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Paysage des variétés hyperboliques

Mémoire présenté par

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Introduction

Variétés, complexes simpliciaux et groupes

Ce manuscrit présente des contributions qui se rattachent toutes plus ou moins directement à la topologie des variétés. C'est un vaste sujet à l'interface de la topologie algébrique, de la géométrie différentielle et de la théorie géométrique des groupes ; on n'en discutera ici qu'une petite partie mais je vais commencer par une dicussion un peu générale bien qu'orientée vers les thèmes plus précis qui seront discutés dans les sections suivantes de cette introduction. Pour présenter vaguement les grandes lignes de ce mémoire on pourrait dire que l'on s'intéresse à certaines familles de variétés et au problème de les classer à isomorphisme près—comme il s'agit d'ensembles dénombrables, on entend par là étudier les manières intelligibles de les énumérer, les comparer entre elles et en étudier divers invariants. On peut aussi de manière plus imagée parler du paysage décrit par ces variétés et des différents angles sous lesquels on peut le regarder.

Les objets centraux seront donc les variétés ; dans le cadre qui nous intéressera cela signifiera variétés lisses. On ne considèrera le plus souvent que celles qui seront compactes ou au moins intérieur d'une variété compacte à bord. Pour nous, les variétés seront réalisées concrètement de deux façons : comme complexes simpliciaux (une description combinatoire) et comme quotients d'un espace explicite par des groupes y agissant discrètement (description que l'on peut qualifier d'algébrique, vu que pour nous les espaces considérés seront des espaces homogènes).

La première description est la plus élémentaire et la plus utilisée pour représenter en pratique les variétés. En effet, il est bien connu que toute variété lisse est homéomorphe à un complexe simplicial, et c'est donc une manière commode de présenter une variété par un ensemble fini de données. La description combinatoire d'une variété permet aussi de calculer très facilement différents invariants d'homéomorphisme (groupe fondamental, groupes d'homologie en particulier). Il est donc important de savoir représenter une variété décrite d'une autre manière comme un complexe simplicial. En revanche l'ensemble des complexes simpliciaux qui sont homéomorphes à des variétés (même lisses) est essentiellement inaccessible à la description ; par exemple il n'existe pas (en dimension au moins 6) d'algorithme permettant de décider si un complexe donné est homéomorphe à une variété. Il est aussi bien connu qu'il ne peut pas exister d'algorithme décidant si deux variétés lisses sont homéomorphes l'une à l'autre. On peut quand même en principe donner une énumération d'une liste de complexes simpliciaux tel que toute variété lisse soit homéomorphe à l'un deux (qui contiendra forcément des doublons).

La seconde description est le mieux illustrée par l'exemple des variétés hyperboliques[1]. Si l'on fixe une dimension $d \geq 2$, le revêtement universel commun à ces variétés, l'espace hyperbolique de dimension d, peut être vu comme l'intérieur de la boule unité dans $\mathbb{R}^d$ muni d'une structure additionnelle que l'on va maintenant décrire. Certaines transformations conformes du bord de la boule (les "transformations de Möbius") qui se prolongent de manière rigide à l'intérieur, forment

[1] Il faudrait préciser "réelles" vu qu'il existe aussi des variétés hyperboliques complexes, quaternioniques voire octonioniques ; cependant, vu que dans ce mémoire je ne traiterai en détail que le cas réel je préfère oublier le qualificatif supplémentaire pour une meilleure lisibilité.

un groupe G agissant proprement et transitivement sur l'espace hyperbolique : les stabilisateurs sont compacts, en fait conjugués dans G au sous-groupe des isométries de la sphère. Tout sous-groupe discret, cocompact et sans torsion de G agit donc librement, proprement discontinûment et de manière cocompacte sur l'espace hyperbolique, c'est-à-dire que le quotient en est une variété lisse compacte. Il ne semble pas évident que de tels groupes existent, mais on peut donner une description plus algébrique du groupe G (et de l'espace hyperbolique) qui permet d'utiliser des méthodes arithmétiques pour construire de tels sous-groupes ; on la détaillera en plus grande généralité dans la section suivante. L'étude des variétés hyperboliques, en particulier de celles obtenues par les méthodes arithmétiques, est l'objet principal de ce mémoire, même si l'on se placera souvent dans un cadre plus général.

On peut aussi commencer à s'intéresser aux variétés d'un point de vue géométrique, c'est-à-dire en étudiant les structures riemanniennes dont elles peuvent être munies. Par exemple, l'espace hyperbolique introduit au paragraphe précédent supporte une métrique à courbure sectionelle constante égale à -1 qui est préservée par l'action des transformations de Möbius. Touts ses quotients sont donc munis d'une telle métrique, et réciproquement toute variété riemannienne complète connexe dont les courbures sectionelles sont toutes égales à -1 est un quotient de l'espace hyperbolique. En général, l'ensemble des variétés supportant une métrique riemanienne à courbures sectionelles négatives (non constantes) est encore mal compris en dimensions 4 et plus. Les principaux exemples sont les espaces localement symétriques qui généralisent les variétés hyperboliques et que l'on décrira en détail dans la section suivante. Plusieurs autres constructions sont néanmoins connues, et la théorie de la structure globale des variétés à courbure négative est assez bien développée. Dans ce mémoire et dans les articles qu'il résume, on se limite aux seuls espaces localement symétriques pour lesquels on a en général des résultats plus précis.

Une propriété importante de ces espaces à courbure négative est qu'ils sont asphériques, c'est-à-dire que leur type d'homotopie est complètement déterminé par leur groupe fondamental ; conjecturalement (conjecture de Borel) c'est vrai aussi pour le type d'homéomorphisme, un point subtil étant que ça ne l'est plus pour leur type de difféomorphisme. Ils sont donc utiles pour l'étude des groupes, en particulier de leur homologie. C'est un aspect qu'on ne développera pas dans la suite mais qui est une motivation importante, en particulier pour ses applications à la théorie géométrique des groupes (principalement comme source d'exemples). Notons que la classe des variétés asphériques est loin de se limiter aux variétés riemaniennes à courbure négative, de nombreux autres exemples sont connus (par exemple quotients compacts de groupes de Lie résolubles, variétés CAT(0) non-lissables). Cette classe, ainsi que la classe des groupes dénombrables qui sont leurs groupes fondamentaux, sont très mal comprises en dimensions 4 et plus.

Espaces localement symétriques et réseaux arithmétiques

Une autre manière de réaliser l'espace hyperbolique est la suivante : on peut montrer que le groupe de transformations de Möbius est isomorphe au groupe de Lie $\mathrm{PO}(d,1)$, en identifiant la sphère de dimension $(d-1)$ à la projectivisation du cône isotrope d'une forme quadratique de signature $(d,1)$ (l'espace hyperbolique est alors identifié au projectivisé du cône des vecteurs négatifs). Le stabilisateur d'un point intérieur de l'espace hyperbolique est alors conjugué à $\mathrm{PO}(d)$, de sorte que l'espace hyperbolique est isomorphe à l'espace homogène G/K avec $G = \mathrm{PO}(d,1)$ et $K = \mathrm{PO}(d)$. On peut aussi retrouver ainsi la métrique invariante, vu qu'il n'existe à proportionalité près qu'une seule métrique euclidienne invariante par l'action adjointe de $\mathrm{PO}(d)$ sur l'orthogonal $\mathfrak{p}$ à $\mathfrak{so}(d)$ pour la forme de Killing sur l'algèbre de Lie $\mathfrak{so}(d,1)$ ($\mathfrak{p}$ est identifié à l'espace tangent à G/K en la classe $e_G K$), que l'on peut alors transporter sur les espaces tangents en d'autres points.

Une classe d'espaces généralisant directement cette construction des variétés hyperboliques est donnée par les espaces localement symétriques, que l'on va maintenant décrire. Étant donné un groupe de Lie connexe G et un sous-groupe fermé K, le quotient $X = G/K$ est une variété lisse, admettant une métrique G-invariante si K est compact. De plus tout sous-groupe Γ de G agit sur X par difféomorphismes ; nous voulons donc trouver de tels Γ dont l'action est de plus propre et libre. C'est le plus simple quand G est semisimple non compact (par exemple $G = \mathrm{SL}_n(\mathbb{R})$ ou G est le groupe orthogonal d'une forme quadratique isotrope) et K est compact maximal dans G ; il faut et suffit alors que Γ soit discret dans G et sans torsion. Dans ce cas X est appelé un espace symétrique riemannien de type non-compact, sans facteur euclidien[2], et les quotients $\Gamma\backslash X$ sont des espaces localement symétriques.

Pour obtenir des variétés fermées il faut de plus que le quotient $\Gamma\backslash X$ soit compact ; avec cette contrainte supplémentaire il n'est pas évident que l'on puisse trouver un tel Γ. Pour diverses raisons (motivées par les applications mais aussi internes à la théorie) il est commode d'affaiblir l'hypothèse que Γ est cocompact en demandant seulement que la mesure de Haar à gauche de G induise une mesure finie sur le quotient $\Gamma\backslash G$ (on dit que Γ est de covolume fini). Un sous-groupe Γ qui est discret et de covolume fini dans G est appelé un réseau de G. Tout réseau contient un sous-groupe d'indice fini sans torsion, et la construction d'espaces localement symétriques de volume fini (respectivement cocompacte) est donc équivalente à celle de réseaux (respectivement de réseaux cocompacts).

Une construction systématique de tels sous-groupes est donnée par les groupes arithmétiques, qui grosso modo généralisent l'exemple $\mathrm{SL}_m(\mathbb{Z}) \subset \mathrm{SL}_m(\mathbb{R})$. Par exemple si q est une forme quadratique (isotrope) sur $\mathbb{R}^n$ à coefficients rationnels, les transformations préservant q et à coefficients entiers forment un réseau[3] dans le groupe orthogonal de q. En général, étant donnés un corps de nombres k (dont on notera $\mathbb{Z}_k$ l'anneau des entiers) et un k-groupe $\mathbf{G}$ muni d'une k-représentation fidèle $\rho : \mathbf{G} \to \mathrm{GL}_N$ on a un groupe des points entiers $\mathbf{G}(\mathbb{Z}_k) = \{g \in \mathbf{G}(k) : \rho(g) \in \mathrm{GL}_N(\mathbb{Z}_k)\}$. On peut alors définir un sous-groupe arithmétique de $\mathbf{G}(k)$ comme n'importe quel sous-groupe dont l'intersection avec $\mathbf{G}(\mathbb{Z}_k)$ est d'indice fini dans les deux groupes (on dit alors qu'ils sont commensurables) ; cette définition ne dépend pas du plongement ρ, contrairement au groupe $\mathbf{G}(\mathbb{Z}_k)$. S'il existe une surjection à noyau compact du groupe de Lie $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$ vers G, on dit alors qu'un sous-groupe de G qui est commensurable à l'image d'un sous-groupe arithmétique de $\mathbf{G}(k)$ dans G est un réseau arithmétique de G, et on appelle le corps k son corps de traces. Cette nomenclature est justifiée par le fait qu'un sous-groupe obtenu de cette manière est bien un réseau (théorème de Borel–Harish-Chandra), et k est le corps engendré par les traces des éléments de Γ dans la représentation adjointe de G (un résultat observé par Vinberg). De plus on sait qu'il est cocompact si et seulement si le groupe $\mathbf{G}$ est k-anisotrope.

En général, il peut exister d'autres réseaux dans G que les réseaux arithmétiques ; cependant ceci ne peut arriver essentiellement que pour une liste assez restreinte de groupes, les groupes orthogonaux $\mathrm{SO}(n, 1)$ et les groupes unitaires $\mathrm{SU}(n, 1)$ (et tous les groupes leur étant isogènes), d'après le théorème d'arithméticité de Margulis et ses extensions par Corlette et Gromov–Schoen. C'est un fait fondamental dans l'étude des espaces localement symétriques, et qui justifie en partie que l'on s'intéresse particulièrement aux espaces localement symétriques arithmétiques. On discutera plus loin des variétés non-arithmétiques, uniquement dans le cas des espaces hyperboliques réels[4].

[2]Cette nomenclature reflète une caractérisation géométrique des métriques riemanniennes G-invariantes sur X ; dans la suite on supposera que tous les espaces symétriques considérés sont tels.

[3]Ce réseau est cocompact si et seulement si la forme q est $\mathbb{Q}$-anisotrope, ce qui ne peut pas être le cas à partir de la dimension 5 ; on peut par exemple construire des réseaux cocompacts en considérant des formes définies sur des extensions finies de $\mathbb{Q}$.

[4]Le cas hyperbolique complexe est encore plus mal compris, en particulier on ne sait pas s'il existe des réseaux

Si l'on fixe l'espace symétrique X, on sait classer assez grossièrement ses quotients arithmétiques. En effet, étant donné un corps de nombres k, la théorie générale des groupes algébriques permet[5] de classer les k-groupes $\mathbf{G}$ tels que $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$ admette une surjection à noyau compact vers le groupe d'isométries G de X et partant, les classes de commensurabilité de réseaux arithmétiques de G de corps de traces k. La question de comprendre la structure de l'ensemble des groupes à l'intérieur d'une classe de commensurabilité est plus nébuleuse. Il y a une sous-classe bien comprise, formée des sous-groupes de congruence qui sont grosso modo les élements de la classe de commensurabilité que l'on peut décrire à partir de la seule donnée de $\mathbf{G}$ (pour les sous-groupes de $\mathbf{G}(k)$ il s'agit exactement de ceux qui sont fermés pour la topologie induite sur $\mathbf{G}(k)$ par le plongement de k dans ses adèles finies). La plupart (mais pas la totalité) des questions qui nous intéressent dans ce mémoire sont le mieux posées dans ce cadre plus limités des réseaux de congruence.

Une propriété importante des espaces localement symétriques de volume fini est que, sauf en dimension 2, la métrique riemannienne localement symétrique sur ces espaces est complètement déterminée par leur type d'homotopie (c'est le théorème de rigidité forte, ou de Mostow–Prasad). De plus, en dimension au moins 4, le volume donne une manière naturelle d'énumérer les espaces localement symétriques (théorème de finitude de Wang). Par rapport aux questions générales un peu vaguement décrites au cours de la section précédente, on peut maintenant formuler une question plus précise : comment cette énumération se compare-t'elle aux manières plus topologiques de le faire? En pratique, ceci veut dire comparer le volume à des invariants numériques définis topologiquement. Les liens entre les invariants riemanniens (en particulier le volume) et les invariants topologiques définis à partir de structures combinatoires dans le cadre des espaces localement symétrique arithmétiques sont le premier thème de ce mémoire, que l'on étudiera en profondeur dans les deux premiers chapitres.

Notons aussi que bien que le volume ne permette pas d'énumérer les variétés hyperboliques en dimension 3 (c'est un phénomène provenant de la construction de telle variétés via les remplissages de Dehn hyperboliques), c'est le cas si on se limite aux variétés arithmétiques (pour lesquels une formule du volume faisant intervenir de manière transparente les invariants arithmétiques permet de le démontrer directement). Cette dimension nous intéressera donc aussi malgré ses particularités, en particulier une autre motivation des travaux que l'on présente est la question suivante posée par Thurston dans sa liste de problèmes[6] :

> *Find topological and geometric properties of quotient spaces of arithmetic subgroups of $PSL_2\mathbf{C}$. These manifolds often seem to have special beauty.*

Notons enfin que la topologie des variétés localement symétriques arithmétiques est liée, via le programme de Langlands et ses dérivées, à la théorie des représentations du groupe de Galois absolu. C'est une motivation importante pour une partie des résultats de ce mémoire mais on n'en parlera pas plus ici.

Volume et complexité topologique

Maintenant que le cadre de ce mémoire est bien posé, on va commencer à discuter des résultats qui y sont présentés. Le premier thème, évoqué ci-dessus, est celui des liens entre divers invariants topologiques combinatoires (accessibles à partir d'une triangulation) et le volume riemannien. Pour une variété lisse, on peut définir de façon grossière sa complexité topologique comme le

non-arithmétiques dans les groupes $\mathrm{SU}(n,1)$ pour $n \geq 4$.

[5] Voir par exemple J. Tits, *Classification of algebraic semisimple groups*, Proc. Symp. Pure Math. 9 (1966).

[6] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Am. Math. Soc. 6 (1982)

nombre minimal de simplexes maximaux dans un complexe simplicial qui lui est homéomorphe. Le principe de proportionalité pour le volume simplicial, et sa positivité pour les espaces localement symétriques, montrent que le volume riemannien est à une constante près une borne inférieure pour cette complexité pour les espaces localement symétriques de dimension fixée. Un des principaux résultats discutés dans ce mémoire est le théorème suivant, obtenu dans un travail en commun avec Sebastian Hurtado et Mikołaj Frączyk.

Théorème A (Voir le théorème 1.30 dans le chapitre 1)**.** *Soient X un espace symétrique de type non-compact sans facteur euclidien et G son groupe d'isométries. Il existe des constantes A, B telles que pour toute variété riemannienne M obtenue comme quotient de X par un réseau arithmétique de G, il existe un complexe simplicial équivalent en homotopie à M ayant au plus $A\,\mathrm{vol}(M)$ simplexes maximaux et où chaque sommet appartient à au plus B simplexes.*

La conclusion légèrement affaiblie que le complexe n'est qu'équivalent en homotopie à M plutôt qu'homéomorphe est largement suffisante en pratique (noter que la conjecture de Borel est connue pour de telles variétés M). En utilisant des résultats sur les triangulations de Delaunay en courbure variable[7], on peut d'ailleurs améliorer le théorème pour obtenir des complexes homéomorphes à M. La question de savoir si un complexe équivalent en homotopie à M doit avoir au moins $a\,\mathrm{vol}(M)$ simplexes (pour une constante a ne dépendant que de X) est encore ouverte en général ; pour les espaces symétriques ayant des invariants L^2 positifs (ce qui inclut en particulier toutes les variétés hyperboliques réelles et complexes) on sait que c'est le cas (voir les remarques suivant le Théorème 1.30). Enfin, notons qu'en dimension 3 il est bien connu que la conclusion de ce théorème est fausse lorsqu'on considère aussi des variétés non-arithmétiques. En dimensions supérieures, rien n'est connu et il semble plausible qu'un résultat semblable soit valable sans l'hypothèse d'arithméticité (cependant les techniques que nous utilisons sont rendues en grande partie caduques par l'absence de celle-ci).

Les invariants topologiques les plus accessibles au calcul pour une variété triangulée M sont sans doute les groupes d'homologie $H_i(M, \mathbb{Z})$. Ce sont aussi des invariants assez puissants en pratique : par exemple, il a été observé empiriquement[8] que l'on peut souvent distinguer deux variétés de dimension 3 en calculant l'homologie de leurs revêtements de petit degré. Il est donc naturel de s'y intéresser dans le cadre des problématiques étudiées ici. Un autre résultat obtenu avec Hurtado et Frączyk donne des bornes a priori sur la taille des nombres de Betti $b_i(M) = \dim(H_i(M, \mathbb{Z}) \otimes \mathbb{Q})$.

Théorème B (Voir le théorème 1.34 dans le chapitre 1)**.** *Soient X, G comme dans l'énoncé précédent ; il existe une constante $\beta^{(2)}(X) \geq 0$ et une fonction $r(v) =_{v \to +\infty} o(v)$ telle que pour tout réseau arithmétique de congruence Γ dans G on ait*

$$\begin{array}{l} b_i(\Gamma\backslash X) - \beta^{(2)}(X)\,\mathrm{vol}(\Gamma\backslash X) \leq r(\mathrm{vol}(\Gamma\backslash X)) \ \textit{si } i = \dim(X)/2; \\ b_i(\Gamma\backslash X) \leq r(\mathrm{vol}(\Gamma\backslash X)) \ \textit{sinon}. \end{array}.$$

Le théorème ne s'applique pas en général aux réseaux arithmétiques qui ne sont pas de congruence ; par exemple beaucoup de variétés hyperboliques arithmétiques ont des suites de revêtements finis où des nombres de Betti en-dehors du degré médian croissent linéairement en le degré. Le théorème précédant sur les triangulations donne une borne en $O(\mathrm{vol}(M))$ pour tous les nombres de Betti d'un espace localement symétrique arithmétique, ce qui est vrai aussi

[7]Voir la proposition 57 de J.-D. Boissonnat, R. Dyer, A. Ghosh et M. Wintraecken, *Local criteria for triangulating general manifolds*, SoCG 2018.

[8]Voir par exemple §3 dans S. Lins, *A tougher challenge to 3-manifold topologists and group algebraists*, `https://arxiv.org/abs/1305.2617`.

pour les espaces symétriques non-arithmétiques et plus généralement pour les variétés à courbure négative pincée normalisée (c'est un théorème assez ancien dû à M. Gromov[9]).

Notons que le théorème B ne dit rien sur des bornes inférieures pour les nombres de Betti en-dehors du degré médian. Dans la généralité où il est énoncé, les comportements possibles sont en effet très variés : on dispose d'exemples de suites de réseaux de congruence où les nombres de Betti sont identiquement nuls, aussi bien que d'exemples où ils croissent comme une puissance (d'exposant < 1) du volume. Les outils utilisés pour démontrer ces bornes inférieures proviennent de la théorie des formes automorphes[10] ou de la géométrie[11], mais sont toujours liés à la structure particulière des espaces symétriques et des réseaux arithmétiques considérés.

Les nombres de Betti ne suffisent pas à décrire complètement les groupes d'homologie $H_i(M, \mathbb{Z})$ à coefficients dans $\mathbb{Z}$ puisqu'ils ont a priori un sous-groupe de torsion non-trivial ; pour un groupe abélien A on notera A_{tors} son sous-groupe de torsion. Comme pour les nombres de Betti, le théorème A implique une borne générale pour l'ordre des groupes de torsion $H_i(M, \mathbb{Z})_{\mathrm{tors}}$. Plus précisément, il implique qu'il existe une constante C, ne dépendant que de X, telle que pour tout quotient arithmétique $M = \Gamma\backslash X$ on ait

$$\log |H_i(M, \mathbb{Z})_{\mathrm{tors}}| \leq C \operatorname{vol}(M),$$

ce qui est aussi connu pour les variétés à courbure négative pincée[12]. En revanche, un résultat précis comme le théorème B n'est pas connu. Une conjecture exactement analogue à ce théorème (formulée par Nicolas Bergeron et Akshay Venkatesh[13] est présentée ci-dessous (cf. §2.2). Cette dernière semble encore hors de portée même dans des cas particuliers ; on discute quand même de résultats analogues lorsqu'on considère une homologie à coefficients dans des systèmes locaux non-triviaux provenant de certaines représentations de G. Ces résultats sont formulés uniquement pour des suites de revêtements de congruence d'un espace de base fixé. On se concentre en particulier sur le cas où ce dernier est une orbi-variété hyperbolique non-compacte en dimension 3, la non-compacité compliquant nettement l'étude par des outils analytiques. Par exemple, on a le résultat suivant (la partie compacte vient d'un travail en commun avec Abért–Bergeron–Biringer–Gelander–Nikolov–Samet, mais l'argument est essentiellement celui de Bergeron–Venkatesh[14]).

Théorème C (Voir les théorèmes 2.1 et 2.4 dans le chapitre 2)**.** *Soient* Γ *un réseau arithmétique dans* $\mathrm{PSL}_2(\mathbb{C})$ *et* $(\Gamma_n)_{n\geq 1}$ *une suite de sous-groupes de congruence de* Γ.

1. *Si* Γ *est cocompact et* L *est un réseau dans* $\mathfrak{sl}_2(\mathbb{C})$ *stable par l'action adjointe de* Γ *(qui existe si et seulement si le corps de traces de* Γ *est quadratique) alors*

$$\lim_{n\to+\infty} \frac{\log |H_i(\Gamma_n, L)_{\mathrm{tors}}|}{\operatorname{vol}(\Gamma_n\backslash\mathbb{H}^3)} = \begin{cases} \frac{13}{6\pi} & \textit{si } i = 1 \\ 0 & \textit{sinon.} \end{cases}$$

2. *Si* Γ *est un groupe de Bianchi* $\mathrm{PSL}_2(\mathbb{Z}_k)$*,* k *un corps quadratique imaginaire,* $(\Gamma_n)_{n\geq 1}$ *est une suite de réseaux de congruence dans* Γ *et* L *un réseau dans* $\mathfrak{sl}_2(\mathbb{C})$ *stable par l'action*

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[9]Voir W. Ballmann, M. Gromov, V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, 61, Birkhäuser 1985.

[10]Voir par exemple S. Marshall, *Endoscopy and cohomology growth on U(3)*, Compos. Math. 150, No. 6 (2014).

[11]Voir par exemple S. Kionke, *Lefschetz numbers of symplectic involutions on arithmetic groups*, Pac. J. Math. 271, No. 2 (2014).

[12]Voir U. Bader, T. Gelander, R. Sauer, *Homology and homotopy complexity in negative curvature*, JEMS 22, No. 8 (2020).

[13]N. Bergeron, A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, JIMJ 12, No. 2 (2013).

[14]Ibid.

adjointe de Γ*, alors*

$$\limsup_{n\to+\infty} \frac{\log|H_i(\Gamma_n, L)_{\mathrm{tors}}|}{\mathrm{vol}(\Gamma_n\backslash\mathbb{H}^3)} \leq \begin{cases} \frac{13}{6\pi} & si\ i=1 \\ 0 & sinon. \end{cases}$$

Le cas cocompact se généralise à tous les corps de traces ; on peut aussi remplacer la représentation adjointe par n'importe laquelle de ses puissances symétriques. L'énoncé plus faible dans le cas non-cocompact (noter que tout réseau arithmétique non-uniforme dans $\mathrm{PSL}_2(\mathbb{C})$ est commensurable à un groupe de Bianchi) est dû à un obstacle de nature arithmétique qui n'apparaît pas dans le cas cocompact. Une borne inférieure pour des suites (Γ_n) particulières a été donnée par J. Pfaff[15].

Une autre manière de considérer la torsion dans l'homologie est de fixer une variété hyperbolique arithmétique et de considérer des systèmes locaux de coefficients de rang tendant vers l'infini. Dans ce cadre, on peut faire des conjectures similaires à celles de Bergeron–Venkatesh. Dans un travail avec Pfaff, on démontre le résultat suivant pour les groupes de Bianchi.

Théorème D (Voir le théorème 2.7 dans le chapitre 2)**.** *Soit k un corps quadratique imaginaire et soit L_n un réseau stable sous $\mathrm{PSL}_2(\mathbb{Z}_k)$ dans la n-ième puissance symétrique de $\mathfrak{sl}_2(\mathbb{C})$. On fixe $\varepsilon > 0$, si Γ est un sous-groupe de congruence principal dans $\mathrm{PSL}_2(\mathbb{Z}_k)$ de niveau assez élevé (en fonction de k, ε) et $M = \Gamma\backslash\mathbb{H}^3$ alors*

$$\liminf_{n\to\infty}\left(\frac{\log|H_1(\Gamma, L_n)_{\mathrm{tors}}|}{n^2}\right) \geq \left(\frac{1}{2} - \varepsilon\right)\frac{\mathrm{vol}(M)}{2\pi}$$

et

$$\limsup_{n\to\infty}\left(\frac{\log|H_1(\Gamma, L_n)_{\mathrm{tors}}|}{n^2}\right) \leq (2+\varepsilon)\frac{\mathrm{vol}(M)}{2\pi}.$$

Le résultat optimal serait que la limite de $\frac{\log|H_1(\Gamma,L_n)_{\mathrm{tors}}|}{n^2}$ existe et vaut $\frac{\mathrm{vol}(M)}{2\pi}$.

Mentionnons aussi quelques résultats sur l'homologie qui ne seront plus évoqués dans ce mémoire. Avant d'être traitées dans le cadre des variétés arithmétiques dans leur ensemble, les questions asymptotiques sur les nombres de Betti ont d'abord été étudiées pour les suites de revêtement d'un complexe simplicial (ou CW) fixé. Le résultat séminal dans ce contexte est le théorème d'approximation de Lück[16], où l'on considère une tour de revêtements telle que l'intersection des sous-groupes correspondants est un sous-groupe distingué et on cherche à relier l'asymptotique des nombres de Betti dans la suite en fonction du degré du revêtement aux nombres de Betti L^2 du revêtement galoisien infini. Des résultats similaires pour la torsion ont été démontrés dans le cas où les revêtements sont abéliens[17], mais au-delà rien n'est connu en général.

Il est aussi naturel de se demander si des résultats similaires au théorème B sont valides pour les nombres de Betti en caractéristique positive. C'est un problème encore largement ouvert, en particulier il n'y a pas de résultat aussi général, ni même dans les tours de revêtements fini. En revanche dans certains contextes particulier on a des résultats plus précis, par exemple pour les nombres de Betti modulo p dans les tours p-adiques analytiques[18].

[15] J. Pfaff, *Exponential growth of homological torsion for towers of congruence subgroups of Bianchi groups*, Ann. Global Anal. Geom. 45, No. 4 (2014).

[16] W. Lück, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. 4, No. 4 (1994).

[17] T. Lê, *Homology torsion growth and Mahler measure*, Comment. Math. Helv. 89, No. 3 (2014).

[18] Voir par exemple N. Bergeron, W. Lück, R. Sauer, *On the growth of Betti numbers in p-adic analytic towers*, Groups Geom. Dyn. 8, No. 2 (2014).

Enfin, il existe quelques autres approches bien étudiées de la notion de complexité topologique, en particulier en petites dimensions[19]. En dimension 3, on peut aussi décrire les variétés en utilisant des constructions alternatives, par exemple les scindements de Heegaard qui définissent le genre de Heegaard d'une variété ; pour les variétés hyperboliques, on peut aussi définir un autre genre en considérant les revêtements finis obtenus comme tores de suspension d'une surface. Ces invariants ne sont pas propres par rapport au volume ou à la complexité topologique, cependant ils le deviennent lorsqu'on ne considère que des variétés hyperboliques arithmétiques de congruence[20]. Une autre complexité "faible" est donnée par le rang (c'est-à-dire le nombre minimal de générateurs) du groupe fondamental. Ce dernier est majoré par une fonction linéaire du volume, mais contrairement aux deux précédents invariants on ne connaît pas de borne inférieure pour les variétés de congruence en général. Notons que toutes ces complexités ne semblent pas aisément calculables en pratique pour une variété donnée, au-delà des exemples de petite taille. En théorie des groupes, le calcul du rang est un problème indécidable pour les présentations finies de groupes[21], et même sur des familles spécifiques, on ne connait en général pas d'algorithme pour le calculer[22].

Géométrie globale des espaces localement symétriques arithmétiques et sous-groupes aléatoires invariants

Les résultats topologiques exposés ci-dessus sont démontrés à l'aide d'outils géométriques. Pour comprendre la structure géométrique globale d'un espace localement symétrique la première étape est d'étudier sa décomposition en parties mince et épaisse. Plus précisément, pour M une variété riemannienne et $R > 0$, on note $M_{\leq R}$ l'ensemble des points de M autour desquels la boule de rayon R n'est pas plongée (la partie R-mince), et $M_{>R}$ son complément (la partie R-épaisse). Si M est un quotient de volume fini d'un expace symétrique X alors il existe un $\varepsilon > 0$ ne dépendant que de X tel que la topologie des composantes de $M_{\leq\varepsilon}$ est bien comprise[23]. La plupart de l'information topologique sur M est alors contenue dans $M_{>\varepsilon}$, dont la structure géométrique locale plus simple permet l'utilisation de certaines techniques de manière plus performante. Par exemple, on y a un bon contrôle sur le noyau de la chaleur, ou on peut construire des triangulations en utilisant le nerf d'un revêtement par boules de rayon constant, ce qui permet déjà de démontrer certaines estimées grossières sur les nombres de Betti, ou sur le rang du groupe fondamental. Ces techniques ne sont cependant pas suffisantes pour démontrer le résultat plus précis des théorèmes B (qui n'est d'ailleurs pas vrai en général) ou A (qui n'est pas vrai au moins en dimension 3 dans le cas non-arithmétique).

Pour l'étude du noyau de la chaleur sur une variété compacte M (et partant de ses nombres de Betti), on a besoin d'étudier les parties $M_{\leq R}$ pour R arbitrairement grand. En effet, en

[19] Par exemple dans S. V. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Appl. Math. 19, No. 2 (1990).

[20] On peut le voir comme une conséquence des propriétés d'expansion de celles-ci, cf. M. Lackenby, *Heegaard splittings, the virtually Haken conjecture and property (τ)*, Invent. Math. 164, No. 2 (2006). C'est aussi un corollaire des résultats de convergence exposés dans la section suivante (qui sont dûs à M Frączyk dans ce cadre).

[21] Ceci suit de l'indécidabilité de la reconnaissance du groupe trivial, vu que si on sait qu'un groupe est de rang 1 ou 0 il est facile de décider s'il est trivial.

[22] C'est quand même possible dans certains cas ; sans surprise, pour les groupes libres et de surface. C'est aussi possible pour les groupes kleinéens, voir I. Kapovich, R. Weidmann, *Kleinian groups and the rank problem*, Geom. Topol. 9 (2005).

