# ON SOME COMPLEX HYPERBOLIC SURFACES OF LARGE VOLUME

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Abstract. We look at the asymptotics injectivity radii of some specific sequences of arithmetic complex hyperbolic surfaces defined over quadratic fields; more precisely we establish that they are convergent "in the Benjamini–Schramm sense" to complex hyperbolic space.

We recall the following notion from [1]: if $X$ is a Hadamard manifold (simply connected with non-positive curvature) and $M_n$ a sequence of finite-volume orbifold quotients of $X$ we say that $M_n$ converges in the Benjamini–Schramm sense (or BS-converges for shortness) to $X$ if we have

$$\forall R > 0 : \frac{\mathrm{vol}(x \in M_n :\ \mathrm{inj}_x(M_n) \leq R)}{\mathrm{vol}\, M_n} \xrightarrow[n\to+\infty]{} 0$$

(here $\mathrm{inj}_x(M_n)$ is the largest radius of an embedded ball around $x \in M_n$, in particular it is zero on the singular locus). The following conjecture was made in [6] (also, it seems, in talks by Peter Sarnak).

**Conjecture.** *Let $G$ is a semisimple Lie group and $X$ the associated symmetric space. If $\Gamma_n$ is a sequence of irreducible congruence lattices in $G$, then the sequence of $X$–orbifolds $\Gamma_n\backslash X$ is BS-convergent to $X$.*

This very short note pulls together a few observations to prove some results in the direction of this conjecture for $G = \mathrm{SU}(2,1)$, Theorems 1 and 5 below. The proofs we give for these results are very similar to that of Theorem B from this paper, in fact most of the contents here are dedicated to describe the constructions of the arithmetic lattices to which our results apply (in this we mostly followed Ben mcReynolds' notes [4]).

## 1. Picard modular surfaces

For a negative discriminant $D$ we let $F_D$ denote the imaginary quadratic field of discriminant $D$, that is $F_D = \mathbb{Q}(\sqrt{-m})$ where $m = -D$ if $D$ is a square-free integer congruent to 2 or 3 modulo 4, and $m = -D/4$ if 4 divides $D$ and $D/4$ is square-free and congruent to 1 modulo 4. The quadratic field $F_D$ is endowed with a Galois automorphism $z \mapsto \overline{z}$; we denote $|z|^2 = z\overline{z}$, and an absolute value $|z|^2 = \overline{z}z$. Let $\mathrm{SU}(h)$ be the subgroup of $\mathrm{SL}_3(F)$ preserving the Hermitian form

$$h(z) = |z_1|^2 + |z_2|^2 - |z_3|^2$$

on $F^3$. We let $\mathcal{O}_D$ denote the ring of integers in $F_D$, and put

$$\Gamma = \mathrm{SU}(h) \cap \mathrm{SL}_3(\mathcal{O}_F).$$

Then $\Gamma$ is a non-uniform lattice in $\mathrm{SU}(h) \otimes \mathbb{C} \cong \mathrm{SU}(2,1)$. We let $\mathbb{H}^2_{\mathbb{C}}$ denote the complex hyperbolic plane, identified with $\mathrm{SU}(2,1)/K$ (where $K$ is a maximal compact subgroup, isomorphic to $U(2)$). The quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is a finite-volume, noncompact orbifold locally isometric to $\mathbb{H}^2_{\mathbb{C}}$ (or to a quotient of it by a finite group), usually called a Picard modular surface after Emile Picard who studied a special case of these. The volume of $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is of order $D^{5/2}$ (see [5, Equation (1)]), and if $F, F'$ are two nonisomorphic imaginary quadratic fields the lattices constructed above for $F, F'$ respectively are non-commensurable (for example because their invariant trace fields are respectively $F$ and $F'$—see [4, 15.2], or also 5.3, loc. cit.).

**Theorem 1.** *For each negative discriminant $D$ let $\Gamma_D$ be a finite-index subgroup in the Picard modular group defined over $F_D$. Then the sequence $\Gamma_D\backslash\mathbb{H}^2_{\mathbb{C}}$ is convergent in the Benjamini–Schramm sense to $\mathbb{H}^2_{\mathbb{C}}$ as $D \to -\infty$.*

*Proof.* The proof rests mainly on the study of hyperbolic elements. They are characterized (amongst isometries) by having three distinct eigenvalues (in the tautological representation $\mathrm{SU}(2,1) \subset \mathrm{SL}_3(\mathbb{C})$) which can be written $\lambda_1 = \lambda e^{i\theta}, \lambda_2 = e^{-2i\theta}, \lambda_3 = \lambda^{-1}e^{i\theta}$ for a $\lambda \in ]1, +\infty[$ and $\theta \in [0, 2\pi[$. If in addition $\gamma$ belongs to $\Gamma_D$ then (since the characteristic polynomial of $\gamma$ has its coefficients in $F_D$) $\lambda_1$ belongs to a cubic extension of $F_D$; we denote by $E(\gamma)$ this extension.

