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A Fast Algorithm for Lacunary Wavelet Bases related to the Solution of PDEs

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Abstract - *We describe a transform for locally refined wavelet bases which employs the cardinal Lagrange function instead of the scaling function. This construction is extended to biorthogonal vaguelettes into which a differential operator is incorporated. The approach is relevant for the solution of nonlinear parabolic PDEs by an explicit or semi-implicit time scheme when the nonlinear term is evaluated in physical space.*

1. INTRODUCTION. – The background of the present work is constituted by an evolutionary PDE where the solution at each time step is developed in an adaptively selected set of wavelet basis functions. This viewpoint is similar to the one of wavelet compression in signal analysis. In contrast to the signal processing field the difficulty when solving a PDE is to avoid the computation of all wavelet coefficients of the solution up to the finest scale before eliminating the irrelevant ones. Just the amplitudes of the relevant set (which is supposed to be known from the previous time step) are to be determined. For this task the classical Mallat algorithm is not suitable as it requires the coefficients of the scaling functions on each scale which do not exhibit the sparsity of the wavelet coefficients and require a preliminary projection. However, for general nonlinear terms as encountered e.g. in [1] the transform between physical space and coefficient space seems to be unavoidable. It is one of the central difficulties for adaptive wavelet algorithms.

The present paper starts with an adaptive wavelet transform which is suitable for a lacunary basis. It employs a collocation projection on successively coarsened grids, similar to [4]. The basic subtraction strategy is applicable to arbitrary generating sets. But in the present context the orthogonality of the wavelet basis can be used as a second ingredience for the solution of differential equations to avoid the inversion of linear systems [3]. This is accomplished by defining biorthogonal vaguelettes that incorporate the differential operator. We therefore generalize the adaptive wavelet transform to the case of positive inhomogeneous elliptic operators by constructing the appropriate bases. This improves the algorithm of [1]. Further details and numerical results can be found in [2] together with the application of the method to the solution of nonlinear parabolic PDEs.

2. PERIODIC MULTIRESOLUTION. – A periodic multiresolution (MRA) of spaces $V_j \subset L^2(\mathbb{T})$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be constructed through periodization from $b^{\mathbb{R}}(x) \in L^2(\mathbb{R})$ with $b^{\mathbb{R}}_{ji}(x) = 2^{j/2}\, b^{\mathbb{R}}(2^j x - i)$ generating an MRA of $L^2(\mathbb{R})$ through [5]

$$b_{ji}(x) = \sum_{n\in\mathbb{Z}} b^{\mathbb{R}}_{ji}(x+n)\,, \quad x \in \mathbb{T} \quad , \qquad \text{or} \qquad \hat{b}(k) = \hat{b}^{\mathbb{R}}(k)\,, \quad k \in \mathbb{Z} \tag{1}$$

with

$$\hat{b}(k) = \int_0^1 b(x)\, e^{-2\pi i k x}\, dx \quad , \qquad \hat{b}^{\mathbb{R}}(\omega) = \int_{-\infty}^{\infty} b^{\mathbb{R}}(x)\, e^{-2\pi i \omega x}\, dx$$

These relations permit to deduce scaling functions ϕ, wavelets ψ and required filters from the nonperiodic case. The cardinal Lagrange functions in V_j are given by

$$\hat{S}_{ji}(k) = 2^{-j}\,\hat{b}_{ji}(k) / \sum_{n\in\mathbb{Z}} \hat{b}_{ji}(k + 2^j n) \tag{2}$$

provided that the denominator is different from zero. Observe that the latter is just the discrete Fourier transform of b_{ji} sampled at the points $n/2^j$.

3. TRANSFORM FOR A LACUNARY BASIS. – Any function $f_J \in V_J$ can be developed in the corresponding wavelet basis

$$f_J(x) = \sum_j \sum_i d_{ji}\, \psi_{ji}(x) \tag{3}$$

(we skip details such as index bounds, $\psi_{-1,0} := \phi_{00}$ etc.). The term "lacunary" is used here to indicate that not the whole set of basis functions in V_J is employed in the representation (3) but just a certain subset, adapted to a given function. Mostly, a so-called cone condition is fulfilled, but this is no prerequisite for the sequel.