[23] Ce sont des cusps (produit d'une variété plate avec une demi-droite) ou des tubes (vosinage tubulaire d'une géodésique fermée), d'après le "lemme de Margulis". Pour les espaces symétriques ce dernier est une conséquence de l'existence de "voisinages de Zassenhaus", voir par exemple le chapitre VIII de M. Raghunathan, *Discrete subgroups of Lie groups*, Springer

utilisant des techniques standard[24] on peut en principe contrôler la trace du noyau de la chaleur par un terme en $\frac{\mathrm{vol}(M_{\leq R})}{\mathrm{vol}(M)} + o_R(1)\,\mathrm{vol}\,M$; si on contrôle le terme $\frac{\mathrm{vol}(M_{\leq R})}{\mathrm{vol}(M)}$ pour n'importe quel R fixé, on a donc des estimées sur les nombre de Betti. En pratique, il y a de nombreuses complications dans la mise en œuvre (surtout liées aux parties à petit rayon d'injectivité) mais ce schéma s'applique plus ou moins tel quel au moins dans le cadre des variétés hyperboliques réelles.

Les résultats principaux dans cette veine que l'on utilise dans la démonstration des théorèmes A et B (et des cas particuliers antérieurs pour le théorème C) sont regroupés dans le théorème suivant, qui est démontré dans cette généralité dans le travail avec Frączyk et Hurtado.

Théorème E (Voir les théorèmes 1.1 et 1.4 dans le chapitre 1)**.** *Soient X un espace symétrique et G son groupe d'isométries.*

1. *Il existe des $\delta, \eta > 0$ tel que pour tout réseau arithmétique Γ dans G de corps de traces k, et $M = \Gamma\backslash X$, on a*
$$\mathrm{vol}\left(M_{\leq \eta[k:\mathbb{Q}]}\right) \leq e^{-\delta[k:\mathbb{Q}]}\,\mathrm{vol}(M).$$
2. *Pour tout $R > 0$ il existe une fonction $t_R : [0, +\infty[\to [0, +\infty[$ telle que $\lim_{v\to+\infty} t_R(v)/v = 0$ et pour tout réseau arithmétique de congruence Γ dans G et $M = \Gamma\backslash X$, on a*
$$\mathrm{vol}\,(M_{\leq R}) \leq t_R(\mathrm{vol}(M)).$$

Le premier résultat donne un contrôle qualitatif sur le volume des parties minces, suffisant pour que l'on puisse l'utiliser en conjonction avec les bornes de Dobrowolski[25] sur la systole des espaces localement symétriques arithmétiques pour construire des revêtements par boules plongées avec un nombre de boules en $O(\mathrm{vol})$, et en déduire par une construction combinatoire standard (le nerf d'un tel recouvrement est équivalent en homotopie à l'espace recouvert) des triangulations qui satisfont les conclusions du théorème A. On peut aussi l'utiliser pour estimer les nombres de Betti en fonction du degré $[k : \mathbb{Q}]$ du corps de traces, en utilisant la théorie fine des représentations unitaires de G.

Le deuxième résultat implique le théorème B via un résultat général sur les nombres de Betti dû à Abért–Bergeron–Biringer–Gelander[26], dont la démonstration utilise la décomposition mince/épaisse de manière plus combinatoire. On peut aussi en principe utiliser les deux parties du théorème E en conjonction pour appliquer les techniques analytiques décrites, ci-dessus mais vu le résultat général déjà existant nous n'avons pas rédigé cette démonstration[27].

La condition que le quotient $\frac{\mathrm{vol}(M_{\leq R})}{\mathrm{vol}(M)}$ tende vers 0 dans une suite d'espaces localement isométriques à X peut être interprétée comme définissant une notion de convergence vers X. Cette dernière est appelée la "convergence de Benjamini–Schramm" d'après le travail[28] de ces auteurs sur une notion correspondante sur les graphes ; dans le cadre des espaces localement symétriques, elle a été introduite dans un travail en commun avec Abért–Bergeron–Biringer–Gelander–Nikolov–Samet (après que son analogue discret ait été introduit par Abért–Glasner–Virág). Une observation importante est que l'on peut placer cette convergence dans un cadre plus général, en plongeant l'ensemble des espaces localement-X de volume fini dans un espace

[24] Voir par exemple H; Donnelly, *On the spectrum of towers*, Proc. Am. Math. Soc. 87 (1983).

[25] E. Dobrowolski, *On a question of Lehmer and the number of irreducible factors of a polynomial*, Acta Arith. 34 (1979).

[26] Voir *Convergence of normalized Betti numbers in nonpositive curvature*, prépublication Arxiv 1811.02520

[27] Néanmoins nous utilisons des techniques bien connues pour déduire de la première partie des bornes explicites en fonction du degré du corps de traces.

[28] I. Benjamini, O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. 6 (2001).

compact. Cet espace est de nature probabiliste ; on peut le décrire de manière algébrique (comme l'ensemble des sous-groupes aléatoires invariants de G) ou géométrique (avec les variétés localement-X unimodulaires[29]). La version algébrique est la plus simple à décrire et c'est donc elle qu'on va retenir ici : un sous-groupe aléatoire invariant de G est une variable aléatoire borélienne à valeurs dans l'espace Sub_G des sous-groupes fermés de G muni de la topologie de Chabauty, dont la loi est invariante sous l'action par conjugaison de G sur Sub_G. Par les théorèmes généraux de compacité l'ensemble de ces sous-groupes aléatoires invariants, muni de la topologie de la convergence faible-étoile des mesures, est un espace compact, que l'on notera $\mathrm{IRS}(G)$. Un réseau Γ de G définit un élément dans $\mathrm{IRS}(G)$ en prenant un conjugué aléatoire de G (ceci se justifie rigoureusement par le fait que la classe de conjugaison de Γ dans G est un G-facteur de l'espace G/Γ qui possède une mesure de probabilité G-invariante). La condition de convergence de Benjamini–Schramm vers X se réinterprète comme la convergence faible-étoile des sous-groupes aléatoires invariants vers un sous-groupe central (la masse de Dirac supportée sur un sous-groupe distingué dans G est un point de $\mathrm{IRS}(G)$).

La compacité de l'espace $\mathrm{IRS}(G)$ permet d'utiliser des arguments non-effectifs pour établir la convergence de Benjamini–Schramm. Pour ceci, on a besoin de propriétés générales des éléments de $\mathrm{IRS}(G)$; ces derniers semblent aussi être des objets d'étude intéressants par eux-mêmes et pour leurs liens avec la théorie des actions ergodiques de G et de ses réseaux. En général, on ne sait pas grand-chose sur l'espace $\mathrm{IRS}(G)$, sauf quand G n'a que des facteurs simples de rang supérieur, auquel cas il n'existe que les sous-groupes aléatoires invariants construits à partir des sous-groupes distingués et des réseaux de G (c'est une conséquence d'un théorème dû à G. Stuck et R. Zimmer[30]). Le dernier chapitre de ce mémoire présente quelques résultats en rang 1, notamment des constructions plus compliquées et des propriétés remarquables. Pour ce qui est des constructions "exotiques", le résultat principal est le suivant, obtenu dans la deuxième partie du travail avec Abért–Bergeron–Biringer–Gelander–Nikolov–Samet.

Théorème F (Voir le théorème 4.5 dans le chapitre 4)**.** *Pour tout $d \geq 2$, il existe un continuum de sous-groupes aléatoires invariants dans $\mathrm{SO}(d,1)$ pour lesquels la mesure des sous-groupes de réseaux est nulle.*

Quelques propriétés remarquables sont données par le résultat suivant (la première partie vient encore du travail avec Abért–Bergeron–Biringer–Gelander–Nikolov–Samet, la deuxième est obtenue dans un article postérieur écrit avec Biringer).

Théorème G (Voir les théorèmes 4.3 et 4.4 dans le chapitre 4)**.** *Soit μ un sous-groupe aléatoire invariant ergodique dans $\mathrm{SO}(d,1)$.*

1. *Si μ n'est pas central alors l'ensemble limite de μ-presque tout sous-groupe est $\mathbb{S}^{n-1}$.*

2. *Si $d = 2$, il existe une liste d'exactement 12 surfaces de type infini telles que si μ n'est pas central, si μ-presque tout groupe est sans torsion et préserve l'orientation, et μ n'est pas supporté sur la classe de conjugaison d'un réseau alors pour μ-presque tout H, la surface $H\backslash\mathbb{H}^2$ est homéomorphe à l'une d'elles.*

[29] On ne discutera pas de cette notion ici, cf. M. Abért, I. Biringer, *Unimodular measures on the space of all Riemannian manifolds*, à paraître à Geom. Topol., prépublication Arxiv 1606.03360

[30] G. Stuck, R. Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, Ann. Math. (2) 139, No. 3 (1994).

Variété hyperboliques génériques

On dispose d'une classification satisfaisante des variétés hyperboliques de congruence, au sens d'une description uniforme via des invariants arithmétiques[31], que l'on peut de plus en principe relier à leur description topologique (par triangulation) et à leurs invariants géométriques (en particulier le volume). Un exemple d'application de cette classification est que l'on sait donner des asymptotes assez précises sur le nombre de variétés hyperboliques arithmétiques de congruence de volume au plus v quand v tend vers l'infini[32].

Pour les variétés hyperboliques arithmétiques qui ne sont pas de congruence (celles-ci existent en toute dimension), on ne dispose pas d'une description uniforme autre que que celle donnée par leur définition, c'est-à-dire qu'elles correspondent aux sous-groupes d'indice fini dans les réseaux arithmétiques maximaux (ces derniers sont de congruence et on sait donc les décrire systématiquement). Un sous-groupe d'indice fini correspond à un morphisme vers un groupe symétrique fini, donné par l'action sur le quotient. Le problème de comptage des sous-groupes est donc fortement lié au problème suivant : étant donné un groupe Γ de type fini, estimer le nombre de morphismes de Γ vers le groupe symétrique S_n quand $n \to +\infty$. Cependant le comptage de tels morphismes est extrêmement difficile pour des groupes de présentation finie généraux : sans information supplémentaire, on ne dispose pas d'autre méthode que celle de compter les solutions dans S_n aux équations données par les relations. On dispose de réponses complète pour les groupes de surface et plus généralement les groupes fuchsiens[33] et ceci a été appliqué au comptage précis du nombres de surfaces hyperboliques arithmétiques[34].

Il n'y a, à ce jour, aucune variété hyperbolique de dimension 3 ou plus pour laquelle un équivalent asymptotique du nombre de sous-groupes d'indice n est connu. Certaines d'entre elles ont des groupes fondamentaux de présentation assez simple[35] mais la structure de la relation ne semble pas adaptée à la solution du problème ci-dessus. Une famille de réseaux hyperboliques plus intéressante de ce point de vue est donnée par les groupes de réflexions à angles droits : ils sont engendrés par des involutions parmi lesquelles les seules relations sont des commutations. Ce sont des groupes de Coxeter à angles droits ; dans un travail avec Baik et Petri, nous avons étudié la croissance des sous-groupes pour cette famille en général.

Nos résultats sont loin de donner une solution complète et ils sont assez longs à énoncer précisément ; on va quand même en donner ici une version condensée. On rappelle que si $\mathcal{G}$ est un graphe fini, on peut lui associer un groupe de Coxeter à angles droits, qui est engendré par des involutions correspondant aux sommets de $\mathcal{G}$, deux générateurs commutant s'ils sont reliés par une arête de $\mathcal{G}$. On définit un invariant de graphes $\gamma(\mathcal{G})$ de la manière suivante : à un sous-graphe induit de $\mathcal{G}$ qui est isomorphe à une union disjointe de graphes complets $\mathcal{C}_1, \dots, \mathcal{C}_n$ (autrement deux sommets du même $\mathcal{C}_i$ sont toujours adjacents dans $\mathcal{G}$ mais si $i \neq j$ alors au cun sommet de $\mathcal{C}_i$ n'est adjacent à un sommet de $\mathcal{C}_j$), on associe le poids $w = \sum_{i=1}^{n}(1 - 2^{-|\mathcal{C}_i|})$, et on définit $\gamma(\mathcal{G})$ comme le maximum de ces poids. Nos résultats ne sont pas valides pour tous les

[31] Plus précisément, c'est exactement le cas pour les réseaux maximaux dans une classe de commensurabilité, voir A. Borel, G. Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semisimple groups*, Publ. Math. IHES 69 (1989). Les sous-groupes de congruence dans ces derniers sont alors en gros associés aux sous-groupes des groupes finis de type Lie et des groupes p-adiques analytiques, et ces derniers sont relativement bien compris.

[32] Voir A. Lubotzky, N. Nikolov, *Subgroup growth of lattices in semisimple Lie groups*, Acta Math. 193, No. 1 (2004).

[33] Voir M. Liebeck, A. Shalev, *Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks*, J. Algebra 276, No. 2 (2004), et

[34] Voir M. Belolipetsky, T. Gelander, A. Lubotzky, A. Shalev, *Counting arithmetic lattices and surfaces*, Ann. Math. (2) 172, No. 3 (2010)

[35] Par exemple le complément du nœud de huit qui a deux générateurs et une relation : $\langle a, b \big| a^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1} \rangle$

graphes, on introduit la classe $\mathcal{AT}$ contenant les arbres finis, les graphes obtenus en leur ajoutant des sommets adjacents uniquement à une ou plusieurs feuilles et ceux obtenus en ajoutant encore des sommets adjacents seulement aux précédents. C'est une classe assez riche mais en aucun cas générique, et malheureusement il n'y a pas de groupes de réflexions de l'espace hyperbolique dans $\mathcal{AT}$.

Théorème H (Voir le théorème 3.9 dans le chapitre 3). *Soit Γ le groupe de Coxeter à angles droits associé à un graphe fini $\mathcal{G}$. On note $s_n(\Gamma)$ le nombre de sous-groupes d'indice n dans Γ.*

1. *Si $\mathcal{G}$ est dans la classe $\mathcal{AT}$, ou si Γ est le groupe engendré par les réflexions par rapport aux faces d'un octaèdre idéal régulier à angles droits alors*

$$\lim_{n\to+\infty} \frac{\log s_n(\Gamma)}{n \log(n)} = \gamma(\mathcal{G}) - 1.$$

2. *Si $\Gamma_1, \ldots, \Gamma_m$ sont des groupes de Coxeter à angles droits virtuellement cycliques[36], $m \geq 2$, et $\Gamma = \Gamma_1 * \cdots * \Gamma_m$ est leur produit libre, alors il existe $a, b > 0, c \in \mathbb{R}$ (effectivement calculables) tels que*

$$s_n(\Gamma) \sim a n^{c+1} e^{b\sqrt{n}} (n!)^{m-1}.$$

On présente aussi un résultat similaire sur le croissance des sous-groupes pour les groupes d'Artin à angles droits, sans limitation sur le graphe définissant le groupe ; mais bien que les liens de ces groupes avec la géométrie hyperbolique soient extrêmement intéressants dans d'autres perspectives, ils sont quasiment sans intérêt pour les problèmes qui nous intéressent ici.

Au-delà des problèmes de comptage, les propriétés topologiques et géométriques des variétés hyperboliques arithmétiques de congruence données par les théorèmes B et E ne sont pas partagées toutes les variétés hyperboliques arithmétiques. Le mieux que l'on puisse espérer est que ces propriétés soient vérifiées presque sûrement au sens suivant : la proportion parmi les variétés hyperboliques arithmétiques de volume au plus v qui vérifient ces estimées (pour une précision fixée arbitrairement proche de l'asymptote) tend vers 1 quand v tend vers l'infini. Étant donné que c'est le cas pour les réseaux arithmétiques maximaux, le problème se réduit essentiellement à étudier les sous-groupes d'indice fini dans un réseau donné. Il a été en grande partie résolu[37] en dimension 2. En dimensions supérieures le problème ne semble pas abordable dans l'état actuel des connaissances.

Enfin, il existe en toute dimension des variétés hyperboliques de volume fini qui ne sont pas arithmétiques. Le cas des dimensions 2 et 3 est à traiter à part et on n'en discutera pas ici. À partir de la dimension 4, il n'existe actuellement qu'une seule construction systématique de telles variétés, dûe originellement à Gromov et Piatetski-Shapiro[38] mais dont de nombreuses variantes adaptées à des problèmes précis ont été introduites depuis[39]. En particulier, on verra au chapitre 3 que cette construction suffit à montrer que les classes de commensurabilité non-arithmétiques sont beaucoup plus nombreuses que les arithmétiques[40].

[36] Ces groupes sont en fait tous de la forme $D_\infty \times (\mathbb{Z}/2\mathbb{Z})^r$ où D_∞ est le groupe diédral infini.

[37] Pour les groupes libres, c'est un résultat classique qui suit par exemple de B. Bollobás, *A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs*, Europ. J. Combinatorics 1 (1980) ; le cas des groupes de surface a été récemment traité dans M. Magee, D. Puder, *The Asymptotic Statistics of Random Covering Surfaces*, arXiv:2003.05892.

[38] M. Gromov, I. Piatetski-Shapiro, *Non-arithmetic groups in Lobachevsky spaces*, Publ. Math. IHES 66 (1988).

[39] Voir par exemple I. Agol, *Systoles of hyperbolic 4-manifolds*, arXiv:math/0612290.

[40] Une version plus forte du résultat ci-dessous a été démontrée peu après, voir T. Gelander, A. Levit, *Counting commensurability classes of hyperbolic manifolds*, Geom. Funct. Anal. 24, No. 5 (2014).

Théorème I (Voir le theorème 3.1 dans le chapitre 3)**.** *Pour $n \geq 1$ et $v > 0$, on note $L_{\mathrm{com}}(v)$ (respectivment $L_{a,\mathrm{com}}$) le nombre de classes de commensurabilité de réseaux (respectivement de réseaux arithmétiques) dans $\mathrm{SO}(n,1)$ qui contiennent un réseau de covolume $\leq v$. Alors on a*

$$\lim_{v \to +\infty} \frac{L_{a,\mathrm{com}}(v)}{L_{\mathrm{com}}(v)} = 0.$$

Une classification à commensurabilité près des variétés hyperboliques de volume fini (y compris non-arithmétiques) en toute dimension ≥ 4 est peut-être possible, mais sans doute hors de portée pour le moment (même des questions simples y afférant sont complètement ouvertes—par exemple on ne sait pas si pour tout réseau il existe un réseau arithmétique avec lequel il ait une intersection Zariski-dense). Pour étudier des variétés "génériques" en-dehors du cadre arithmétique, on peut essayer d'utiliser d'autres modèles probabilistes, au moins en petite dimension. En dimension 2, on sait construire des surfaces aléatoires à partir de triangulations, qui sont presque sûrement hyperboliques et dont les propriétés géométriques et topologiques génériques sont bien comprises[41]. Dès la dimension 3, cette construction pose problème. En effet, les complexes simpliciaux obtenus en collant deux à deux des faces de tétraèdres ne sont pas automatiquement des variétés, vu que le lien d'un sommet dans le complexe est une surface triangulée qui est rarement homéomorphe à une sphère[42]. Il est donc difficile de construire un modèle aléatoire reposant sur les triangulations qui soit accessible à l'étude. Pour finir le chapitre 3, on présente un travail avec Petri, où on étudie les variétés à bord obtenues en excisant les sommets. Nous y établissons entre autres les propriétés suivantes, qui illustrent dans le contexte aléatoire le lien du volume avec le complexité topologique.

Théorème J. *Pour un $n \geq 5$ soit M_n la 3–variété compacte à bord obtenue en excisant les sommets d'un 3–complexe simplicial aléatoire (dans le modèle de configurations, en conditionnant pour que le graphe dual soit sans boucles ou bigones). Alors avec probabilité tendant vers 1 quand $n \to +\infty$ on a que M_n admet une unique métrique hyperbolique à bord totalement géodésique et $\mathrm{vol}(M_n) \sim_{n \to +\infty} v_O \cdot n$ (où v_O est le volume de l'octaèdre régulier idéal à angles droits dans $\mathbb{H}^3$).*

En plus du cas des surfaces mentionnés plus haut, ce travail a aussi été motivé par des modèles aléatoires déjà existant en dimension 3. Ceux-ci utilisent les constructions propres à la dimension 3, notamment les scindements de Heegaard aléatoires[43]. Dans ce cas un invariant topologique (le genre de Heegaard) est fixé au départ. La construction est donc conditionnée par cet invariant ; ceci signifie qu'elle présente un biais intrinsèque. Les modèles de triangulations aléatoires ne devraient pas présenter de tel biais, et nous le démontrons d'ailleurs dans le cas du modèle à bord.

[41]Voir par exemple R. Brooks, E. Makover, *Random construction of Riemann surfaces*, J. Differ. Geom. 68, No. 1 (2004).

[42]En fait, le nombre de telles triangulations est super-exponentiel (en le nombre de tétraèdres), alors que le nombres de variétés triangulées est exponentiel, voir par exemple G. Chapuy, G. Perarnau, *On the number of coloured triangulations of d-manifolds*, Discrete Comput. Geom. 65, No. 3 (2021).

[43]Voir N. Dunfield, W. Thurston, *Finite covers of random 3-manifolds*, Invent. Math. 166, No. 3 (2006)

Directions futures

La première question relative aux théorèmes B et E est d'en obtenir des versions quantitatives. Les estimées optimales que l'on peut espérer obtenir pour la convergence de Benjamini–Schramm des variétés de congruence seraient de la forme

$$\mathrm{vol}(M_{\leq \delta \log \mathrm{vol}(M)}) \ll (\mathrm{vol}\, M)^{1-\eta} \tag{1}$$

pour des $\eta, \delta > 0$. Ceci semble possible en utilisant les méthodes utilisées pour les démontrer dans le travail avec Frączyk et Hurtado, mais les nombreux approfondissements requis représentent un travail substantiel nécessitant de nouvelles idées à certains points. Plus précisément, deux problèmes principaux se posent. Le premier est que la démonstration de la convergence de Benjamini–Schramm dans le cas d'une suite de réseaux à corps de traces de degré borné repose sur des estimées d'intégrales orbitales, qui ne sont menées à bien que pour des éléments suffisament réguliers (en un sens technique précisé dans les démonstrations du chapitre 1). Un argument non-effectif est ensuite utilisé. Pour obtenir des bornes quantitatives du type (1), il faudrait pouvoir estimer de la même manière les intégrales pour des éléments non-réguliers, pour lesquelles de nouveaux arguments doivent être utilisés. Il semble raisonnable de commencer à étudier ces questions sur des groupes de petite dimension (par exemple les groupes de rang réel 1 et de type A_2, A_3 ou B_2).

Le second problème est qu'il faudrait étendre la démonstration utilisant les techniques ci-dessus au cas d'une suite de réseaux dont les corps de traces sont de degré tendant vers l'infini, notre démonstration actuelle utilisant des techniques complètement différentes ne faisant intervenir que le degré du corps de traces, qui est très insuffisant pour obtenir une estimée asymptotique du volume. Pour le cas de type A_1 ceci a été fait dans le travail de Frączyk cité plus haut ; une partie de ses arguments s'adapte sans trop de difficultés (au moins dans le cas d'intégrales orbitales pour des éléments réguliers) mais certains éléments pourraient poser problème dans le cas des types exceptionnels et des formes trialitaires.

Mentionnons que toute estimée de la forme (1) implique, par des arguments standards[44], des estimées pour les nombres de Betti de la forme $b_i(M) \ll (\mathrm{vol}\, M)^{1-\alpha}$ pour les degrés suffisament éloignés de la dimension moitié (par exemple $i < \frac{d-1}{2}$ pour les d-variétés hyperboliques de congruence), et $b_i(M) \ll \mathrm{vol}(M)/\log(\mathrm{vol}\, M)^\alpha$ pour les degrés restants[45] (par exemple $i = \frac{d-1}{2}, \frac{d+1}{2}$ pour les variétés hyperboliques de dimension d impaire).

Un problème sans doute plus accessible serait d'étendre les applications topologiques du résultat géométrique E (qui est aussi valide pour des réseaux Γ ayant des éléments de torsion) aux orbi-variétés. Ceci nécèssiterait en particulier une étude spécifique du lieu singulier de ces dernières ; ceci a été fait dans un travail antérieur avec Frączyk[46] pour les orbi-variétés de dimension 3, sans arriver jusqu'au résultat sur les triangulations. Il existe[47] par ailleurs déjà des méthodes pour construire des triangulations sur les orbi-variétés hyperboliques.

Une question dans la même veine que les résultats de comparaison entre volume et complexité topologique est la suivante : étant donné une 3–variété N, existe-t'il une infinité d'entrelacs dans N dont le complémentaire est difféomorphe à une variété hyperbolique arithmétique de

[44] Voir par exemple la section 7 dans la référence [1] de la bibliographie ci-dessous.

[45] En dehors du degré milieu dans le cas où le groupe G admet des représentation de série discrète. On ne définira pas cette notion ici mais cette condition équivaut à ce qu'il existe des tores maximaux dans G qui soient compacts. Par exemple un tore maximal de $\mathrm{SO}(d)$ est maximal dans $\mathrm{SO}(d,1)$ pour d impair mais pas pour d pair.

[46] Voir la référence [8] ci-dessous.

[47] Voir par exemple H. Alpert, M. Belolipetsky, *Thickness of skeletons of arithmetic hyperbolic orbifolds*, à paraître au J. Top. Anal., arxiv:1811.05280

congruence?[48] On s'attend à une réponse négative, mais la solution ne peut pas être donnée en utilisant uniquement l'expansion ou la convergence de Benjamini–Schramm des variétés de congruence. Dans un travail en cours avec Steffen Kionke, nous démontrons des résultats dans cette direction, et nous espérons arriver sous peu à une solution complète. De plus, on peut espérer que les méthodes utilisées se prêtent (modulo un travail conséquent) à une implémentation effective, permettant par exemple de lister tous les entrelacs dans la sphère $\mathbb{S}^3$ dont le complément est hyperbolique de congruence. Plus précisément, nos méthodes devraient aboutir sur des bornes effectives pour le co-volume d'un réseau de congruence sans torsion pour lequel la variété hyperbolique associée est difféomorphe à un complémet d'entrelacs dans $\mathbb{S}^3$. On peut espérer, moyennant un travail conséquent, réduire ensuite suffisament ces bornes pour pouvoir examiner un par un[49] tous les sous-groupes de congruence les satisfaisant pour déterminer lesquels sont bien des compléments d'entrelacs.

Le cas des coefficients triviaux dans la conjecture de Bergeron–Venkatesh sur la torsion homologique (c'est-à-dire une extension du théorème C à la suite des groupes d'homologie entiers $H_1(\Gamma_n, \mathbb{Z})$) semble hors de portée pour le moment, même si les problèmes auxilaires qu'elle pose semblent accessibles à un progrès incrémental[50]. La conjecture de Bergeron–Venkatesh est formulée pour n'importe quel autre espace symétrique, et dans le cas de réseaux cocompacts et de certains coefficients non-triviaux la partie (1) du théorème C se généralise. En petites dimensions, il est possible de vérifier la conjecture pour des coefficients triviaux de manière assez précise, en calculant numériquement l'homologie de revêtements de congruence de haut degré et en comparant les résultats avec l'asymptote attendue. Ceci a été fait de manière concluante[51] en dimension 3. La conjecture de Bergeron–Venkatesh prédit que l'ordre des sous-groupes de torsion dans l'homologie des variétés localement symétriques de dimension paire est toujours sous-exponentielle. Les premiers cas intéressants après la dimension 3 sont donc de dimension au moins 5. La conjecture prédit que pour les variétés hyperboliques arithmétiques de dimension 5 et les espaces localement isométriques à $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$, le sous-groupe de torsion homologique en degré 2 devrait croître exponentiellement en le volume dans des suites de revêtements de congruence.

Dans le cadre d'un projet avec Aurel Page, nous avons commencé une telle vérification pour certaines orbi-variétés hyperboliques de dimension 5. La difficulté est nettement plus relevée qu'en dimension 3 vu la paucité de revêtements de petit degré pour les petites orbi-variétés arithmétiques. En effet, les sous-groupes de congruence de petit indice correspondent à des sous-groupes des groupes finis de type Lie qui sont des groupes orthogonaux en dimension 6, et de tels groupes ont un indice croissant rapidement avec le niveau[52]. Dans les cas les plus favorables, le plus petit indice d'un tel sous-groupe est p^3 pour un groupe de niveau p (pour certains "bons" nombres premiers p, dépendant du choix initial du réseau Γ mais ayant une densité asymptotique positive), à contraster avec le cas des réseaux arithmétiques dans $\mathrm{SL}_2(\mathbb{C})$

[48]Voir M. Baker, A. Reid, *Congruence link complements – a 3-dimensional Rademacher conjecture*, dans *Cohomology of arithmetic groups. On the occasion of Joachim Schwermer's 66th birthday*, Springer Proc. Math. Stat. 245 (2018).

[49]En utilisant des méthodes similaires à celles de M. Baker, M. Goerner, A. Reid, *All principal congruence link groups*, J. Algebra 528 (2019). La topologie des variétés de dimension 3 est bien fournie en outils de calcul adaptés à ce type de problèmes, en particulier le logiciel Snappy.

[50]Voir par exemple M. Lipnowski, M. Stern, *Geometry of the smallest 1-form Laplacian eigenvalue on hyperbolic manifolds*, Geom. Funct. Anal. 28, No. 6 (2018), et N. Bergeron, M. H. Şengün, *Torsion homology growth and cycle complexity of arithmetic manifolds*, Duke Math. J. 165, No. 9 (2016).

[51]Voir entre autres H. Şengün, *On the integral cohomology of Bianchi groups*, Exp. Math. 20, No. 4 (2011) et les dernières sections de J. Brock, N. Dunfield, *Injectivity radii of hyperbolic integer homology 3-spheres*, Geom. Topol. 19, No. 1 (2015).

[52]Voir par exemple S. Guest, J. Morris, C. Praeger, P. Spiga, *On the maximum orders of elements of finite almost simple groups and primitive permutation groups*, Trans. Am. Math. Soc. 367 No. 11 (2015)

à corps de traces quadratique, où pour la moitié (asymptotiquement) des p il existe un sous-groupe d'indice p. Nous espérons cependant des résultats concluants sous peu, en utilisant des techniques d'accélération des calculs d'homologie[53] pour pouvoir accéder à des sous-groupes de très grand indice. Nous envisageons aussi d'étudier (y compris en dimension 3) d'autres questions plus fines, aussi posées par Bergeron et Venkatesh, sur la structure des groupes d'homologie, en particulier sur les statistiques du rang et des exposants des p-sous-groupes en fonction de p.

Il est naturel d'essayer de généraliser les résultats sur la croissance des sous-groupes de Coxeter à angles droits donnés dans le théorème H. Même sur des exemples concrets les difficultés combinatoires apparaissent énormes, en particulier les méthodes utilisées pour les estimées les plus fines ne peuvent pas se généraliser de manière directe au-delà du cadre virtuellement abélien. On peut quand même espérer y parvenir pour quelques cas où la structure du graphe définissant le groupe est particulièrement simple. Le cas-test serait celui des graphes cycliques, qui définissent des groupes représentables comme groupes fuchsiens engendrés par des réflexions. Pour ceux-ci, de bonnes asymptotes pour la croissance des sous-groupes sont connues[54], mais il serait intéressant de les retrouver par des méthodes différentes[55]. On peut de plus espérer obtenir des résultats de convergence des sous-groupes génériques comme corollaire. Notons pour finir qu'au delà des groupes fuchsiens, l'extension aux groupes de Coxeter généraux semble encore plus difficile ; nous n'avons même pas un énoncé conjectural pour les groupes de réflexions des polyèdres de Coxeter dans les espaces hyperboliques réels de dimension au moins 3.

Pour ce qui est des sous-groupes aléatoires invariants dans $\mathrm{PSL}_2(\mathbb{R})$, le théorème G est optimal en ce qui concerne la topologie des surfaces associées. En revanche, on sait peu de choses sur leurs invariants métriques : l'espace des structures hyperboliques sur une surface de type infini est immense, mais il est clair que celles qui apparaissent dans le support d'une surface aléatoire obtenue comme quotient du plan hyperbolique par un sous-groupe aléatoire invariant n'en occupent qu'une petite partie. Dans ce cadre il semble naturel d'étudier les invariants métriques asymptotiques ; par exemple il est connu[56] que l'exposant critique d'une surface aléatoire unimodulaire est $> 1/2$ et on peut se demander si le type topologique de la surface implique des contraintes supplémentaires sur l'exposant critique ou la croissance volumique. Par exemple, si μ est un sous-groupe aléatoire invariant ergodique de $\mathrm{SL}_2(\mathbb{R})$ (supporté sur les sous-groupes sans torsion) et on sait que $H\backslash\mathbb{H}^2$ a μ-presque sûrement deux bouts de volume infini, est-ce-que la croissance du volume est au plus[57] linéaire? Si μ-presque sûrement il y a un nombre infini de bouts et le rayon d'injectivité est positif, est-ce-que la croissance volumique est exponentielle? Il existe des exemples au comportement "sauvage" dans le cas des graphes, par exemple un graphe unimodulaire dont la croissance oscille entre exponentielle et sous-exponentielle[58].

