Exercice 1.

1) $IPP \left| \begin{array}{ll} u = t^2+t+1 & u' = 2t+1 \\ v = -\cos t & v' = \sin t \end{array} \right.$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (t^2+t+1)\sin t dt &= \left[-(t^2+t+1)\cos t\right]_0^{\frac{\pi}{2}} + \underbrace{\int_0^{\frac{\pi}{2}} (2t+1)\cos t dt}_{\substack{u=2t+1 \quad u'=2 \\ v=\sin t \quad v'=\cos t}} \\
&= 1 + [(2t+1)\sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin t dt \\
&= 1 + (\pi+1) - 2\times \underbrace{[-\cos t]_0^{\frac{\pi}{2}}}_{=1} \\
&= \boxed{\pi}
\end{aligned}$$

2) $IPP \left| \begin{array}{ll} u = \sin t & u' = \cos t \\ v = \dfrac{1}{2}e^{2t} & v' = e^{2t} \end{array} \right.$

$$\begin{aligned}
I = \int_0^{\frac{\pi}{2}} e^{2t}\sin t dt &= \left[\frac{1}{2}e^{2t}\sin t\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}} e^{2t}\cos t dt}_{\substack{u=\cos t \quad u'=-\sin t \\ v=\frac{1}{2}e^{2t} \quad v'=e^{2t}}} \\
&= \frac{e^{\pi}}{2} - \frac{1}{2}\left(\left[\frac{1}{2}e^{2t}\cos t\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\times I\right) \\
I &= \frac{e^{\pi}}{2} + \frac{1}{4} - \frac{1}{4}\times I \\
\Longrightarrow I &= \boxed{\frac{1+2e^{\pi}}{5}}
\end{aligned}$$

3) $IPP \left| \begin{array}{ll} u = \ln t & u' = \frac{1}{t} \\ v = t^3+t & v' = 3t^2+1 \end{array} \right.$

$$\begin{aligned}
\int_1^2 (3t^2+1)\ln t dt &= \left[(t^3+t)\ln t\right]_1^2 - \int_1^2 \frac{t^3+t}{t}dt \\
&= 10\ln 2 - \int_1^2 (t^2+1)dt \\
&= 10\ln 2 - \left[\frac{t^3}{3}+t\right]_1^2 \\
&= 10\ln 2 - \frac{8}{3} - 2 + \frac{1}{3} + 1 \\
&= \boxed{10\ln 2 - \frac{10}{3}}
\end{aligned}$$

4) $IPP \left| \begin{array}{ll} u = \arctan t & u' = \frac{1}{1+t^2} \\ v = \frac{t^2}{2} & v' = t \end{array} \right.$

$$\begin{aligned}
\int_0^1 t\arctan t dt &= \left[\frac{t^2}{2}\arctan t\right]_0^1 - \frac{1}{2}\int_0^1 \frac{t^2}{1+t^2}dt \\
&= \frac{\pi}{8} - \frac{1}{2}\left(\int_0^1 \left(1-\frac{1}{1+t^2}\right)dt\right) \\
&= \frac{\pi}{8} - \frac{1}{2}\left[t-\arctan t\right]_0^1 \\
&= \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right) \\
&= \boxed{\frac{\pi}{4} - \frac{1}{2}}
\end{aligned}$$

5) Calcul d'une primitive de $(t^2+1)e^{2t}$:

$$IPP\left|\begin{array}{ll} u=t^2+1 & u'=2t \\ v=\frac{1}{2}e^{2t} & v'=e^{2t}\end{array}\right.$$

$$\begin{aligned}\int^x (t^2+1)e^{2t}dt &= \frac{1}{2}(x^2+1)e^{2x} - \underbrace{\int^x te^{2t}dt}_{\substack{u=t \quad u'=1 \\ v=\frac{1}{2}e^{2t} \quad v'=e^{2t}}} \\ &= \frac{1}{2}(x^2+1)e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int^x e^{2t}dt\right) \\ &= \frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} \\ &= \frac{e^{2x}}{4}\times\left(2x^2-2x+3\right)\end{aligned}$$

Exercice 2.

