MODELS OF THE HYPERBOLIC SPACE

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1. Introduction

We survey the usual models of the hyperbolic space $\mathbb{H}^n$. The presentation of the materials is mostly based on [Rat06, Chapter 3 and §6.1], see also [BP92, Chapter A], [AVS93] and [Dav08, Chapter 6].

2. Lorentzian and hyperbolic spaces

Let V be a real vector space of dimension $n+1$ equipped with a symmetric bilinear form B. We will denote by $q(\cdot) = B(\cdot,\cdot)$ the quadratic form associated to B and by $Q := \{x \in V \mid q(x) = 0\}$ the *isotropic cone of* B, or equivalently, of q.

2.1. **Lorentzian spaces.** Suppose from now on that the signature of B is $(n, 1)$. The couple (V, B) is then called a *Lorentzian* $(n+1)$*-space* and Q is called the *light cone*. Moreover, the elements in the set $Q^- := \{v \in V \mid q(v) < 0\}$ are said to be *time-like*, while the elements in $Q^+ := \{v \in V \mid q(v) > 0\}$ are *space-like*[1]; see the top picture in Figure 1 for an illustration.

A *Lorentzian transformation* is a map on V that preserves B. So, in particular, a Lorentz transformation preserves Q, Q^+ and Q^-. It turns out that Lorentzian transformations are linear isomorphisms on V (by the Mazur-Ulam theorem). We denote by $\mathrm{O}_B(V)$ the set of Lorentzian transformations of V:

$$\mathrm{O}_B(V) := \{f \in \mathrm{GL}(V) \mid B(f(u), f(v)) = B(u, v),\ \forall u, v \in V\}.$$

The well-known *Cartan-Dieudonné Theorem* states that, since B is non-degenerate, an element of $\mathrm{O}_B(V)$ is a product of at most $(n+1)$ B*-reflections*: for a non-isotropic vector $\alpha \in V \setminus Q$, the B*-reflection associated to* α (or simply reflection since B is clear) is defined by the equation[2]

$$s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha,\alpha)}\,\alpha, \quad \text{for any } v \in V. \tag{1}$$

We denote by $H_\alpha := \{v \in V \mid B(\alpha, v) = 0\}$ the orthogonal of the line $\mathbb{R}\alpha$ for the form B. Since $B(\alpha,\alpha) \neq 0$, we have $H_\alpha \oplus \mathbb{R}\alpha = V$. It is straightforward to check that s_α fixes H_α pointwise and that $s_\alpha(\alpha) = -\alpha$.

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[1] This vocabulary is borrowed from the theory of relativity, where $n = 4$.

[2] Observe that if B was a scalar product, this equation would be the usual formula for an Euclidean reflection.

2.2. **Hyperbolic spaces.** We fix a basis $\mathcal{B} = (e_1, \dots, e_{n+1})$ of V such that $q(v) = x_1^2 + x_2^2 + \dots x_n^2 - x_{n+1}^2$, for any $v \in V$ with coordinates $(x_1, \dots, x_{n+1})$ in the basis $\mathcal{B}$. With this basis, V is often denoted by $\mathbb{R}^{n,1}$. The quadratic hyper-surface $\{v \in V \mid q(v) = -1\}$, called the *hyperboloid*, consists in time-like vectors and has two sheets. It is interesting to note that it is a differentiable surface (as the preimage of a regular value by the differentiable map $q : V \longrightarrow \mathbb{R}$) and is naturally endowed with a Riemannian metric because $B(.,.)$ restricted to the tangent spaces of each sheet is definite positive. A *vector v is positive* if $x_{n+1} > 0$; the positive sheet,

$$\mathbb{H}^n := \{v \in V \mid q(v) = -1 \text{ and } x_{n+1} > 0\}$$

turns out to be a simply connected complete Riemannian manifold with constant sectional curvature equal to -1 (cf. [BP92, Theorem A.6.7]). This is *the hyperboloid model of the hyperbolic n-space*, see Figure 1. The distance function d on $\mathbb{H}^n$ satisfies the equation $\cosh d(x,y) = -B(x,y)$.