Une question adjacente est de donner une définition géométrique de moyennabilité pour les surfaces unimodulaires qui forcerait en particulier l'exposant critique à être égal à 1 (pour des revêtements galoisiens de surfaces hyperboliques de volume fini[59], le groupe de Galois est moyennable si et seulement si l'exposant critique est 1).

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[53] Voir L. Lampret, *Chain complex reduction via fast digraph traversal*, arXiv:1903.00783.

[54] Voir M. Liebeck, A. Shalev, *Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks*, J. Algebra 276 No. 2 (2004).

[55] Nous travaillons en particulier sur une approche combinatoire directe inspirée de celle utilisée dans la section 3.3 de notre travail [5].

[56] Voir I. Gekhtman, A. Levit, *Critical exponents of invariant random subgroups in negative curvature*, Geom. Funct. Anal. 29, No. 2 (2019).

[57] On peut construire des exemples ou la croissance est sous-linéaire.

[58] Voir A. Timár, *A stationary random graph of no growth rate*, Ann. Inst. Henri Poincaré, Probab. Stat. 50 No. 4 (2014).

[59] Voir R. Brooks, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helv. 56 (1981) et D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Differ. Geom. 25 (1987).

0.1 Articles présentés dans ce mémoire

[1] Mikołaj Frączyk, Sebastian Hurtado, and Jean Raimbault. *Homotopy type and homology versus volume for arithmetic locally symmetric spaces.* preprint. 2022.

[2] Bram Petri and Jean Raimbault. *A model for random 3–manifolds.* prépublication 2020, à paraître à Com. Math. Helv. 2022.

[3] Stefan Friedl, Junghwan Park, Bram Petri, Jean Raimbault, and Arunima Ray. "On distinct finite covers of 3-manifolds". In: *Indiana Univ. Math. J.* 70.2 (2021), pp. 809–846.

[4] Miklós Abért, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. "On the Growth of L2-Invariants of Locally Symmetric Spaces, II: Exotic Invariant Random Subgroups in Rank One". In: *International Mathematics Research Notices* 2020.9 (2020), pp. 2588–2625.

[5] Hyungryul Baik, Bram Petri, and Jean Raimbault. "Subgroup growth of right-angled Artin and Coxeter groups". In: *J. Lond. Math. Soc., II. Ser.* 101.2 (2020), pp. 556–588.

[6] Jonathan Pfaff and Jean Raimbault. "The torsion in symmetric powers on congruence subgroups of Bianchi groups". In: *Trans. Am. Math. Soc.* 373.1 (2020), pp. 109–148.

[7] Hyungryul Baik, Bram Petri, and Jean Raimbault. "Subgroup growth of virtually cyclic right-angled Coxeter groups and their free products". In: *Combinatorica* 39.4 (2019), pp. 779–811.

[8] Mikołaj Frączyk and Jean Raimbault. "Betti numbers of Shimura curves and arithmetic three-orbifolds". In: *Algebra Number Theory* 13.10 (2019), pp. 2359–2382.

[9] Jean Raimbault. "Analytic, Reidemeister and homological torsion for congruence three-manifolds". In: *Ann. Fac. Sci. Toulouse, Math. (6)* 28.3 (2019), pp. 417–469.

[10] Miklós Abért, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. "On the growth of L^2-invariants for sequences of lattices in Lie groups". In: *Ann. Math. (2)* 185.3 (2017), pp. 711–790.

[11] Ian Biringer and Jean Raimbault. "Ends of unimodular random manifolds". In: *Proc. Am. Math. Soc.* 145.9 (2017), pp. 4021–4029.

[12] Jean Raimbault. "On the convergence of arithmetic orbifolds". In: *Ann. Inst. Fourier* 67.6 (2017), pp. 2547–2596.

[13] Jean Raimbault. "A note on maximal lattice growth in SO(1,n)". In: *Int. Math. Res. Not.* 2013.16 (2013), pp. 3722–3731.

0.2 Autres articles non présentés

[1] Léo Bénard and Jean Raimbault. "Twisted L^2-torsion on the character variety". In: *Publ. Mat., Barc.* 66.2 (2022), pp. 857–881.

[2] Jean Raimbault. "Analytic Torsion, Regulators and Arithmetic Hyperbolic Manifolds". In: *Arithmetic L-Functions and Differential Geometric Methods.* Ed. by P. Charollois, G. Freixas i Montplet, and V. Maillot. Vol. 338. Progress in Mathematics. Birkhäuser, Cham, 2021.

[3] Holger Kammeyer, Steffen Kionke, Jean Raimbault, and Roman Sauer. "Profinite invariants of arithmetic groups". In: *Forum Math. Sigma* 8 (2020). Id/No e54, p. 22.

[4] Jean Raimbault. "Asymptotics of analytic torsion for hyperbolic three-manifolds". In: *Comment. Math. Helv.* 94.3 (2019), pp. 459–531.

[5] Steffen Kionke and Jean Raimbault. "On geometric aspects of diffuse groups. With an appendix by Nathan Dunfield." In: *Doc. Math.* 21 (2016), pp. 873–915.

[6] Jean Raimbault. "Géométrie et topologie des variétés hyperboliques de grand volume". In: *Actes de Séminaire de Théorie Spectrale et Géométrie. Année 2012–2014*. St. Martin d'Hères: Université de Grenoble I, Institut Fourier, 2014, pp. 163–195.

[7] Jean Raimbault. "Exponential growth of torsion in abelian coverings". In: *Algebr. Geom. Topol.* 12.3 (2012), pp. 1331–1372.

Parmi ces textes, [7, 4] sont contenus dans ma thèse ; [6, 2] sont des articles de survols parus dans des compte-rendus de séminaires ou de conférences. [5, 3, 1] sont des articles contenant des résultats originaux mais qui ne s'inscrivent pas naturellement dans le cheminement que j'ai retenu pour écrire cette thèse.

Chapter 1

Geometry and topology of arithmetic locally symmetric spaces

1.1 Introduction

Let us start by describing in some detail the construction of arithmetic lattices in semisimple connected linear Lie groups and arithmetic locally symmetric spaces. Given such a group G, which we assume to be without compact factors, and a maximal compact subgroup K of G the quotient $X = G/K$ has a natural G-invariant Riemannian metric. Such spaces are called symmetric spaces (of noncompact type without Euclidean factors) as they can be characterised in purely geometric terms. All their sectional curvatures are nonpositive and they are simply-connected. Locally symmetric spaces are connected complete Riemannian manifolds which are locally isometric to symmetric spaces. Equivalently they are the quotients of X by discrete, torsion-free subgroups of G (these subgroups are exactly those acting properly discontinuously and freely on X). If X is fixed, we call such a manifold a locally-X manifold. The manifold $M = \Gamma\backslash X$ is compact (respectively of finite volume) if and only if Γ is cocompact (resp. of finite covolume) in G; such Γ are called uniform lattices (resp. lattices) in G.

If G is a semisimple Lie group without compact factors, one can always construct lattices in G as follows (see [21]). Call an embedding of a number field k into the complex numbers an infinite place of k; the set of such embeddings is usually denoted by $\mathrm{V}_{k,\infty}$. If v is an infinite place of k the field k_v is the completion of k associated with v: this is the closure of its image inside $\mathbb{C}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. Take a totally real[1] number field k with ring of integers $\mathbb{Z}_k$, a k-subgroup $\mathbf{G}$ of a general linear group over k, and a set S of infinite places such that $\prod_{v\in S}\mathbf{G}(k_v)$ is isogenous to G and $\mathbf{G}(k_v)$ is compact for all infinite places v of k outside S. Then the group of integer points $\mathbf{G}(\mathbb{Z}_k)$ is discrete in $\prod_{v\in \mathrm{V}_{k,\infty}}\mathbf{G}(k_v)$ and so is its projection to $\prod_{v\in S}\mathbf{G}(k_v)$; moreover a theorem of A. Borel and Harish-Chandra states that it has cofinite volume in there, and another of G. Mostow and T. Tamagawa that it is cocompact if and only if $\mathbf{G}$ is k-anisotropic. We can then take its image in G via the isogeny we assumed to exist and get a lattice there. Any discrete subgroup of G which is commensurable to a group obtained this way is called an arithmetic lattice in G. It is said to be irreducible if the group $\mathbf{G}$ in the construction is k-simple. An obvious necessary condition for a semisimple group to have an irreducible arithmetic lattice is that all its simple factors should have the same absolute type; it is also a sufficient condition by [86, Proposition 6.17].

The commensurability clause in the definition of an arithmetic lattice leaves a degree of vagueness. A more rigid definition is that of a congruence arithmetic lattice[2]: a lattice in the commensurability class of the arithmetic lattice $\mathbf{G}(\mathbb{Z}_k)$ in $\prod_{v\in S}\mathbf{G}(k_v)$ is said to be congruence if its intersection with the image of $\mathbf{G}(k)$ satisfies the property that it is equal to the intersection of its closure in $\mathbf{G}(\mathbb{A}_{k,f})$ (where $\mathbb{A}_{k,f}$ denotes the ring of finite adèles of k) with $\mathbf{G}(k)$. In particular, if Γ is a congruence lattice in the commensurability class of $\mathbf{G}(\mathbb{Z}_k)$ and Δ the smallest normal subgroup of $\mathbf{G}(\mathbb{Z}_k)$ contained in Γ then $\mathbf{G}(\mathbb{Z}_k)/\Delta$ is a finite quotient of the profinite group $\prod_{v\in \mathrm{V}_{k,f}}\mathbf{G}(\mathbb{Z}_{k_v})$. This shows that this definition of congruence subgroup generalises the classical example of the congruence subgroups given by matrices congruent to the identity mod n in $\mathrm{SL}_2(\mathbb{Z})$. Note that the congruence lattices need not be contained in $\mathbf{G}(\mathbb{Z}_k)$, in fact there are infinitely many maximal lattices in the commensurability class. Unless $\mathbf{G}$ is adjoint, they need not even be contained in $\mathbf{G}(k)$.

In this chapter, we study the geometry of the family of all congruence lattices in a fixed semisimple group G as above. We will be concerned mostly with the geometry in terms of

[1] Meaning that the closure at all infinite places is $\mathbb{R}$.

[2] In most textbooks the classical definition in terms of reduction modulo n is given , before the adelic setting is introduced and used for giving a more uniform description; compare for instance Chapters 4 and 5 in [86]. Papers on automorphic forms usually use the adelic language directly, see for instance [14], and we followed this practice here.

Benjamini–Schramm convergence to the symmetric space X. This notion was first studied in the context of locally symmetric spaces in the joint work [2] with M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov and I. Samet, and it roughly means that congruence locally-X manifolds of large volume look locally (at any fixed scale) like X at most points; i.e. for any fixed $R > 0$, the proportion of points x where the ball of radius R around x is not embedded (the R-thin part) is much smaller than the volume.

The most general result about this is Theorem 1.1, from the joint work [42] with M. Frączyk and S. Hurtado, which establishes this property for the family of all congruence arithmetic lattices in a given group G. This theorem is the combination of several special cases, which are proven using very different techniques. The first one is the case of congruence subgroups in a fixed arithmetic lattices, which was dealt with in [2]; in this setting, we obtain by essentially group-theoretical methods a much more precise result. The second result, from [42], gives an estimate for the volume of the R-thin part in terms of the degree of the trace field of an arithmetic lattice (the field k occuring in the construction detailed above). The proof is purely geometrical, the arithmeticity only mattering through a result of E. Breuillard [24] which implies a refined version of the Margulis lemma taking into account the degree of the trace field. The last case, also from [42], is that of maximal lattices with trace fields of fixed degree, for which we use the advanced structure theory of arithmetic lattices, the Prasad volume formula and we prove strong estimates for orbital integrals and the number of conjugacy classes of small displacement. In the end, this is not sufficient to imply what we want and we resort to a non-effective criterion for Benjamini–Schramm convergence.

We also include an effective result from [93] about the family of Bianchi groups which uses quite explicitely classical reduction theory for SL_2. The estimates in this result are quite better than those that can be obtained in the general case (even restricting to hyperbolic 3–manifolds where convergence in general was proven by M. Frączyk with effective bounds) and the methods of proof are completely different from those of the other results.

The motivation for studying Benjamini–Schramm convergence in [2] was initially to generalise asymptotic results about Betti numbers of finite covers (namely the Lück Approximation Theorem) to the more general setting of locally symmetric space with fixed local geometry. The result on congruence arithmetic lattices implies such a result for the family of congruence arithmetic lattices. In low dimensions, it can also be applied to other topological invariants; we present a result from the joint work [43] with M. Frączyk where we generalise an asymptotic Gauss–Bonnet formula for the genus of modular surfaces due to J. G. Thompson to the family of all compact congruence arithmetic 2–orbifolds. Finally, the estimates on thin parts of arithmetic locally symmetric spaces with trace field of large degree from [42] implies a result about optimal triangulations of such spaces which in turn gives bounds for the torsion homology of such spaces.

1.2 Statement of results

1.2.1 A general result

The following result from [42, Theorem D], establishes the Benjamini–Schramm convergence of congruence arithmetic locally symmetric spaces.

Theorem 1.1. *Let G be a connected linear semisimple Lie group, without compact factors, and let X be its symmetric space. For any $R > 0$ there exists a function $t_R :]0, +\infty[\to]0, +\infty[$ such that $\lim_{v\to+\infty} (t_R(v)/v) = 0$ and for any irreducible locally-X congruence arithmetic orbifold M we have*

$$\mathrm{vol}(M_{\le R}) \le t_R(\mathrm{vol}\, M).$$

This was known previous to [42] in a few particular cases: when the group G is of higher rank and has property (T) (see Theorem 1.5 below), and in the complementary higher-rank case where G is a product (proven in [59]); note that the first result uses no assumption of arithmeticity or congruence, while the second does through the use of property (τ) for congruence arithmetic lattices from [28]. For rank 1 groups, we detail some previously known results below.

The proof of this theorem consists in collating various specialisations: first we deal with lattices whose trace field has degree going to infinity, see Theorem 1.4, then with maximal lattices whose trace field has fixed degree, see Theorem 1.20, and finally with congruence subgroups in a fixed arithmetic lattice. For the last point, Theorem 1.2 gives a quantitative result but only for uniform lattices. The case of a non-uniform lattice can be dealt with using non-effective arguments, which we explain in §1.4.3; a generalisation of the effective result was given by T. Finis and E. Lapid in [39, 40].

The arguments for each case are essentially disjoint; in principle it should be possible to give a unified proof but this would require many nontrivial extensions of our arguments for the second case. We note that in the case of lattices in groups with factors of type A_1, such a proof was given by M. Frączyk in [41], which is the foremost inspiration for our proof of the second case.

1.2.2 Explicit results in some special cases

For some families of lattices, we can say more about the volume of the thin part. That is, we can get a more or less explicit form for the function t_R in Theorem 1.1.

Congruence covers

In the setting of congruence subgroups of a fixed uniform arithmetic lattice (which need not be congruence) a more precise version of Theorem 1.1 is proven in [2, Theorem 5.2].

Theorem 1.2. *Let Γ be a uniform irreducible arithmetic lattice in a semisimple group G. There are constants $\alpha > 0$ (a priori depending on Γ) and C (depending on R, and a priori also on Γ), such that for any congruence subgroup $\Delta \leq \Gamma$ and $M = \Delta\backslash X$ we have*

$$\mathrm{vol}(M_{\leq R}) \leq C(\mathrm{vol}\, M)^{1-\alpha}.$$

We present the proof of this in §1.3.1 below. This has been generalised to nonuniform lattices by T. Finis–E. Lapid in [39, 40] (the case where $G = \mathrm{SL}_2(\mathbb{C})$ was treated shortly before in [91, Theorem B]).

Bianchi groups

Bianchi groups are arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ of the form $\mathrm{SL}_2(\mathbb{Z}_k)$ where k is an imaginary quadratic field. The following result on this family of arithmetic groups is [93, Theorem 1.3].

Theorem 1.3. *For any $\alpha < 1/3$ and $R > 0$, there exists $C > 0$ such that for any Bianchi group $\Gamma = \mathrm{SL}_2(\mathbb{Z}_k)$ and $M = \Gamma\backslash\mathbb{H}^3$ we have*

$$\mathrm{vol}(M_{\leq R}) \leq C(\mathrm{vol}\, M)^{1-\alpha}.$$

We can use the geometric form of the Jacquet–Langlands correspondance to deduce from this result its analogue for compact orbifolds defined over quadratic fields. On the other hand M. Frączyk has proven the most general possible result about hyperbolic 3–manifolds in his thesis (published in [41]), using techniques that were partly generalised in the preprint [42] which we describe in 1.5 below. We should also cite J. Matz's paper [69] where she generalizes the statement above to groups $\mathrm{SL}_2(\mathbb{Z}_k)$ with k a number field with arbitrary fixed signature (r_1, r_2) (so they are lattices in the same Lie group $\mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$).

Trace field of unbounded degree

A very general explicit result (which however does not apply to the two cases above) is the following one, from [42, Theorem C].

Theorem 1.4. *Let G be a semisimple Lie group as in Theorem 1.1 and X its symmetric space. There exist $C, \epsilon, \delta > 0$ depending only on G such that for any arithmetic lattice Γ in G with trace field k_Γ and $M = \Gamma\backslash X$, we have*

$$\operatorname{vol}(M_{\leq \epsilon [k_\Gamma:\mathbb{Q}]}) \leq C e^{-\delta [k_\Gamma:\mathbb{Q}]} \operatorname{vol} M.$$

1.2.3 Higher rank

In the case where G is higher rank and has no factors locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, a much simpler result can be proven [2, Theorem 1.5].

Theorem 1.5. *Let G be a semisimple Lie group of real rank at least 2 and with Kazhdan's property (T). For any $R > 0$, there is a function $t_R :]0, +\infty[\to]0, +\infty[$ such that $\lim_{v\to+\infty} (t_R(v)/v) = 0$ and for any irreducible locally-X orbifold M, we have*

$$\operatorname{vol}(M_{\leq R}) \leq t_R(\operatorname{vol} M).$$

This is a priori more general than Theorem 1.1 when restricted to these groups G. On the other hand, for such G the Margulis superridity theorem implies that all lattices are arithmetic, and the congruence subgroup problem is expected to have a positive solution, so in fact it may well be that all lattices in such a G are congruence arithmetic. Since the congruence subgroup problem remains open in some cases, the statement above remains interesting in its own right, maybe even as a slight additional piece of evidence that the congruence subgroup property should hold for these G.

1.2.4 A note

In the exposition below, we will often restrict to the case where G is simple when this is convenient. The proofs in the general case make a few more difficulties appear but they can be overcome without much difficulty.

1.3 Direct approaches

In this section, we present various methods to estimate directly the volume of the thin part, each proving one of the quantitative results above.

1.3.1 Congruence subgroups

We sketch here the proof of Theorem 1.2. First we notice that studying the ratio $\operatorname{vol}(M'_{\leq R})/\operatorname{vol}(M')$ for M' a finite cover of a fixed orbifold of non-positive curvature and finite volume translates to a question in pure group theory.

Lemma 1.6. *For any cocompact arithmetic lattice Γ in G, there exists $C, c > 0$ depending only on Γ such that for any $R > 0$ and any finite-index subgroup $\Delta \leq \Gamma$ and $M = \Delta\backslash X$ we have*

$$\operatorname{vol}(M_{\leq R}) \leq C e^{cR} \sum_{[\gamma]\in\mathcal{C}_{\leq R}(\Gamma)} \left|\operatorname{Fix}_{\Gamma/\Delta}(\gamma)\right|$$

where

- $\mathcal{C}_{\le R}(\Gamma)$ *is the set of conjugacy classes* $[\gamma] \neq \{\mathrm{Id}\}$ *of* Γ *such that there exists* $x \in X$ *with* $d_X(x, \gamma x) \le R$*;*

- $\mathrm{Fix}_{\Gamma/\Delta}(\gamma)$ *is the subset of* Γ/Δ *constisting of those cosets* $g\Delta$ *for which* $\gamma g\Delta = g\Delta$*.*

A proof is given in [2, Section 5.7]. We sketch a slightly different and more explicit argument in the cas where G is of real rank 1 (which generalises to higher rank in straightforward fashion). We also restrict to the case when Δ is torsion-free (similar arguments apply to deal with the subset of the thin part coming from the singular subset in the case when M is an orbifold).

Let $M_0 = \Gamma\backslash X$; for each primitive conjugacy class $[\gamma] \in \mathcal{C}_{\le R}(\Gamma)$ there is a closed geodesic $c_\gamma \subset M_0$, of length ℓ_γ. Its preimage under the covering map $M \to M_0$ is a union $\tilde{c}_\gamma$ of closed geodesics indexed by $N_\Gamma(\gamma)\backslash\Gamma/\Delta$ which may be of length $k \cdot \ell_\gamma$ for $1 \le k \le [\Gamma : \Delta]$. The number of components of length $k \cdot \ell_\gamma$ is equal to at most $\left|\mathrm{Fix}_{\Gamma/\Delta}(\gamma^k)\right|$; indeed, if the geodesic corresponding to $N_\Gamma(\gamma)g\Delta$ is of length $k \cdot \ell_\gamma$ then $g\gamma^k g^{-1} \in \Delta$ so the coset $g^{-1}\Delta$ is fixed by γ^k. So there is a map from $\bigcup_{[\gamma]\in\mathcal{C}_{\le R}(\Gamma)} \mathrm{Fix}_{\Gamma/\Delta}(\gamma^k)$ onto the set of closed geodesics of length $\le R$ in M.

The R-thin part $M_{\le R}$ is contained in the $(R + R_0)$-neighbourhood of the union of all these closed geodesics, where R_0 depends only on the systole of M_0. The volume of the $(R + R_0)$-neighbourhood of a closed geodesic of length l is at most $C_0 l e^{cR}$ where c depends on X, R_0 on X and R_0. It follows that

$$\mathrm{vol}(M_{\le R}) \le \sum_{[\gamma]\in\mathcal{C}_{\le R}(\Gamma)} C_0 R e^{cR} \left|\mathrm{Fix}_{\Gamma/\Delta}(\gamma)\right|$$

and this proves the lemma.

With this lemma, Theorem 1.2 follows from the next purely group-theoretical statement [2, Theorem 5.6].

Theorem 1.7. *Let* Γ *be an arithmetic lattice; there are* $\alpha, c > 0$ *depending only on* Γ *such that for any* $\gamma \in \Gamma$*,* $\gamma \neq \mathrm{Id}$*, and any congruence subgroup* Δ *in* Γ*, we have*

$$\left|\mathrm{Fix}_{\Gamma/\Delta}(\gamma)\right| \le e^{c\ell_\gamma}[\Gamma : \Delta]^{1-\alpha}.$$

We give a short sketch of the proof of this theorem; we denote by $\mathbf{G}$ the k-algebraic group defining Γ and assume for simplicity that $\Gamma \subset \mathbf{G}(k)$. The first steps are to reduce the general statement to its local version. That is, for p a rational prime and G_p the closure of Γ in $\prod_{v|p} \mathbf{G}(k_v)$, H_p an open subgroup of G_p and $\gamma \in G_p$, we have

$$\mathrm{Fix}_{G_p/H_p}(\gamma) \le p^{cn_p(\gamma)}[G_p : H_p]^{1-\alpha} \tag{1.1}$$

where $\alpha > 0$ does not depend on p, the integer $n_p(\gamma)$ is the largest n such that, for some prime ideal $\mathfrak{p}_v$ of $\mathbb{Z}_k$ dividing p, we have that $\mathfrak{p}_v^n$ divides all coefficients of $\mathrm{Ad}_{\mathbf{G}(k_v)}(\gamma) - 1$, and c is a constant depending only on the absolute type of $\mathbf{G}$. As $\ell_\gamma \asymp \prod_p p^{n_p(\gamma)}$ (with constants depending only on Γ), the general result is a consequence of this and the fact that there is a $c_1 \in]0, 1[$ such that any congruence subgroup H of Γ is contained in a congruence subgroup H' such that the closure of H' in $\mathbf{G}(\mathbb{A}_{k,f})$ is the product of its p-factors, and $[\Gamma : H'] \ge [\Gamma : H]^{c_1}$ (see [2, 5.9 and 5.10]).

The proof of (1.1) is quite involved, so we won't give a detailed account. The index $[G_p : H_p]$ is controlled by a power of the level p^{m_p} of H_p so the proof reduces to showing that

$$\mathrm{Fix}_{G_p/H_p}(\gamma) \ll_\gamma p^{-c_2 m_p}[G_p : H_p]$$

for some $c_2 > 0$. The difficulty lies mainly in dealing with large values of m_p (for instance, for groups of level p the proof is quite easier than the general case; see also [2, p. 747]). For this, one uses the structure theory of analytic pro-p groups (of which G_p is an instance) to reduce the problem to counting solutions of polynomial congruences modulo prime powers; then a general lemma gives the result.

1.3.2 Bianchi groups

The proof of Theorem 1.3 is based on explicit reduction theory for SL_2 over an imaginary quadratic field k. We use the following notation:

- $\Gamma_k = \mathrm{SL}_2(\mathbb{Z}_k)$;
- $\mathbf{N}$ is the unipotent k-subgroup of upper triangular matrices in SL_2;
- $K = \mathrm{SU}(2)$ is a maximal compact subgroup in $\mathrm{SL}_2(\mathbb{C})$;
- for $c \in]0, +\infty[$ let
$$T_\infty(c) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^\times,\ |t|^2 \geq c \right\}.$$

Moreover we choose $\alpha_1, \beta_1, \ldots, \alpha_{h_k}, \beta_{h_k} \in \mathbb{Z}_k$ such that the ideals (α_i, β_i) are representatives for the class-group of k and put
$$\xi_i = \frac{\alpha_i}{\beta_i},\ \gamma_i = \begin{pmatrix} 0 & 1 \\ -1 & \xi_i \end{pmatrix},\ \mathfrak{c}_i = \frac{(\beta_i)}{(\alpha_i, \beta_i)}.$$

Define also
$$\gamma_k = \sup_{g \in \mathrm{GL}_2(\mathbb{A}_k)} \inf_{x \in k^2 \setminus \{(0,0)\}} \left(\frac{\|gx\|}{|\det(g)|^{1/2}} \right), \qquad c_i = \gamma_k^{-1} \cdot N(\mathfrak{c}_i)$$

where $\|\cdot\|$ is the product of the standard norms on the completions k_v for v in the set of equivalence classes of valuations on k. Then we have
$$\mathrm{SL}_2(\mathbb{A}_k) = \bigcup_{i=1}^{h_k} \mathrm{SL}_2(k)\mathbf{N}(\mathbb{A}_k)T_\infty(c_i)K\gamma_i U$$

for any compact-open subgroup U in $\mathrm{SL}_2(\mathbb{A}_{k,f})$. See [93, Eq. (4.6)] for more details. Moreover, a result of Ohno–Watanabe [78] gives that
$$\gamma_k \ll |\Delta_k|^{1/2} \tag{1.2}$$

where Δ_k is the discriminant of the number field k. It follows from the strong approximation property for SL_2 that the projection $\mathrm{SL}_2(\mathbb{A}_k) \to \mathrm{SL}_2(\mathbb{C})$ induces a bijection from $\mathrm{SL}_2(k)\backslash\mathrm{SL}_2(\mathbb{A}_k)/KU$ to $\Gamma_k\backslash\mathbb{H}^3$. We denote by B_i the image of $\mathrm{SL}_2(k)\mathbf{N}(\mathbb{A}_k)T_\infty(c_i)K\gamma_i U$ in $\Gamma_k\backslash\mathbb{H}^3$ under this map (this is a neighbourhood of the cusp associated with the orbit of α_i/β_i under Γ_k).

There is also a classical description of closed geodesics in the orbifolds $\Gamma_k\backslash\mathbb{H}^3$. They are in bijection with the Γ_k-equivalence classes of binary quadratic forms $ax^2 + bxy + cz^2$, $a, b, c \in \mathbb{Z}_k$, whose discrimimnant $d = b^2 - 4ac$ is square-free in $\mathbb{Z}_k$ (these correspond to embeddings of the split torus $\mathrm{SO}(2)$ into SL_2). Moreover the length of a geodesic is a function of the discriminant d. We have the following counting lemma [93, Lemma 5.2].

Lemma 1.8. *Let $N_{i,d}$ be the number of closed geodesics in $\Gamma_k\backslash\mathbb{H}^3$ which come from quadratic forms of discriminant d and intersect the cusp neighbourhood B_i. Then for any $\varepsilon > 0$*

$$N_{i,d} \ll_\varepsilon c_i^{-2-\varepsilon} \cdot N(\mathfrak{c}_i)^{2+\varepsilon} \cdot |\Delta_k|^{-1/2}.$$

With this estimate, it is easy to estimate the volume of the subset of the R-thin part coming from hyperbolic elements: since we are in rank 1, it suffices to estimate the total length of all closed geodesics of length at most R, plus the volume of cusps (note that while there is a singular locus which a priori contributes to the thin part, it is contained in closed geodesics associated with quadratic forms of bounded determinant so we do not need it to evaluate it separately). The cusps in the R-thin part have volume $\ll_R \sum_{i=1}^{h_k} N(\mathfrak{c}_i)$ which is $O(|\Delta_k|\log|\Delta_k|)$ (see also [93, Eq. (5.4)]). For closed geodesics, since there are only finitely many discriminants d corresponding to lengths at most R, it suffices for a fixed d to estimate the number N_d of geodesics corresponding to quadratic forms of discriminant d. Since the horoball quotients B_i cover $\Gamma_k\backslash\mathbb{H}^3$, we have $N_d \le \sum_{i=1}^{h_k} N_{i,d}$. By Lemma 1.8, and using that

$$c_i^{-2-\varepsilon} \cdot N(\mathfrak{c}_i)^{2+\varepsilon} = c_k^{2+\varepsilon} \ll |\Delta_k|^{1+\varepsilon}$$

according to (1.2) we have for any $\varepsilon > 0$ that

$$N_d \ll_\varepsilon \sum_{i=1}^{h_k} |\Delta_k|^{\frac{1}{2}+\varepsilon} \ll \Delta_k^{1+2\varepsilon}$$

where the second inequality follows from the estimate $h_k \ll \Delta_k^{1/2}\log|\Delta_k|$. By Humbert's volume formula [37, Eq. (8.1) in Section 8.8] for $\Gamma_k\backslash\mathbb{H}^3$ it finally follows that $N_d \ll_\varepsilon (\operatorname{vol}\Gamma_k\backslash\mathbb{H}^3)^{2/3+\varepsilon}$, which is essentially the statement of Theorem 1.3.

Sketch of proof of Lemma 1.8

Assume that $\xi \neq \infty$ and let ξ be the point at infinity of an horoball covering B_i. Then it is easy to see that the closed geodesic corresponding to the quadratic form $ax^2+bxy+cy^2$ goes through B_i if and only if $|a\xi^2+b\xi+c| \le C_{k,d}c_i^2$ (see [93, Lemma 5.1]). We can then use counting arguments for the solutions of such an inequation (modulo the group of unipotent transformations fixing ξ) to get the estimate. The case $\xi = \infty$ is dealt with separately with an easier argument.

1.3.3 An arithmetic Margulis lemma and its application

The following theorem is proven in [42, Theorem 3.1, Remark 3.2], as a consequence of a result of Breuillard [24, Corollary 1.7].

Theorem 1.9. *Let G be a semisimple Lie group. There exists a constant $\varepsilon_G > 0$ such that if $\Gamma \subset G$ is a uniform arithmetic lattice with trace field k, then for any $x \in X$, the subgroup*

$$\langle \gamma \in \Gamma :\ d(x,\gamma x) \le \varepsilon_G \cdot [k:\mathbb{Q}]\rangle$$

is virtually abelian.

This is an essential ingredient in the proof of Theorem 1.4, which we now present. For any $R > 0$ and any semisimple non-compact isometry $\gamma \in \Gamma$, we can define a "model Margulis tube" as follows: let

$$S_\gamma^R = \{x \in X :\ d(x,\gamma x) \le R\}$$

and

$$T_\gamma^R = Z_\Gamma(\gamma)\backslash S_\gamma^R$$

which is always compact when Γ is uniform. For R larger than a usual Margulis constant for G, there is no reason that the T_γ^R should embed in $\Gamma\backslash X$ and not intersect between them. However with the arithmetic Margulis lemma we can prove the following quantitative approximation to this ideal solution.