1) Changement de variable $x=\phi(t)=\cos t$; $\phi'(t)=-\sin t$:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+\cos^2 t}dt &= -\int_0^{\frac{\pi}{2}} \frac{\phi'(t)}{1+\phi(t)2}dt = -\int_{\cos 0}^{\cos\frac{\pi}{2}} \frac{1}{1+x^2}dx \\ &= [\arctan x]_0^1 = \boxed{\frac{\pi}{4}}\end{aligned}$$

2) Calcul de primitive par le changement de variable $x=\phi(t)=\cos t$; $\phi'(t)=-\sin t$:

$$\begin{aligned}\int^y \sin^3 t dt &= \int^y (1-\cos^2 t)\sin t dt = -\int^y (1-\phi(t)^2)\phi'(t)dt \\ &= -\int^{\phi(y)} (1-x^2)dx = \boxed{\frac{\cos^3(y)}{3} - \cos y}\end{aligned}$$

3) Calcul de primitive par le changement de variable $x=\phi(t)=\ln t$; $\phi'(t)=\frac{1}{t}$:

$$\int^y \frac{1}{t(\ln t)^n}dt = \int^y \frac{\phi'(t)}{\phi(t)^n}dt = \int^{\phi(y)} \frac{1}{x^n}dx = \frac{1}{(1-n)\phi(y)^{n-1}} = \boxed{\frac{1}{(1-n)\ln(y)^{n-1}}}$$

4) Changement de variable : $x=\phi(t)=\tan t$; $\phi'(t)=\dfrac{1}{\cos^2 t}$:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{1+2\cos^2 t}dt &= \int_0^{\frac{\pi}{4}} \frac{1}{\sin^2 t+3\cos^2 t}dt = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 t}\times\frac{1}{\tan^2 t+3}dt \\ &= \int_0^{\frac{\pi}{4}} \phi'(t)\times\frac{1}{3+\phi^2(t)}dt = \int_{\phi(0)}^{\phi(\pi/4)} \frac{1}{3+x^2}dx \\ &= \int_0^1 \frac{1}{3+x^2}dx = \frac{1}{3}\int_0^1 \frac{1}{1+\left(\frac{1}{\sqrt{3}}x\right)^2}dx \\ &= \frac{\sqrt{3}}{3}\times\int_0^1 \frac{\frac{1}{\sqrt{3}}}{1+\left(\frac{1}{\sqrt{3}}x\right)^2}dx \\ &= \frac{\sqrt{3}}{3}\times\left[\arctan\frac{1}{\sqrt{3}}x\right]_0^1 \\ &= \frac{\sqrt{3}}{3}\times\arctan\frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\times\frac{\pi}{6} = \boxed{\frac{\pi}{6\sqrt{3}}}\end{aligned}$$

5) Changement de variable $x=\phi(t)=\sin t$;

$$\frac{dx}{dt} = \phi'(t) = \cos t \implies dx = \cos t\, dt$$

et $\sin 0 = 0, \sin\frac{\pi}{2} = 1$:

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\times\cos t\, dt = \int_0^{\frac{\pi}{2}} |\cos t|\times\cos t\, dt \\ &= \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt \\ &= \frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{4}}\end{aligned}$$

6) Changement de variable : $x=\phi(t)=\sqrt{2+t}$; $\phi'(t)=\dfrac{1}{2\sqrt{2+t}}$:

$$\begin{aligned}\int_0^1 \frac{\sqrt{2+t}}{1+t}dt &= \int_0^1 \frac{1}{2\sqrt{2+t}}\times\frac{2(2+t)}{(2+t)-1}dt = \int_0^1 \phi'(t)\times\frac{2\phi^2(t)}{\phi^2(t)-1}dt\\ &= 2\int_{\phi(0)}^{\phi(1)}\frac{x^2}{x^2-1}dx = 2\int_{\sqrt{2}}^{\sqrt{3}}\left(1+\frac{1}{x^2-1}\right)dx\end{aligned}$$