2.2.1. *Group of isometries.* Observe that the group $\mathrm{O}_B(V)$ acts on the quadratic hyper-surface $\{v \in V \mid q(v) = -1\}$. A Lorentz transformation is *a positive Lorentz transformation* if it maps time-like positive vectors to time-like positive vectors. So the group $O_B^+(V)$ of positive Lorentz transformations preserves $\mathbb{H}^n$ and its distance, and the group of isometries $\mathcal{I}(\mathbb{H}^n)$ of $\mathbb{H}^n$ is isomorphic to $O_B^+(V)$: any isometry of $\mathbb{H}^n$ is the restriction to $\mathbb{H}^n$ of a positive Lorentz transformation. Moreover, it is well known that $\mathcal{I}(\mathbb{H}^n)$ is generated by *hyperbolic reflections across hyperbolic hyperplanes*, of which we now recall the definition.

2.2.2. *Hyperbolic reflections.* A linear subspace F of V is said to be *time-like* if $F \cap Q^- \neq \varnothing$, otherwise it is *space-like*. An *hyperbolic hyperplane* is the intersection of $\mathbb{H}^n$ with a time-like hyperplane of V. Let H be a linear hyperplane in V and $\alpha \in V$ be a normal vector to H for the form B. Since $H \oplus \mathbb{R}\alpha = V$, we have necessarily that H is time-like if and only if $\alpha \in V$ is a space-like vector. A reflection $s_\alpha \in \mathrm{O}_B(V)$ is an *hyperbolic reflection* if $\alpha^\perp = H$ is a time-like hyperplane or, equivalently, if α is a space-like vector of V. In this case, $s_\alpha \in \mathrm{O}_B^+(V)$ and it restricts to an isometry of $\mathbb{H}^n$.

Remark 2.1. The fact that $s_\alpha \in \mathrm{O}_B^+(V)$ for a space-like vector α follows from the fact that a reflection s_α is continuous and that it exchanges the two sheets (i.e. connected components) of the quadratic surface $\{v \in V \mid q(v) = -1\}$ if and only if H_α is space-like, i.e., if and only if α is a time-like vector.

2.3. **The projective model.** Consider the unit open (Euclidean) n-ball embedded in the affine hyperplane $\mathbb{R}^n \times \{1\}$ of V:

$$D_1^n = \{v \in V \mid x_{n+1} = 1 \text{ and } x_1^2 + \dots + x_n^2 < 1\}$$

and the map p from D_1^n to $\mathbb{H}^n$, called the *radial projection*

$$p : D_1^n \to \mathbb{H}^n$$

where $p(v)$ is the intersection point of the line $\mathbb{R}v$ with $\mathbb{H}^n$ (see Figure 1). A simple calculation shows that

$$p(v) = \frac{v}{\sqrt{|q(v)|}}.$$

The unit ball D_1^n endowed with the pullback metric with respect to p, i.e. which makes p an isometry, is a (non conformal) model $\mathbb{H}_p^n$ for $\mathbb{H}^n$ called the *projective ball model*[3], see [Rat06, §6.1].

First, observe that using the equation for q in the basis $\mathcal{B}$, we have that $D_1^n \subseteq Q^-$. Let $\mathcal{H}$ be the affine hyperplane directed by $\mathrm{span}(e_1, \ldots, e_n)$ and passing through the point e_{n+1}, then we get

$$D_1^n = Q^- \cap \mathcal{H},$$

with boundary $Q \cap \mathcal{H}$. The next proposition follows from the previous discussion and [Rat06, Equation (6.1.2)].

