Averaged models for compressible two-phase stratified flows on thin domains

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Abstract. This paper deals with the derivation of compressible two-phase flow models. We use a thin domain approximation of a two-layer configuration governed by the Navier-Stokes equations, following the works of Ransom and Hicks or Stewart and Wendroff in the 1980s. In order to recover source terms and two-velocity models directly from this asymptotic analysis, we remove the continuity of the tangential component of the velocity at the interface, and properly scale friction and viscosity coefficients. We are then able to derive two-velocity one-pressure models, first in the barotropic case and then in the full Navier-Stokes-Fourier case.

1 Introduction

In this paper, we present a derivation of averaged models for compressible two-phase flows. The usual process consists in applying averaging operators on a complete local description of the flow, see for instance the classical textbooks [DP99] and [IH11]. The application of this process to bifluid formulations of stratified compressible two-phase flows led to a variety of different averaged models in the 1980s, see for instance [SW84, RH84, RH88]. The averaging process being incomplete, the previous papers provide additional heuristic arguments to close the models. They also discuss the influence of the closure arguments on the resulting models, in particular when focusing on interaction terms between phases. The *ad hoc* closure conditions introduced in these papers — especially at the interface — make the derivation of two-velocity models possible. Such approaches have been followed in [DH17] to include gravity effects as well as mass and momentum transfers.

Recent applications of mathematical homogenization to multiphase compressible Navier-Stokes equations provided the first complete, and rigorous, derivation of one-dimensional averaged models including the mechanical interaction source terms [BH11, BH15, BH19, Hil19, BBL22, HMS23]. Indeed, provided that the velocity field is smooth enough — especially through the interfaces between fluids —, this method enables the extraction of averaged variables and equations by letting the frequency of density oscillations tend to infinity. Since

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this procedure has yet only been carried in one space dimension, the inherent smoothness of the velocity field causes the averaged model to be described by a unique velocity field, making it difficult to obtain two-velocity models.

In this paper, we propose an asymptotic derivation of two-velocity, one-pressure averaged models. They are composed of two equations for phasic mass conservation, two nonconservative equations for phasic momentum, and two nonconservative equations for phasic energies:

$$\begin{aligned} \partial_t(\alpha_k\rho_k) + \nabla\cdot(\alpha_k\rho_k\mathbf{u}_k) &= 0, \\ \partial_t(\alpha_k\rho_k\mathbf{u}_k) + \nabla\cdot(\alpha_k\rho_k\mathbf{u}_k\otimes\mathbf{u}_k) + \nabla(\alpha_k p_k) - p_i\nabla\alpha_k &= S_k^m, \\ \partial_t(\alpha_k\rho_k E_k) + \nabla\cdot(\alpha_k(\rho_k E_k + p_k)\mathbf{u}_k) + p_i\partial_t\alpha_k &= S_k^e, \end{aligned} \tag{1.1}$$

where α_k is the volume fraction of phase $k\in\{1,2\}$, ρ_k is its density, $\mathbf{u}_k$ its velocity, p_k its pressure and E_k its energy. The source terms in the momentum and energy equations include the contribution of external forces, drag forces and heat exchanges. These models are closed by the definitions of total energies,

$$E_k = \frac{1}{2}\left|\mathbf{u}_k\right|^2 + e_k, \tag{1.2}$$

where e_k is the specific internal energy of phase k, the equations of state

$$\begin{aligned} p_k &= p_k(\rho_k, \rho_k e_k), \\ \theta_k &= \theta_k(\rho_k, \rho_k e_k) \end{aligned} \tag{1.3}$$

where θ_k is the temperature of phase k (which may appear in S_k^e), and the equality of pressures

$$p_1(\rho_1, \rho_1 e_1) = p_2(\rho_2, \rho_2 e_2) \tag{1.4}$$

which permits to deduce the void fractions since $\alpha_1 + \alpha_2 = 1$.

It is well-known that this model is not well-posed in the sense of Hadamard, since the convective part entails complex eigenvalues [RT78, Ste79]. Several directions have been explored to stabilize this model, like the addition of viscous phenomena, surface tension [RT78], virtual mass [Bes90, Ndj07] or internal capillary forces [BDGG10]. It is also possible to consider two-velocity and two-pressure models [BN86, RH84], which are well-posed for smooth solutions [CHSS14].