**Lemma 2.** *With the notation above, $\lambda_1$ cannot belong to a proper cubic subfield of $E(\gamma)$.*

*Proof.* Suppose this is not the case. Then $E' = \mathbb{Q}(\lambda_1)$ is of degree 3 over $\mathbb{Q}$, in particular it is either totally real or of signature $(1,1)$. Also, the $\mathbb{Q}$-conjugates in $\mathbb{C}$ of $\lambda_1$ have to be $\lambda_2$ and $\lambda_3$. In any case one of the three $\lambda_i$ has to be real, and whichever it is this implies the conclusion that $e^{i\theta} \in \mathbb{R}$, which implies that $\lambda_2 = \pm 1$, a contradiction. □

It follows from Lemma 2 that for every $D$ and every $\gamma \in \Gamma_D$ the eigenvalues of which are not real, if we let $\lambda(\gamma)$ be any of its eigenvalues then the field $\mathbb{Q}(\lambda(\gamma))$ contains $F_D$. In particular its discriminant is larger than $D$. It follows that for any finite set $S$ of algebraic numbers, for $D$ large enough any $\lambda(\gamma)$ as above does not belong to $S$. Now $\lambda(\gamma)$ is in fact an algebraic unit, and thus the following lemma implies that for any $R > 0$ there are only finitely many $D$ such that $\Gamma_D$ contains an hyperbolic element $\gamma$ with an eigenvalue $\lambda_1$ such that $1 < |\lambda_1| < e^R$ and $\lambda_1$ is not real quadratic (we recall that the Mahler measure of an algebraic number $a$ is the sum over all embeddings $\sigma$ of $\mathbb{Q}(a)$ in $\mathbb{C}$ of $\min(0, c_\sigma \log|a^\sigma|)$ where $c_\sigma$ is 1 if $a^\sigma \in \mathbb{R}$ and 2. otherwise).

**Lemma 3.** *Fix an integer $r \geq 1$ and a real $R > 0$; there are only finitely many algebraic units $\varepsilon \in \overline{\mathbb{Z}}^\times$ such that the degree $[\mathbb{Q}(\varepsilon) : \mathbb{Q}] \leq r$ and the Mahler measure $m(\varepsilon) \leq R$.*

*Proof.* If $\varepsilon$ is an in the statement then the coefficients of the minimal polynomial of $\varepsilon$ are bounded by polynomials (depending only on $r$) in $e^R$. $\square$

The following lemma then allows us to conclude the proof of Theorem 1.

**Lemma 4.** *If $\Gamma_n$ is a sequence of lattices in $\mathrm{SU}(2,1)$ such that for any $R > 0$ and large enough $n$, the only hyperbolic elements $\gamma \in \Gamma_n$ with $|\lambda_1| \leq e^R$ have $\lambda_1 \in \mathbb{R}$, then the orbifolds $\Gamma_n \backslash \mathbb{H}^2_{\mathbb{C}}$ are BS-convergent to $\mathbb{H}^2_{\mathbb{C}}$.*

*Proof.* By [1, Theorem 2.6] and the main result of [2] an invariant random subgroup of $\mathrm{SU}(2,1)$ has almost surely not all of the eigenvalues of his non-trivial elements real. Using this we can conclude as in the proof of [6, Lemma 4.2], since the minimal displacement of $\gamma$ is, up to constants, $\log |\lambda_1|$. $\square$

$\square$

## 2. Some compact orbifolds

It is not possible to construct cocompact lattices in $\mathrm{SU}(2,1)$ by using Hermitian forms over quadratic fields (see [4, Theorem 6.5.1]). There is however another construction of lattices which uses division algebras over quadratic fields, which we will now shortly describe (a much more detailed account is given in Chapters 7 to 9 of loc. cit.).

Let $D$ be a negative discriminant and $F = F_D$ the imaginary quadratic field of discriminant $D$. Let $L$ be a Galois cubic extension of $F$ and $\theta$ agenerator for the Galois group. For any $\alpha \in F^\times$ one defines a $F$-algebra $A$ generated over $F$ by $L$ and an element $X$, with the relations

$$X^d = \alpha, \quad X\beta = \beta^\theta X(\text{ for } \beta \in L).$$

This is a nine-dimensional central simple algebra over $F$. If moreover $\alpha, \alpha^2$ are not equal to the norm of an element of $L$ then $A$ is a division algebra. Under an aditional condition on $\alpha$ (see [4, Theorem 8.1.3]) $A$ can be endowed with an involution $z \mapsto z^*$ extending the non-trivial Galois automorphism of $F/\mathbb{Q}$.

