Aix–Marseille Université
Faculté des Sciences
Master M1 MAS
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TD-2 Representation and approximation of structured data

Exercise 1: Inner product and norm

We consider the complex-valued vector space $\mathbb{C}^N$ with $N = 2$ and consider the four vectors $u = [3, -1]^T$, $v = [2, -5]^T$, $x = [2 + 3i, -1 - 2i]^T$ and $y = [-2 + 5i, 3 + 6i]^T$.

a) Compute the norms ($|| \cdot ||_2$) of the four vectors and construct the corresponding unit vectors.

b) Compute the following inner products: $\langle u, v\rangle$ and $\langle x, y\rangle$.

c) Compute the following distances: $d(u, v)$ and $d(x, y)$.

Note that $i = \sqrt{-1}$.

Exercise 2: Projection, bases and frames

A) We consider the unit vector $\phi = [1/\sqrt{2}, 1/\sqrt{2}]^T$ and the two vectors $x = [2, 1]^T$ and $[-2, 1]^T$ in $\mathbb{R}^2$.

a) Compute the orthogonal projection $\widehat{x} = \langle x, \phi\rangle\phi$ of x and y onto ϕ.

b) Show that $x - \widehat{x}$ is orthogonal to ϕ

c) Make a graphical illustration.

B) Show that the vectors $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$ and $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$ are linearly independent, i.e. $\sum_{k=1}^{3} \alpha_k \, e_k = 0$, if and only if $\alpha_k = 0$ for $k = 1, 2, 3$. Show also that any vector $x \in \mathbb{R}^3$ can be represented using these three vectors.

C) Show that the vectors $\phi_1 = [1, 0]^T$, $\phi_2 = [0, 1]^T$ and $\phi_3 = [-1, -1]^T$ are linearly dependent and that any vector $x \in \mathbb{R}^2$ can be represented using the three vectors. Construct the corresponding frame.

D) We consider the set of vectors $\Phi = \{\phi_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^{[-1/2,1/2]}$, where $\phi_0(t) = 1$ and $\phi_k(t) = \sqrt{2}\cos(2\pi kt)$ for $k = 1, 2, ...$

a) Make a graphical illustration of ϕ_0, ϕ_1 and ϕ_2.

b) Show that the set of vectors Φ is orthogonal, i.e. $\langle\phi_k, \phi_\ell\rangle = \delta_{k,\ell}$.

c) Show that the set of vectors is orthogonal to the set of odd functions $S_{odd} = \{s|s(t) = -s(-t) \text{ for } t \in [-1/2, 1/2]\}$

Exercise 3: Norms

Verify if $||x||_\alpha$ satisfies for $\alpha = 0, 1, 2$ and ∞ the properties of a norm.

Note that the ℓ^0 'norm' is defined by $||x||_0 = \lim_{p\to 0} \sum_k |x|^p$ and counts the number of non zero entries of a vector, here we define $0^0 = 0$. For $1 < p < \infty$ the corresponding ℓ^p norms are defined as $||x||_p = \sum_k |x_k|^p$ and for $p = \infty$ we have $||x||_\infty = \sup_k |x_k|$.