Proposition 2.2. *The* projective model $\mathbb{H}_p^n$ *has underlying space* $D_1^n = Q^- \cap \mathcal{H}$ *and its boundary* $\partial\mathbb{H}_p^n$ *is* $Q \cap \mathcal{H}$. *Moreover,* $p : \mathbb{H}_p^n \to \mathbb{H}^n$ *is an isometry whose inverse is*

$$p^{-1}(v) = \frac{v}{x_{n+1}} = (x_1/x_{n+1}, \ldots, x_n/x_{n+1}, 1).$$

This proposition is illustrated for $n+1 = 2$ and $n+1 = 3$ in Figure 1.

2.3.1. *Hyperplanes, reflections and isometries.* The projective model gives us an easy description of hyperplanes: an *hyperbolic hyperplane in* $\mathbb{H}_p^n$ is simply the intersection of a time-like linear hyperplane of V with $\mathbb{H}_p^n$. Let $\mathcal{I}(\mathbb{H}_p^n)$ be the group of isometries of $\mathbb{H}_p^n$.

Corollary 2.3. *The conjugation by* p *is an isomorphism from* $\mathcal{I}(\mathbb{H}^n)$ *to* $\mathcal{I}(\mathbb{H}_p^n)$*: for* $\varphi \in \mathcal{I}(\mathbb{H}^n)$ *and a point* $v \in \mathbb{H}_p^n$, $\varphi \cdot v := p^{-1} \circ \varphi \circ p(v)$ *defines the isometric action of* φ *on* $\mathbb{H}_p^n$. *Moreover* $\varphi \cdot v$ *is the intersection point of the linear line* $\mathbb{R}\varphi(v)$ *with the ball* D_1^n.

In particular, if $\alpha \in V$ *is a space-like vector, then* $s_\alpha \cdot v = p^{-1} \circ s_\alpha \circ p(v)$ *is the hyperbolic reflection in* $\mathcal{I}(\mathbb{H}_p^n)$ *of* v *across the time-like hyperplane* $\alpha^\perp$.

Proof. This result follows immediately from Proposition 2.2" Let us detail the "moreover part". Let $(x_1, \ldots, x_{n+1})$ be the coordinates of v, and $(y_1, \ldots, y_{n+1})$ the coordinates of $\varphi(v)$, in the basis $\mathcal{B}$. Since $v \in D_1^n$ and $\varphi \in \mathrm{O}_B^+(V)$, we have that $x_{n+1} = 1$ and $y_{n+1} > 0$. Therefore, $\varphi(v) \in Q^-$ and $\varphi(v)/y_{n+1} \in Q^- \cap \mathcal{H} = D_1^n$ is the intersection point of $\mathbb{R}\varphi(v)$ with the ball D_1^n. Now remember the formula for p^{-1} in Proposition 2.2: we have

$$\begin{aligned}\varphi \cdot v &= p^{-1} \circ \varphi \circ p(v) = p^{-1} \circ \varphi \left(\frac{v}{\sqrt{|q(v)|}} \right) \\ &= p^{-1} \left(\frac{1}{\sqrt{|q(v)|}} \varphi(v) \right) = p^{-1}(\varphi(v)) = \frac{\varphi(v)}{y_{n+1}}.\end{aligned}$$

□

[3] This model is also sometimes called the *Beltrami-Klein model* in the literature.