The algebra $A$ splits over $\mathbb{C}$ (actually over $L$—and any cubic extension splitting $A$ can be used to define $A$ as above, see [4, 7.4]), i.e. $A \otimes_F \mathbb{C} \cong M_3(\mathbb{C})$. We may assume that when we extend scalars to $\mathbb{C}$ the involution $*$ becomes the standard involution on $\mathbb{C}$ (the one given by complex conjugation and transposition). Let $h \in A$ be any element such that $h^* = h$, and suppose that the hermitian form defined by the matrix $h \otimes 1_{\mathbb{C}}$ is of signature $(2,1)$. Then the algebraic $\mathbb{Q}$-group $\mathbf{G}$ defined by

$$\mathbf{G}(\mathbb{Q}) = \{g \in A^\times : g^*hg = h\}$$

is a $\mathbb{Q}$-form of $\mathrm{SU}(2,1)$. If $\mathcal{O}_A$ is an order in $A$ then $\mathcal{O}_A \cap \mathbf{G}(\mathbb{Q})$ is a cocompact lattice in $\mathbf{G}(\mathbb{R}) \cong \mathrm{SU}(2,1)$ ([4, Theorem 9.1.1]). Any finite-index subgroup of such a lattice we will call an arithmetic group derived from the algebra $A$.

We can prove the following theorem for these lattices.

**Theorem 5.** *For every negative discriminant $D$ let $A_D$ be a cyclic division algebra of degree 3 over $F_D$ and $\Gamma_D$ a lattice in* $\mathrm{SU}(2,1)$ *derived from $A_D$. Then the sequence of hyperbolic surfaces $\Gamma_D\backslash\mathbb{H}^2_{\mathbb{C}}$ is BS-convergent to $\mathbb{H}^2_{\mathbb{C}}$ as $D \to -\infty$.*

*Proof.* In what follows we fix $D$ and forgot about the indices. Let $L$ be a cubic extension splitting $A$ a,d $\rho$ be the isomorphism from $A \otimes_F L$ to $M_3(L)$. To be able to apply the arguments in the proof of Theorem 1 it suffices to see that $\operatorname{tr}\rho(\gamma) \in F$ for all $\gamma \in \Gamma$.

Let $\operatorname{tr}_{A/F}$ be the application from $A$ to $F$ given by

$$\operatorname{tr}_{A/F}\left(\beta_0 + \beta_1 X + \beta_2 X^2\right) = \operatorname{tr}_{L/F}(\beta_0) = \beta_0 + \beta_0^\theta + \beta_0^{\theta^2} \in F.$$

We claim that

$$\operatorname{tr}(\rho(a)) = \operatorname{tr}_{A/F}(a) \tag{2.1}$$

for all $a \in A$, from which the conclusion follows at once. This last claim is checked directly from the construction of $\rho$ in [4, Proposition 7.4.1]: if $\beta \in L$ we have

$$\rho(\beta) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^\theta & 0 \\ 0 & 0 & \beta^{\theta^2} \end{pmatrix}$$

and hence we see that (2.1) holds for these elements. It remains to see that $\operatorname{tr}(\beta X) = 0 = \operatorname{tr}(\beta X^2)$ for all $\beta \in L$. We have

$$\rho(\beta X) = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & \beta^\theta \\ \alpha\beta^\theta & 0 & 0 \end{pmatrix}, \quad \rho(\beta X^2) = \begin{pmatrix} 0 & 0 & \beta \\ \alpha\beta^\theta & 0 & 0 \\ 0 & \alpha\beta^{\theta^2} & 0 \end{pmatrix}$$

so we see that it is indeed the case. □

In this case we are able (because of compactness) to apply the limit multiplicity results of [1]. For example loc. cit., Corollary 1.4 (together with the same argument used in the proof of [6, 1.3.4, Corollary]) implies the following result (recall that $b_p(\Gamma) = \dim H^p(\Gamma;\mathbb{Q})$ for a group $\Gamma$).

**Theorem 6.** *Let $\Gamma_D$ be as in the statement of Theorem 5. For $p \neq 2$ we have*

$$\frac{b_p(\Gamma_D)}{\operatorname{vol}(\Gamma_D\backslash\mathbb{H}^2_{\mathbb{C}})} \xrightarrow[D\to-\infty]{} 0$$

*and*

$$\frac{b_2(\Gamma_D)}{\operatorname{vol}(\Gamma_D\backslash\mathbb{H}^2_{\mathbb{C}})} \xrightarrow[D\to-\infty]{} b_2^{(2)}(\mathbb{H}^2_{\mathbb{C}}) > 0.$$

The constant $b_2^{(2)}(\mathbb{H}^2_{\mathbb{C}})$ is equal to $3/\operatorname{vol}(\mathbb{P}^2_{\mathbb{C}})$ where $\mathbb{P}^2_{\mathbb{C}}$ is endowed with the $\mathrm{SU}(3)$-invariant metrix dual to that of $\mathbb{H}^2_{\mathbb{C}}$ (see [3, Theorem 5.12]).

## References


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