The classical wavelet transform (WT) consists of a first projection step onto V_J and a subsequent decomposition in terms of scaling functions and wavelets. For the solution of PDEs a collocation projection is mostly applied in the first step for diverse reasons. Then, however, it is natural to use the cardinal function which is at the origin of this projection in every decomposition step $j = J, \dots, 0$ instead of the scaling function, i.e.

$$f_j(x) = \sum_i f_j(\frac{i}{2^j})\, S_{ji}(x) = \sum_i f_{j-1}(\frac{i}{2^{j-1}})\, S_{j-1,i}(x) + \sum_i d_{j-1,i}\, \psi_{j-1,i}(x) \tag{4}$$

Due to the orthogonality of the functions ψ_{ji} and since $\langle S_{j-1,i}\,, \psi_{j-1,n}\rangle = 0$ the wavelet amplitudes d_{ji} are computed by the filters $D_i^j = \langle S_{ji}\,, \psi_{j-1,0}\rangle$ where $\langle .,. \rangle$ is the usual scalar product. Subsequently, the contribution of the second sum in the rhs of (4) is subtracted at the even grid points to get

$$f_{j-1}(\frac{i}{2^{j-1}}) = f_j(\frac{i}{2^{j-1}}) - \sum_n d_{j-1,n}\, \psi_{j-1,n}(\frac{i}{2^{j-1}}) \tag{5}$$

When working with the entire set of basis functions in V_J all operations are conveniently carried out in Fourier space by FFT. However, the WT of f_J does not seem to have any advantage for a direct (i.e. non-iterative) solution of a differential equation. In that case employing the functions ϕ_{Ji}, S_{Ji}, or b_{Ji} is much simpler and leads to the same result.

The above WT has been set up to be executed in physical space for a lacunary basis set (where FFT is inapplicable), as it works with the values at grid points in physical space. The filters D^j and the functions ψ_{ji} will generally have non-compact support. But in that case they exhibit fast decay which allows truncation in space up to a given precision. The successive coarsening of the employed grids leads to an $O(n\,M)$ operation count if the resulting filters have length M and n entries are retained in (3). The price to be payed for the finite filter length is a slight error on each level which can be controlled, however. On the other hand, the evaluation of f_J which is often costly in the PDE context is not required at all points $n/2^J$ but just at a subset defined by the selected wavelet functions. The inverse transform is analogeous and again based on (4) with x replaced by $n/2^j$. Note that for even n the first sum on the rhs contains only one entry. In summary, the cardinal function can be a convenient means to accomplish the simultaneous projections with locally varying finest grid and to relate the amplitudes to values at these grid points which is required in the PDE context.

4. OPERATOR–ADAPTED DECOMPOSITION. – Consider a linear operator L of order s with constant coefficients and positive symbol $\sigma(\xi) = \sum_{m=0}^{s} a_m (2\pi i\xi)^m > 0$ (i.e. $a_0 > 0$). For the periodic case ξ is replaced by $k \in \mathbb{Z}$. The aim now is to solve the differential equation

$$Lu(x) = f(x) \quad , \quad x \in \mathbb{T} \tag{6}$$

We set $u(x) = \sum_j \sum_i d_{ji}\, \psi_{ji}(x)$ with ψ belonging to a sufficiently smooth multiresolution and restrict $j < J$ for some large $J \in \mathbb{N}$. Then the image of V_J is $V_{L;J} = span\{Lb_{Ji}\}$. In many cases the related cardinal Lagrange functions $S_{L;ji}$ can be constructed as with eq. (2) replacing $\hat{b}_{ji}(k)$ with $\sigma(k)\hat{b}_{ji}(k)$. This allows to project the rhs of (6) onto $V_{L;J}$ by collocation

$$f_{L;J}(x) = \sum_i f(\frac{i}{2^J})\; S_{L;Ji}(x) \tag{7}$$

A Petrov-Galerkin method with test functions θ_{ji} is now used to determine the amplitudes d_{ji} of the solution. Solving (6) can thereby be made equivalent to representing the rhs as