Calcul de $\int \dfrac{1}{x^2-1}dx$: décomposition en élément simple : déterminons a et b tels que :

$$\frac{1}{x^2-1}=\frac{a}{x-1}+\frac{b}{x+1}=\frac{a(x+1)+b(x-1)}{x^2-1}=\frac{(a+b)x+a-b}{x^2-1}$$

$$\iff \begin{cases}a+b=0\\ a-b=1\end{cases} \iff \begin{cases}a=1/2\\ b=-1/2\end{cases} \implies \frac{1}{x^2-1}=\frac12\times\frac{1}{x-1}-\frac12\times\frac{1}{x+1}$$

Donc :

$$\begin{aligned}\int_0^1\frac{\sqrt{2+t}}{1+t}dt &= 2\times\left[x+\frac12\ln|x-1|-\frac12\ln|x+1|\right]_{\sqrt{2}}^{\sqrt{3}}\\ &= 2(\sqrt{3}-\sqrt{2})+\ln(\sqrt{3}-1)-\ln(\sqrt{2}-1)-\ln(\sqrt{3}+1)+\ln(\sqrt{2}+1)\\ &= \boxed{2(\sqrt{3}-\sqrt{2})+\ln\frac{(\sqrt{3}-1)(\sqrt{2}+1)}{(\sqrt{3}+1)(\sqrt{2}-1)}}\end{aligned}$$

Deuxième méthode (pour le changement de variable) : $\phi(t)=\sqrt{2+t}$ est strictement croissante sur $[0,1]$ comme composée d'applications strictement croissantes. Ainsi elle réalise une bijection sur son image, et donc le changement de variable $x=\phi(t)$ est équivalent au changement de variable $t=x^2-2$;

$$\frac{dt}{dx}=2x \implies dt=2x\,dx$$

De plus pour $x=\sqrt{2}$ et $x=\sqrt{3}$, $t=x^2-2=0$ ou 1 ; ainsi :

$$\int_0^1\frac{\sqrt{2+t}}{1+t}dt=\int_{\sqrt{2}}^{\sqrt{3}}\frac{\sqrt{2+x^2-2}}{1+x^2-2}\times 2x\,dx=\int_{\sqrt{2}}^{\sqrt{3}}\frac{2x^2}{x^2-1}dx$$

Le reste du calcul est similaire.

7) Changement de variable $x=\phi(t)=e^t$; $\phi'(t)=e^t$.

$$\int_0^1\frac{e^t}{1+e^{2t}}dt=\int_0^1\frac{\phi'(t)}{1+\phi^2(t)}dt=\int_{\phi(0)}^{\phi(1)}\frac{1}{1+x^2}dx=[\arctan x]_1^e=\boxed{\arctan e-\frac{\pi}{4}}$$

TD n° 22 - Calcul intégral - Corrigé des exercices restant

Ex3 ∘ $I = \int_0^1 \frac{1}{t^2+\frac{1}{2}}\,dt$

$$= 2 \times \int_0^1 \frac{1}{2t^2+1}\,dt = \frac{2}{\sqrt{2}} \times \int_0^1 \underbrace{\frac{\sqrt{2}}{1+(\sqrt{2}t)^2}}\,dt$$

$$= \sqrt{2} \times \left[\arctan(\sqrt{2}t)\right]_0^1 \qquad \int \frac{u'}{1+u^2} = \arctan(u)$$

$$= \sqrt{2} \times \left[\arctan(\sqrt{2}) - \arctan 0\right]$$

$$\boxed{I = \sqrt{2} \times \arctan(\sqrt{2})}$$

∘ $J = \int_0^1 \frac{1}{(t^2+\frac{1}{2})^2}\,dt$

On procède à une IPP sur I pour faire apparaître J et en déduire son calcul.

$$I = \int_0^1 \frac{1}{t^2+\frac{1}{2}}\,dt$$

IPP : $u = \frac{1}{t^2+\frac{1}{2}}$ $\quad u' = \frac{-2t}{(t^2+\frac{1}{2})^2}$