Lemma 1.10. *Let G, Γ, k be as above; there exist ε_0, d_0, m depending only on G such that the following holds. Let*

$$\overline{T}_\gamma^R = \Gamma\backslash \bigcup_{g\in\Gamma} gS_\gamma^R$$

(the image of T_γ^R in $\Gamma\backslash X$). Then for Γ such that $[k:\mathbb{Q}] \geq d_0$ and for any $R < \varepsilon_0[k:\mathbb{Q}]$, we have

1. $\operatorname{vol}\overline{T}_\gamma^R \geq [k:\mathbb{Q}]^{-m}\operatorname{vol}(T_\gamma^R)$ *;*

2. *if $N \geq [k:\mathbb{Q}]^m$ and $\gamma_1, \dots, \gamma_N \in \Gamma$ are pairwise non-conjugate in Γ then $\bigcap_{i=1}^N \overline{T}_{\gamma_i}^R$ is empty.*

An immediate corollary of this lemma is that, writing $M = \Gamma\backslash X$, for any $R \leq \varepsilon_0[k:\mathbb{Q}]$ we have

$$\operatorname{vol}(M_{\leq R}) \geq [k:\mathbb{Q}]^{-2m} \sum_{[\gamma]\in\mathcal{C}_{\leq R}(\Gamma)} \operatorname{vol}(T_\gamma^R). \tag{1.3}$$

With this, Theorem 1.4 follows from the next estimate: there exists $\eta > 0, C \geq 1$ (depending only on G) such that for any $R, S > 0$ we have

$$\operatorname{vol}(T_\gamma^{CR+S}) \geq e^{\eta S}\operatorname{vol}(T_\gamma^R). \tag{1.4}$$

Indeed, if we apply this estimate with $S = \varepsilon_G[k:\mathbb{Q}] - CR$ (which we may for any $R < (\varepsilon_G/C)[k:\mathbb{Q}]$), we get that

$$\begin{aligned}
\operatorname{vol}(M) \geq \operatorname{vol}(M_{\leq\varepsilon_G[k:\mathbb{Q}]}) &\geq [k:\mathbb{Q}]^{-2m} \sum_{[\gamma]\in\mathcal{C}_{\leq R}(\Gamma)} \operatorname{vol}(T_\gamma^{\varepsilon_G[k:\mathbb{Q}]}) \\
&\geq [k:\mathbb{Q}]^{-2m} \sum_{[\gamma]\in\mathcal{C}_{\leq R}(\Gamma)} e^{\eta(\varepsilon_G[k:\mathbb{Q}]-CR)}\operatorname{vol}(T_\gamma^R) \\
&\geq [k:\mathbb{Q}]^{-2m} e^{\eta(\varepsilon_G[k:\mathbb{Q}]-CR)}\operatorname{vol}(M_{\leq R})
\end{aligned}$$

so in the end

$$\operatorname{vol}(M_{\leq R}) \ll_G [k:\mathbb{Q}]^{-2m} e^{-\frac{\eta\varepsilon_G}{2}[k:\mathbb{Q}]}\operatorname{vol}(M)$$

for any $R < \frac{\varepsilon_G}{2C}[k:\mathbb{Q}]$, which proves Theorem 1.4 (we can take any $\delta < \eta\varepsilon_G/2$ in the statement).

Proof of Formula (1.4)

The first step is to express $\operatorname{vol}(T_\Gamma^R)$ using orbital integrals which are defined as follows: for a compactly supported locally integrable function f on G and a semisimple element $\gamma \in G$, let $G_\gamma = Z_G(\gamma)$ be the centraliser of γ in G and define

$$O(\gamma, f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx. \tag{1.5}$$

The integral is convergent since the map $G_\gamma\backslash G \to G$, $x \mapsto x^{-1}\gamma x$ is proper (see e.g. [42, Proposition 2.6]). Then

$$\mathrm{vol}(T_\gamma^R) = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)O(\gamma, 1_{B(R)})$$

where $B(R)$ is defined as the preimage in G of the ball of radius R around the identity coset in $X = G/K$ and $1_{B(R)}$ is its indicator function. As $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$ is independent of R, the estimate (1.4) follows directly from the corresponding estimate for orbital integrals which is stated in the next theorem [42, Theorem 4.4].

Theorem 1.11. *There exist $C, \delta > 0$ (depending only on G) such that for every $R, S \geq 0$ and every semisimple $\gamma \in G$:*

$$O(\gamma, 1_{B(R)}) < Ce^{-\delta S}O(\gamma, 1_{B(R+S)}).$$

The proof of this theorem relies on the spectral gap property of G and its semi-regular representations. Namely, using property (T) when it holds, and the spectral gap for convex-cocompact subgroups with limit set of codimension at least 1 in real or complex hyperbolic spaces otherwise, it follows that there exists $\delta_0 > 0$ such that for a function φ supported on $B(1)$, which is bi-K-invariant and such that $\int_G \varphi(x)dx = 1$, for any reductive subgroup $H \subsetneq G$ and $f \in L^2(H\backslash G)$ we have

$$\|\varphi * f\|_{L^2(H\backslash G)} \leq e^{-\delta_0}\|f\|_{L^2(H\backslash G)}$$

(see Lemma 5.1 in [42]). From this inequality and the other properties of φ, it follows after a short computation using the function $x \mapsto x^{-1}\gamma x$ (which belongs to $L^2(H\backslash G)$ for $H = G_\gamma$) that $O(\gamma, 1_{B(R)}) < e^{-\delta}O(\gamma, 1_{B(R+2)})$. Theorem 1.11 is an immediate consequence of this last inequality (we get $\delta = \delta_0/2$ and $C = e^{2\delta}$).

1.4 Non-effective methods: Benjamini–Schramm convergence

1.4.1 Setup

So far we have only studied the condition that

$$\mathrm{vol}\left((M_n)_{\leq R}\right) = o(\mathrm{vol}\, M_n) \tag{1.6}$$

for a sequence of locally-X manifolds (or orbifolds). This fits in a broader framework as we explain now.

The set Sub_G of closed subgroups of G is endowed with a natural topology called Chabauty topology, which is metrisable compact (see e.g. [17]). Note that there is a forgetful map from the subset of discrete subgroups in Sub_G to the set of pointed locally-X orbifolds (which sends a discrete subgroup Λ to the pointed orbifold $(\Lambda\backslash X, x_0)$ where x_0 is the image of the identity coset in $\Lambda\backslash G/K$), and that the pushforward of the Chabauty topology on this set is the geometric topology or pointed Gromov–Hausdorff topology (see [2, Section 3]).

We want to consider the set $\mathrm{Prob}(\mathrm{Sub}_G)^G$ of Borel probability measures on Sub_G which are invariant under the action of G by conjugation. Such a measure can be interpreted as the distribution of a random variable with values in Sub_G that is invariant under conjugating it by an element of G, which justifies that they be given the name of "invariant random subgroups" (IRS) of G; we will use the notation $\mathrm{Prob}(\mathrm{Sub}_G)^G = \mathrm{IRS}(G)$. The set $\mathrm{Prob}(\mathrm{Sub}_G)$ is a compact space for the topology of weak-star convergence of measures. As $\mathrm{IRS}(G)$ is closed therein, it is also compact for this topology.

A locally-X orbifold of finite volume can be interpreted as an element of IRS as follows. Let $M = \Gamma\backslash X$ be such an orbifold; note that Γ is a discrete subgroup of G of finite covolume which is well-defined up to conjugation by an element of G. There are a unique G-invariant probability measure μ on G/Γ and a G-equivariant continuous map $\Phi : G/\Gamma \to \mathrm{Sub}_G$ given by $\Phi(g\Gamma) = g\Gamma g^{-1}$. The pushforward $\mu_M = \Phi_*\mu$ is an element of $\mathrm{IRS}(G)$ which does not depend on the choice of Γ.

Other elements of $\mathrm{IRS}(G)$ are given by normal subgroups: if $N \lhd G$ then the Dirac mass δ_N supported on $\{N\}$ is an IRS; if Z is the centre of G this applies in particular to subgroups of Z. With this notation we have the following result [2, Corollary 3.8].

Lemma 1.12. *For a sequence $(M_n)_{n\geq 1}$ of locally-X orbifolds of finite volume, the condition* (1.6) *holds if and only if any accumulation point of the sequence $(\mu_{M_n})_{n\geq 1}$ in* $\mathrm{IRS}(G)$ *is supported on central subgroups.*

1.4.2 Main criterion for convergence

By compacity of $\mathrm{IRS}(G)$, we can use Lemma 1.12 to prove that the condition (1.6) holds for a sequence by proving that it cannot have any other limit point. We present here a general criterion using this principle; we will apply it to concrete situations in the next subsection. For the sake of simplicity, we will restrict to the case where G is simple but, pending necessary additions, the criterion applies in the more general setting as well (see [42, Theorem 7.3]). The statement is the following.

Theorem 1.13. *Let G be a simple Lie group. Let W be a Borel subset of G such that W is invariant by conjugation by any element of G and for any discrete, Zariski-dense subgroup $\Lambda \subset G$, we have $\Lambda \cap W \neq \{1\}$ or $\emptyset$.*

Let (Γ_n) be a sequence of cocompact lattices in G; then $\Gamma_n\backslash X$ satisfies (1.6) *if and only if for any compactly supported locally integrable function f on G, we have*

$$\lim_{n\to+\infty}\left(\frac{1}{\mathrm{vol}(\Gamma_n\backslash X)}\sum_{[\gamma]_{\Gamma_n}\subset W}\mathrm{vol}\left((\Gamma_n)_\gamma\backslash G_\gamma\right)\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx\right) = 0.$$

We will give examples of sets W in the applications of this criterion in §1.4.3 and Theorem 1.20 below.

Borel density for IRS

The main ingredient for the proof of Theorem 1.13 is the following generalisation of the Borel density theorem from lattices to IRS [2, Theorem 2.9]. The proof is an easy generalisation of Furstenberg's argument [45].

Theorem 1.14. *Let G be a simple Lie group. If $\mu \in \mathrm{IRS}(G)$ is ergodic (with respect to the action of G by conjugation) and not supported on subgroups of the center or on $\{G\}$, then μ-almost every subgroup is discrete and Zariski-dense in G.*

We note that in many cases this is sufficient to prove that the condition (1.6) holds without needing to appeal to the more involved Theorem 1.13; see for example §1.4.3, §1.4.3 below.

The proof of Theorem 1.13 proceeds straightforwardly from this result: the condition on the sequence $(\Gamma_n)_{n\geq 1}$ implies by standard arguments that any invariant random subgroup which is an accumulation point of the sequence $(\mu_{\Gamma_n})_{n\geq 1}$ almost surely does not intersect W. Borel density then forces it to be central.

1.4.3 Non-effective proofs in special cases

The tools above can be used to give much shorter proofs of convergence in all three cases dealt with in the previous section. Of course the drawback is that they give no explicit estimates on the volume of the thin part.

Congruence subgroups

We give here the short argument proving a non-quantitative version of Theorem 1.2 from [2, Theorem 5.3]. It relies only on Borel density and the strong approximation property of Zariski-dense subgroups in semisimple simply connected algebraic groups. The general case reduces immediately to that of a sequence of congruence subgroups $\Gamma_n \leq \Gamma_0$ where Γ_0 is an arithmetic subgroup of $\mathbf{G}(k)$ with k a number field, $\mathbf{G}$ an absolutely simple, simply connected k-group; for simplicity let us assume that $\mathbf{G}$ is embedded in some GL_N and $\Gamma_0 = \mathbf{G}(\mathbb{Z}_k)$. Then the pro-congruence completion of Γ_0 is $\mathbf{G}(\widehat{\mathbb{Z}}_k) = \prod_{v \in \mathrm{V}_{k,f}} \mathbf{G}(\mathbb{Z}_{k_v})$ and the strong approximation property (see for instance [64, Theorem 1 on p. 391]) states that if $\Lambda \leq \Gamma_0$ is Zariski-dense the closure of its image in $\mathbf{G}(\widehat{\mathbb{Z}}_k)$ has finite index. In particular any congruence-closed Zariski dense subgroup of Γ_0 must be a congruence subgroup. Congruence-closedness is preserved under Chabauty limits, and since the subgroups Γ_n are pairwise distinct they do not have Chabauty limits that are congruence subgroups, any Chabauty limit point of $(\Gamma_n)_{n\geq 1}$ is not Zariski-dense. It follows that any limit point in $\mathrm{IRS}(G)$ of $(\mu_{\Gamma_n})_{n\geq 1}$ must be supported on non-Zariski dense subgroups of Γ_0, hence by Theorem 1.14 it must be supported on subgroups of the center of G. By Lemma 1.12, this means that the sequence $(\Gamma_n \backslash X)_{n\geq 1}$ satisfies the condition (1.6).

Bianchi groups

We present a slight variation on the proof of a non-quatitative version of Theorem 1.3 given in [93, Section 3]. For this, we apply Theorem 1.13 (for $G = \mathrm{SL}_2(\mathbb{C})$) with

$$W = \{g \in \mathrm{SL}_2(\mathbb{C}) : \ \mathrm{tr}(g) \notin \mathbb{R}\}.$$

We note that in the case when G is of real rank 1, the centraliser volumes $\mathrm{vol}\,(\Gamma_\gamma \backslash G_\gamma)$ and the orbital integrals $\int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx$ in Theorem 1.13 depend only on the minimal displacement ℓ_γ and the function f, so we have

$$\sum_{[\gamma]_{\Gamma_n} \subset W} \mathrm{vol}\,(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx \ll_f \mathcal{N}_\Gamma^W(R)$$

where f is supported in the ball $B_G(R)$ and $\mathcal{N}_\Gamma^W(R)$ counts the number of Γ-conjugacy classes in $W \cap \Gamma$ with minimal displacement $\ell_\gamma \leq R$. So if $(\Gamma_n)_{n\geq 1}$ is a sequence of lattices in $\mathrm{SL}_2(\mathbb{C})$, we must only prove that

$$\lim_{n\to+\infty} \left(\frac{N_{\Gamma_n}^W(R)}{\mathrm{vol}(\Gamma_n \backslash X)} \right) = 0$$

to deduce that the condition (1.6) holds for $(\Gamma_n \backslash X)$.

For the sequence of Bianchi groups $\mathrm{SL}_2(\mathbb{Z}_k)$, with k ranging over the imaginary quadratic fields, we now prove that in fact $N_{\Gamma_n}^W(R) = 0$ for Δ_k large enough. Indeed, if $\gamma \in W \cap \mathrm{SL}_2(\mathbb{Z}_k)$ then $2\cosh(\ell_\gamma) = |\,\mathrm{tr}(\gamma)|$. Since $\mathrm{tr}(\gamma) \in \mathbb{Z}_k \setminus \mathbb{Z}$ and $\mathbb{Z}_k$ is a 2-dimensional lattice we see that $\mathbb{Z} + \mathbb{Z}\,\mathrm{tr}(\gamma)$ is a finite-index sub-lattice in $\mathbb{Z}_k$. But Δ_k is comparable with the covolume of $\mathbb{Z}_k$ in $\mathbb{C}$ and by Minkowski's theorem it follows that $|\,\mathrm{tr}(\gamma)| \gg |\Delta_k|$ so $\ell_\gamma > R$ for Δ_k large enough.

Unbounded trace field degree

We present the short proof of a non-quantitative version of Theorem 1.4 using only the arithmetic Margulis lemma 1.9 and Borel density theorem 1.14, which is written up in [42, Section 7.3].

We need an additional lemma [42, Lemma 7.5]

Lemma 1.15. *For any $m, k \geq 1$, there exists $A > 0$ such that if Δ is a finitely generated subgroup of $\mathrm{GL}_m(\mathbb{C})$ which contains an abelian subgroup of finite index and rank at most k composed only of semisimple elements then it has an abelian subgroup of index at most A.*

Let $(\Gamma_n)_{n\geq 1}$ be a sequence of arithmetic lattices in G with trace fields k_n and $[k_n : \mathbb{Q}] \to +\infty$. With this and the arithmetic Margulis lemma, it is essentially immediate to deduce that any Chabauty limit of $(\Gamma_n)_{n\geq 1}$ must be virtually abelian. Since a virtually abelian group cannot be Zariski-dense in a semisimple Lie group, the Borel density theorem 1.14 implies that any limit of $(\mu_{\Gamma_n})_{n\geq 1}$ in $\mathrm{IRS}(G)$ must be supported on central subgroups of G, and we can conclude by Lemma 1.12.

1.4.4 Stuck–Zimmer theorem and higher rank lattices

In this section, we explain the proof of Theorem 1.5. We note that according to Wang's finiteness theorem [102], we can order conjugacy classes of lattices in G by volume, so it is enough to prove that for a sequence $(\Gamma_n)_{n\geq 1}$ the condition (1.6) holds. This is done by ineffective means, using the following rigidity theorem, a special case of [2, Theorem 4.2]. We say that an invariant random subgroup μ of G is irreducible if for every simple factor G_i of (the simply connected cover of) G its action on (Sub_G, μ) is ergodic. In particular irreducible lattices (in the classical sense) give irreducible invariant random subgroups.

Theorem 1.16. *Let G be a semisimple Lie group of real rank at least 2 and with Kazhdan's property (T). Let μ be an irreducible invariant random subgroup of G without atoms, then $\mu = \mu_\Gamma$ for some lattice Γ in G.*

The following result was proven by T. Gelander in [48, Proposition 1.9]; its proof follows immediately from local (Calabi–Weil) rigidity and finite presentability of lattices in Lie groups.

Lemma 1.17. *If G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ then the conjugacy orbit of a lattice is isolated from that of any other lattice.*

The next lemma is implied by a theorem of E. Glasner and B. Weiss who prove that for a group with property (T) acting on a Borel space, the set of ergodic probability measures is closed under weak-star convergence [51].

Lemma 1.18. *The subset of irreducible invariant random subgroups is closed in $\mathrm{IRS}(G)$.*

Finally, we will need the following lemma, which is essentially a consequence of the existence of Zassenhaus neighbourhoods.

Lemma 1.19. *If H is a closed subgroup in G which is a Chabauty limit of a sequence of lattices of G then its identity component is nilpotent.*

With these lemmas we can conclude the proof of Theorem 1.5: let $(\Gamma_n)_{n\geq 1}$ be a sequence of irreducible lattices in G, then any limit point in $\mathrm{IRS}(G)$ of the sequence $(\mu_{\Gamma_n})_{n\geq 1}$ is an irreducible invariant random subgroup of G by Lemma 1.18. It follows from Lemma 1.17 that its support does not contain any lattice of G, so from Theorem 1.16 we can conclude that it must be atomic. An atom must contain a normal subgroup with finite index; by Lemma 1.19, the latter must have a nilpotent identity component, and since it is normal it must be trivial. Any discrete normal subgroup in G is central; hence we have proven that any limit point of $(\mu_{\Gamma_n})_{n\geq 1}$ is supported on central subgroups and we can once more use Lemma 1.12 in order to conclude.

Proof of Theorem 1.16

The theorem for simple groups is essentially a consequence of a result of G. Stuck–R. Zimmer [98, Theorem 4.3]. It implies that under the hypotheses of the theorem there exists a lattice Γ such that for μ-almost all $H \in \mathrm{Sub}_G$ we have $N_G(H) = \Gamma_0$. By Margulis' normal subgroup theorem and the fact that μ is noncentral (since it has no atoms), it follows that μ-almost every H must be a lattice (in fact a finite-index subgroup of Γ_0). By ergodicity, and since the subset of lattices is made up of countably many separated components in Sub_G, it follows that in fact there is a single Γ such that μ is supported on its orbit, so that necessarily $\mu = \mu_\Gamma$.

1.5 Arithmetic lattices with trace field of bounded degree

In this section, we explain the proof of the following theorem, which together with Theorems 1.4 and 1.2 finishes the proof of Theorem 1.1.

Theorem 1.20. *Let $d \in \mathbb{N}$ and let $(\Gamma_n)_{n\geq 1}$ be a sequence of irreducible maximal arithmetic lattices in G such that the trace field of each Γ_n is an extension of degree d of $\mathbb{Q}$, and let $M_n = \Gamma_n\backslash X$. Then the condition* (1.6) *holds for the sequence $(M_n)_{n\geq 1}$.*

The proof relies on Theorem 1.13, but it is much more involved than the other applications of this theorem we have seen so far. The sufficiently dense subset W is chosen so that we have a good control of the arithmetic invariants of the centralisers of its elements, under the hypothesis that they correspond to isometries of small displacement. This allows us to get good bounds on the orbital integrals corresponding to them, as well as on the number of conjugacy classes of such elements. In the end, we inject these in the geometric side of Selberg's trace formula for compact quotient and we compare the result to the estimates for the volume given by Prasad's volume formula, which allows the application of Theorem 1.13. We describe some of the details below, including the finer description of maximal arithmetic lattices; we work exclusively in the adélic setting which is much more convenient for this.

1.5.1 Arithmetic lattices in the adélic setting

For the proof of Theorem 1.20, we need a precise description of the arithmetic lattices in the group G. For this section, we will assume, in addition to our standing hypotheses, that G is simply-connected.

A congruence arithmetic subgroup in G is constructed as follows: let k be a number field and $\mathbf{G}$ a simply connected semisimple k-group such that $\prod_{v\in \mathrm{V}_{k,\infty}} \mathbf{G}(k_v)$ is isomorphic to the product of G with a compact group. Then for any compact-open subgroup $U \subset \mathbf{G}(\mathbb{A}_{k,f})$, let $\Gamma_U = \mathbf{G}(k) \cap U$ (where we view $\mathbf{G}(k)$ as diagonally embedded in $\mathbf{G}(\mathbb{A}_{k,f})$). By definition, the image of Γ_U in G (via the map $\mathbf{G}(k) \to \prod_{v\in \mathrm{V}_{k,\infty}} \mathbf{G}(k_v) \to G$) is a congruence arithmetic lattice in G.

Since G is noncompact and $\mathbf{G}$ is simply connected, the latter satisfies strong approximation with respect to infinite places [86, Theorem 7.12], that is, $\mathbf{G}(k)$ is dense in $\mathbf{G}(\mathbb{A}_{k,f})$. Let k_∞ be the product of all Archimedean completions of k and let dg_∞ be the standard Haar measure on the Lie group $\mathbf{G}(k_\infty)$. Let dg_f be the Haar measure on $\mathbf{G}(\mathbb{A}_{k,f})$ normalised so that $\mathrm{vol}_{dg_f}(U) = 1$. Then we have an isomorphism Ψ of measured spaces from $(\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)/U, dg_\infty \otimes dg_f)$ to $(\Gamma_U\backslash G, dg_\infty)$ (where we define quotient measures by taking counting measure on the discrete subgroups $\mathbf{G}(k), \Gamma_U$ and Haar probability measure on the compact subgroup U); Ψ is simply induced by the first projection $\mathbf{G}(\mathbb{A}) = \mathbf{G}(k_\infty) \times \mathbf{G}(\mathbb{A}_{k,f}) \to \mathbf{G}(k_\infty)$.

Bruhat–Tits theory

In order to describe maximal lattices, we will need a specific class of congruence lattices which are described using the Bruhat–Tits theory of the local groups $\mathbf{G}(k_v)$. Our references for the following are [100] and [88, Sections 0.5, 0.6].

We fix $v \in \mathrm{V}_{k,f}$ and a maximal k_v-split k_v-torus $\mathbf{T}$ in $\mathbf{G}$. Bruhat–Tits theory constructs a building $X(\mathbf{G}, k_v)$, which is a polysimplicial complex of dimension equal to $\dim(\mathbf{T})$, with an action of $\mathbf{G}(k_v)$ which is transitive on the top-dimensional products of simplices. Moreover $\mathbf{T}(k_v)$ stabilises an apartment, which is a subcomplex A of $X(\mathbf{G}, k_v)$ isomorphic to the tiling of $\mathbb{R}^{\dim(\mathbf{T})} = X^*(\mathbf{T}, k_v) \otimes \mathbb{R}$ (where $X^*(\mathbf{T}, k_v)$ is the group of k_v-rational characters of $\mathbf{T}$) by the Coxeter group generated by the Weyl group of $(\mathbf{G}(k_v), \mathbf{T}(k_v))$ together with the translations coming from the action of $\mathbf{T}(k_v)$ on the valuations on the unipotent subgroups. The root system of this group is called the affine root system of $(\mathbf{G}(k_v), \mathbf{T}(k_v))$, we denote it by $\Phi_a(\mathbf{G}(k_v), \mathbf{T}(k_v))$. We will also denote by $\Delta_{a,v}$ its Coxeter diagram.

We fix a maximal product of simplices C of A and let I_v be the subgroup of $\mathbf{G}(k_v)$ which fixes all vertices of C; this is called a Iwahori subgroup and it is unique up to conjugation in $\mathbf{G}(k_v)$. The compact subgroups containing I_v are the stabilisers of the facets of C, which are in bijection with proper subsets of the diagram $\Delta_{a,v}$. If θ_v is a proper subset of the vertices $\Delta_{a,v}$ we denote by U_{θ_v} the corresponding compact-open subgroup (which is unique up to conjugation by $\mathbf{G}(k_v)$ independently of the choices of $\mathbf{T}, C$). These subgroups are called parahoric subgroups of $\mathbf{G}(k_v)$; if U_{θ_v} is a parahoric we call the subset θ_v its type.

Maximal parahoric subgroups are stabilisers of vertices of C, they correspond to θ_v containing all vertices of $\Delta_{a,v}$ but one. If the sub-diagram of $\Delta_{a,v}$ on the vertices of θ_v is isomorphic to the Dynkin diagram of the the linear root system $\Phi(\mathbf{G}(k_v), \mathbf{T}(k_v))$ then we say that θ_v, or the corresponding parahoric subgroup U_{θ_v}, is special. If furthermore $\mathbf{G}$ splits over an unramified extension L_v of k_v then we say that θ_v or U_{θ_v} is hyperspecial if the corresponding vertex of $X(\mathbf{G}, k_v)$ is also special in $X(\mathbf{G}, L_v)$.

Description of maximal lattices

A congruence lattice Γ_U is said to be a principal arithmetic lattice (following the terminology in [88]) if $U = \prod_{v \in \mathrm{V}_{k,f}} U_v$ with U_v being parahoric for all v (and hyperspecial for almost all v, which is a necessary condition for being open in $\mathbf{G}(\mathbb{A}_{k,f})$). The following fact is contained in [22, Proposition 1.4].

Lemma 1.21. *If Γ is a maximal arithmetic lattice in G, there exists a principal arithmetic lattice Γ_U in G such that $\Gamma = N_G(\Gamma_U)$.*

The conjugacy class in G of the lattice $\Gamma = N_G(\Gamma_U)$ is completely described by the number field k, the k-isomorphism class of the k-group $\mathbf{G}$ and the $\mathbf{G}(k)$-conjugacy class of the compact-open subgroup $U = \prod_{v \in \mathrm{V}_{k,f}} U_v$. We will only use the following data:

1. The number field k.

2. The number field ℓ defined as follows: unless $\mathbf{G}$ is of type D_4, ℓ is the smallest field containing k such that $\mathbf{G}$ is an inner form of its split form over ℓ; this is a quadratic extension of k. If G is of type D_4 we take ℓ to be either this extension if it has degree 2 or 3, or a cubic subfield (well-defined up to automorphism) if it has degree 6.

3. The finite set S_U of places v of k where one of the following conditions is met:

   - $\mathbf{G}$ is not quasi-split over k_v;

- $\mathbf{G}$ does not split over an unramified extension of k_v;
- U_v is not hyperspecial in $\mathbf{G}(k_v)$.

This does not completely determine Γ_U but there can be only finitely many congruence lattices which share the same data, which is sufficient for our purposes.

Prasad's volume formula

We will describe here part of the formula obtained by G. Prasad [88] for the covolume $\mathrm{vol}(\Gamma_U \backslash G)$ of a principal arithmetic lattice in G with respect to the appropriate Haar measure on G. The following theorem is a consequence of his results, which will be sufficient for us. In the sequel we use C_G to denote a constant depending only on G, if ℓ/k we denote by $\Delta_{\ell/k}$ its relative discriminant (an ideal in $\mathbb{Z}_k$) and by $N_{k/\mathbb{Q}}(\Delta_{\ell/k})$ its norm (a rational integer). The next result follows immediately from [88, Theorem 3.7] together with the following further facts from this reference: in the infinite product $\mathscr{E}$ appearing in this theorem all factors are > 1 and they are $\geq 2q_v/3$ at places in S_U (Proposition 2.10(iv) in loc. cit.), and if $\mathbf{G}$ is not an inner form then the exponent $\mathfrak{s}(\mathbf{G})$ of $N_{k/\mathbb{Q}}(\Delta_{\ell/k})^{1/2}$ is at least 2 for all types (this can be checked case-by-case using the formulas given in loc. cit., 0.4).

Theorem 1.22. *Let k be a number field, let $\mathbf{G}$ be a (simply connected) k-group such that $\mathbf{G}(k_\infty)$ is isomorphic to the product of G with a compact factor, and U a compact-open subgroup which is a product of parahoric groups (so that Γ_U is a principal arithmetic lattice). Then*

$$\mathrm{vol}(\Gamma_U \backslash G) \geq C_G |\Delta_k|^{\frac{\dim(\mathbf{G})}{2}} \cdot N_{k/\mathbb{Q}}(\Delta_{\ell/k}) \cdot \prod_{v \in S_U} 2q_v/3$$

where S_U, ℓ are defined as above.

Index estimates

In order to estimate covolumes of maximal arithmetic lattices we will use the following result, which is proven in [42, Proposition 9.3]. We will not detail the argument here, as it follows from standard arguments from [67] and [22]; in [12, Section 4] M. Belolipetsky–V. Emery deal with the case of even orthogonal groups and their argument generalises to all groups with no problem.

Lemma 1.23. *Let Γ_U be a principal arithmetic lattice in G and $\Gamma = N_G(\Gamma_U)$ its normaliser. Let k, ℓ, S_U be as in 1, 2, 3 above. Then*

$$[\Gamma : \Gamma_U] \leq C_G^{|S_U|} N_{k/\mathbb{Q}}(\Delta_{\ell/k})^{\frac{1}{2}} \Delta_k^{\frac{3}{2}}.$$

Depending on S_U, the group Γ/Γ_U can be arbitrarily large even for trace fields of bounded degree, but in this latter case the exponent is bounded as given in the following lemma; the first part is essentially [67, Proposition 2.6(i)], the second is immediate.

Lemma 1.24. *Both*

1. *the exponent of the finite group Γ/Γ_U and*
2. *for any $m \in \mathbb{N}$, and for any regular semisimple element $g \in G$, the number of elements $h \in G$ such that $h^m = g$*

are bounded by a constant depending only on G and the degree $[k : \mathbb{Q}]$.

In the sequel we will denote the constants appearing in this lemma also by C_G since we assume that the degree $[k : \mathbb{Q}]$ is fixed.

1.5.2 Proof of Theorem 1.20

Regularity conditions

The set of semisimple elements for which we are able to get good estimates on the corresponding summands in the trace formula is defined by the following conditions.

- Let $g \in G$ be semisimple, T a maximal torus of G containing g, $\|\cdot\|$ the norm on $X^*(T)\otimes\mathbb{R}$ associated with the convex hull of the root system of (G,T). For $m \in \mathbb{N}$ we say that g is m-regular if $\xi(g) \neq 1$ for all $\xi \in X^*(\mathbf{T}), \xi \neq 1$ such that $\|\xi\| \leq m$, and strongly m-regular if $\xi(g) \neq \zeta(g)$ for all $\xi, \zeta \in X^*(\mathbf{T})$ with $\|\xi\| \leq m$. We will abbreviate "strongly 1-regular" to "strongly regular" (as 1-regular elements are regular elements in the usual sense).
- A semisimple $g \in G$ is said to be $\mathbb{R}$-regular if it is regular (in the usual sense, that is 1-regular) and its centraliser has maximal $\mathbb{R}$-split rank; equivalently, g has the maximal possible number of adjoint eigenvalues which are not of modulus 1 (for G of real rank 1, this is equivalent to $g^{\mathbb{Z}}$ being unbounded).

The following lemma [42, Lemma 11.8] will let us apply our criterion for Benjamini–Schramm convergence.

Lemma 1.25. *Let G be a semisimple Lie group and $m \in \mathbb{N}$. The set W of strongly m-regular, $\mathbb{R}$-regular elements of G satisfies the assumptions of Theorem 1.13.*

The proof of this lemma essentially follows from arguments of Prasad [87].

The main argument

Let f be a smooth function on G and W a conjugacy-invariant subset of semisimple elements in G. We put[3]

$$\operatorname{tr} R_{\Gamma_U}^W f = \sum_{[\gamma]_{\Gamma_n} \subset W} \operatorname{vol}\left((\Gamma_n)_\gamma \backslash G_\gamma\right) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx$$

For a principal arithmetic lattice Γ_U, we put $f_{\mathbb{A}} = f \otimes 1_{U_\infty} \otimes 1_U$ where U is the closure of Γ_U in $\mathbf{G}(\mathbb{A}_{k,f})$ and U_∞ is the product of $\mathbf{G}(k_v)$ for $v \in \mathrm{V}_{k,\infty}$ with $\mathbf{G}(k_v)$ compact. Then for f locally integrable and compactly supported we have the following adelic formula for the distribution $\operatorname{tr} R_{\Gamma_U}^W$, which follows immediately from the isomorphism $\Gamma\backslash G \cong \mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)/U$ given by strong approximation:

$$\operatorname{tr} R_{\Gamma_U}^W f = \sum_{[\gamma]_{\mathbf{G}(k)} \subset W} \operatorname{vol}(\mathbf{G}_\gamma(k)\backslash\mathbf{G}_\gamma(\mathbb{A}_k)) \int_{\mathbf{G}_\gamma(\mathbb{A}_k)\backslash\mathbf{G}(\mathbb{A}_k)} f_{\mathbb{A}}(x^{-1}\gamma x)dx \tag{1.7}$$

We can use this in order to estimate $\operatorname{tr} R_{\Gamma_U}^W f$: the adélic orbital integrals $\int_{\mathbf{G}(\mathbb{A}_k)/\mathbf{G}_\gamma(\mathbb{A}_k)} f_{\mathbb{A}}(x\gamma x^{-1})dx$ are estimated in Theorem 1.27 below, the volume terms $\operatorname{vol}(\mathbf{G}_\gamma(k)\backslash\mathbf{G}_\gamma(\mathbb{A}_k))$ can be estimated using a formula of E. Ullmo–A. Yafaev [101] and standard estimates on value of L-functions for fields of bounded degree. Finally, the number of nonzero terms in (1.7) is estimated in Theorem 1.28 below. Altogether this gives the following result [42, Theorem 10.7].