This paper carries a derivation of model (1.1)-(1.4). We start from a thin three-dimensional domain: the vertical length is assumed to be much smaller than the horizontal lengths, the ratio being described by the thickness parameter $\varepsilon = D/L \ll 1$ (see figure 1). The usual averaging processes [DP99, IH11] are replaced by an integration with respect to the vertical variable, resulting in a two-dimensional averaged model. The standard two-phase framework prescribes the continuity of the velocity field through the interface, resulting in a single smooth velocity field to describe the flow. In order to obtain a two-velocity model, our derivation circumvents the smoothness of the velocity field by making two complementary choices on the three-dimensional model. First, instead of imposing the continuity of the velocity field through the interface between phases, we prescribe a slip condition between the fluids tangentially to the interface. Second, in order to concentrate the efforts on the interaction of this interface condition with the averaging process, we study a simplifed configuration: we assume an initial and persistent stratified configuration, in the

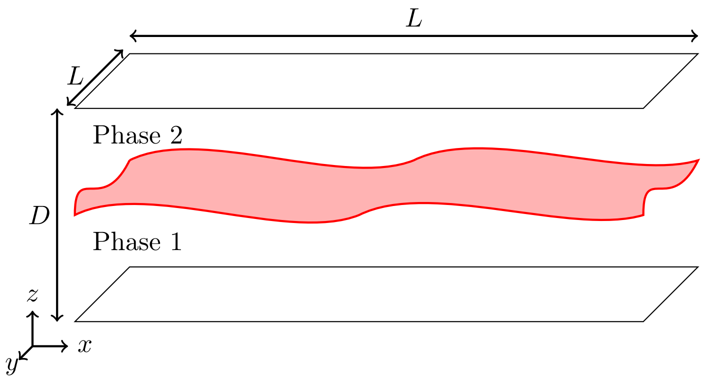

Figure 1: Flow configuration

sense that one fluid remains below the other, and the position of the interface can be parametrized by a function which only depends on time and the horizontal space variables (x, y). We finally set walls at the bottom and at the top. After rescaling the equations, we perform an asymptotic expansion of the microscopic model with respect to ε and average the equations to obtain the two-dimensional model (1.1)-(1.4) at the principal order. Thanks to this particular setting, it is possible to understand the influence of the different parameters of the initial model on the averaged equations, with respect to the thickness parameter. Compared to the works of Stewart and Wendroff and Ransom and Hicks [SW84, RH84], the introduction of Navier conditions and the asymptotic analysis enable the deduction of a two-velocity averaged model without additional heuristic arguments. Moreover, we are able to derive momentum and heat exchange terms. In order to achieve this formal derivation, we will only manipulate strong solutions to the considered Navier-Stokes systems.

This paper is divided into three parts. Section 2 introduces the configuration of the flow and the three-dimensional microscopic model in the barotropic case. Section 3 presents the averaged two-velocity, one-pressure model derived from the equations of the first section and develops the proof of this result. Section 4 is dedicated to the extension to the non-barotropic case, therefore adding two energy equations in the three-dimensional model. The resulting averaged model is described by two velocities, one pressure and two temperatures.

2 Barotropic two-phase stratified flows

This section presents the configuration of the flow, with a particular care for the description of the stratification. Then, it introduces the compressible barotropic Navier-Stokes systems which describe the flow. These systems are completed by Navier boundary conditions, which model the friction of the fluids on the top and bottom of the domain. Navier conditions are also prescribed on the relative velocity of the fluids at the interface. They depict the friction between both fluids tangentially to the interface. Finally, the different scalings taking place in our configuration are presented, which enables the deduction of a dimensionless version of the three-dimensional viscous model.

2.1 Stratified configuration and definition of the interface

Consider a stratified two-phase flow in a thin domain

$$\Omega_D = \{(\mathbf{x}, z) = (x, y, z) \in \mathbb{T}_L^2 \times (0, D)\}, \quad L,\, D > 0$$

where $\mathbb{T}_L^2$ denotes the two-dimensional torus of period L.