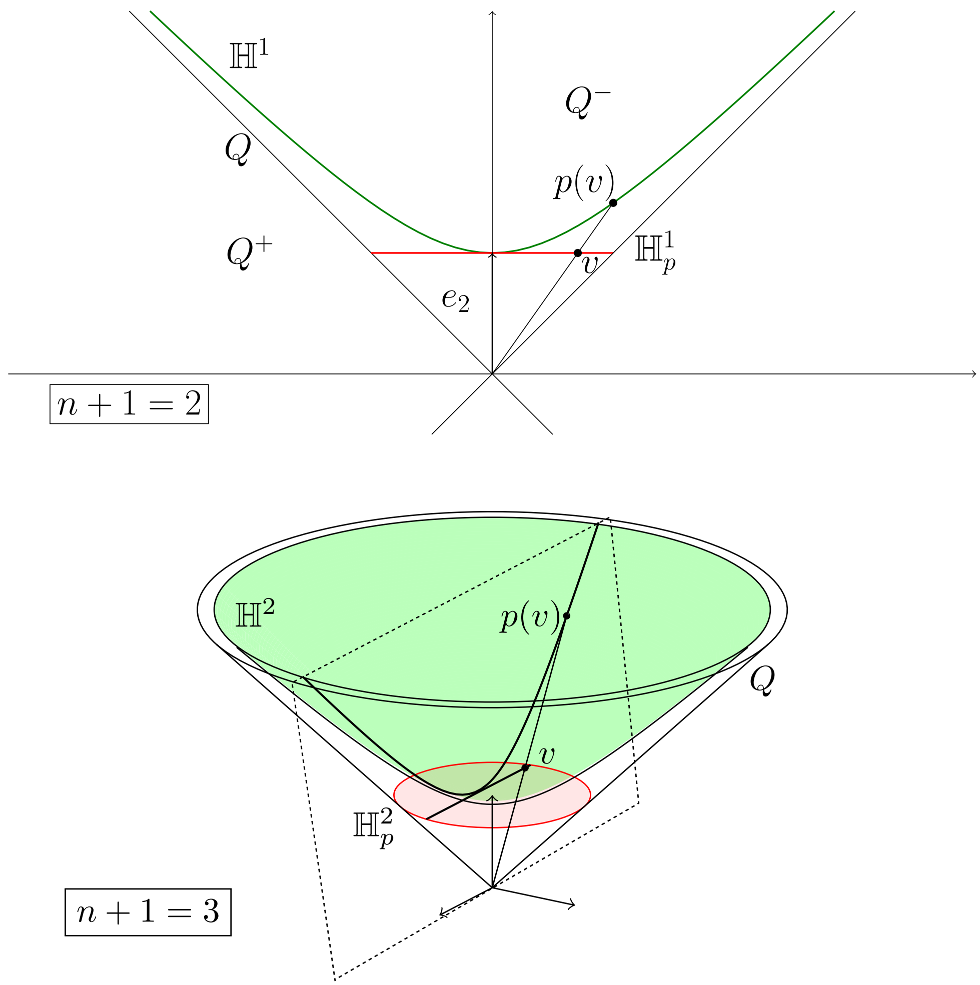

FIGURE 1. Pictures of Lorentzian spaces of dimension $n+1=2$ and $n+1=3$ with the hyperbolic spaces $\mathbb{H}^n$, the projective model $\mathbb{H}^n_p$, the radial projection p and a time-like plane H cutting $\mathbb{H}^2$ and $\mathbb{H}^2_p$.

2.4. **Limit sets of discrete groups of hyperbolic isometries.** The notion of *limit set* is central to study the dynamics of discrete groups of hyperbolic isometries: it provides an interesting topological space on which the group naturally acts, and the properties of the limit set characterize some properties of the group. We recall below the definition and some basic features of limit sets; see for example [Rat06, §12.1], [VS93, §2] or [Nic89, §1.4] for details.

Given a hyperbolic isometry φ of $\mathbb{H}^n_p$ (with underlying space D^n_1), we extend the action of φ to the closed ball $\overline{D^n_1}$. Note that φ preserves the boundary $\partial\mathbb{H}^n_p = Q \cap \mathcal{H}$ of $\mathbb{H}^n_p$.

Let $\Gamma \subseteq \mathcal{I}(\mathbb{H}^n_p)$ be a discrete group of hyperbolic isometries. For x a point in the closed ball $\overline{D^n_1}$, the following are equivalent:

- x is an accumulation point of the orbit $\Gamma \cdot x_0$ for some $x_0 \in \mathbb{H}^n_p$;

- x is an accumulation point of the orbit $\Gamma \cdot x_0$ for any $x_0 \in \mathbb{H}^n_p$;
- x is in $\partial \mathbb{H}^n_p \cap \overline{\Gamma \cdot x_0}$ for some $x_0 \in \mathbb{H}^n_p$.