Theorem 1.26. *Let G be a simply connected Lie group and f a continous, compactly supported function on G. Let W be a subset of the set of strongly regular, $\mathbb{R}$-regular semisimple elements in G. For fixed $d \in \mathbb{N}$, for all number fields k with $[k:\mathbb{Q}] = d$ and $\mathbf{G}(k_\infty) \simeq G$, we have*

$$\sum_{[\gamma]_{\mathbf{G}(k)} \subset W} \operatorname{vol}(\mathbf{G}_\gamma(k)\backslash\mathbf{G}_\gamma(\mathbb{A}_k)) \int_{\mathbf{G}(\mathbb{A}_k)/\mathbf{G}_\gamma(\mathbb{A}_k)} f_{\mathbb{A}}(x\gamma x^{-1})dx \ll_{f,d} \Delta_k^{\frac{\dim(\mathbf{G})}{2}-\delta}$$

[3] This is the trace of an operator $R_{\Gamma_U}^W f$ given by convolution with a smooth kernel on $\Gamma\backslash G$, hence the notation.

for any $\delta < \frac{\dim(\mathbf{G})-\dim(\mathbf{T})}{2}$ *where* $\mathbf{T}$ *is a maximal* k*-torus in* $\mathbf{G}$.

Together with Formula (1.7) this implies that

$$\operatorname{tr} R^W_{\Gamma_U} f \ll_{f,d} \Delta_k^{\frac{\dim(\mathbf{G})}{2}-\delta}$$

for any $\delta > \dim(\mathbf{T})/2$ where $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$ and $d = [k : \mathbb{Q}]$.

From here on we use the estimates on index and volume to deduce a similar upper bound for $\operatorname{tr} R^W_\Gamma 1_{B(R)}$ where $B(R)$ is the ball of radius R in G in the semi-distance lifted from the Riemannian distance on X. The first step is that for W the set of strongly m-regular elements in G (where m is the bound on the exponent of Γ/Γ_U given in the first part of Lemma 1.24) and W' the set of strongly regular elements we have

$$\operatorname{tr} R^W_\Gamma 1_{B(R)} \le C_G \cdot \operatorname{tr} R^{W'}_{\Gamma_U} 1_{B(mR)}. \tag{1.8}$$

where C_G is given by the second part of Lemma 1.24; this follows from an immediate comparison of orbital integrals between Γ and Γ_U. Now we estimate the right-hand side of (1.8) using Theorem 1.26, and we compare between the lower bound for the covolume of Γ_U given by Theorem 1.22 and the upper bound for the index $|\Gamma/\Gamma_U|$ given by Lemma 1.23 in order to prove that it is $o(\operatorname{vol}(\Gamma\backslash G))$. By Lemma 1.25 and Theorem 1.13, we can conclude that the condition (1.6) must hold for a sequence of maximal lattices with trace fields of constant degree, which finishes the proof of Theorem 1.20.

Orbital integrals for regular elements

The main new contribution from [42] regarding the proof of Theorem 1.20 are the technical results on orbital integrals and counting conjugacy classes. The first result is the following [42, Theorem 12.1].

Theorem 1.27. *Let* k *be a number field,* $\mathbf{G}$ *a simply connected semisimple* k*-group,* U *a compact-open subgroup in* $\mathbf{G}(\mathbb{A}_{k,f})$ *which is a product of parahoric subgroups. Let* $f \in C_c^\infty(\mathbf{G}(k_\infty)$ *and* $f_\mathbb{A} = f \otimes 1_U$*. There exists* C_o *depending on* R *and* $[k : \mathbb{Q}]$ *such that for* $\gamma \in \mathbf{G}(k) \cap U$ *a strongly regular,* $\mathbb{R}$*-regular element such that* $m(\gamma) \le R$ *we have*

$$\mathcal{O}(\gamma, f_\mathbb{A}) \le C_o \|f\|_\infty$$

for the normalisation of Haar measure such that $\operatorname{vol}(U) = 1$.

This is proven by local computations. Namely, writing $U = \prod_{v \in \mathrm{V}_f} U_v$ we have to estimate the orbital integrals

$$\mathcal{O}(\gamma, 1_{U_v})) \int_{\mathbf{G}_\gamma(k_v)\backslash\mathbf{G}(k_v)} 1_{U_v}(x^{-1}\gamma x) dx$$

and the archimedean integral

$$\mathcal{O}(\gamma, f) = \int_{\mathbf{G}_\gamma(k_\infty)\backslash\mathbf{G}(k_\infty)} f(x^{-1}\gamma x) dx.$$

The dependency on γ will be seen through the Weyl discriminant

$$\Delta(\gamma) = \prod_{\lambda \in \Phi} (1 - \lambda(\gamma))$$

where Φ is a root system for $(\mathbf{G}, \mathbf{T})$ with $\mathbf{T}$ a maximal k-torus containing γ ; note in particular that the product is an element of k. We prove that there exists $a > 0$ depending only on the absolute type of $\mathbf{G}$ such that the following estimates hold when $d_G(1, \gamma) \leq R$

$$\forall v \in \mathrm{V}_f, \quad \mathcal{O}(\gamma, 1_{U_v}) \leq |\Delta(\gamma)|_v^{-a} \tag{1.9}$$
$$\mathcal{O}(\gamma, f) \ll_R \|f\|_\infty \prod_{v \in \mathrm{V}_\infty} |\Delta(\gamma)|_v. \tag{1.10}$$

We will concentrate on the proof of Formula (1.9), as Formula (1.10) is essentially a straightforward computation in the appropriate Iwasawa decomposition (though there is a subtlety as we get estimates in terms of the real roots, and we must estimate the impact of the missing roots on the Weyl discriminant, see [42, Lemma 12.6]).

In order to prove Formula (1.9) the first step is to convert it into a more geometric statement about orbits on the Bruhat–Tits building $X(\mathbf{G}, k_v)$, namely we have the following [42, Lemma 12.3]

$$\mathcal{O}(\gamma, 1_{U_v}) = \sum_{x \in \mathbf{G}_\gamma(k_v) \backslash (X_\Theta)^\gamma} |\mathbf{G}_\gamma(k_v)^b \cdot x| \tag{1.11}$$

where $\mathbf{G}_\gamma(k_v)^b$ is the unique maximal bounded subgroup in the torus $\mathbf{G}_\gamma(k_v)$, X_Θ is the set of simplices in $X(\mathbf{G}, k_v)$ which are stabilised by a conjugate of U (Θ denotes the type of U) and $(X_\Theta)^\gamma$ are those fixed by γ.

In order to estimate the right-hand side in the inequality (1.11) it is easy to see that it suffices to prove that there exists $c > 0$ depending on the absolute type of $\mathbf{G}$ such that any fixed point of γ in $X(\mathbf{G}, k_v)$ lies at distance at most $c \cdot v(\Delta(\gamma))$ from the apartment A_γ associated with $\mathbf{G}_\gamma(k_v)$. For this we first reduce to the case where $\mathbf{G}$ is split using standard descent arguments (see [42, Section 11.1.3]). Then, assuming that $\mathbf{G}$ splits over k_v, we compute distances in the decomposition associated with a choice of maximal chamber in A_γ, which is a bit involved but not difficult (see the proof of [42, Lemma 11.4]), and the result follows.

Conjugacy classes of small displacement

The second result needed for the proof of Theorem 1.20 is the following.

Theorem 1.28. *For all semisimple k-groups $\mathbf{G}$ with $\mathbf{G}(k_\infty) \simeq G$ and $R > 0$ the number of conjugacy classes of strongly regular elements $\gamma \in \mathbf{G}(k)$ such that $m(\gamma) \leq R$, and which generate a compact subgroup at all non-Archimedean places, is $O(1)$ with the constant depending on $R, [k : \mathbb{Q}]$ and the absolute type of $\mathbf{G}$.*

We give a short account of the proof. First it is essentially immediate that the number of conjugacy classes of such γ over the algebraic closure $\overline{k}$ is bounded by a constant depending only on R (see [42, Lemma 13.2]). So the proof boils down to estimating the number of integral $\mathbf{G}(k)$-classes inside a single $\mathbf{G}(\overline{k})$-class. This can be interpreted as a problem in Galois cohomology, which we will use here without definitions (see [42, Section 9] for notations and basic facts). Namely, the classes we are interested in are in bijection with a subset $\mathcal{S}_U(\gamma)$ of $\ker(H^1(k, \mathbf{G}_\gamma) \to H^1(k, \mathbf{G}))$. Using a local-global principle and standard estimates on the size of a Tate–Shafarevich group, the estimation of $|\mathcal{S}_U(\gamma)|$ boils down to that of $\prod_{v \in \mathrm{V}_f} \mathcal{S}_{U_v}(\gamma)$ where $\mathcal{S}_{U_v}(\gamma) \subset H^1(k_v, \mathbf{G}_\gamma)$ are its local analogues. This in turn depends on the behaviour of v with respect to $\mathbf{G}$ and γ. At "good" places where $\mathbf{G}_\gamma$ splits over an unramified extension of k_v and $|\Delta(\gamma)|_v = 1$, we have $|\mathcal{S}_{U_v}(\gamma)| = 1$ (see [42, Proposition 13.5]); this part of the argument is quite delicate to carry out as we need to specifically care about which classes are integral at v. So these places will not contribute to the size of the global set $\mathcal{S}_U(\gamma)$. On the other hand, at the remaining places, we can afford to just bound $|H^1(k_v, \mathbf{G}_\gamma)|$ by a constant depending only on the absolute type of $\mathbf{G}$ (see [42, Lemma 13.8]), which is essentially immediate, and the global estimate then follows by brutally estimating the number of "bad" places.

1.6 Topological applications

1.6.1 Genus of Shimura curves

A Shimura curve is a quotient of $\mathbb{H}^2$ by a congruence arithmetic lattice of $\mathrm{SL}_2(\mathbb{R})$ (the underlying Riemann surface can be obtained as an algebraic curve defined over a number field which is where the name "curve" comes from). These objects are orientable 2-orbifolds, therefore their singularities can only be cone points, which have neighbourhoods that are homeomorphic to discs. So topologically, a Shimura curve S is just a surface of finite type which has a well-defined genus $g(S)$. However, for such a surface the Gauss–Bonnet formula does not apply to compute the genus directly in terms of the volume: one must take the singularities into account. The following result, which is [43, Theorem B] shows that the relation still holds asymptotically, i.e. we can ignore the additional terms coming from singularities when the volume is large.

Theorem 1.29. *There is a function $s :]0, +\infty[\to]0, +\infty[$ such that $\lim_{v\to+\infty} s(v) = 0$ and for any Shimura curve S*

$$\frac{g(S)}{4\pi \operatorname{vol}(S)} = s(\operatorname{vol}(S)).$$

The proof follows easily from the specialisation of Theorem 1.1 to $G = \mathrm{SL}_2(\mathbb{R})$ (which is due to M. Frączyk [41] for torsion-free lattices and extended to lattices with torsion in [43, Theorem A]). If an orientable hyperbolic 2-orbifold O has k cusps and s cone points of order $m_1, \dots, m_s$ then its genus $g = g(O)$ is given by the formula

$$\frac{\operatorname{vol}(O)}{2\pi} = 2g - 2 + k + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right). \tag{1.12}$$

By the Margulis lemma (see [43, Lemma 3.2]), cusps and cone points of order $m_i > 2$ all contribute a fixed quantity to the volume of the thin part, hence $k + \sum_{i,\, m_i>2} \left(1 - \frac{1}{m_i}\right) = o(\operatorname{vol}(O))$. Cone points of order 2 can be arbitrarily close to each other since a pair of rotations of order 2 generate an infinite dihedral group, but the Margulis lemma still implies that at worst pairs of them contribute a fixed quantity to the volume of the thin part, hence $\sum_{i,\, m_i=2} \left(1 - \frac{1}{m_i}\right) = o(\operatorname{vol}(O))$ as well. So the theorem follows from the formula (1.12).

1.6.2 Triangulations of arithmetic locally symmetric spaces

Beyond Benjamini–Schramm convergence, a major consequence of the explicit estimates on the thin part given in Theorem 1.4 is the solution to a conjecture of T. Gelander from [49] about the homotopy type of arithmetic locally symmetric spaces. The following result is [42, Theorem A]: it was proven for non-compact manifolds by Tsachik Gelander [49, Theorem 1.5(1)], and we proved it for compact manifolds.

Theorem 1.30. *There are constants A, B dependent only on X, such that every arithmetic locally-X manifold M is homotopy equivalent to a simplicial complex $\mathcal{N}$ with at most $A \operatorname{vol} M$ simplices, where every vertex is incident to at most B simplices.*

A consequence of this is the following rough bound on the size of torsion homology for torsion-free arithmetic lattices (we will discuss the finer behaviour of torsion homology in the next chapter).

Theorem 1.31. *There is a constant C depending only on X such that for any arithmetic locally-X manifold M of finite volume and $0 \leq i \leq \dim(X)$ we have*

$$\log |H_i(M, \mathbb{Z})_{\mathrm{tors}}| \leq C \operatorname{vol}(M)$$

where for an abelian group A we denote by A_{tors} its torsion subgroup.

The second result, though not the first one, was known for rank 1 symmetric spaces by work of U. Bader–T. Gelander–R. Sauer (see [7] which gives their most general result). They prove a weaker result about homotopy type which is nevertheless sufficient to imply the bounds on torsion homology; on the other hand their result also apply to non-arithmetic manifolds (it is not valid for hyperbolic 3–manifolds for which it is well-known that the above bounds do not hold in general). For arithmetic hyperbolic 3–manifolds, Theorem 1.30 was proven by M. Frączyk in [41].

Note that the converse to Theorem 1.30, that is the statement that for any B there exists an $A > 0$ such that any simplicial complex of degree at most B which is homotopy equivalent to a locally-X manifold M must have at least $A \operatorname{vol}(M)$ vertices, is not known in general. For complexes which are homeomorphic to X-manifolds it follows immediately from positivity and the proportionality principle of the simplicial volume for X-manifolds. When X has pisitive L^2-betti numbers or L^2-torsion (for instance for real or complex hyperbolic spaces) the converse can be proven for homotopy equivalence (see [42, Section 6.2]).

Proof of Theorem 1.30

If M is a closed Riemannian manifold and $x \in M$ we denote

$$\varepsilon_x = \min(1, \operatorname{inj}_M(x)).$$

The proof of Theorem 1.30 uses the usual procedure of constructing the nerve complex of a well-chosen net of points inside a manifold. This argument is not specific to the locally symmetric context, a general result we can use is the following one (a detailed proof of which can be found in [42, Proposition 4.6]).

Lemma 1.32. *Let X be a symmetric space of non-compact type without Euclidean factors. There exist A, B depending only on X such that any compact X-manifold is homotopy equivalent to a simplicial complex with at most*

$$A \int_M \varepsilon_x^{-\dim(X)} dx$$

vertices and where every vertex belongs to at most B simplices.

To estimate the integral in this lemma we use the following classical bound on the injectivity radius of an arithmetic manifold (which follows from a result of E. Dobrowolski on Mahler measures; an explanation can be found in [42, Proposition 2.5]): if Γ is a cocompact torsion-free arithmetic lattice in G with trace field k and $M = \Gamma \backslash X$ then

$$\operatorname{inj}(M) \gg_G \frac{1}{(\log[k : \mathbb{Q}])^3}.$$

It follows from this and Theorem 1.4 (applied with $R = 1$) that

$$\begin{aligned}\int_M \varepsilon_x^{-\dim(X)} dx &\ll_G \mathrm{vol}(M_{\geq 1}) + (\log[k:\mathbb{Q}])^{3\dim X}\,\mathrm{vol}(M_{\leq 1})\\ &\ll_G \mathrm{vol}(M)\left(1 + e^{-\delta[k:\mathbb{Q}]}(\log[k:\mathbb{Q}])^{3\dim X}\right)\\ &\ll_G \mathrm{vol}(M)\end{aligned}$$

so Theorem 1.30 now follows from Lemma 1.32.

Bounds on torsion homology

For the sake of exposition, we explain how to bound the size of torsion homology for a simplicial complex of bounded degree. Let Y be a simplicial complex such that every vertex belongs to at most B simplices. Let $C_*(Y)$ be its chain complex (with $\mathbb{Z}$-coefficients) and $\partial_k : C_{k+1}(Y) \to C_k(Y)$ be the kth differential. As $C_k(Y)/\ker(\partial_{k-1})$ is torsion-free, the torsion subgroup of $H_k(Y)$ is isomorphic to that of $\mathrm{coker}(\partial_k)$. Now observe that since Y is simplicial all nonzero coefficients of ∂_k are ± 1, and each of its columns has at most B nonzero coefficients. So the estimate $|H_k(Y)_{\mathrm{tors}}| \ll_B |Y^0|$ that we want follows immediately from the next lemma.

Lemma 1.33. *Let $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ be a linear map, and let C be the maximum of Euclidean norms over all basis vectors of $\mathbb{Z}^n$. Then $|\,\mathrm{coker}(\phi)_{\mathrm{tors}}| \leq C^n$.*

This lemma is [97, Lemma 1] where the author gives a proof which he attributes to O. Gabber. The level of mathematical language in this reference is completely out of line with the degree of difficulty of this proof; instead we note that it is easily seen that the proof reduces to the case where ϕ is injective of full rank, and the lemma then follows immediately from Hadamard's inequality ($\det(A) \ll \prod_{i=1}^n \|A_i\|$ for a square real matrix A with columns $A_1, \dots, A_n$).

1.6.3 Betti numbers

The following theorem is an immediate consequence of Theorem 1.1 and a result of M. Abért, N. Bergeron, I. Biringer and T. Gelander [1, Corollary 1.4].

Theorem 1.34. *Let $(\Gamma_n)_{n\geq 1}$ be a sequence of pairwise distinct, torsion-free irreducible congruence arithmetic lattices or pairwise non-commensurable irreducible arithmetic lattices in a semisimple Lie group G. Then*

$$\lim_{n\to\infty} \frac{b_i(\Gamma_n\backslash X)}{\mathrm{vol}(\Gamma_n\backslash X)} = \begin{cases}\beta_i^{(2)}(X) & \text{if } i = \dim(X)/2;\\ 0 & \text{otherwise.}\end{cases}$$

The nonzero L^2-Betti numbers $\beta^{(2)}_{\dim(X)/2}(X)$ appearing in these theorems can be computed explicitely: they are equal to $\chi(X^d)/\mathrm{vol}(X^d)$ where X^d is the the "compact dual" of G endowed with the Haar measure compatible with that of G. A list (up to isogeny) of all simple Lie groups G of non-compact type for which $\beta^{(2)}_{\dim(X)/2}(G) \neq 0$ is as follows:

- All unitary groups $\mathrm{SU}(n,m)$;
- All orthogonal groups $\mathrm{SO}(n,m)$ with nm even;
- All symplectic groups $\mathrm{Sp}(n)$ and all groups $\mathrm{Sp}(p,q)$ (isometries of quaternionic hermitian forms);

- some exceptional groups (at least one in each absolute type).

This is well–known, and can be recovered as follows: by [79, Theorem 1.1], a group is on the list if and only if it has a compact maximal torus. In terms of the classification by Vogan diagrams [57, p. VI.8], this means that the action of the Cartan involution on the Dynkin diagram is trivial. The list of such diagrams is given in this book, in Figure 6.1, p. 414 and Figure 6.2, p. 416 for classical and exceptional types respectively.

Chapter 2

Torsion homology of arithmetic hyperbolic manifolds

2.1 Introduction

In the previous chapter, there are two results about the asymptotic behaviour of homology for congruence arithmetic lattices, Theorem 1.31 which is a not too precise result about torsion homology and Theorem 1.34 which is a sharper result about Betti numbers. In this chapter, we discuss what is conjectured and what is known about sharp asymptotic results on the torsion homology of congruence arithmetic lattices. This was initiated by N. Bergeron and A. Venkatesh in [16] where they made a conjecture similar to Lück's approximation theorem for the torsion homology in congruence covers of compact arithmetic locally symmetric spaces. They also proved results towards this conjecture, establishing a version of it for homology with coefficients in certain natural modules. In the joint work [2], it was observed that their method of proof would work for any sequence of Benjamini–Schramm convergent locally symmetric spaces, pending the existence of the coefficient modules they used. The scheme of proof of Bergeron–Venkatesh, which rests on analytic torsion and the Cheeger–Müller theorem, runs into major obstacles when applied to trivial coefficients, the problem of small eigenvalues and that of regulators. Both of these have focused a lot of effort since the appearance of their paper but to this day there is no case where a definitive solution has been found. Much of the evidence for the Bergeron–Venkatesh conjecture thus comes from extensive computations made from low-dimensional spaces.

In [16], the focus was on uniform lattices since analytic torsion is only defined for compact Riemannian manifolds. An attempt to generalise their result to non-uniform lattices in $\mathrm{SO}(3,1)$, or rather in the isogenous group $\mathrm{SL}_2(\mathbb{C})$, was made in the papers [92, 91]. The first one deals with the general analytic aspects, defining analytic torsion for cusped hyperbolic manifolds using the Selberg trace formula and providing asymptotic results for this torsion and some results towards a Cheeger–Müller equality. The second one applies this to congruence lattices in $\mathrm{SL}_2(\mathbb{C})$ and works out some applications to torsion homology. Ultimately this attempt fell short of extending the Bergeron–Venkatesh result for these lattices, because of a number-theoretical obstacle that is very precisely identified. We should note that in parallel to this work, W. Müller–J. Pfaff developed a similar theory of analytic torsion for cusped hyperbolic manifolds and J. Pfaff obtained an analogue of the Cheeger–Müller in this setting [77], [83], both valid in arbitrary dimensions. Their result are slightly less general than mine in dimension 3 but they are much more precise when they apply; however, since the approaches are essentially the same though different in form, they cannot allow any better results about torsion homology without dealing with the same problems arising in [91].

Parallel to [16] and its sucessors, there is also an orthogonal problem which appeared in the 1990's in quite different context from that we are interested in, and which was revived by results of W. Müller and S. Marshall–W. Müller [68] around the same time as when [16] appeared. These results concern the torsion homology of a fixed arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$, with coefficients in a sequence of modules with rank going to infinity; they were generalised to lattices in other groups in a series of paper by W. Müller and J. Pfaff. The scheme of proof is the same as the one alluded to above, in particular it does not apply to non-uniform lattices. For non-uniform lattices in $\mathrm{SL}_2(\mathbb{C})$ a similar result is proven in the joint work [85] with J. Pfaff. The difficulties occurring in the last steps in order to prove torsion growth are the same as those discussed in the previous paragraph but in this different context existing tools in number theory can be applied to surmount them.

2.2 Conjectural context and results in the cocompact case

The results in this chapter are much less conclusive in range and strength then in the previous section so we motivate them by presenting the conjectural picture in which they fit. Similarly to what happens with Betti numbers (see Theorem 1.34), it is expected that specifically for congruence arithmetic lattices it should be possible to say much more about the torsion subgroups of homology groups than the general bounds given in Theorem 1.31. Namely, given a semisimple Lie group G and its symmetric space X, we should have that (with uniform bounds over all congruence lattices $\Gamma \subset G$) :

1. if $i \neq \frac{\dim(X)-1}{2}$ then $\log|H_i(\Gamma\backslash X)_{\mathrm{tors}}| = o(\mathrm{vol}(\Gamma\backslash X)$;

2. for $i = \frac{\dim(X)-1}{2}$, $\log|H_i(\Gamma\backslash X)_{\mathrm{tors}}| = \mathrm{vol}(\Gamma\backslash X) \cdot |t^{(2)}(X)| + o(\mathrm{vol}(\Gamma\backslash X)$.

Here $t^{(2)}(X)$ is the L^2-torsion of the symmetric space X. It is computed in [79], which shows in particular that it is nonzero if and only if maximal compact tori in G have dimension exactly equal to the absolute rank of G minus 1. Among simple Lie groups this is the case only for $G = \mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$ and $\mathrm{SO}(p,q)$ with p odd and $p+q$ even (in other words pq odd); these correspond to Vogan diagrams with an involution exchanging exactly one pair of vertices, which is possible only in types A_2, A_3 and $D_n, n \geq 2$ (see also [16, Section 1.2]).

This conjecture was originally stated by N. Bergeron and A. Venkatesh in [16, Conjecture 1.3] only for a nested sequence $\Gamma_1 \supset \Gamma_2 \supset \cdots$ of congruence subgroups inside a fixed arithmetic lattice. The restriction to nested sequences is certainly not necessary. It seems more hasardous to generalise their conjecture to sequences with non-commensurable lattices but from a topologist's point of view this is natural in view of the convergence results from the previous chapter. Another motivation is given by extensions of the Langlands program from automorphic forms to torsion cohomology classes, which we shall not discuss here.

The conjecture as stated above is completely open in all cases; namely there is not a single example of a group G and sequence $(\Gamma_n)_{n\geq 1}$ of congruence arithmetic lattices in G such that the conclusion holds. However, there is an approach to it also introduced in [16] which does succeed in proving some analogous results for homology with coefficients and seems to have a chance of applying also for homology with $\mathbb{Z}$-coefficients. We shall present it in the remainder of this section. Discussion about the obstacles to going through with this argument in the case of trivial coefficients can be found in particular in [15].

Finally, we note that the conjecture in the case $G = \mathrm{SO}(3,1)$ is well-supported by numerical computations of the homology groups of congruence subgroups up to relatively high level of many arithmetic lattices of small covolume (see [96, 25] and [90, Section 3.3.1]). For higher-dimensional groups, numerical evidence is scant; some computations for nonuniform lattices in $G = \mathrm{SL}_3(\mathbb{R}), \mathrm{SL}_4(\mathbb{R})$ are given in [6].

2.2.1 Exponential growth for nontrivial coefficients in the uniform case

We present here the arguments for the proof of the following theorem.

Theorem 2.1. *Let $(\Gamma_n)_{n\geq 1}$ be a sequence of uniform, torsion-free congruence arithmetic lattices in G. Let ρ be a representation of G on a real vector space V such that the G-representations ρ and $\rho\circ\theta$ are not isomorphic, where θ is a Cartan involution of G. Assume that for each n there is a lattice $L_n \subset V$ such that $\rho(\Gamma_n)\cdot L_n = L_n$. Then*

$$\sum_{i=0}^{\dim(X)} (-1)^i \log|H_i(\Gamma_n, L_n)_{\mathrm{tors}}| = t^{(2)}(X,\rho)\cdot \mathrm{vol}(\Gamma_n\backslash X) + o(\mathrm{vol}\,\Gamma_n\backslash X).$$

This is a generalisation of [16, Theorem 1.4] and the arguments are due to these authors, with the observation that Benjamini–Schramm convergence of the sequence $(\Gamma_n\backslash X)_{n\geq 1}$ to X is sufficient to extend it being used in [2, Theorem 8.4]. In the statement $t^{(2)}(X,\rho)$ is the L^2-torsion of X with coefficients in the equivariant flat bundle induced[1] by ρ; the groups G for which this theorem gives a nontrivial exponential growth are the same ones as those in the conjecture of the previous subsection, since the condition given there for $t^{(2)}(X) > 0$ is also necessary and sufficient for $t^{(2)}(X, \rho) > 0$. Moreover, an explicit computation of the exact value of $t^{(2)}(X, \rho) > 0$ is given in [16, Section 5]. Morover they prove that when this condition is met, representations ρ with $\rho \circ \theta \ncong \rho$ always exist. See §2.2.2 below for some comments on the use of this condition.

We note that this result is more naturally stated for a sequence of congruence subgroups inside a fixed arithmetic lattice as one can then consider a single stable lattice. In general, the hypothesis that such lattices exist for all n implies that the trace field of Γ_n has bounded degree. Finally, in general we do not know how to extract the exponential growth rate for a given degree from this result, but it does imply that at least one degree with the parity of $(\dim X - 1)/2$ has exponential growth of torsion since the L^2-torsion has the same sign as $(-1)^{(\dim X - 1)/2}$. In particular, in dimension 3 (for $G = \mathrm{SL}_2(\mathbb{C})$), this theorem gives the exact exponential growth rate of the torsion subgroup in H_1 since H_3 is torsion-free and it can be shown that H_0, H_2 have sub-exponential growth rate (see [16, p. 8.6], [91, p. 6.2]).

Analytic torsion and the Cheeger–Müller theorem

The following theorem allows to use analytic methods to study torsion. Originally it is due to Cheeger and Müller (independently) for trivial coefficients, the extension to unimodular coefficients is due (independently) to Müller and Bismut–Zhang. We use the notation $t(M, \rho)$ to denote the analytic torsion[2] of a closed, odd-dimensional Riemannian manifold M with coefficients in the flat bundle induced by a representation ρ of $\pi_1(M)$.

Theorem 2.2. *Let Γ be a uniform torsion-free lattice in G, ρ a representation of G and $L \subset V$ a Γ-stable lattice. If $H^*(\Gamma, V) = 0$ then*

$$\sum_{i=0}^{\dim(X)} (-1)^i \log |H_i(\Gamma, L)_{\mathrm{tors}}| = t(\Gamma \backslash X, \rho).$$

Note that unimodularity is implied by the holonomy representation being a representation of the semisimple group G. The usual formulation does not assume acyclicity and uses a Reidemeister torsion rather than homological torsion ; the hypothesis that $H^*(\Gamma, V) = 0$ ensures that both are equal.

Approximation for analytic torsion

With Theorem 2.2 above, Theorem 2.1 is an immediate consequence of its analogue for analytic torsion.

Theorem 2.3. *If $\rho \circ \theta \ncong \rho$ then $H^*(\Gamma, V) = 0$ for any uniform lattice Γ in G and*

$$t(\Gamma \backslash X, \rho) = t^{(2)}(X, \rho) \cdot \mathrm{vol}(\Gamma \backslash X) + o(\mathrm{vol}\, \Gamma \backslash X)$$

uniformly for congruence lattices.

This follows from Theorem 1.1 and [2, Theorem 8.4]. The proof is the same as the of [16, Theorem 4.5].

[1] That is, the quotient $\Gamma \backslash (X \times V_\rho)$ by the action $\gamma \cdot (x, v) = (\gamma \cdot x, \rho(\gamma)v)$, with the natural map to $\Gamma \backslash X$.

[2] We will not discuss the definition here; it depends a priori on the choice of a Euclidean or Hermitian metric on the bundle to define Laplace operators on differential forms with coefficients in the bundle. The analytic torsion is then defined using the discrete spectra of these operators. For odd-dimensional manifolds, the choice of metric on the bundle does not bear on the resulting number. For locally symmetric spaces, there is a canonical choice of such a metric.

2.2.2 Strong acyclicity

The condition that $\rho \circ \theta \not\cong \rho$ for a G-representation is used through its consequence of "strong acyclicity". It means that there is a uniform positive lower bound for all eigenvalues of all Laplacians on differential forms on a X-manifold with coefficients in the flat bundle associated with ρ (this was observed by Bergeron–Venkatesh, the mere acyclicity under this condition goes back further). This implies immediately that $H^*(\Gamma, V) = 0$ for any uniform lattice Γ. It is also used in the proof of the other statement in Theorem 2.3. This is because arbitrarily small eigenvalues of the Laplace operators make controlling the analytic torsion difficult; such eigenvalues may well appear in a sequence of covers without the assumption of strong acyclicity on the bundle. For instance, in a sequence of covers of a closed hyperbolic 3–manifold there will always be arbitrarily small Laplacian eigenvalues on 1-forms with trivial coefficients.

2.3 Exponential growth for congruence subgroups of Bianchi groups

In this section, we present the proof of the following theorem, which is an improvement on [91, Theorem A].

Theorem 2.4. *Let k be an imaginary quadratic field and let $(\Gamma_n)_{n\geq 1}$ be a sequence of pairwise distinct torsion-free congruence subgroups in $\mathrm{SL}_2(\mathbb{Z}_k)$. Let $V = \mathrm{Sym}^m(\mathbb{C}^2)$ and $L = \mathrm{Sym}^m(\mathbb{Z}_k^2)$ $(m \geq 1)$. Then*

$$\limsup_{n\to+\infty} \left(\frac{\log |H_1(\Gamma_n; L)_{\mathrm{tors}}|}{\mathrm{vol}\, M_n} \right) \leq -t^{(2)}(V).$$

Here $t^{(2)}(V)$ is the L^2-torsion $t^{(2)}(\mathbb{H}^3, \rho)$ where ρ is the natural representation[3] of $\mathrm{SL}_2(\mathbb{C})$ on V. This result by itself is not our main concern; a purely topological proof has been given by T. Le [58], which works for all coefficient systems (in particular the trivial one). However the proof we present, which attempts to reproduce the scheme used by Bergeron–Venkatesh in the compact case, almost succeeds in replacing the upper limit in the statement by a genuine limit; it is only at the end that the argument encounters an obstacle, which is of a purely number-theoretical nature.