$v = t \quad v' = 1$

$u, v \in \mathcal{C}^1([0,1], \mathbb{R})$

$$\Rightarrow I = \left[\frac{t}{t^2+\frac{1}{2}}\right]_0^1 + \int_0^1 \frac{2t^2}{(t^2+\frac{1}{2})^2}\,dt$$

$$\Rightarrow I = \frac{2}{3} + \int_0^1 \left[\frac{2t^2+1}{(t^2+\frac{1}{2})^2} - \frac{1}{(t^2+\frac{1}{2})^2}\right]dt$$

$$I = \frac{2}{3} + 2\int_0^1 \frac{dt}{t^2+\frac{1}{2}} - \int_0^1 \frac{dt}{(t^2+\frac{1}{2})^2}$$

$$\Rightarrow \boxed{I = \frac{2}{3} + 2I - J}$$

Finalement, on obtient :

$$\underline{J = \frac{2}{3} + I = \frac{2}{3} + \sqrt{2}\arctan(\sqrt{2})}$$

- $K = \int_0^{\frac{\pi}{4}} \frac{1}{(1+\sin^2 x)^2}\,dx$

Changement de variable
$u = \tan x$
$du = \frac{dx}{\cos^2 x} = (1+\tan^2 x)\,dx$

$$\frac{dx}{(1+\sin^2 x)^2} = \frac{dx}{(\cos^2 x + \sin^2 x + \sin^2 x)^2}$$

$$= \frac{dx}{(\cos^2 x + 2\sin^2 x)^2} = \frac{dx}{\cos^4 x} \times \frac{1}{(1+2\tan^2 x)^2}$$

$$= \frac{dx}{\cos^2 x} \times \frac{1+\tan^2 x}{(1+2\tan^2 x)^2}$$

$$= du \times \frac{1+u^2}{(1+2u^2)^2}$$

$$\Rightarrow K = \int_0^1 \frac{1+u^2}{(1+2u^2)^2}\,du$$

$$= \int_0^1 \frac{u^2}{(1+2u^2)^2}\,du + \int_0^1 \frac{du}{(1+2u^2)^2}.$$

$$\boxed{K = \int_0^1 \frac{u^2}{(1+2u^2)^2}\,du + \frac{1}{4} \times J.}$$

Il suffit donc de calculer $L = \int_0^1 \frac{u^2}{(1+2u^2)^2}\,du$ dont on ne voit pas une primitive évidente.

Par IPP : $L = \int_0^1 u \times \frac{u}{(1+2u^2)^2}\,du$

$$f = u \qquad f' = 1$$

$$g = \frac{-1}{4} \times \frac{1}{(1+2u^2)} \qquad g' = \frac{u}{(1+2u^2)^2} \qquad f, g \in \mathcal{C}^1([0,1], \mathbb{R})$$

$$\Rightarrow L = \left[\frac{-u}{4(1+2u^2)} \right]_0^1 + \frac{1}{4} \int_0^1 \frac{du}{1+2u^2}$$

$$\boxed{L = -\frac{1}{3} + \frac{1}{8} \times I}$$

Ainsi $K = L + \frac{1}{4} J = -\frac{1}{3} + \frac{1}{8} \times I + \frac{1}{4} \times J$.

$$\Rightarrow K = -\frac{1}{3} + \frac{\sqrt{2}}{8} \arctan(\sqrt{2}) + \frac{1}{4} \left[\frac{2}{3} + \sqrt{2} \arctan(\sqrt{2} \right]$$

$$\boxed{K = \frac{3\sqrt{2}}{8} \arctan(\sqrt{2}) - \frac{1}{6}}$$

Ex4 $\quad I_n(x) = \int_0^x \frac{dt}{(1+t^2)^n}$.