Such a point is called a *limit point* of Γ.

Definition 2.4. The set $\Lambda(\Gamma)$ of limit points is called *the limit set of* Γ.

The fact that an orbit has no accumulation points inside the open ball D^n_1 is clear, since the group is discrete. The fact that the limit set of an orbit does not depend on the chosen point follows from the relation between hyperbolic and Euclidean distances, see [Rat06, Theorem 12.1.2].

The limit set $\Lambda(\Gamma)$ is clearly closed and Γ-stable. Many general properties are known for limit sets of discrete groups of hyperbolic isometries. For example, either $\Lambda(\Gamma)$ is finite, in which case $|\Lambda(\Gamma)| \leq 2$ and Γ has a finite orbit in the closed ball $\overline{D^n_1}$, see [Rat06, Theorem 12.2.1], or $\Lambda(\Gamma)$ is uncountable and the action of Γ on $\Lambda(\Gamma)$ is minimal.

3. Conformal models of the Hyperbolic space

We now consider the two conformal models of the Hyperbolic spaces.

3.1. **Conformal models of the hyperbolic space.** We now introduce two conformal models of the hyperbolic space, the *conformal ball model* and the *upper halfspace model* that turn out to be more practical to deal with the geometry because their isometries are Möbius transformations. For details we refer the reader to [Rat06, Chapter 4] and [BP92, Chapter A]. We use the notation $\|.\|$ for the Euclidean norm of $\mathbb{R}^n$.

3.1.1. *Inversions and the Möbius group.* In the Euclidean space $\mathbb{R}^n$ endowed with its standard scalar product, let $\mathcal{S}(a,r)$ denotes a sphere with center a and radius r. The *inversion* with respect to $\mathcal{S}(a,r)$ is the map:

$$\begin{array}{rccc} i_{a,r}: & \mathbb{R}^n \setminus \{a\} & \longrightarrow & \mathbb{R}^n \setminus \{a\} \\ & x & \longmapsto & a + r^2 . \frac{x-a}{\|x-a\|^2} \end{array}$$

It is an involutive diffeomorphism that is conformal and changes spheres into spheres. It extends to an involution $I_{a,r}$ of the one point compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ by setting $I_{a,r}(a) = \infty$ and $I_{a,r}(\infty) = a$, which is a conformal involutive diffeomorphism once $\overline{\mathbb{R}^n}$ is given its standard diffeomorphic and conformal structures. Then $I_{a,r}$ changes spheres/hyperplanes into spheres/hyperplanes. Its set of fixed points is the whole sphere $\mathcal{S}(a,r)$, and conformality implies that a sphere/hyperplane H is stable under $I_{a,r}$ if and only if H intersects $\mathcal{S}(a,r)$ orthogonally. Whenever $I_{a,r}$ changes the sphere $\mathcal{S}_{b,\rho}$ into an hyperplane H then $I_{a,r} \circ I_{b,\rho} \circ I^{-1}_{a,r}$ is the Euclidean (orthogonal) reflection with respect to H. The *Möbius group* of $\overline{\mathbb{R}^n}$ (or $\mathbb{R}^n$) is defined as the group generated by all inversions and reflections in $\overline{\mathbb{R}^n}$.