We should note that a lower bound for the exponential growth (which is strictly smaller than the conjectured limit) has been given by J. Pfaff [84] (the statement in loc. cit. does not concern all sequences of congruence lattices but the argument should be adaptable to encompass this greater generality).

2.3.1 Reidemeister torsion for manifolds with cusps

Let us explain the difficulties in adapting the arguments from the previous section to the non-compact case. The analytic torsion of compact manifolds can be defined via Mellin transform using the trace of the heat kernels. On a cusped manifold, heat kernels are not trace-class, but there is a natural way to define analytic torsion in the same way for them, using the Selberg trace formula which gives a way to define the trace (see [75] and [92, Section 5]). There is also a natural way to define a Reidemeister torsion $\tau(M, L)$, for M a cusped manifold and L a local system over M, which we detail in §2.3.1 below, and appropriate extensions of the Cheeger–Müller theorem, for instance [83] and the weaker [92, Theorem B]. There are additional difficulties to extending Theorem 2.3 to this setting but it has been done for certain exhaustive sequences of covers in [77] and for congruence covers in [92, 91]. The result is the following one.

[3] Note that such V exhaust the non-trivial finite-dimensional, complex-linear representations of $\mathrm{SL}_2(\mathbb{C})$. The theorem also holds for most real representations—the condition that $\rho \not\cong \rho \circ \theta$ can be made very explicit for representations of $\mathrm{SL}_2(\mathbb{C})$.

Theorem 2.5. *Let $(\Gamma_n)_{n\geq 1}$ be as in Theorem 2.4 and $M_n = \Gamma_n\backslash\mathbb{H}^3$. Then for any $m \geq 1$ and $L = \mathrm{Sym}^m(\mathbb{Z}_k^2)$ and for ρ the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $V = \mathrm{Sym}^m(\mathbb{C}^2)$ we have*

$$\lim_{n\to+\infty} \frac{\tau(M_n, L)}{\mathrm{vol}(M_n)} = t^{(2)}(\mathbb{H}^3, \rho).$$

The proof of this theorem is contained in [92] and [91, Sections 3, 4, 5]. We shall not present it here in detail as it was in large part (all of [92] and some special cases of [91]) contained in the author's PhD thesis. The main ingredients are [92, Theorem A, B], the Maass–Selberg relations (see [91, Section 5.3]) and ideas from [26, Chapter 6]. The complete proof is given in [91, Sections 5.4, 5.5]. We note that the statement here is substantially more general than in these references; this can be obtained using the same arguments with a slightly more precise version of a crucial lemma that we present in 2.3.2 below.

Cuspidal homology and Reidemeister torsion

We explain here how to define a Reidemeister torsion for cusped hyperbolic manifolds similar to that used in the statement of the Cheeger–Müller theorem [27, Section 1]. The difference between the compact and cusped cases comes from the fact that not all cohomology classes are represented by square-integrable harmonic forms in the latter case.

In order to overcome this difficulty we use the Borel–Serre compactification. Let M be a cusped hyperbolic 3–manifold M, then it is the interior of a compact manifold $\overline{M}$ whose boundary $\partial\overline{M}$ is a disjoint union of tori, and the inclusion $M \subset \overline{M}$ is a homotopy equivalence. The manifold $\overline{M}$ is called the Borel–Serre compactification of M (though in this case the construction is immediate and predates their work which is mostly useful in the case of locally symmetric spaces of higher $\mathbb{Q}$-rank). Then we can consider the pair $(\overline{M}, \partial\overline{M})$ which gives restriction maps in cohomology $\iota^* : H^i(M, V) \to H^i(\partial\overline{M}, V)$ and a long exact sequence (we will use V to denote the coefficient system $\mathrm{Sym}^m(\mathbb{C}^2)$). In this case the kernel $\ker(\iota^*) \subset H^i(M, V)$ is called the cuspidal cohomology and it is isomorphic to the space of square-integrable harmonic forms on M.

On the other hand, each torus T in $\partial\overline{M}$ has a flat structure[4] coming from the hyperbolic structure of M. We can take the induced Euclidean structure on $\iota^* H^i(M, V)$ and we endow the space $\ker(\iota^*) \oplus \iota^* H^i(M, V) \cong H^i(M, V)$ with the Euclidean structure where the summands are orthogonal. With this the following definition for Reidemeister torsion with coefficients in a Γ-stable lattice $L \subset V$ is equivalent to that from [26, Chapter 5] (see also [91, (5.6)].

$$\tau(M; L) = \frac{|H^1(M; L)_{\mathrm{tors}}|}{\mathrm{vol}\, H^1(M; L)_{\mathrm{free}}} \times \frac{\mathrm{vol}\, H^2(M; L)_{\mathrm{free}}}{|H^2(M; L)_{\mathrm{tors}}|}. \tag{2.1}$$

We have the following lemma which simplifies matters when considering nontrivial local coefficients.

Lemma 2.6. *For $m \geq 1$, $V = \mathrm{Sym}^m(\mathbb{C}^2)$ and $i = 0, 1, 2$, the restriction maps ι^* are injective.*

The proof uses the same argument as the vanishing of cohomology for cocompact lattices: an analysis based on representation theory of G shows that there cannot be square-integrable harmonic forms with coefficients in V.

[4] A priori this is just defined up to homothety, but as we need a parametrisation of the cusps of M to write down the trace formula and define analytic torsion, we can use the same parametrisation to identify the torus T with a cross-section, which has then a well-defined Riemannian structure.

2.3.2 Addendum

The statements in [92] and [91] are significantly weaker than that of Theorem 2.5 in that they ask for an additional condition on the geometry of the M_n. However, while writing this thesis, I realised that the need for this condition can be removed in a very simple way that I will explain below. Lemma 2.2 in [92] states that if $\mathcal{N}^*_\lambda(R)$ counts the number of nonzero vectors of length $\leq R$ in a 2-dimensional Euclidean lattice Λ, and $\alpha_1(\Lambda) \leq \alpha_2(\Lambda)$ are the successive minima of Λ, then

$$\left|\mathcal{N}^*_\lambda(R) - \frac{\pi R^2}{\operatorname{vol}(\Lambda)}\right| \leq C\left(\frac{R}{\alpha_1(\Lambda)} + \frac{\alpha_2(\Lambda)}{\alpha_1(\Lambda)}\right)$$

for some constant C independent of Λ. In fact, this can be replaced for $0 \leq R \leq \sqrt{\alpha_1(\Lambda)\alpha_2(\Lambda)}$ by the following more precise estimate:

$$\left|\mathcal{N}^*_\lambda(R) - \frac{\pi R^2}{\operatorname{vol}(\Lambda)}\right| \leq C\frac{R}{\alpha_1(\Lambda)}.$$

In order to see this we need just observe that in the range $0 \leq R \leq \alpha_2(\Lambda)$ (so in particular $R \leq \sqrt{\alpha_2(\Lambda)\alpha_1(\Lambda)}$), we have $\mathcal{N}^*_\lambda(R) = \left\lfloor \frac{R}{\alpha_1(\Lambda)} \right\rfloor$ so that

$$\left|\mathcal{N}^*_\lambda(R) - \frac{\pi R^2}{\operatorname{vol}(\Lambda)}\right| \leq \left\lfloor \frac{R}{\alpha_1(\Lambda)} \right\rfloor + \frac{\pi R^2}{\operatorname{vol}(\Lambda)} \ll \frac{R}{\alpha_1(\Lambda)}$$

where the second inequality follows from Minkoswki's theorem which states that $\operatorname{vol}(\Lambda) \asymp \alpha_1(\Lambda)\alpha_2(\Lambda) \geq R^2$.

This improved estimate allows to ignore hypothesis (3) in Theorem A of [92] and replace the corresponding hypothesis in Theorem B by $\operatorname{vol}(M_n)_{\leq R} \ll \operatorname{vol}(M_n)/\log(\operatorname{vol} M_n)^{20}$. In particular these two theorems apply to a sequence $(\Gamma_n)_{n\geq 1}$ of congruence subgroups in a Bianchi group, by [91, Theorem B].

2.3.3 Growth of regulators and Eisenstein cohomology

According to (2.1) we have that

$$\begin{aligned}\log|H^2(M_n, L)_{\text{tors}}| = \tau(M_n, L) &+ \log\operatorname{vol} H^2(M_n, L)_{\text{free}} \\ &+ \log|H^1(M_n, L)_{\text{tors}}| - \log\operatorname{vol} H^1(M_n, L)_{\text{free}}.\end{aligned}$$

We can thus deduce Theorem 2.4 from Theorem 2.5 together with the following estimates:

$$\log|H^1(M_n, L)_{\text{tors}}| = o(\operatorname{vol} M_n); \tag{2.2}$$

$$\log\operatorname{vol} H^2(M_n, L)_{\text{free}} = o(\operatorname{vol} M_n); \tag{2.3}$$

$$\liminf_{n\to+\infty} \frac{\log\operatorname{vol} H^1(M_n, L)_{\text{free}}}{\operatorname{vol} M_n} \geq 0. \tag{2.4}$$

The estimate (2.2) is proven in [91, Lemma 6.6], and (2.3) is proven in loc. cit., Lemma 6.8. Both proofs use only elementary estimates on the boundary and on co-invariants and duality tricks. We will spend a bit more time discussing (2.4), and how to attempt a proof of its converse inequality.

Eisenstein cohomology and the regulator in degree 1

We give a short account (limited to our setting of $\mathrm{SL}_2(\mathbb{C})$) of G. Harder's theory of Eisenstein cohomology. We do so in classical language; let M be an orientable hyperbolic 3–manifold with h cusps. Let $T_1, \dots, T_h$ be cross-sections of the cusps, so that each T_j is a flat torus and $\bigsqcup_{j=1}^h T_j$ is identified with the boundary $\partial\overline{M}$ of the Borel–Serre compactification of M. Recall that V is the space $\mathrm{Sym}^m(\mathbb{C}^2)$; we define $H^{1,0}(T_j, V)$ to be the 1-dimensional space of holomorphic forms on T_j with coefficients in V, and $H^{0,1}(T_j, V)$ the space of anti-holomorphic forms. By classical Hodge theory, we have $H^1(T_j, V) = H^{1,0}(T_j, V) \oplus H^{0,1}(T_j, V)$. Using Eisenstein series, it is possible to construct :

- A map E from the space $H^{1,0}(\partial\overline{M}, V)$ of holomorphic 1-forms on $\partial\overline{M} = \bigsqcup_{j=1}^h T_j$ (with coefficients in V) to $H^1(M, V)$;
- A map $\Phi : H^{1,0}(\partial\overline{M}, V) \to H^{0,1}(\partial\overline{M}, V)$ (the space of anti-holomorphic forms on $\partial\overline{M}$)

such that for all $\omega \in H^{1,0}(\partial\overline{M}, V)$, we have

$$\iota^* E(\omega) = \omega + \Phi(\omega).$$

In particular it follows from Lemma 2.6 and duality arguments that E is a surjective map and that, if π is the projection from $H^1(\partial\overline{M}, V) \to H^{1,0}(\partial\overline{M}, V)$, then $\pi \circ \iota$ is an isomorphism. Proving a lower bound for $\mathrm{vol}\, H^1(M_n, L)_{\mathrm{free}}$, from which the estimate (2.4) follows at once, is then quite easy as it boils down to understanding the index of $H^{1,0}(\partial\overline{M}, L) \oplus H^{0,1}(\partial\overline{M}, L)$ in $H^1(\partial\overline{M}, L)$. See [91, Lemma 6.7]. This finishes the proof of Theorem 2.4.

On the other hand, to give an upper bound for $\mathrm{vol}\, H^1(M_n, L)_{\mathrm{free}}$ (which would be necessary to arrive at a limit in the statement of Theorem 2.4) we need to estimate:

1. the product of singular values of the orthogonal projection $\pi : \iota^* H^1(M, V) \to H^{1,0}(\partial\overline{M}, V)$;
2. the index of $\pi\iota^* H^1(M, L)$ in $H^{1,0}(\partial\overline{M}, L)$.

To deal with the problem in 1, we need only to use a bound on the determinant of Φ which is provided by [91, Proposition 4.1]. On the other hand, the issues in 2 are much harder to deal with and we could only succeed in reducing the estimate to a number-theoretical statement for which we do not know whether it holds or not (see Proposition C in [91]).

In order to perform this reduction, we estimate the denominator of a class $E(\omega)$ for $\omega \in H^{1,0}(\partial\overline{M}, L)$, that is the smallest $d \in \mathbb{N}$ such that $dE(\omega) \in H^1(M, L)$ (this exists because the map Φ is defined over $\mathbb{Q}$, see for instance [54, Corollary 4.2.1]). Let N be the lowest common multiple of the denominators of $E(\omega)$, for ω in a basis of the free part of $H^{1,0}(\partial\overline{M}, L)$. Then $N \cdot H^{1,0}(\partial\overline{M}, L) \subset \pi\iota^* H^1(M, L)$ so we get an estimate on the index. This boils down (in part) to estimating the denominator of the intertwining map Φ, and as the latter can be expressed using L-functions, we arrive at a problem about algebraicity properties of special L-values. We will detail this in a slightly different setting where it can be resolved in the next section (see 2.4.1 below).

2.4 Exponential growth in symmetric powers of the standard representation

In a different direction one can ask for the asymptotic behaviour of $H_1(\Gamma, L_n)_{\mathrm{tors}}$ where Γ is a fixed arithmetic lattice in a semisimple group G and L_n are Γ-stable lattices in a sequence of representations ρ_n of G. To expect resonably precise results one might take the ρ_n to be the representations associated with the weights along a half-line in the weight lattice; in this setting, and for cocompact torsion-free Γ a reasonable analogue of the results of Bergeron–Venkatesh have been obtained by S. Marshall–W. Müller [68] (using analytic results of W. Müller [73]) and W. Müller–J. Pfaff (the analytic part is [74] and the application to torsion homology is given in [76]).

In this section we present a similar result about non-cocompact lattices in $\mathrm{SL}_2(\mathbb{C})$ obtained with J. Pfaff. The result is the following [85, Theorem A].

Theorem 2.7. *Let $\varepsilon > 0$ and k be an imaginary quadratic field. Let Γ be a principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_k)$ of sufficiently large level (depending on k, ε) and let L_n be the lattice $\mathrm{Sym}^n(\mathbb{Z}_k^2)$. Then*

$$\liminf_{n\to\infty} \left(\frac{\log |H_1(\Gamma, L_n)_{\mathrm{tors}}|}{n^2} \right) \geq \left(\frac{1}{2} - \varepsilon \right) \frac{\mathrm{vol}(M)}{2\pi} \tag{2.5}$$

and

$$\limsup_{n\to\infty} \left(\frac{\log |H_1(\Gamma, L_n)_{\mathrm{tors}}|}{n^2} \right) \leq (2+\varepsilon) \frac{\mathrm{vol}(M)}{2\pi}. \tag{2.6}$$

Conjecturally, there should be a limit and it should be equal to $\frac{\mathrm{vol}(M)}{2\pi}$. The scheme of proof is the same one as in the previous section, using a result of Pfaff–Müller for the growth of analytic torsion [75]. Then a generalisation of the Cheeger–Müller theorem for manifolds with cusps due to Pfaff [83] gives us exponential growth for Reidemeister torsion (not with tight bounds), see [85, Proposition 3.2].

The upper bound (2.6) in this theorem is deduced using the same method as for Theorem 1.31. A difference is that the matrix norms of the differentials in the cellular complex grow with n, but we can estimate this growth precisely and deduce that $\log |H_1(\Gamma, L_n)_{\mathrm{tors}}| = O(n^2)$ (see [85, Proposition 8.3]). Then we use a balancing trick to get the more precise bound (2.6), at the cost of having to take smaller congruence subgroups in the statement.

2.4.1 Estimating the denominator of Eisenstein classes

To deduce the lower bound in Theorem 2.7 from the exponential growth of the Reidemeister torsion defined by the formula (2.1), we need to deal with the term $\mathrm{vol}\, H^1(M, L_n)$. The method is the same one as that in §2.3.3 but we are able to (mostly) carry it to term in this setting. Namely, we prove the following result (see [85, Proposition 6.1]).

Theorem 2.8. *Let $d(n)$ be the smallest $d \in \mathbb{N}$ such that $d \cdot \Phi(\omega) \in H^{0,1}(\partial\overline{M}, L_n)$ for all $\omega \in H^{1,0}(\partial\overline{M}, L_n)$. Then $\log d(n) = O(n \log n)$.*

This is not sufficient to estimate the index of $\pi\iota^* H^1(M, L_n)$ in $H^{1,0}(\partial\overline{M}, L_n)$, as this helps only to check integrality against 1-cycles which are homologous to the boundary; in principle one also needs to check it for 1-cycles relative to the boundary (which could be done using techniques from [26, Chapter 5.7]). However we use another balancing trick to deduce a slightly weaker estimate from this theorem (see [85, Lemma 4.1]).

We finish this chapter with a few words on the proof of Theorem 2.8. It reduces (non-trivially) to proving a statement about certain particular 1-forms which, when viewed as functions on the adèle group associated with the arithmetic lattice $\mathrm{SL}_2(\mathbb{Z}_k)$, are eigenfunctions of the group of diagonal matrices; see loc. cit., sections 5 and 6.1. For such functions, the intertwining operator Φ essentially reduces to multiplying by the quotient of L-values

$$\frac{L(\chi, n+2)}{\pi \cdot L(\chi, n)}$$

where χ is the eigencharacter associated with the function on which we compute the intertwining operator; see loc. cit., Sections 5.5 and 6.3. Now estimating the denominator of the algebraic number above can be done along the lines of the proof of a result of Damerell [29, 30] which gives a direct proof (without Eisenstein cohomology) that it is algebraic. This is done in the Appendix to [85].

Chapter 3

Generic hyperbolic manifolds and related topics

3.1 Introduction

In the previous chapters, we described results solely about congruence arithmetic lattices in semisimple Lie groups. We now discuss the position of these particular lattices (and of their properties described there) with respect to the set of all lattices. There is in this regard a well-known trichotomy (which might reduce to a dichotomy) among semsisimple Lie group regarding the nature of their lattices:

- In groups of real rank at least 2 all (irreducible) lattices are arithmetic, and conjecturally they are all essentially congruence, meaning that for a given G there is a universal bound for the index of a lattice in its congruence closure;

- In the real rank 1 groups $\mathrm{Sp}(n,1)$, $n \geq 2$ and in the exceptional rank 1 group, all lattices are arithmetic and nothing is conjectured one way or the other about them being congruence;

- For the remaining rank 1 groups $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ knowledge is still patchy regarding this problem:

  - In the real rank 1 groups $\mathrm{SU}(n,1)$ with $n \geq 4$ no examples of non-arithmetic lattices are known, and there are examples of lattices with arbitrarily large index in their congruence closure (which also exist for $n = 2,3$);
  - In each of $\mathrm{SU}(2,1)$ and $\mathrm{SU}(3,1)$ there are finitely many commensurability classes of non-arithmetic lattices known (cf. [34, 32] for the most recent results on the problem);
  - In each group $\mathrm{SO}(n,1)$ there are infinitaly many known commensurability classes of non-arithmetic lattices [52], and in each arithmetic commensurability class (except for a specific family for $n = 7$) there are examples of arithmetic lattices with arbitrarily large index in their congruence closure (this follows from [70, 60] in dimensions > 3, and [5] in dimension 3; see [13] for relevant information on the exceptional lattices in $\mathrm{SO}(7,1)$).

It is not clear at present whether all $\mathrm{SU}(n,1)$ should have the same behaviour as $\mathrm{SO}(n,1)$ or whether arithmetic lattices should be essentially congruence in $\mathrm{Sp}(n,1)$ and the exceptional rank 1 group. In any case, in this chapter we concentrate on $\mathrm{SO}(n,1)$, especially in small dimensions.

The examples which establish the failure of the congruence subgroup property for $\mathrm{SO}(n,1)$ also show that the conclusion of Theorem 1.1 does not hold for more general lattices. More precisely, the papers of J. Millson and Millson–J.-S. Li [70, 60] quoted above show that any arithmetic manifold M of dimension $n \geq 4$ (except for the exceptional manifolds of dimension 7) has positive virtual first Betti number. That is, there exists a finite cover $M' \to M$ such that $\pi_1(M')$ admits a nontrivial morphism φ to $\mathbb{Z}$. Thus there is an infinite sequence of cyclic covers $M_k \to M'$ associated with the morphisms $\pi_1(M') \to \mathbb{Z}/k\mathbb{Z}$ obtained by composing φ with reduction modulo k. Obviously, the sequence $(M_k)_{k\geq 1}$ cannot be Benjamini–Schramm convergent to $\mathbb{H}^n$, and it follows by applying Theorem 1.2 that for k large enough they cannot be congruence either (and in fact their index inside their congruence closure must go to infinity with k). For $n = 3$, it is actually known by a result of I. Agol that all hyperbolic manifolds of finite volume have virtual positive first Betti numbers. In the complex hyperbolic case only arithmetic manifolds of the simplest type are known to satisfy this property, as follows from a theorem of D. Kazhdan [56], while for other arithmetic manifolds it is known that no congruence covers can have positive first Betti number [95].

In view of this, we ask whether Benjamini–Schramm convergence holds "generically". For $n \geq 4$ there is a natural way to make this precise: the set of volumes of lattices is locally finite

(Wang's finiteness theorem), so we can filter the set of hyperbolic manifolds by increasing volume[1] and ask whether the probability that an event holds is asymptotically almost sure. However, questions of this nature seem out of reach with our current understanding of non-arithmetic lattices.

The first result in this chapter, from [89], is that the known class of examples suffice to prove that among commensurability classes ordered by minimal volume, non-arithmetic classes predominate by a large margin. We note that the examples produced to this end do not satisfy the conclusion of Theorem 1.1. However, even barring the possibility of a still unknown more abundant source of non-arithmetic commensurability classes, this does not imply anything one way or the other regarding the generic behaviour, as lattices from a few arithmetic commensurability class may still account for a comparable number.

Since arithmetic lattices are not all congruence, and in fact it is known that generically in the sense above they are not[2], the question of generic behaviour is still interesting when restricted to arithmetic lattices. In view of Theorem 1.1, the main focus should be on studying generic finite-index subgroups in a single arithmetic lattice. The question then admits a purely group-theoretical reformulation: given a finitely generated group Γ, if $g \in \Gamma \setminus \{1\}$, does the proportion of subgroups of index n in Γ which contain g tend to 0 as $n \to +\infty$? This is known to be true when Γ is a non-compact lattice in $\mathrm{PGL}_2(\mathbb{R})$ (a result of E. Baker–B. Petri [10], generalising a result of B. Bollobás on free groups) and for surface groups by a result of M. Magee and D. Puder [65] (and it seems likely that the arguments on the latter paper could be generalised to include all cocompact lattices in $\mathrm{PGL}_2(\mathbb{R})$).

For lattices in higher-dimensional groups, starting with $\mathrm{SO}(3,1)$, essentially nothing is known about their generic finite-index subgroups. The most basic question, without an answer to which further progress seems unlikely, is to get a precise asymptotic information about the number of such groups. This fits within the topic of subgroup growth, which we now describe briefly. If Γ is a finitely generated group, let $s_n(\Gamma)$ denote the number of its subgroup of index n; we are interested in the asymptotic behaviour of $s_n(\Gamma)$ as $n \to +\infty$. In order to study this, the first step is to observe that it is enough to get a sufficiently fast-growing and well-behaved asymptotic equivalent for the number $h_n(\Gamma)$ of morphisms from Γ to the symmetric group $\mathfrak{S}_n$. The number of such morphisms is at most $(n!)^{d_\Gamma}$ where d_Γ is the minimal number of generators of Γ, so it is natural to first study whether the limit

$$\lim_{n \to +\infty} \frac{\log s_n(\Gamma)}{n \log(n)} \tag{3.1}$$

exists and to evaluate it. There is no answer to this in general, as the study of $h_n(\Gamma)$ must often involve a direct combinatorial approach searching for solutions in $\mathfrak{S}_n$ to the defining relations of Γ, which is very hard even when a rather simple presentation of Γ (say with 1 relation) is available. One general fact is that for groups which are large (including all 3-dimensional hyperbolic lattices by the work of Agol–Wise, and many arithmetic lattices in higher dimensions), the upper limit is positive—but it is not known whether the limit even exists.

Among lattices in hyperbolic spaces, perhaps the ones with the simplest presentations are right-angled Coxeter groups. For Fuchsian groups, a result of M. Liebeck and A. Shalev [61] gives a precise asymptotic equivalent to $s_n(\Gamma)$ which can be interpreted in a simple way from

[1] Note that in view of Theorem 1.30 and its conjectural generalisation to non-arithmetic manifolds, this filtration also makes topological sense.

[2] At least for lattices in $\mathrm{SO}(n,1)$ of the simplest type, or lattices in $\mathrm{SO}(3,1)$: for these, a result of Lubotzky [62] in the first case, and the results of Agol [5] in the second, show that they virtually surject onto a nonabelian free group, hence the number of subgroups of index $\leq n$ is asymptotically $\geq (n!)^a$ for some $a > 0$. On the other hand, the number of congruence subgroups is much smaller by [63].

the Coxeter diagram for right-angled reflection groups. In dimension 3, the only such group for which even the value for the limit (3.1) is known is the reflection group of the regular right-angled ideal octahedron (cf. [8, § 2.5.2]). For general right-angled Coxeter groups, there is a conjecture (generalising the two known facts above) for this value due to H. Baik, B. Petri and myself, which we proved for a large family of groups which unfortunately does not include any other hyperbolic reflection groups. On the other hand, for right-angled Artin groups the value of the limit (3.1) is known in general. For finer asymptotic information nothing is known outside of the simplest examples, virtually cyclic right-angled Coxeter groups (products of an infinite dihedral group with an elementary abelian 2-group), and the answer in this case is already quite complicated (see Theorem 3.11 below).

In the discussion above, we considered only subgroups of finite index in lattices of $\mathrm{SO}(n,1)$, but if we are ultimately interested in properties of the corresponding hyperbolic manifolds we should rather study isomorphism classes of such subgroups. When $n=2$, there is a dramatic difference between these two quantities: if $e_n(\Gamma)$ denotes the number of isomorphism classes of subgroups of index n in Γ and Γ is a torsion-free lattice in $\mathrm{SO}(2,1)$ then $e_n(\Gamma)=1$ for all $n \geq 1$. For lattices with torsion, the behaviour is slightly more complicated and can be related to the geometry of the quotient. As far as counting goes, for $n \geq 3$, there is no large difference between s_n and e_n but the finer relation between the two is still not very clear. For example, if Γ is a lattice in $\mathrm{SO}(3,1)$, it is not quite easy to produce infinitely many examples of non-isomorphic normal subgroups with the same quotient.

Finally, in dimensions 2 and 3, the situation with regard to the motivation of this chapter (understanding generic hyperbolic manifolds) is completely different. In dimension 2, the volume spectrum is still discrete (by the Gauss–Bonnet formula) but for each volume there is a coninuous moduli space of hyperbolic surfaces realising it. The flavour of our question in this setting is thus completely different but it has a positive answer by the work of M. Mirzakhani [71]. In dimension 3 the volume spectrum is not discrete (see for instance [53]). However there is a purely topological characterisation of hyperbolicity in this dimension, given by G. Perelman's solution to the Geometrisation conjecture (see [99, Conjecture 1.1] for the statement, and [81, 80] for the solution). Hence it is feasible to study hyperbolic 3–manifolds via a topological model for random manifolds, by proving that hyperbolicity is generic in this model. This has been done for random Heegaard splittings by J. Maher [66] following work by N. Dunfield and W. Thurston on this model [36]; a finer description of the generic geometric properties of these manifolds is given by P. Feller, A. Sisto and G. Viaggi in [38].

On the other hand, the constraint brought by fixing Heegaard genus seems unnatural with regards to what we expect of a random manifold. For our purposes it is better to study random triangulation on a fixed number (going to infinity) of tetrahedra. Unfortunately such triangulations are not (in the natural random model) homeomorphic to manifolds, and studying conditional properties on the subset of those which are seems extremely difficult. Nevertheless, removing the singular locus (the vertices of the triangulation) gives a random manifold with boundary which turns out to be hyperbolic with totally geodesic boundary and a well-understood geometry. This has a priori no bearing on the original problem but still seems interesting in its own right and perhaps as a stepping stone towards it.

3.2 Generic commensurability classes

The following result is [89, Corollary 1.3].

Theorem 3.1. *Let $n \geq 3$ and for $v > 0$, let $L_{\mathrm{com}}(v)$ (respectively $L_{a,\mathrm{com}}$) be the number of commensurability classes of lattices (respectively arithmetic lattices) in* $\mathrm{SO}(n,1)$ *which contain a* *lattice of covolume at most v. Then*

$$\lim_{v \to +\infty} \frac{L_{a,\mathrm{com}}(v)}{L_{\mathrm{com}}(v)} = 0.$$

The proof shows that $\log L_{\mathrm{com}}(v) \gg v$ (Theorem 1.1 in loc. cit.); this implies the statement above by a result of M. Belolipetsky which states that $\log L_{a,\mathrm{com}}(v) = o(v)$ [11]. A strenghening of this theorem is given in [50], where T. Gelander and A. Levit prove that $\log L_{\mathrm{com}}(v) \gg v \log(v)$.

The proof of the estimate $\log L_{\mathrm{com}}(v) \gg v$ is by a direct construction. Namely, we generalise the well-known construction on non-arithmetic hyperbolic manifolds by M. Gromov and I. Piatetski-Shapiro [52] as follows. Take, as they do, two closed arithmetic hyperbolic n-manifolds M_0, M_1 which are non-commensurable with each other, such that each contain a hyperbolic $(n-1)$-manifold Σ as an embedded totally geodesic hypersurface, and assume additionally that Σ is non-separating in both. For $i = 0, 1$ let N_i be the completion of $M_i \setminus \Sigma$, which is a compact hyperbolic n-manifold with totally geodesic boundary isometric to two copies of Σ; we label these arbitrarily with ± 1. Then any sequence $\alpha \in \{0,1\}^{\mathbb{Z}/n\mathbb{Z}}$ gives a hyperbolic manifold N_α obtained by gluing N_0, N_1 according to the pattern (identifying via the identity map the $+1$ boundary component of N_{α_i} to the -1 boundary component of $N_{\alpha_{i+1}}$). The claim is that these manifolds include sufficiently many commensurability classes to account for the estimate $\log L_{\mathrm{com}}(v) \gg v$, which is an immediate consequence of the following lemma, since there are 2^n manifolds obtained by the construction above and their volume is at most $n \cdot \max(\mathrm{vol}\, M_0, \mathrm{vol}\, M_1)$.

Lemma 3.2. *The manifolds N_α, N_β are commensurable to each other if and only if the sequences α, β can be obtained from each other by shifting or mirrorring the indices (that is $\alpha_i = \beta_{\pm i + i_0}$ for some $i_0 \in \mathbb{Z}/n\mathbb{Z}$). In particular there are exactly $2n$ manifolds in any given commensurability class among the N_α for $\alpha \in \{0,1\}^{\mathbb{Z}/n\mathbb{Z}}$.*

The proof of this is quite simple, resting on the following two lemmas from [52, Section 1.6] and [89, Lemma 3.2] respectively.

Lemma 3.3. *If Γ, Γ' are two arithmetic lattices in a semisimple Lie group G and $\Gamma \cap \Gamma'$ is Zariski-dense in G then Γ is commensurable with Γ'.*

Lemma 3.4. *Let M be a closed hyperbolic n-manifold. If U, U' are two submanifolds of M with totally geodesic boundary then either $U \cap U'$ has Zariski-dense monodromy in* $\mathrm{SO}(n,1)$ *or its interior is empty.*

From this it is immediate to deduce that if N_α, N_β are isometric then we must have $\alpha_i = \beta_{\pm i + i_0}$; the lemmas essentially imply that any isometry must send N_{α_0} inside a $N_{\beta_{i_0}}$ with $i_0 = 0$, and actually induce an isometry between them because they have the same volume. Then looking at the pieces on the right one by one, we prove by induction that $\alpha_i = \beta_{\pm i + i_0}$ (the sign $\pm$ depending on where the isometry sends the boundary components of N_{α_0}). The proof for non-commensurability is essentially the same though a bit more involved (see the proof of Proposition 2.1 in [89]).

Note that the proof for non-isometry sketched above generalises immediately to the situation where the pieces have d boundary components and we glue them along an (edge-colored) d-regular graph. However the argument for non-commensurability does not go through since it really uses the simple structure of the cyclic graph. T. Gelander and A. Levit in [50] manage to side-step this problem by adding "buffers" between N_0, N_1, that is, a third arithmetic piece which is not commensurable to any of them. This allows them to prove that the number of commensurability classes is comparable to the number of 4-regular graphs, which gives the lower estimate that $\log L_{\mathrm{com}}(v) \gg v \log(v)$.

