in order to understand such manifolds (even ``knotted curves in space") one had to solve these problems in the special case of the fundamental groups of the manifolds concerned. Dehn also noted the geometric significance of the problems --- e.g. the conjugacy problem in the fundamental group of a space is equivalent to the problem of determining whether the loops represented by the given words are freely homotopic.

Remarkably, it was not until the revolutionary work of Thurston on the existence of geometric structures for 3-manifolds that the fundamental problems of group theory became tractible in their original setting. Indeed, in the absence of a complete proof of Thurston's geometrization conjecture, one still cannot solve even the word problem for arbitrary 3-manifold groups.

I include these contextual remarks for three reasons: first, to emphasise the classical pedigree of the problem solved in this thesis; second, to

[1] I apologise sincerely to colleagues for the fact that pressure of time forces me to write in English in order to be clear and accurate.

emphasise that the hypothesis on geometrisability of the manifold is essential for the moment; and third, to point out that although Préaux adopts a strongly algebraic tone in his articulation of his solution to the conjugacy problem, he
is *de facto* solving the geometric problem of determining when two loops in a geometrizable 3-manifold are connected by an incompressible annulus.

The highlights of the previous progress on the conjugacy problem for 3-manifold groups are as follows: the groups of alternating knots and links were shown to be small-cancellation groups in the 1960s and this yields a straightforward solution to the conjugacy problem (this is a rather elementary
argument, and is essentially the only progress that does note require the characteristic sub-manifold theory and Thurston geometrization); Sela (1993) solved the conjugacy problem for all knot groups; many 3-manifolds are now known to support metrics of non-positive curvature, and this leads to a straightforward solution to the conjugacy problem (but showing the existence of metrics of non-positive curvature on ``most" 3-manifolds is a non-trivial application of Thurston's theorems); the work of Epstein-Thurston and Neumann-Reeves on (bi)automaticity of many geometrizable 3-manifolds, particularly Seifert fibred spaces, solves the conjugacy problem for those groups.

Préaux's work builds on that of Sela, but there are many new difficulties to be overcome. The basic strategy is as follows: one decomposes the manifold into its geometric pieces, thinks about how incompressible annuli realising conjugacies must intersect the pieces, and then devises algorithms to determine if such annuli can exist connecting two given loops. (This is a geometric reformulation --- Préaux uses more algebraic language.) Since the ends of the annuli for which we are searching are not specified in each piece, the search involves problems more general than the conjugacy problem --- e.g. one needs to understand incompressible annuli connecting pairs of toral boundary components, and one needs to know if a given loop can be homotoped into a boundary torus. Algebraically, these problems amount to determining whether elements can be conjugated into free abelian subgroups, and whether elements belong to double cosets of such subgroups. Thus Préaux is obliged to examine the algorithmic solubility of these auxillary problems in the geometric pieces. (He calls them ``algorithmes élémentaires", since they correspond to les pièces élémentaires of the geometric decomposition.)

Since the formation of free products preserves solubility of the conjugacy problem, and the Sol summands can be dealt with easily (Préaux, in his unrelenting style, gives a detailed argument in 4-5 pages), the essence of the problem lies with the Seifert and hyperbolic pieces of the geometric decomposition. The treatment of the Seifert pieces is almost 30 pages, much of which is detailed case-by-case analysis. Much of this could be avoided by quoting the automatic groups literature, but that would have the disadvantage of introducing a black-box to the algorithm, whereas Préaux's explicit solution is something that one could easily implement and analyse for complexity.

The most delicate and elegant part of the argument comes in dealing with the hyperbolic pieces. Here, following Sela, Préaux makes delicate and beautiful use of Thurston's Dehn surgery theorem (with considerations of geometric convergence) to translate his basic problems with finite volume manifolds into equivalent problems for certain closed manifolds. These last problems are easily solved because the fundamental groups at hand are now hyperbolic in the sense of Gromov.

This seems an appropriate point to mention that an impressive aspect of Préaux's work is that in it one sees many of the key developments of 3-manifold geometry/topology and geometric group theory coming together, each playing a natural role. Thus elements of Gromov's theory of hyperbolic groups enter, as does Thurston's Dehn Surgery Theorem and the existence of the Margulis Constant, the JSJ theory is required, and the Bass-Serre Theory of graphs of groups is needed to assemble the solutions to the auxillary "elementary" problems into a global solution.

The use of the Bass-Serre theory involves a somewhat lengthy generalisation to arbitrary graphs of groups of standard facts concerning conjugacy and centralizers in amalgamated free products and HNN extensions. The treatment here is thorough and others may find it useful, but it is clearly influenced by the combinatorial approach in the texts of Magnus *et al* and Lyndon-Schupp, and since it is the statement of these facts rather than their proofs that feed into the ultimate algorithm, this is a point at which a more geometric approach would certainly streamline the treatment. For example, thinking in terms of the Bass-Serre tree and elliptic versus hyperbolic isometries, the complicated case analysis for centralizers becomes much more natural.

Préaux's handling of the details of the Bass-Serre theory are, however, very carefully done. His introduction of the idea of graphs that are "presque sans cycle" is useful and well-explained.

There are various points in the thesis where the geometrically inclined reader
feels that the treatment would be cleaner and more natural if an algebraic argument were replaced by a geometric one. But such criticism is almost always ill-founded, because when one reflects on what is actually being achieved one realises that what Préaux is doing is producing, in a remarakably explicit manner, an actual algorithm for solving the conjugacy problem. This is an important point: he gives an algorithm, he does not simply prove that one exists. Indeed one can imagine a real implementation of most of his solution, based on a combination of existing software packages, except for two points in the argument, whose practical implementation seem to require new ideas: the Haken search for

incompressible tori, and the search for suitable sequences of Dehn surgeries to deal with the hyperbolic pieces. Further investigation to streamline these procedures and estimate their complexity would be very interesting (although far from straightforward).

With this desire for explicitness in mind, the geometric shortcuts that one thinks of at various stages in Préaux's proof lose much of their appeal. For example, although his lengthy arguments about generalised word problems being soluble can be replaced by the simple remark that the subgroups in question are quasi-isometrically embedded, such a remark would shorten the proof of the existence of an algorithm only at the expense of introducing the black-box of an unkown constant of quasi-isometry. Thus Préaux's relentless and detailed arguments often have a considerable intellectual and practical advantage over their slicker geometric cousins.

In summary, I find this to be an extremely good thesis. The arguments are written clearly in a detailed and convincing manner. The essential features of the various bodies of knowledge required (e.g. graphs of groups, geometric convergence, geometric decomposition etc.) are introduced in a concise and intelligently-distilled manner. Despite its length (200 pages) and its many layers, the proof is conceptually clear; it is executed in a thoroughly convincing manner. The candidate has solved a difficult problem with considerable style, and I have no hesitation in recommending that this thesis be accepted.

Yours faithfully

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