1.a) L'application $t \longmapsto \frac{1}{(1+t^2)^n}$ est définie et continue sur $\mathbb{R}$. Donc $\int_0^x \frac{dt}{(1+t^2)^n}$ est bien définie pour tout $x \in \mathbb{R}$

1.b) IPP sur $I_n(x)$: pour $n \geq 1$

$u = \frac{1}{(1+t^2)^n} \qquad u' = \frac{-2nt}{(1+t^2)^{n+1}}$

$u, v \in \mathcal{C}^1([0,x], \mathbb{R}) \qquad v = t \qquad v' = 1$

$$\Rightarrow I_n(x) = \left[\frac{t}{(1+t^2)^n}\right]_0^x + n\int_0^x \frac{2t^2}{(1+t^2)^{n+1}}\,dt$$

$$\Rightarrow I_n(x) = \frac{x}{(1+x^2)^n} + n\left[\int_0^x \frac{2+2t^2}{(1+t^2)^{n+1}}\,dt - \int_0^x \frac{2\,dt}{(1+t^2)^{n+1}}\right]$$

$$I_n(x) = \frac{x}{(1+x^2)^n} + 2n\,I_n(x) - 2n\,I_{n+1}(x)$$

D'où la relation de récurrence pour tout $n \geq 1$:

$$\boxed{2n\,I_{n+1}(x) = (2n-1)I_n(x) + \frac{x}{(1+x^2)^n}}$$

1.c) • Pour $n=0$: $\boxed{I_0(x)} = \int_0^x dt = \boxed{x}$

• Pour $n=1$: $\boxed{I_1(x)} = \int_0^x \frac{dt}{1+t^2} = \boxed{\arctan x}$

• Pour $n=2$: $I_2(x) = \frac{1}{4}\left[3I_1(x) + \frac{x}{(1+x^2)^2}\right]$

$$\boxed{I_2(x) = \frac{3}{4}\arctan(x) + \frac{x}{4(1+x^2)^2}}$$

Ex 5 1) $\forall n \in \mathbb{N}^*$, $u_n = \sum_{k=0}^{n} \frac{1}{n+k}$

$$u_n = \sum_{k=0}^{n} \frac{1}{n} \times \frac{1}{1+\frac{k}{n}}$$

En posant $f(x) = \frac{1}{1+x}$

$$u_n = \frac{1}{n} \times \sum_{k=0}^{n} f\left(\frac{k}{n}\right)$$
$$= \frac{1}{n} \times \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) + \frac{1}{n} \times f(1)$$
$$= \frac{1}{n} \times \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) + \frac{1}{2n}$$

On $\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \int_0^1 f(t)\,dt = \left[\ln|1+x|\right]_0^1 = \ln 2$

Ainsi : $\boxed{\lim_{+\infty} u_n = \ln 2}$

2.) $\forall n \in \mathbb{N}^*$, $v_n = \sum_{k=0}^{n} \frac{1}{(n+k)^\alpha (n+k+1)^\beta}$ $\qquad \alpha+\beta = 1$, $\alpha, \beta \geq 0$

Puisque $\alpha, \beta \geq 0$ et $\alpha + \beta = 1$:

$$(n+k) = (n+k)^\alpha \times (n+k)^\beta$$
$$\leq (n+k)^\alpha \times (n+k+1)^\beta \leq (n+k+1)^\alpha \times (n+k+1)^\beta$$

$$\Rightarrow \frac{1}{(n+k+1)^\alpha (n+k+1)^\beta} \leq \frac{1}{(n+k)^\alpha (n+k+1)^\beta} \leq \frac{1}{n+k}$$

En sommant pour k variant de 0 à n :

$$\Rightarrow \underbrace{\sum_{k=0}^{n} \frac{1}{(n+k+1)^\alpha (n+k+1)^\beta}} \leq v_n \leq u_n$$

chgt d'indice $\Rightarrow \underbrace{\sum_{k=1}^{n+1} \frac{1}{(n+k)^\alpha (n+k)^\beta}} \leq v_n \leq u_n$

$\alpha+\beta=1$ $\Rightarrow \sum_{k=1}^{n+1} \frac{1}{(n+k)} \leq v_n \leq u_n$

$$\underset{\text{d'encadrement}}{\Longrightarrow} \quad \underset{\substack{\downarrow +\infty \\ \ln 2}}{u_n} + \underbrace{\frac{1}{2n+1} - \frac{1}{n}}_{\substack{\downarrow n \to +\infty \\ 0}} \leq v_n \leq \underset{\substack{\downarrow +\infty \\ \ln 2}}{u_n}$$

Ainsi, d'après le théorème des gendarmes :

$$\boxed{\lim v_n = \lim u_n = \ln 2}$$