3.1.2. *The Conformal Ball Model* $\mathbb{H}^n_c$. Consider the open unit ball embedded in the hyperplane $\mathbb{R}^n \times \{0\}$ of V:

$$D^n = \{(x_1, \ldots, x_{n+1}) \in V \mid x_{n+1} = 0 \text{ and } x_1^2 + \cdots + x_n^2 = 1\}$$

and the *stereographic projection* c with respect to $-e_{n+1}$ of $\mathbb{R}^n \times \mathbb{R}_+^*$ onto $\mathbb{R}^n \times \{0\}$:

$$c: \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}_+^* & \longrightarrow & \mathbb{R}^n \times \{0\} \\ (x_1, \dots, x_{n+1}) & \longmapsto & \frac{(x_1,\dots,x_n,0)}{1+x_{n+1}} \end{array}$$

One verifies that c restricted to the hyperboloid model $\mathbb{H}^n$ is a diffeomorphism onto D^n (cf. [Rat06, BP92]). Once D^n is endowed with the pull-back metric (the Riemannian metric $\mathrm{d}s = \frac{\mathrm{d}x}{1-\|x\|^2}$) one obtains the *conformal ball model* of the hyperbolic space, that we denote by $\mathbb{H}_c^n$ (cf. Figure 2).

The (hyperbolic) hyperplanes in $\mathbb{H}_c^n$ are the intersections with D^n of the Euclidean spheres and hyperplanes in $\mathbb{R}^n \times \{0\}$ that are perpendicular to the boundary sphere $\partial\mathbb{H}_c^n := \partial\overline{D^n}$. The hyperbolic reflection with axis the hyperplane H is the restriction of the inversion with respect to the Euclidean sphere or hyperplane in $\mathbb{R}^n \times \{0\}$ containing H. It turns out that the group of isometries $\mathcal{I}(\mathbb{H}_c^n)$ is the subgroup of the Möbius group

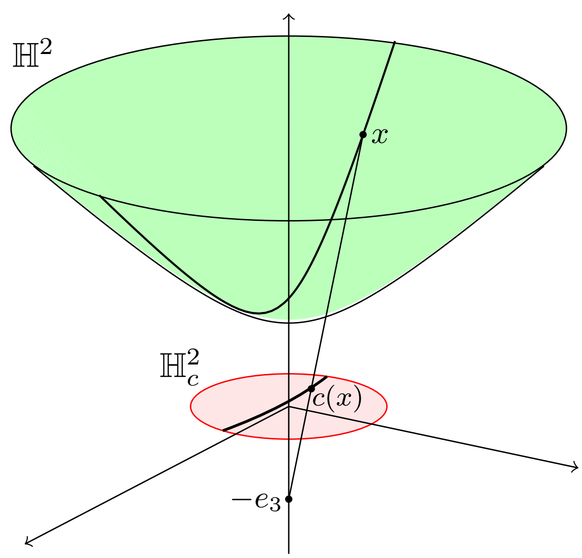

FIGURE 2. The conformal disk model.

of $\mathbb{R}^n \times \{0\}$ that leaves invariant D^n or, equivalently, generated by inversions/reflections with respect to hyperplanes/spheres that are perpendicular to the boundary. The model is conformal: the hyperbolic and Euclidean angles are the same.

The map $c \circ p : \mathbb{H}_p^n \longrightarrow \mathbb{H}_c^n$ is an isometry from the projective to the conformal ball models and a simple computation shows that:

$$c \circ p(x_1, \dots, x_n, 1) = \frac{1 - \sqrt{1 - x_1^2 - \cdots - x_n^2}}{x_1^2 + \cdots + x_n^2}.\,(x_1, \dots, x_n, 0)$$

so that it obviously extends to an homeomorphism from $\overline{\mathbb{H}_p^n} = \mathbb{H}_p^n \cup \partial\mathbb{H}_p^n$ to $\overline{\mathbb{H}_c^n} = \mathbb{H}_c^n \cup \partial\mathbb{H}_c^n$ that restricted to $\partial\mathbb{H}_p^n \longrightarrow \partial\mathbb{H}_c^n$ is the translation with vector $-e_{n+1}$.