3.3 Subgroup growth of right-angled groups

3.3.1 Subgroup growth and permutation representations

In order to study the function $n \mapsto s_n(\Gamma)$, the most useful fact is that

$$s_n(\Gamma) = \frac{t_n(\Gamma)}{(n-1)!}$$

where $t_n(\Gamma)$ is the number of transitive actions of Γ on the set $\{1, \dots, n\}$. This follows immediately from the fact that a subgroup of index n in Γ corresponds to an action of Γ (on its left cosets) which is transitive, and has a distinguished point (the identity coset); there are $(n-1)!$ possible labelings which do not change the distinguished point.

It often happens that $t_n(\Gamma) \sim_{n\to+\infty} h_n(\Gamma)$ where h_n is the number of all actions, in other words, morphisms from Γ to the symmetric groups $\mathfrak{S}_n$. Most of the known results on subgroup growth beyond polynomial growth amount to an estimate of $h_n(\Gamma)$.

3.3.2 Artin groups

For a finite graph $\mathcal{G}$ let $\Gamma^{\mathrm{Art}}(\mathcal{G})$ be the right-angles Artin group associated with $\mathcal{G}$. That is, $\Gamma^{\mathrm{Art}}(\mathcal{G})$ is defined by the presentation

$$\Gamma^{\mathrm{Art}}(\mathcal{G}) = \langle s_v,\, v \text{ a vertex of } \mathcal{G} \mid [s_v, s_w] = 1 \text{ if } v \text{ is adjacent to } w \rangle .$$

We also denote by $\alpha(\mathcal{G})$ the independence number of $\mathcal{G}$, that is the maximal cardinality of an independent set in $\mathcal{G}$—a set of vertices which are pairwise not adjacent in $\mathcal{G}$. The following result is [8, Theorem A].

Theorem 3.5. *For $\Gamma = \Gamma^{\mathrm{Art}}(\mathcal{G})$ we have*

$$\lim_{n\to+\infty} \frac{\log s_n(\Gamma)}{n \log(n)} = \alpha(\mathcal{G}) - 1.$$

The proofs that $\liminf_{n\to+\infty} \frac{\log s_n(\Gamma)}{n\log(n)} \geq \alpha(\mathcal{G}) - 1$ and $\limsup_{n\to+\infty} \frac{\log s_n(\Gamma)}{n\log(n)} \leq \alpha(\mathcal{G}) - 1$ are separate.

The lower limit is essentially immediate to deduce from the case of the free groups: for the free group F_r of rank r, it is well–known (see for instance [64, Theorem 2.1]) that the theorem holds, that is $\lim_{n\to+\infty} \frac{\log s_n(F_r)}{n\log(n)} = r - 1$. If $\alpha(\mathcal{G}) = r$, let $\{v_1, \dots, v_r\}$ be an independent set. There is a morphism from $\Gamma^{\mathrm{Art}}(\mathcal{G})$ to F_r sending the generator corresponding to v_i to the ith generator of F_r, and the remaining generators to 1. This is surjective, so $s_n(\Gamma^{\mathrm{Art}}(\mathcal{G})) \geq s_n(F_r)$, hence $\liminf_{n\to+\infty} \frac{\log s_n(\Gamma)}{n\log(n)} \geq r - 1 = \alpha(\mathcal{G}) - 1$.

The upper bound is proven by recurrence on the number of vertices of $\mathcal{G}$. To do so we fix an arbitary vertex $v_0 \in \mathcal{G}$. Let $v_1, \dots, v_d$ be its neighbours in $\mathcal{G}$ and $\sigma_0, \dots, \sigma_d$ the corresponding generators of Γ. We enumerate (arbitrarily) the remaining generators of Γ as $\sigma_{d+1}, \dots, \sigma_r$.

The upper bound on $h_n(\Gamma)$ is obtained by reducing it to more manageable summands as follows: for $\ell, K \geq 0$ let

$$L(\ell, K) = \{\rho \in \mathrm{Hom}(\Gamma, \mathrm{Sym}_\ell) : \text{all orbits of } \langle \rho(\sigma_1), \dots, \rho(\sigma_d)\rangle \text{ are of size } > K\}$$

and

$$S(\ell, K) = \{\rho \in \mathrm{Hom}(\Gamma, \mathrm{Sym}_\ell) : \text{all orbits of } \langle \rho(\sigma_1), \dots, \rho(\sigma_d)\rangle \text{ are of size } \leq K\}.$$

Then we have the following result [8, Lemma 4.2].

Lemma 3.6. *There is $C_0 > 1$ (depending only on $\mathcal{G}$) such that for all $n \geq 1$ we have :*

$$h_n(\Gamma) \leq C_0^n \sum_{m=0}^{n} \binom{n}{m} |L(n-m, K)| \cdot |S(m, K)|. \tag{3.2}$$

On the other hand the quantities L and S are not very hard to control directly; by using the induction hypothesis on $\mathcal{G} \setminus \{v_0\}$, it is proven in [8, Lemma 4.3] that:

Lemma 3.7. *There exists a constant $C_1 > 1$ (depending only on $\mathcal{G}$), such that for any $n \geq 2$ and $0 \leq m \leq n$, we have for $K = \lfloor \log(n) \rfloor$ that:*

$$|L(n-m, K)| \leq C_1^{n \log\log(n)} \left((n-m)!\right)^{\alpha(\mathcal{G})}. \tag{3.3}$$

Using the induction hypothesis on $\mathcal{G} \setminus \{v_0, \ldots, v_d\}$, we have [8, Lemma 4.4]:

Lemma 3.8. *There exists a constant $C_2 > 1$ (depending only on $\mathcal{G}$), such that for any $n \geq 2$ and $0 \leq m \leq n$, we have for $K = \lfloor \log(n) \rfloor$ that:*

$$|S(m, K)| \leq C_2^{n \log\log(n)} (m!)^{\alpha(\mathcal{G})}. \tag{3.4}$$

Together these three lemmas imply the upper bound on $h_n(\Gamma)$.

3.3.3 Coxeter groups

The right-angled Coxeter group $\Gamma^{\mathrm{Cox}}(\mathcal{G})$ associated with a graph $\mathcal{G}$ is defined like the Artin group but with the generators being involutions. In other words it has the following presentation:

$$\Gamma^{\mathrm{Cox}}(\mathcal{G}) = \left\langle s_v,\ v \text{ a vertex of } \mathcal{G} \;\middle|\; s_v^2,\ [s_v, s_w] = 1 \text{ if } v \text{ is adjacent to } w \right\rangle.$$

For right-angled Coxeter groups, a lower bound for the lower limit can be proven as for Artin groups: if $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are pairwise non-adjacent cliques in $\mathcal{G}$, then there is a surjection from $\Gamma^{\mathrm{Cox}}(\mathcal{G})$ to the free product $(\mathbb{Z}/2\mathbb{Z})^{|\mathcal{C}_1|} * \cdots * (\mathbb{Z}/2\mathbb{Z})^{|\mathcal{C}_r|}$. For the latter it is easily seen that

$$\lim_{n \to +\infty} \frac{\log s_n((\mathbb{Z}/2\mathbb{Z})^{|\mathcal{C}_1|} * \cdots * (\mathbb{Z}/2\mathbb{Z})^{|\mathcal{C}_r|})}{n \log n} = \sum_{i=1}^{r} \left(1 - \frac{1}{2^{|\mathcal{C}_i|}}\right) - 1$$

whenever the right-hand side is non-negative (i.e. as soon as $r \geq 2$), so we get that

$$\lim_{n \to +\infty} \frac{\log s_n(\Gamma^{\mathrm{Cox}}(\mathcal{G}))}{n \log n} \geq \max_{\mathcal{C}_1, \ldots, \mathcal{C}_r} \sum_{i=1}^{r} \left(1 - \frac{1}{2^{|\mathcal{C}_i|}}\right) - 1.$$

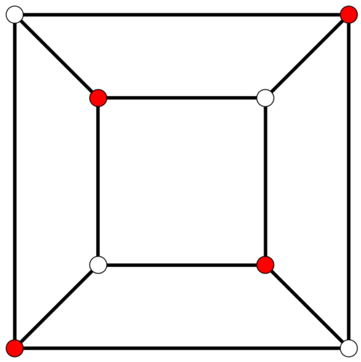
We will denote $\gamma(\mathcal{G}) = \max_{\mathcal{C}_1, \ldots, \mathcal{C}_r} \sum_{i=1}^{r} \left(1 - \frac{1}{2^{|\mathcal{C}_i|}}\right)$. We define a class $\mathcal{AT}$ of finite graphs as follows: it contains all finite trees; all graphs obtained by adding to a tree new vertices, each of which is adjacent to only leaves in the tree, and no two new vertices are adjacent to the same leaf; and all graphs obtained from the previous ones by adding vertices which are adjacent only to the new vertices (and no two are adjacent to the same vertex). For graphs in $\mathcal{AT}$, we can prove that the limit is actually equal to $\gamma(\mathcal{G}) - 1$, see [8, Theorem C].

Theorem 3.9. *Let $\mathcal{G}$ be a graph in the class $\mathcal{AT}$. Then*

$$\lim_{n \to +\infty} \frac{\log s_n(\Gamma)}{n \log(n)} = \gamma(\mathcal{G}) - 1.$$

A simple example

Let Γ be the subgroup of $\mathrm{SO}(3,1)$ generated by the reflections in the faces of a regular right-angled ideal octahedron. It is naturally isomorphic to the right-angled Coxeter group associated with the 1-skelethon $\mathcal{G}$ of the dual of the octahedron, namely the cube. It has $\gamma(\mathcal{G}) = 2$, realised by an independent set of 4 vertices as illustrated below.



In this case we can prove that $\lim(\log s_n(\Gamma)/n \log n) = 1$ by a simple argument. Removing all white vertices from the graph above we get a surjection from Γ to the free product $(\mathbb{Z}/2\mathbb{Z})^{*4}$ of four copies of $\mathbb{Z}/2\mathbb{Z}$, for which we have $\lim \log s_n((\mathbb{Z}/2\mathbb{Z})^{*4})/n \log n = 1$. On the other hand, removing the edges joining the two squares in $\mathcal{G}$ we get a surjection from the group

$$\Gamma' = (D_\infty \times D_\infty) * (D_\infty \times D_\infty)$$

to Γ (where D_∞ is the infinite dihedral group), for which it is also easy to show that $\lim \log s_n(\Gamma')/n \log n = 1$. It follows that we must have $\lim \log s_n(\Gamma)/n \log n = 1$ as well.

The case of trees

If $\mathcal{G}$ is a graph, S a set of vertices of $\mathcal{G}$ and $\underline{\pi}$ a function from S to the set of partitions of $\{1, \dots, n\}$ we define

$$h_n^{\underline{\pi}}(\Gamma^{\mathrm{Cox}}(\mathcal{G})) = |\{\rho \in \mathrm{Hom}(\Gamma^{\mathrm{Cox}}(\mathcal{G}), \mathrm{Sym}_n) : \forall v \in S,\ \rho(s_v) \in \mathrm{Stab}(\pi_v)\}|.$$

For trees, we can prove the following theorem [8, Proposition 5.5], which implies in particular that they satisfy the conclusion of Theorem 3.9.

Theorem 3.10. *Let $\mathcal{T}$ be a tree and S a set of leaves of $\mathcal{T}$. Let $\underline{\pi} = (\pi_v, v \in S)$ as above, such that every partition π_v has only blocks of size 1 or 2; we denote by $\pi_{v,2}$ the number of those of size 2. Then there exists $C > 1$ such that we have*

$$h_n^{\underline{\pi}}(\Gamma^{\mathrm{Cox}}(\mathcal{G})) \le C^{n \log\log(n)} \frac{(n!)^{\gamma(\mathcal{G})}}{(n!)^{\frac{1}{2n}\sum_{v \in S} \pi_{v,2}}}.$$

This is proven by induction starting with $|\mathcal{T}| = 4$. For $|\mathcal{T}| = 1, 2$ it is elementary (see loc. cit., Lemma 5.2 and Lemma 5.3); for $|\mathcal{T}| = 3$ it is quite harder and we use a result of Liebeck and Shalev [61] (see loc. cit., Lemma 5.4). From there it is quite easy to proceed by removing well-chosen subsets of leaves from the tree and applying the induction hypothesis.

Completing the proof

It is quite easy to see using Theorem 3.10 that if $v_0 \in \mathcal{G}$ verifies that the subgraph $\mathcal{G} \setminus \{v_0\}$ is a tree $\mathcal{T}$, and all vertices of $\mathcal{T}$ which are adjacent in $\mathcal{G}$ to v_0 are leaves, then the conclusion of Theorem 3.9 holds for $\mathcal{G}$; the main remark is that if $\mathcal{G}$ is not a tree itself then $\gamma(\mathcal{G}) = \gamma(\mathcal{T})$. This works also when we add vertices to a tree which are adjacent to disjoint subsets of leaves.

In order to deal with the case where a second layer of vertices is added, we use the same strategy as for the induction steps in the proof of the result for right-angled Artin groups presented above; though we need to take into account the constraints on the vertices as well for this case, because the behaviour of the invariant γ when removing a 1-neighbourhood of a vertex is more complicated than that of the independence number. See [8, Section 5.4] for details.

3.3.4 Virtually cyclic groups and their free products

The only virtually cyclic Artin group is the infinite cyclic group. On the other hand there are infinitely many virtually cyclic righ-angled Coxeter groups: these are exactly the groups $D_\infty \times (\mathbb{Z}/2\mathbb{Z})^r$ for $r \geq 0$, where $D_\infty \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is the infinite dihedral group. For these groups, there is an explicit formula for $s_n(\Gamma)$ which is not very hard to prove though its statement is a bit cumbersome [9, Corollary 4.2].

On the other hand, the asymptotic formula for $h_n(D_\infty \times (\mathbb{Z}/2\mathbb{Z})^r)$ is much more involved, but it gives the following asymptotic for the subgroup growth of free products [9, Theorem 5.9].

Theorem 3.11. *Let $m \geq 2$ and $r_1, \dots, r_m \in \mathbb{N}$. Let Γ be the free product of the groups $D_\infty \times (\mathbb{Z}/2\mathbb{Z})^{r_i}$, then*

$$s_n(\Gamma) \sim a\, n^{c+1} e^{b\sqrt{n}} (n!)^{m-1}$$

for some $a, b > 0$ and $c \in \mathbb{R}$.

The constants a, b, c have very explicit though quite complicated formulas in terms of the r_i.

The main ingredient for the proof of Theorem 3.11 is a precise asymptotic for the number of permutation representations of a virtually cyclic Coxeter group; if $\Gamma = D_\infty \times (\mathbb{Z}/2\mathbb{Z})^r$ we prove that

$$h_n(\Gamma) \sim a_r n^{c_r+1} e^{b_r\sqrt{n}} n!.$$

The proof of this is completely disjoint from the arguments in the previous subsection, and it uses more advanced techniques from analytic combinatorics. Namely we compute an explicit form for the exponential generating function for the sequence $h_n(\Gamma)$ [9, Theorem 4.1].

Theorem 3.12. *Let $\Gamma = D_\infty \times (\mathbb{Z}/2\mathbb{Z})^r$ and $G_r(x) = \sum_{n\geq 1} \frac{h_n(\Gamma)}{n!} x^n$. There are coefficients s_j, $1 \leq j \leq r$ depending only on r such that*

$$G_r(x) = \prod_{j=0}^{r} \left(\left(1 - x^{2^{j+1}}\right)^{-s_{2j}/2} \exp\left(-2^j\, s_{2j} + \frac{2^j\, s_{2j}}{1 - x^{2^j}} \right) \right).$$

The proof of this is by direct combinatorial methods, which give an explicit expression for $h_n(\Gamma)$; it is not quite manageable but can be used to decompose G_r as a product of functions $F_j^{(2)}$. The coefficients for these functions are more manageable and it is possible to derive a differential equation satisfied by each $F_j^{(2)}$, from which the expression in the theorem follows.

Using this theorem and techniques similar to those in [55], we derive the asymptotic equivalent for $h_n(\Gamma)$; from its form we can deduce that for $t_n(\Gamma)$ and hence the asymptotic formula appearing in Theorem 3.11.

3.4 Finite covers of 2–orbifolds and 3–manifolds

Beyond subgroup growth, one can study the number $e_n(\Gamma)$ of isomorphism types of subgroups of index n in Γ. This makes more sense in a topological context: if $\Gamma = \pi_1(M)$ for an aspherical smooth manifold M, then $e_n(\Gamma)$ is equal to the number of homeomorphism types of degree n connected covers of M, at least if the Borel conjecture is true. This may seem like a big "if" but in many cases it is nonconditional (for example for nonpositively curved manifolds). Given Mostow's rigidity theorem, for hyperbolic manifolds in dimensions at least 3 (and other locally symmetric spaces), this also gives the number of isometry classes of finite covers.

3.4.1 Distinct subgroups in a Fuchsian group

For Fuchsian groups the situation is quite different; for example any surface or free group has only 1 isomorphism class of subgroups of index n for a given n, while a hyperbolic surface of which it is the fundamental group has many more non-isometric covers of degree n. When there is torsion the same gap persists between subgroups and covers but it turns out that the number of isomorphism classes has a more interesting behaviour. A very nice example is given by the following result which is mentioned without proof in the introduction of [44].

Theorem 3.13. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Then*

$$e_n(\Gamma) \sim \frac{n^2}{6}.$$

The proof of this is mostly given by a result of T. Müller and J. C. Schlage-Puchta [72] which states that all groups of the form $(\mathbb{Z}/2\mathbb{Z})^{*k} * (\mathbb{Z}/3\mathbb{Z})^{*l} * \mathbb{Z}^{*r}$ with $k/2 + 2l/3 + r > 1$ are realisable as subgroups of finite index in $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. If Λ is such a subgroup its Euler characteristic is given by

$$\chi(\Lambda) = \frac{k}{2} + \frac{2l}{3} + r - 1$$

and in particular $\chi(\Gamma) = \frac{1}{6}$, so we must have

$$[\Gamma : \Lambda] = \frac{\chi(\Lambda)}{\chi(\Gamma)} = 3k + 4l + 6r - 6.$$

By Müller and Schlage–Puchta's result, for n large enough, the number of isomorphism classes among subgroups in Γ of index n is thus equal to the number of non-negative integer solutions (k, l, r) to the equation $3k + 4l + 6r = n + 6$. For convenience, we will estimate the number of such for index $(n - 6)$; then it is equal to the number of points of the lattice $\frac{1}{n}\mathbb{Z}^3$ in the triangle given by the intersection of the positive octant in $\mathbb{R}^3$ with the plane $3x + 4y + 6z = 1$; the area of the triangle is equal to $\sqrt{61}/6$ while the co-area of the intersection of $\frac{1}{n}\mathbb{Z}^3$ with the plane is equal to $\sqrt{61}/n^2$, hence the number of points is asymptotically $n^2/6$.

For general Fuchsian groups, a similar result is true with regard to degree of growth. We did not carry out the computation of the full asymptotic estimate, which should be feasible at least for non-cocompact groups. This is [44, Proposition 2.2], which is proven by methods quite similar to the above result, generalising enough of the result of Müller–Schlage-Puchta to cocompact groups.

Theorem 3.14. *Let Γ be a cofinite Fuchsian group, let $m_1, \dots, m_s$ be the cardinalities of a family of representatives of the conjugacy classes of its maximal finite subgroups (geometrically this corresponds to the orders of the cone points on the orbifold $\Gamma\backslash\mathbb{H}^2$). Let d be the total number of divisors of all m_i. Then*

$$e_n(\Gamma) \asymp n^{d-1}.$$

3.4.2 Regular covers of hyperbolic 3–manifolds

Even for groups whose subgroup growth is well-understood (which is not the case of fundamental groups of hyperbolic manifolds), their normal subgroup growth (the asymptotic behaviour of the number of normal subgroups of index n) seems to be much more complicated. For example, for the free group of rank d, this amounts to counting the number of finite groups with at most d generators, so the function must be very irregular (for instance it takes value 1 at each prime number). We can still prove the following result about hyperbolic 3–manifolds, which implies in particular that they have infinitely many non-isometric pairs of finite covers with the same degree [44, Proposition 4.5].

Theorem 3.15. *Let M be a connected closed hyperbolic 3–manifold. There are $C, \alpha > 0$ (depending on M) such that if c_n denotes the number of non-isometric manifolds which are regular covers of M whose Galois group contains a cyclic subgroup of index at most C then*

$$\limsup_{n \to +\infty} \frac{c_n}{n^\alpha} > 0.$$

The proof is quite straightforward; if $b_1(M) \geq 2$ we can produce many non-isomorphic finite-index subgroup with cyclic quotient. By the solution to the Virtually Haken conjecture we can always take a finite index subgroup Δ in Γ such that $b_1(\Delta \backslash \mathbb{H}^3) \geq 2$; then we take the normal cores in Γ of the subgroups with abelian quotients in Δ and we check that they give enough subgroups to prove the proposition. This is not completely obvious and we use the fact that the quotients of Δ are abelian in an essential way.

3.5 Random 3-dimensional triangulations and manifolds

3.5.1 A random model for 3–manifolds with boundary

Let T be a complex obtained by gluing finitely many tetrahedra (4-simplices) along their faces so that each face belongs to exactly 2 tetrahedra. Then T is not necessarily a 3–manifold but each point in T which is not a vertex of a tetrahedron has a neighbourhood in T which is homeomorphic to a 3–ball, so removing from T a conical neighbourhood of every vertex gives a manifold with boundary. The manifold is orientable if the gluings were made by orientation-reversing maps.

In [82] we introduce a model for 3–manifolds by taking a random simplicial complex satisfying the properties above and excising the vertices (it is well-known that generically, the random complex T_n obtained by gluing n tetrahedra along a 4-valent graph is not a manifold as $n \to +\infty$, see [36, §2.1]). The model for the complex is natural: we choose uniformly at random an involution σ on $\{1, \ldots, 4n\}$, without fixed points. Viewing $\{1, \ldots, 4n\}$ as the set of faces of n tetrahedra, we choose for each transposition in the cycle decomposition of σ an orientation-reversing identification between the two corresponding faces, uniformly at random among the 3 possibilities. For technical reasons, we also ask that in the dual graph to the complex there be no loops or bigons (this condition has positive probability as $n \to +\infty$).

We denote by M_n the resulting random manifold. It is oriented (by construction) and asymptotically almost surely connected.

3.5.2 Generic properties

It turns out that despite its purely combinatorial construction the random manifold M_n has a very rigid geometric behaviour, as illustrated from the following properties from [82, Theorem 1.2].

Theorem 3.16. *Let M_n be the asymptotically almost surely connected random manifold described above. Then asymptotically almost surely as $n \to +\infty$ the manifold M_n has an hyperbolic structure with totally geodesic boundary, and its hyperbolic volume is $\sim v_O \cdot n$ where v_O is the volume of a regular right-angled ideal octahedron in $\mathbb{H}^3$.*

In [82, Theorems 1.1, 1.2] we also prove more topological and geometric properties of M_n: the boundary component is connected has genus $\sim n$, the first Betti numbers are $\sim n$, the hyperbolic structure has a uniform spectral gap, and it converges in Benjamini–Schramm topology to an explicit invariant random subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Here we only explain the proof of the hyperbolisation theorem above, which contains all the main ingredients for the proof of the other results.

The proof of Theorem 3.16 proceeds via explicit hyperbolisation of the manifold M_n under certain combinatorial conditions on the triangulation T_n; namely (in addition to excluding loops and bigons in the dual graph) we ask that its dual graph has few short cycles, and that cycles which loop around an edge of the triangulation are disjoint up to length $n^{1/3}$. These conditions are proven to be generic in [82, Theorem 2.4].

We can also use T_n to construct a hyperbolic manifold of finite volume with totally geodesic boundary as follows: pick a vertex in the middle of each edge of T_n and truncate the tetrahedra along the triangles spanned by these vertices. Each tetrahedron is replaced by an octahedron whose vertices correspond to the edges of the tetrahedron. Identifying each octahedron with the regular right-angled ideal tetrahedron in $\mathbb{H}^3$, we obtain a hyperbolic manifold X_n with totally geodesic boundary and finite volume.

The cusps of X_n correspond to the edges of T_n. We can use this to describe X_n as the interior of a compact manifold, whose boundary contains annuli each corresponding to a cusp of X_n. Indeed, the neighbourhood of the cusp of X_n corresponding to the edge e of T_n is made up of the cusps of the ideal octahedra coming from the tetrahedra of T_n around e. Each of these octahedral cusps has a cross section which is a square; hence, when we remove the neighbourhood, we add to the boundary a surface made up by gluing squares cyclically, which is an annuli.

We can recover M_n from X_n as follows: if we glue cylinders $[0, 1] \times \mathbb{D}^2$ to the annuli on the boundary[3] of this compactification of X_n, we obtain a compact manifold homeomorphic to M_n. Indeed, this amounts to blowing up each ideal vertex of X_n to an edge, and each octahedron in X_n becomes a truncated tetrahedron, which glue together to form M_n.

If we exclude triangles from the dual graph we can apply the 2π-theorem [20, Theorem 9 on p. 816] to conclude that M_n, as a Dehn filling of X_n, must be generically hyperbolic. Under our slightly more general condition, and in order to have a better control over the distortion in the hyperbolic metric on the thick part induced by the filling, we use a more explicit procedure. We refer to [82, Lemma 3.6] and its proof for detailed statements. First we replace all octagons which are adjacent to cusps of X_n corresponding to an edge of T_n around which there are at most $n^{1/4}$ tetrahedra (we call such cusps *small cusps* in the sequel) by compact polygons realising the new hyperbolic metric. This is not obvious to perform : first we need to invoke Andreev's theorem to construct the polygons. Then, to ensure that we can perform the gluing isometrically, we observe that generically small cusps correspond to edges lying far apart in T_n. This procedure affects a relativaly small part of the manifold by the generic hypothesis that there are few short cycles, and we have a bilipschitz control with uniform constant over the distortion in the thick part. We call the resulting manifold Z_n; the observation above shows that in particular $\mathrm{vol}(Z_n) \sim \mathrm{vol}(X_n)$.

After that we use explicit drilling/filling estimate of D. Futer, J. Purcell and S. Schleimer [46] to control the distortion when filling the remaining cusps in Z_n; this is carried out in [82, Lemma 3.7]. In short we get that (in addition to M_n being hyperbolic), for an arbitrarily small

[3]This also amounts to performing the only mirror-invariant Dehn filling on the double of X_n.

$\delta > 0$ the distortion on the δ-thick part between Z_n and its filling M_n, tends to zero as $n \to +\infty$. It follows that $\mathrm{vol}(M_n) \sim \mathrm{vol}(Z_n)$, so $\mathrm{vol}(M_n) \sim \mathrm{vol}(X_n) = nv_O$ as well.

Chapter 4

Invariant random subgroups

4.1 Introduction

In this last chapter, we study in some depth invariant random subgroups in semisimple Lie groups, which were introduced in the first chapter as a tool for proving results on Benjamini–Schramm convergence. As we saw there, in most higher-rank groups the Stuck–Zimmer theorem applies so the space of invariant random subgroups is completely understood (see Theorem 1.16). In those which have factors of rank 1 without property (T), nothing is known one way or the other (for irreducible invariant random subgroups) and we will not discuss the situation here.

We will therefore limit ourselves here to studying invariant random subgroups in rank 1 Lie groups, mostly $\mathrm{SO}(n, 1)$. It is known that there are many such in these groups beyond the trivial ones and the μ_Γ for Γ a lattice because cocompact lattices in a rank 1 group are Gromov-hyperbolic groups and these always have infinite normal subgroups of infinite index. In $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$ (and in the exceptional rank 1 group) no other constructions are known at present. In [3] several other constructions are presented of invariant random subgroups in the groups $\mathrm{SO}(n, 1)$ whose support does not intersect the set of subgroups of all lattices in $\mathrm{SO}(n, 1)$. These constructions essentially follow known constructions of hyperbolic manifolds: in dimensions 2 and 3, the geometric tools at disposal allow for specific constructions, while for dimensions $n \geq 4$ there is a construction inspired by Gromov and Piatetski-Shapiro's examples of nonarithmetic manifolds.

On the other hand, discrete subgroups of semisimple Lie groups appearing in the support of invariant random subgroups should be somewhat constrained. The first instance of this is the generalisation of Borel density theorem we encountered in the first chapter (see Theorem 1.14). Beyond that result, invariant random subgroups share more properties with the special case of normal subgroups of lattices: their limit sets must be the whole boundary at infinity [3], the associated quotient spaces must have either zero, one, two or infinitely many infinite-volume ends [18] (this generalises the Hopf–Freudenthal theorem on ends of groups).

For surfaces (that is, invariant random subgroups in $\mathrm{SL}_2(\mathbb{R})$), these properties imply much more detailed results. The result on limit sets implies that any invariant random subgroup in $\mathrm{SL}_2(\mathbb{R})$ which is supported on finitely generated subgroups[1] must be supported on lattices. The result on ends gives a complete list of surfaces of infinite type that can appear as quotients of $\mathbb{H}^2$ by invariant random subgroups. In higher dimensions not much is known; a result of M. Abért–I. Biringer [4] shows that invariant random subgroups in $\mathrm{SL}_2(\mathbb{C})$ supported on torsion-free, finitely generated subgroups must in fact be supported on doubly degenerate surface groups and lattices. A result of I. Gekhtman–A. Levit [47] gives more information about the behaviour at infinity of invariant random subgroups in rank-1 groups via their critical exponents.

4.2 Properties of invariant subgroups in rank 1

4.2.1 The no-core principle

In this subsection we present some results on invariant random subgroups due to M. Abért–I. Biringer and I. Biringer–O. Tamuz [4, 19]. The no-core principle is the following theorem, which gives a precise meaning to the statement that it is not possible to measurably pick a finite-volume region in a conjugacy class in Sub_G unless it is itself of finite (G-invariant) volume. We give an algebraic statement, a more geometric version[2] in a slightly different setting is given in [4, Theorem 1.15].

[1] Note that finite generation is a Chabauty-closed property.

[2] They are equivalent though the correspondence of discrete subgroups of G with pointed X-orbifolds, given by $\Lambda \mapsto (\Lambda\backslash G/K, \Lambda K)$.

Theorem 4.1. *Let G be a semisimple Lie group and A a Borel subset in Sub_G. If ν is a discrete, torsion-free invariant random subgroup in G then for ν-almost every $H \in \mathrm{Sub}_G$, either*

$$\mathrm{vol}_{G/H}\{gH : gHg^{-1} \in A\} = 0 \text{ or } +\infty$$

or H is cofinite in G.

The proof of this is not hard given the following theorem due to I. Biringer and O. Tamuz [19, Theorem 1.4]. We denote by Cos_G the space of (left or right) cosets of closed subgroups in G[3]; it has a Chabauty topology as a closed subset of the set of closed subsets of G.

Theorem 4.2. *Let G be a unimodular locally compact topological group and ν an invariant random subgroup in G. Then for any Borel function f on Cos_G we have*

$$\int_{\mathrm{Sub}_G} \int_{H\backslash G} f(g^{-1}H) dg d\nu(H) = \int_{\mathrm{Sub}_G} \int_{H\backslash G} f(Hg) dg d\nu(H)$$

We sketch the argument following the one given in a geometric setting in [4, Section 2.1]. Assume that $A \subset \mathrm{Cos}_G$ is such that $0 < \mathrm{vol}_{H\backslash G}(A \cap (H\backslash G)) < +\infty$ with positive ν-probability. Then there is a $R > 0$ such that $0 < \mathrm{vol}_{H\backslash G}(A \cap (H\backslash G)) \le R$, still with positive ν-probability. Let $\overline{A}$ be the closure in Chabauty topology of all conjugates gAg^{-1} for $g \in G$; then we still have that $0 < \mathrm{vol}_{H\backslash G}(\overline{A} \cap (H\backslash G)) \le R$ with positive probability, and since $\overline{A}$ is G-invariant, we may assume that ν is supported on the subgroups in $\overline{A}$. It follows that

$$\int_{\mathrm{Sub}_G} \int_{H\backslash G} 1_A(gHg^{-1}) dg d\nu(H) \le R$$

in particular it is finite. On the other hand

$$\int_{\mathrm{Sub}_G} \int_{H\backslash G} 1_A(H) dg d\nu(H)$$

is infinite unless ν-almost every H is cofinite, since the integrand equals $\mathrm{vol}(H\backslash G)$ and we assume $\nu(A) > 0$. By the mass transport principle (Theorem 4.2) applied to the function on Cos_G defined by $Hg \mapsto 1_A(H)$, $gH \mapsto 1_A(gHg^{-1})$, these two integrals are equal, and Theorem 4.1 follows.

4.2.2 Limit sets

If H is a discrete subgroup in G, its limit set is the subset of the visual boundary $\partial_\infty X$ given by limit points of some (or any) orbit of H in X. If it is infinite (which is the case for example if H is Zariski-dense in G) then it is also the unique minimal closed nonempty H-invariant subset in the boundary. The following result can be seen as a geometric strengthening of the Borel density theorem 1.14; it is a particular case of [3, Proposition 11.3].