$$f_{L;J}(x) = \sum_i \sum_j \; d_{ji}\; L\,\psi_{ji}(x) = \sum_i \sum_j \; \langle f_{L;Ji} \, , \, \theta_{ji} \rangle \; \mu_{ji}(x) \tag{8}$$

with the biorthogonal vaguelettes $\theta_{ji} = L^{-1*}\,\psi_{ji}$ and $\mu_{ji} = L\,\psi_{ji}$ constructed from the symbol. Several properties of these functions are reported in [6] where they are used with a different transform. In particular it can be shown that even if θ_{ji} is equivalent to the convolution of ψ_{ji} with the Greens function of the operator, it decays rapidly if the wavelets have sufficient vanishing moments.

With the above tools we now extend the WT of the previous section to the operator adapted case. The central equation is

$$f_{L;j}(x) = \sum_i f_{L;j}(\frac{i}{2^j}) S_{L;ji}(x) = \sum_i f_{L;j-1}(\frac{i}{2^{j-1}})\; S_{L;j-1,i}(x) + \sum_i d_{j-1,i}\; \mu_{j-1,i}(x) \tag{9}$$

Hence, the filters $D^j_{L;i} = \langle S_{L;ji} \, , \, \theta_{j-1,0} \rangle$ have to be employed to determine the amplitudes of the solution through

$$d_{j-1,i} = \sum_n f_{L;j}(\frac{n}{2^j})\; D^j_{L;n-2i}$$

(recall that in general $S_{L;ji} \neq L\, S_{ji}$ so that this expression does not simplify). Furthermore, ψ is replaced by μ in the subtraction analogeous to (5).

5. NUMERICAL RESULTS. – In this section we present results for Meyer wavelets ($b^{\mathbb{R}} = \phi_{Mey}$). The required computations in Fourier space are straightforward and furthermore their numerical support in physical space with low precision is even slightly smaller than the one of quintic spline wavelets [1]. The formulae for the exact filters in the operator adapted spline wavelet case ($b^{\mathbb{R}} = N_m$) are more involved and reported in [2] together with the related results.

The truncation of the filters advocated in the previous section introduces an error. It does not alter the perfect reconstruction property of the decomposition–recomposition scheme but the orthogonality which is relevant for the inversion of the differential operator. In Table I we report as an example $E = \max\{\langle \psi_{j0} \, , \, \psi_{km} \rangle_Q - \delta_{j0}\delta_{km}\}$ with different truncations ($J = 10$, double precision, full index set) for the collocation transform, its inverse, and the operator adapted decomposition. In the later case E is set up with θ and μ where $L = \lambda - \partial_{xx}$ with $\lambda = 150$. We observe that an asymmetric truncation improves the result, e.g. to 5.2E-4 for $K_Q = 40, K_S = 20$.

We furthermore solved a Helmholtz equation under the conditions of [1]. The new transform results in less than half the amount of work compared to the older method which employed the scaling function in the intermediate decomposition step.

6. CONCLUSION. The proposed operator adapted wavelet transform for a lacunary basis constitutes an appropriate framework for the solution of PDEs by adaptive wavelet-type bases. Generalization to higher dimensions is immediate. Its efficiency depends on the sizes of the resulting filters. Future work will be concerned with reducing their lengths by using MRAs with compactly supported cardinal functions and biorthogonal wavelets.

K_Q, K_S	WT	WT^{-1}	operator adapted
full grid	8.7 E-14	5.5 E-14	1.5 E-12
50, 50	2.7 E-6	3.9 E-6	1.7 E-4
30, 30	1.6 E-4	9.1 E-5	1.5 E-3
20, 20	8.6 E-4	6.3 E-4	1.1 E-2

Table I: Error in orthogonality relations when quadrature and substraction occuring in the transform (evaluation of S_{jl} and ψ_{jl} for WT^{-1}) are stopped at K_Q and K_S grid points from the wavelet center, respectively.

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