3.1.3. *The Upper Half Space Model* $\mathbb{H}^n_u$. Consider the differentiable map:

$$u: \begin{array}{ccc} D^n & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & 2\,\dfrac{x+e_n}{\|x+e_n\|^2} - e_n \end{array}$$

One verifies (cf. [BP92], Chapter A) that u is a diffeomorphism from D^n onto the open upper half-space: $\mathbb{R}^{n-1} \times \mathbb{R}^*_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$, which, in fact, is the inversion with respect to the sphere with radius $\sqrt{2}$ and center $-e_n$ (cf. §3.1.1 and Figure 3). Once D^n is identified with the conformal ball model $\mathbb{H}^n_c$ and $\mathbb{R}^{n-1} \times \mathbb{R}^*_+$ is endowed with the pull-back metric with respect to u^{-1}, we obtain the *upper half-space model* $\mathbb{H}^n_u$ of the hyperbolic space with Riemannian metric $\mathrm{d}s^2 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2 + \mathrm{d}x_n^2/x_n^2$. The hyperplanes in $\mathbb{H}^n_u$ are euclidean half-spheres with centers on the boundary $\mathbb{R}^{n-1} \times \{0\}$ as well as vertical affine hyperplanes. The model is conformal: hyperbolic angles agree with Euclidean ones. A reflection with respect to an hyperplane H is a Euclidean reflection with respect to H (when H is a 'vertical' Euclidean hyperplane) or an inversion with respect to H (when H is a 'half-sphere').

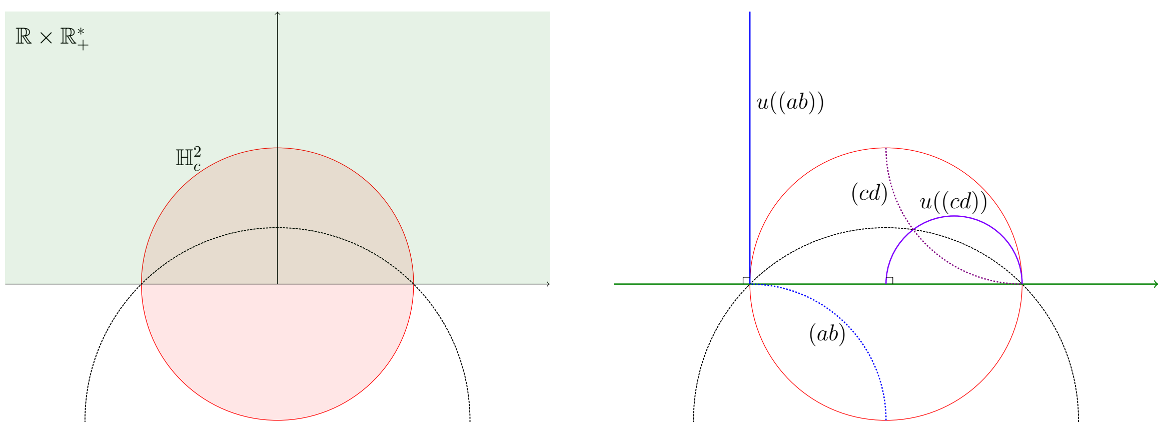

FIGURE 3. The inversion u with respect to the sphere with center $-e_n$ and radius $\sqrt{2}$ (in dash) sends the conformal disk model (in red) onto the upper half-plane model (in green). On the right side two infinite geodesics in $\mathbb{H}^2_c$, (ab) and (cd), and their images by u, which are geodesics of $\mathbb{H}^2_u$.

The group of isometries of $\mathbb{H}^n_u$ is the subgroup of the Möbius group of $\mathbb{R}^n$ that stabilizes $\mathbb{R}^{n-1} \times \mathbb{R}^*_+$, or equivalently, that one generated by inversions/reflections with respect to spheres/hyperplanes perpendicular to the boundary $\mathbb{R}^{n-1} \times \{0\}$.

The hyperbolic boundary $u(\partial\mathbb{H}^n_c)$ of $\mathbb{H}^n_u$ is the one point compactification $\partial\mathbb{H}^n_u := (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$.

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