Theorem 4.3. *Let G be a simple Lie group of real rank 1 and ν an invariant random subgroup in G, such that $\nu(\{G\}) = 0$. Then for ν-almost every $H \in \mathrm{Sub}_G$ either H is central in G or it has full limit set in $\partial_\infty X$.*

A generalisation to invariant random subgroup in isometry groups of CAT(0) spaces is proven by B. Duchesne, Y. Glasner, N. Lazarovich and J. Lécureux [35].

We describe how to deduce this from the no-core principle; a different (detailed) argument is given in [3], which is not much longer or complicated. Fix a point $\xi \in \partial_\infty X$. Let A be the set

[3]They correspond to doubly pointed X-orbifolds.

of discrete, torsion-free, non-central subgroups H of G such that ξ does not belong to the limit set of H, and the root $He_GK \in H\backslash G/K$ is at distance at most 1 of the projection of ξ onto the convex hull in G/K of the limit set. This is obviously a Borel subset of Sub_G. Moreover, for any H in the support of ν such that ξ is not in the limit set of H, the intersection of A with the conjugacy class of H lifts (via the covering map $H\backslash G \to N_G(H)\backslash G$) to a finite-volume subset of $H\backslash G$ (since its preimage maps to a ball of radius 1 in $H\backslash G/K$). By the no-core principle it follows that ν-almost surely ξ belongs to the limit set of H, or ν is supported on central subgroups. Considering only the first possibility, and taking a dense countable subset of $\partial_\infty X$ we see that ν-almost surely the limit set of H contains a dense subset of $\partial_\infty X$ so it must be equal to $\partial_\infty X$.

4.2.3 Topology of unimodular random hyperbolic surfaces

An end of a locally compact, second-countable connected topological space M is an element of the projective limit $\lim_{n\to+\infty} \pi_0(M \setminus C_n)$ where C_n is an increasing exhaustion of Y by compact subsets (it does not depend on such an exhaustion). The set of ends has a natural topology for which it is compact.

If M is a Riemannian manifold an end is said to have finite volume if for some n the corresponding component of $Y \setminus C_n$ has finite Riemannian volume (in case where Y is hyperbolic this means the end corresponds to a cusp of M). If M is a surface, and we take the C_n to be subsurfaces then we say that an end has infinite genus if for all n the corresponding component of $M \setminus C_n$ has positive genus (i.e. contains a nonseparating closed curve). The following theorem is then [18, Theorem 1.1].

Theorem 4.4. *Let ν be an invariant random subgroup in $\mathrm{SL}_2(\mathbb{R})$ and assume that a ν-random subgroup is almost surely discrete and torsion-free. Then for ν-almost every H, the quotient $S = H\backslash\mathbb{H}^2$ satisfies the following properties:*

1. *the space of infinite-volume ends of S has at most two elements or it is a Cantor set;*
2. *if S has a cusp then every infinite-volume end of S is a limit of cusps;*
3. *if S has positive genus then every infinite-volume end of S has infinite genus.*

All three properties are trivially satisfied by any finite-volume surface. On the other hand they restrict dramatically the topological types for surfaces of infinite volume. Topological surfaces of infinite type are classified by their end space and the genus function on this space [94]; it is then easy to see that up to homeomorphism there are exactly 12 surfaces of infinite type satisfying the three conditions above. Those without cusps[4] are pictured in Figure 4.1.

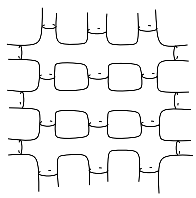
We note that the statements 1 and 2 in the theorem are also valid for torsion-free invariant random subgroups[5] in any G.

Each of the statements in the theorem can be proven using the no-core principle (the proof in [18] is essntially the same but does not appeal explicitely to the principle). Namely, each of them follows from applying the theorem to one of the subsets of $\mathrm{Sub}_{\mathrm{SL}_2(\mathbb{R})}$ in the following list:

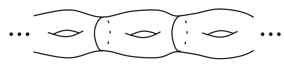
- For 1, the discrete subgroups H of $\mathrm{SL}_2(\mathbb{R})$ such that $H\backslash\mathbb{H}^2$ has one isolated infinite-volume end and at least two others infinite-volume ends, and the ball of radius R around the root

[4]Except the plane, which occurs only for the trivial IRS, and the cylinder, which does not occur for IRSs in $\mathrm{PSL}_2(\mathbb{R})$. To obtain the complete list one needs to add the 6 surfaces with cusps accumulating on every end.

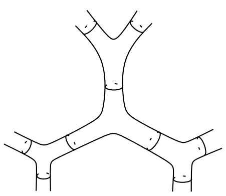
[5]In [18, Theorem 1.1] these properties are stated for unimodular random manifolds; the quotient of a symmetric space by a discrete, torsion-free invariant random subgroup in its isometry group is a unimodular random manifold so the theorem applies to it.



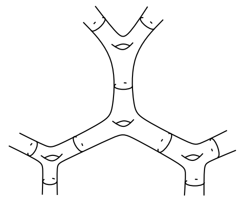
the infinite prison window,



Jacob's ladder,



the Cantor tree,



the Cantor tree in bloom,

Figure 4.1: Surfaces without cusps

$He\mathrm{SO}(2)$ in $H\backslash\mathbb{H}^2$ separates the first end from at least two other ends (where R is chosen large enough so that this happens with positive probability); the subsets in all $H\backslash\mathbb{H}^2$ cotaining all roots $He\mathrm{SO}(2)$ (with H satisfying the conditions) have finite volume because there are at most finitely many isolated ends, see [18, Lemma 3.1].

- For 2, the discrete subgroups H of $\mathrm{SL}_2(\mathbb{R})$ such that the ball of radius R around the root in $H\backslash\mathbb{H}^2$ separates all cusps from an infinite-volume end, see [18, Lemma 3.4].

- For 3, the discrete subgroups H of $\mathrm{SL}_2(\mathbb{R})$ such that the ball of radius R around the root in $H\backslash\mathbb{H}^2$ separates a genus-zero end from all subsurfaces with positive genus, and the ball itself contains such a surface; see [18, Lemma 3.5]

4.3 Constructions of invariant random subgroups in rank 1 groups

4.3.1 Invariant random subgroups of lattices and induction

If Γ is a lattice in G and $\Lambda \lhd \Gamma$ is a normal subgroup then the G-conjugacy class of Λ is a G-equivariant quotient of G/Γ and so it carries a unique G-invariant probability measure. Hence there is an invariant random subgroup in G supported on the conjugacy class of Λ; in particular, such an example shows that the conclusion of Theorem 1.16 cannot hold for G since cocompact lattices in G are non-elementary Gromov-hyperbolic groups and as such they always contain infinite-index infinite normal subgroups (cf. [31]).

Note that since not much is known about the quotient Γ/Λ, it is still possible that the conclusion of Theorem 1.5 holds for those rank 1 groups in which we do not know the answer to the congruence subgroup problem for their arithmetic lattices (so $\mathrm{Sp}(n,1)$ and the exceptional group; in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$, as was observed in the introduction to Chapter 3, there are arithmetic lattices with an infinite cyclic quotient and the associated finite cyclic covers give a sequence of manifolds which converge to an infinite index infinite normal subgroup).

More generally it is possible to induce invariant random subgroups from lattices to the ambient group as follows. If G is a locally compact group and Γ a lattice in G then Γ acts on the right on $G \times \mathrm{Sub}_\Gamma$ by $(g,H)\cdot\gamma = (g\gamma, \gamma^{-1}H\gamma)$. We see that there is a map $(G\times\mathrm{Sub}_\Gamma)/\Gamma \to \mathrm{Sub}_G$ given by $(g,H)\mapsto gHg^{-1}$. If ν is an invariant random subgroup in Γ, by taking the product with the Haar measure on G, we obtain a Γ-invariant measure on $G\times\mathrm{Sub}_\Gamma$. Then the quotient

$(G \times \mathrm{Sub}_\Gamma)/\Gamma$ is a Borel space, and the quotient measure is a finite measure. We can thus normalise this measure to be a probability measure, and its pushforward by the map Φ is an invariant random subgroup in G.

L. Bowen has constructed examples of ergodic invariant random subgroups in non-abelian free groups which are supported on uncountable subsets [23]. As there are lattices in every $\mathrm{SO}(n,1)$ and in $\mathrm{SU}(2,1)$ which have non-abelian free quotients [62, 33], this gives examples of invariant random subgroups in these groups which are supported on uncountably many conjugacy classes. It is not known whether these exist in $\mathrm{Sp}(n,1)$ or in the exceptional rank 1 group.

4.3.2 Non-lattice examples in $\mathrm{PO}(n,1)$

All examples previously constructed of non-trivial invariant random subgroup in a simple Lie group G rely on lattices in G or their subgroups. In $\mathrm{PO}(n,1)$ more complicated examples can be constructed. We state this in the following theorem.

Theorem 4.5. *Let $n \geq 2$. There exists uncountably many invariant random subgroups in $\mathrm{PO}(n,1)$ for which the measure of the Borel set of all subgroups of lattices in $\mathrm{PO}(n,1)$ is zero.*

All constructions will be of a geometric nature. Recall that the connected component $\mathrm{PO}(n,1)^\circ$ is $\mathrm{PO}(n,1)^\circ$-equivariantly diffeomorphic to the direct orthonormal frame bundle of $\mathbb{H}^n$, so that if (M,ϕ) is an orientable framed hyperbolic n-manifold there is a unique torsion-free discrete subgroup Γ in $\mathrm{PO}(n,1)$ such that $\Gamma\backslash\mathbb{H}^n$ is isometric to M and the framing ϕ is the image of the framing of $\mathbb{H}^n$ given by the trivial element of $\mathrm{PO}(n,1)$; we call this subgroup Γ the holonomy group of (M,ϕ).

Dimension 2

In this case the construction is quite elementary. It uses the following construction of infinite-type surfaces: a 3-valent tree T, each of whose edges is marked by a length $\ell > 0$ and a twist parameter $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ determines a surface by putting at each vertex a pair of pants whose cuff lengths are given by the lengths marking the edges adjacent to the vertex, and gluing the pants corresponding to two adjacent vertices using the twist marking of the edge between the two. We randomise this construction by taking the lengths to be i.i.d. random variables supported in $]0,+\infty[$ (with arbitrary law) and the twists to be i.i.d. uniform random variables. Then we fix a root in T and we choose uniformly randomly a direct orthonormal frame inside the corresponding pair of pants. The holonomy of the resulting framed surface gives an ergodic invariant random subgroup in $\mathrm{PO}(2,1)$ [3, Proposition 11.2]. As soon as the measure determining the length markings have a non locally-finite support, it will almost surely not be a subgroup of a lattice, as a random surface will almost surely contain countably many closed geodesics with lengths which are pairwise distinct and smaller than some constant, which is not possible for a cover of a finite-type surface.

Dimension 3

For dimension 3, the construction relies on doubly degenerate surface groups in $\mathrm{PGL}_2(\mathbb{C})$. It is quite technical and we will only describe it briefly; details can be found in [3, Theorem 12.8]. A doubly degenerate surface group $\Lambda \cong \pi_1(S)$ can be associated to certain pair of points in the Thurston boundary of hyperbolic space; such a pair corresponds to the ending laminations of the hyperbolic manifold $\Lambda\backslash\mathbb{H}^3$, which is diffeomorphic to $S \times \mathbb{R}$. In particular, if F is a Schottky subgroup of the mapping class group $\mathrm{Mod}(S)$ freely generated by pseudo-Anosov mapping classes $\phi_1,\dots,\phi_n$ every complete geodesic line in the Cayley tree of F with respect to the generators

$\phi_1, \ldots, \phi_n$ corresponds to a doubly degenerate surface group. A vertex in this geodesic gives a basepoint in the manifold, well-defined up to a bounded distance.

Roughly, the invariant random subgroups we construct are described as follows: pick a random subsequence in $\{1, \ldots, n\}^{\mathbb{Z}}$ according to an ergodic, non-periodic shift-invariant probability measure. This gives a bi-infinite geodesic in F through the root (the identity element); we obtain a framed manifold by taking the doubly degenerate surface group associated with the geodesic, rooted at a random frame chosen near the coarsely defined basepoint. The last step is not rigorous and it does not seem to be quite obvious how to formalise it. Instead there is an easier way: the shift-invariant measure can be approximated by periodic measures. For these the doubly-degenerate manifolds are infinite cyclic covers of fibred hyperbolic manifolds, so they correspond to a well-defined invariant random subgroup. Taking a weak limit point of this sequence of invariant random subgroups we get another invariant random subgroups which is supported on the holonomies of doubly degenerate manifolds associated with sequences in the support of our original shift-invariant measure. It can be checked that these manifolds do not cover any finite-volume hyperbolic 3–manifold [3, Proposition 12.26], hence the invariant random subgroup is almost surely not a subgroup of a lattice in $\mathrm{PGL}_2(\mathbb{C})$.

All dimensions

The construction in any dimension $n \geq 3$ uses a similar idea to the construction above but implemented with quite different tools so it is easy to make it rigorous. We start as in §3.2 with N_0, N_1 two manifolds with totally geodesic boundary obtained from two arithmetic manifolds by removing a common totally geodesic hypersurface. We label the boundary components in each N_i by ± 1 and we fix isometries between the components with the same lable, so that any sequence $\alpha \in \{0, 1\}^{\mathbb{Z}}$ gives a hyperbolic manifold by gluing N_0, N_1 according to the sequence (identifying the $+1$ boundary component of N_{α_i} to the -1 boundary component of $N_{\alpha_{i+1}}$).

Then picking the sequence α randomly according to an ergodic shift-invariant probability measure on $\{0, 1\}^{\mathbb{Z}}$ and a random frame uniformly in the component N_{α_0}, we get an ergodic invariant random subgroup in $\mathrm{PO}(n, 1)$ (note that the measure on $\{0, 1\}^{\mathbb{Z}}$ needs to be weighted according to the respective volumes of N_0, N_1 to ensure invariance, see [3, p. 13.2]).

If the shift-invariant measure is periodic then the invariant random subgroup comes from a cyclic cover of a closed hyperbolic manifold. On the other hand, if it is not then almost all subgroups are not subgroups of lattices in $\mathrm{PO}(n, 1)$ [3, Theorem 13.6]. Note that we can, as in the previous paragraph, approximate a non-periodic measure by periodic ones and this shows that the invariant random subgroups constructed here are the limits of the invariant random subgroups associated with the non-arithmetic lattices in the proof of Theorem 3.1.

Bibliography

[1] Miklos Abert, Nicolas Bergeron, Ian Biringer, and Tsachik Gelander. *Convergence of normalized Betti numbers in nonpositive curvature.* 2018. eprint: `arXiv:1811.02520`.

[2] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. "On the growth of L^2-invariants for sequences of lattices in Lie groups". In: *Ann. Math. (2)* 185.3 (2017), pp. 711–790.

[3] Miklós Abért, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. "On the Growth of L2-Invariants of Locally Symmetric Spaces, II: Exotic Invariant Random Subgroups in Rank One". In: *International Mathematics Research Notices* 2020.9 (May 2018), pp. 2588–2625.

[4] Miklós Abért and Ian Biringer. *Unimodular measures on the space of all Riemannian manifolds.* 2016. eprint: `arXiv:1606.03360`.

[5] Ian Agol. "The virtual Haken conjecture (with an appendix by Ian Agol, Daniel Groves and Jason Manning)." In: *Doc. Math.* 18 (2013), pp. 1045–1087.

[6] A. Ash, P. E. Gunnells, M. Mcconnell, and D. Yasaki. "On the growth of torsion in the cohomology of arithmetic groups". In: *J. Inst. Math. Jussieu* 19.2 (2020), pp. 537–569.

[7] Uri Bader, Tsachik Gelander, and Roman Sauer. "Homology and homotopy complexity in negative curvature". In: *J. Eur. Math. Soc. (JEMS)* 22.8 (2020), pp. 2537–2571.

[8] Hyungryul Baik, Bram Petri, and Jean Raimbault. "Subgroup growth of right-angled Artin and Coxeter groups". In: *J. Lond. Math. Soc., II. Ser.* 101.2 (2020), pp. 556–588.

[9] Hyungryul Baik, Bram Petri, and Jean Raimbault. "Subgroup growth of virtually cyclic right-angled Coxeter groups and their free products". In: *Combinatorica* 39.4 (2019), pp. 779–811.

[10] Elizabeth Baker and Bram Petri. *Statistics of finite degree covers of torus knot complements.* 2020. eprint: `arXiv:2005.11956`.

[11] Mikhail Belolipetsky. "Counting maximal arithmetic subgroups. Appendix by Jordan Ellenberg and Akshay Venkatesh". In: *Duke Math. J.* 140.1 (2007), pp. 1–33.

[12] Mikhail Belolipetsky and Vincent Emery. "On volumes of arithmetic quotients of $\mathrm{PO}(n,1)^\circ$, n odd". In: *Proc. Lond. Math. Soc. (3)* 105.3 (2012), pp. 541–570.

[13] Nicolas Bergeron and Laurent Clozel. "On the cohomology of trialitary hyperbolic varieties of dimension 7". In: *Isr. J. Math.* 222.1 (2017), pp. 333–400.

[14] Nicolas Bergeron and Laurent Clozel. *Spectre automorphe des variétés hyperboliques et applications topologiques.* Vol. 303. Astérisque. Paris: Société Mathématique de France, 2005.

[15] Nicolas Bergeron, Mehmet Haluk Şengün, and Akshay Venkatesh. "Torsion homology growth and cycle complexity of arithmetic manifolds". In: *Duke Math. J.* 165.9 (2016), pp. 1629–1693.

[16] Nicolas Bergeron and Akshay Venkatesh. "The asymptotic growth of torsion homology for arithmetic groups". In: *Journal of the Institute of Mathematics of Jussieu* 12.2 (2013), pp. 391–447.

[17] Ian Biringer. "Metrizing the Chabauty topology". In: *Geom. Dedicata* 195 (2018), pp. 19–22.

[18] Ian Biringer and Jean Raimbault. "Ends of unimodular random manifolds". In: *Proc. Am. Math. Soc.* 145.9 (2017), pp. 4021–4029.

[19] Ian Biringer and Omer Tamuz. "Unimodularity of invariant random subgroups". In: *Trans. Am. Math. Soc.* 369.6 (2017), pp. 4043–4061.

[20] Steven A. Bleiler and Craig D. Hodgson. "Spherical space forms and Dehn filling". In: *Topology* 35.3 (1996), pp. 809–833.

[21] Armand Borel. "Compact Clifford-Klein forms of symmetric spaces". In: *Topology* 2 (1963), pp. 111–122.

[22] Armand Borel and Gopal Prasad. "Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups". In: *Publications Mathématiques de l'IHÉS* 69 (1989), pp. 119–171.

[23] Lewis Bowen. "Invariant random subgroups of the free group". In: *Groups Geom. Dyn.* 9.3 (2015), pp. 891–916.

[24] Emmanuel Breuillard. "A height gap theorem for finite subsets of $\mathrm{GL}_d(\overline{\mathbb{Q}})$ and nonamenable subgroups". In: *Ann. of Math. (2)* 174.2 (2011), pp. 1057–1110.

[25] Jeffrey F. Brock and Nathan M. Dunfield. "Injectivity radii of hyperbolic integer homology 3-spheres". In: *Geom. Topol.* 19.1 (2015), pp. 497–523.

[26] Frank Calegari and Akshay Venkatesh. *A torsion Jacquet-Langlands correspondence*. Vol. 409. Paris: Société Mathématique de France (SMF), 2019, pp. x + 226.

[27] Jeff Cheeger. "Analytic torsion and the heat equation". In: *Ann. Math. (2)* 109 (1979), pp. 259–322.

[28] Laurent Clozel. "Démonstration de la conjecture τ". In: *Invent. Math.* 151.2 (2003), pp. 297–328.

[29] R. M. Damerell. "L-functions of elliptic curves with complex multiplication. I". In: *Acta Arith.* 17 (1970), pp. 287–301.

[30] R. M. Damerell. "L-functions of elliptic curves with complex multiplication. II". In: *Acta Arith.* 19 (1971), pp. 311–317.

[31] Thomas Delzant. "Distinguished subgroups and quotients of hyperbolic groups". In: *Duke Math. J.* 83.3 (1996), pp. 661–682.

[32] Martin Deraux. "A new nonarithmetic lattice in $\mathrm{PU}(3,1)$". In: *Algebr. Geom. Topol.* 20.2 (2020), pp. 925–963.

[33] Martin Deraux. "Forgetful maps between Deligne-Mostow ball quotients". In: *Geom. Dedicata* 150 (2011), pp. 377–389.

[34] Martin Deraux, John R. Parker, and Julien Paupert. "New nonarithmetic complex hyperbolic lattices. II". In: *Mich. Math. J.* 70.1 (2021), pp. 133–205.

[35] Bruno Duchesne, Yair Glasner, Nir Lazarovich, and Jean Lécureux. "Geometric density for invariant random subgroups of groups acting on CAT(0) spaces." In: *Geom. Dedicata* 175 (2015), pp. 249–256.

[36] Nathan M. Dunfield and William P. Thurston. “Finite covers of random 3-manifolds”. In: *Invent. Math.* 166.3 (2006), pp. 457–521.

[37] J. Elstrodt, F. Grunewald, and J. Mennicke. *Groups acting on hyperbolic space*. Springer Monographs in Mathematics. Harmonic analysis and number theory. Springer-Verlag, Berlin, 1998, pp. xvi+524.

[38] Peter Feller, Alessandro Sisto, and Gabriele Viaggi. *Uniform models and short curves for random 3-manifolds*. 2019. eprint: `arXiv:1910.09486`.

[39] Tobias Finis and Erez Lapid. “An approximation principle for congruence subgroups”. In: *J. Eur. Math. Soc. (JEMS)* 20.5 (2018), pp. 1075–1138.

[40] Tobias Finis and Erez Lapid. “An approximation principle for congruence subgroups. II: Application to the limit multiplicity problem”. In: *Math. Z.* 289.3-4 (2018), pp. 1357–1380.

[41] M. Frączyk. “Strong limit multiplicity for arithmetic hyperbolic surfaces and 3-manifolds”. In: *Invent. math.* (2020).

[42] Mikołaj Frączyk, Sebastian Hurtado, and Jean Raimbault. *Homotopy type and homology versus volume for arithmetic locally symmetric spaces*. preprint. eprint: `arxiv:2202.13619`.

[43] Mikołaj Frączyk and Jean Raimbault. “Betti numbers of Shimura curves and arithmetic three-orbifolds”. In: *Algebra Number Theory* 13.10 (2019), pp. 2359–2382.

[44] Stefan Friedl, Junghwan Park, Bram Petri, Jean Raimbault, and Arunima Ray. “On distinct finite covers of 3-manifolds”. In: *Indiana Univ. Math. J.* 70.2 (2021), pp. 809–846.

[45] Harry Furstenberg. “A note on Borel’s density theorem”. In: *Proc. Am. Math. Soc.* 55 (1976), pp. 209–212.

[46] David Futer, Jessica S. Purcell, and Saul Schleimer. *Effective bilipschitz bounds on drilling and filling*. To appear in Geometry & Topology. 2019. eprint: `arXiv:1907.13502`.

[47] Ilya Gekhtman and Arie Levit. “Critical exponents of invariant random subgroups in negative curvature”. In: *Geom. Funct. Anal.* 29.2 (2019), pp. 411–439.

[48] Tsachik Gelander. “A view on invariant random subgroups and lattices”. In: *Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures*. Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1321–1344.

[49] Tsachik Gelander. “Homotopy type and volume of locally symmetric manifolds”. In: *Duke Math. J.* 124.3 (2004), pp. 459–515.

[50] Tsachik Gelander and Arie Levit. “Counting commensurability classes of hyperbolic manifolds”. In: *Geom. Funct. Anal.* 24.5 (2014), pp. 1431–1447.

[51] E. Glasner and B. Weiss. “Kazhdan’s property T and the geometry of the collection of invariant measures”. In: *Geom. Funct. Anal.* 7.5 (1997), pp. 917–935.

[52] M. Gromov and Ilya I. Piatetski-Shapiro. “Non-arithmetic groups in Lobachevsky spaces”. In: *Publ. Math., Inst. Hautes Étud. Sci.* 66 (1988), pp. 93–103.

[53] Michael Gromov. *Hyperbolic manifolds according to Thurston and Jorgensen*. Semin. Bourbaki, 32e annee, Vol. 1979/80, Exp. 546, Lect. Notes Math. 842, 40-53 (1981). 1981.

[54] G. Harder. “Eisenstein cohomology of arithmetic groups. The case GL_2”. In: *Invent. Math.* 89 (1987), pp. 37–118.

[55] W. K. Hayman. “A generalisation of Stirling’s formula”. In: *J. Reine Angew. Math.* 196 (1956), pp. 67–95.

[56] David Kazhdan. “Some applications of the Weil representation”. In: *J. Anal. Math.* 32 (1977), pp. 235–248.

[57] Anthony W. Knapp. *Lie groups beyond an introduction.* Second. Vol. 140. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002.

[58] Thang T. Q. Lê. “Croissance de la torsion homologique dans les revêtements finis et le volume hyperbolique”. In: *Ann. Inst. Fourier* 68.2 (2018), pp. 611–645.

[59] Arie Levit. “On Benjamini-Schramm limits of congruence subgroups”. In: *Isr. J. Math.* 239.1 (2020), pp. 59–73.

[60] Jian-Shu Li and John J. Millson. “On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group”. In: *Duke Math. J.* 71.2 (1993), pp. 365–401.

[61] Martin W. Liebeck and Aner Shalev. “Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks”. In: *J. Algebra* 276.2 (2004), pp. 552–601.

[62] Alexander Lubotzky. “Free quotients and the first Betti number of some hyperbolic manifolds”. In: *Transform. Groups* 1.1-2 (1996), pp. 71–82.

[63] Alexander Lubotzky and Nikolay Nikolov. “Subgroup growth of lattices in semisimple Lie groups”. In: *Acta Math.* 193.1 (2004), pp. 105–139.

[64] Alexander Lubotzky and Dan Segal. *Subgroup growth.* Vol. 212. Progress in Mathematics. Birkhäuser Verlag, Basel, 2003, pp. xxii+453.

[65] Michael Magee and Doron Puder. *The Asymptotic Statistics of Random Covering Surfaces.* 2020. eprint: `arXiv:2003.05892`.

[66] Joseph Maher. “Random Heegaard splittings”. In: *J. Topol.* 3.4 (2010), pp. 997–1025.

[67] G. A. Margulis and J. Rohlfs. “On the proportionality of covolumes of discrete subgroups”. In: *Math. Ann.* 275 (1986), pp. 197–205.

[68] Simon Marshall and Werner Müller. “On the torsion in the cohomology of arithmetic hyperbolic 3-manifolds”. In: *Duke Math. J.* 162.5 (2013), pp. 863–888.

[69] Jasmin Matz. “Limit multiplicities for $\mathrm{SL}_2(\mathcal{O}_F)$ in $\mathrm{SL}_2(\mathbb{R}^{r_1} \oplus \mathbb{C}^{R_2})$”. In: *Groups Geom. Dyn.* 13.3 (2019), pp. 841–881.

[70] John J. Millson. “On the first Betti number of a constant negatively curved manifold”. In: *Ann. Math. (2)* 104 (1976), pp. 235–247.

[71] Maryam Mirzakhani. “Growth of Weil-Petersson volumes and random hyperbolic surface of large genus”. In: *J. Differ. Geom.* 94.2 (2013), pp. 267–300.

[72] T. W. Müller and J. -C. Schlage-Puchta. “Classification and statistics of finite index subgroups in free products”. In: *Adv. Math.* 188.1 (2004), pp. 1–50.

[73] Werner Müller. “The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds”. In: *Metric and differential geometry. The Jeff Cheeger anniversary volume. Selected papers based on the presentations at the international conference on metric and differential geometry, Tianjin and Beijing, China, May 11–15, 2009.* Berlin: Springer, 2012, pp. 317–352.

[74] Werner Müller and Jonathan Pfaff. “Analytic torsion and L^2-torsion of compact locally symmetric manifolds”. In: *J. Differ. Geom.* 95.1 (2013), pp. 71–119.

[75] Werner Müller and Jonathan Pfaff. “Analytic torsion of complete hyperbolic manifolds of finite volume”. In: *J. Funct. Anal.* 263.9 (2012), pp. 2615–2675.

[76] Werner Müller and Jonathan Pfaff. "On the growth of torsion in the cohomology of arithmetic groups". In: *Math. Ann.* 359.1-2 (2014), pp. 537–555.

[77] Werner Müller and Jonathan Pfaff. "The analytic torsion and its asymptotic behaviour for sequences of hyperbolic manifolds of finite volume". In: *J. Funct. Anal.* 267.8 (2014), pp. 2731–2786.

[78] Shin Ohno and Takao Watanabe. "Estimates of Hermite constants for algebraic number fields". In: *Comment. Math. Univ. St. Paul.* 50.1 (2001), pp. 53–63.

[79] Martin Olbrich. "L2-invariants of locally symmetric spaces". In: *Doc. Math* 7 (2002), pp. 219–237.

[80] Grisha Perelman. "Ricci flow with surgery on three-manifolds". In: *arXiv e-print service* 2003 (2003). Id/No 0303109, p. 22.

[81] Grisha Perelman. "The entropy formula for the Ricci flow and its geometric applications". In: *arXiv e-print service* 2002 (2002). Id/No 0211159, p. 39.

[82] Bram Petri and Jean Raimbault. "A model for random 3–manifolds". In: (2020).

[83] Jonathan Pfaff. "A gluing formula for the analytic torsion on hyperbolic manifolds with cusps". In: *J. Inst. Math. Jussieu* 16.4 (2017), pp. 673–743.

[84] Jonathan Pfaff. "Exponential growth of homological torsion for towers of congruence subgroups of Bianchi groups". In: *Ann. Global Anal. Geom.* 45.4 (2014), pp. 267–285.

[85] Jonathan Pfaff and Jean Raimbault. "The torsion in symmetric powers on congruence subgroups of Bianchi groups". In: *Trans. Am. Math. Soc.* 373.1 (2020), pp. 109–148.

[86] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory.* Vol. 139. Pure and Applied Mathematics. Translated from the 1991 Russian original by Rachel Rowen. Academic Press, Inc., Boston, MA, 1994, pp. xii+614.

[87] Gopal Prasad. "$\mathbf{R}$-regular elements in Zariski-dense subgroups". In: *The Quarterly Journal of Mathematics* 45.4 (1994), pp. 542–545.

[88] Gopal Prasad. "Volumes of S-arithmetic quotients of semi-simple groups". In: *Inst. Hautes Études Sci. Publ. Math.* 69 (1989). With an appendix by Moshe Jarden and the author, pp. 91–117.

[89] Jean Raimbault. "A note on maximal lattice growth in $\mathrm{SO}(1,n)$". In: *Int. Math. Res. Not.* 2013.16 (2013), pp. 3722–3731.

[90] Jean Raimbault. "Analytic Torsion, Regulators and Arithmetic Hyperbolic Manifolds". In: *Arithmetic L-Functions and Differential Geometric Methods.* Ed. by Maillot V. Charollois P. Freixas i Montplet G. Vol. 338. Progress in Mathematics. Birkhäuser, Cham, 2021.

[91] Jean Raimbault. "Analytic, Reidemeister and homological torsion for congruence three-manifolds". In: *Ann. Fac. Sci. Toulouse, Math. (6)* 28.3 (2019), pp. 417–469.

[92] Jean Raimbault. "Asymptotics of analytic torsion for hyperbolic three-manifolds". In: *Comment. Math. Helv.* 94.3 (2019), pp. 459–531.

[93] Jean Raimbault. "Sur la convergence des orbi-variétés arithmétiques". In: *Ann. Inst. Fourier* 67.6 (2017), pp. 2547–2596.

[94] Ian Richards. "On the classification of noncompact surfaces". In: *Trans. Amer. Math. Soc.* 106 (1963), pp. 259–269.

[95] Jonathan D. Rogawski. *Automorphic representations of unitary groups in three variables.* Vol. 123. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 1990.

[96] Mehmet Haluk Sengün. "On the integral cohomology of Bianchi groups". In: *Exp. Math.* 20.4 (2011), pp. 487–505.

[97] C. Soulé. "Perfect forms and the Vandiver conjecture". In: *J. Reine Angew. Math.* 517 (1999), pp. 209–221.

[98] Garrett Stuck and Robert J. Zimmer. "Stabilizers for ergodic actions of higher rank semisimple groups". In: *Ann. of Math. (2)* 139.3 (1994), pp. 723–747.

[99] William P. Thurston. "Three dimensional manifolds, Kleinian groups and hyperbolic geometry". In: *Bull. Am. Math. Soc., New Ser.* 6 (1982), pp. 357–379.

[100] J. Tits. "Reductive groups over local fields". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1.* Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

[101] Emmanuel Ullmo and Andrei Yafaev. "Nombre de classes des tores de multiplication complexe et bornes inférieures pour les orbites galoisiennes de points spéciaux". In: *Bull. Soc. Math. France* 143.1 (2015), pp. 197–228.

[102] Hsien Chung Wang. "Topics on totally discontinuous groups". In: *Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970).* New York: Dekker, 1972, 459–487. Pure and Appl. Math., Vol. 8.