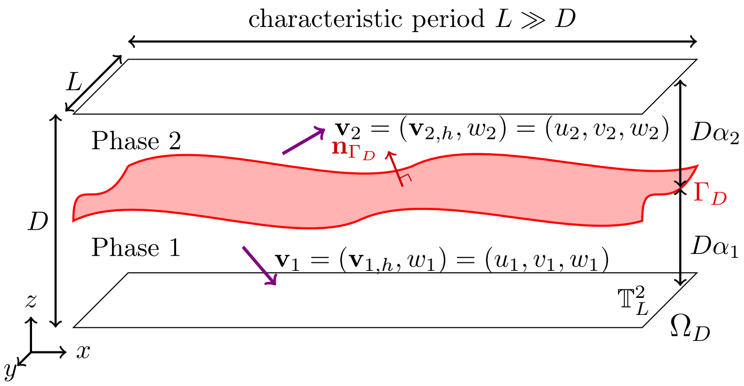

Figure 2: Flow configuration

The model studied in this work represents a compressible two-phase flow where the fluids are supposed immiscible and stratified (see figure 2). The domain filled by each fluid is defined thanks to the top and bottom boundary and the interface between each fluid. We assume that the interface Γ_D can be described by the use of continuous function $\alpha_1 : \mathbb{R}_+ \times \mathbb{T}_L^2 \to (0, 1)$ in the following way:

$$\forall\, t \in \mathbb{R}_+, \qquad \Gamma_D(t) = \{\ (\mathbf{x}, D\alpha_1(t, \mathbf{x})),\ \mathbf{x} \in \mathbb{T}_L^2\}.$$

The domain $\Omega_{D,1}(t)$ filled by the first phase is described by the characteristic function

$$\forall\, t \in \mathbb{R}_+, \quad \forall\, \mathbf{x} \in \mathbb{T}_L^2, \qquad \chi_{D,1}(t, \mathbf{x}, z) = \begin{cases} 1 & \text{for } 0 < z < D\alpha_1(t, \mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the function $D\alpha_1$ describes the height of the domain filled by the first phase, which means that α_1 can also be interpreted as the volume fraction of the first phase.
Similarly, the domain $\Omega_{D,2}(t)$ filled by the second phase is described by the characteristic function $\chi_{D,2} = 1 - \chi_{D,1}$. Denote $D\alpha_2 = D - D\alpha_1$ the height of the domain filled by the second phase. The function α_2 thus also describes the volume fraction of the second phase. The domain Ω_D is assumed to be saturated, which means

$$\alpha_1 + \alpha_2 = 1. \tag{2.1}$$

2.2 The compressible barotropic stratified model

Let $k \in \{1, 2\}$. Fluid k is described by its density $\rho_k : \mathbb{R}_+ \times \Omega_{D,k} \to \mathbb{R}_+$, its velocity $\mathbf{v}_k = (u_k, v_k, w_k) : \mathbb{R}_+ \times \Omega_{D,k} \to \mathbb{R}^3$ and its pressure $p_k : \mathbb{R}_+ \times \Omega_{D,k} \to$

$\mathbb{R}$. The flows being viscous, we introduce μ_k and λ_k the shear and bulk viscosity of the fluid. This section describes the microscopic Navier-Stokes model, which is the starting point of the derivation. In addition to the fluid equations and boundary conditions, particular care will be given to the presciption of interface conditions.

2.2.1 Fluid equations

This flow is described by two compressible, barotropic and viscous Navier-Stokes systems — one for each fluid. Let $h_0 = 0$, $h_1 = \alpha_1$ and $h_2 = 1$. Consider the following system for

$$k \in \{1,2\}, \quad t > 0, \quad (\mathbf{x}, z) \in \Omega_{D,k}(t) = \mathbb{T}_L^2 \times (Dh_{k-1}(t,\mathbf{x}), Dh_k(t,\mathbf{x})) :$$

$$\partial_t \rho_k + \nabla \cdot (\rho_k \mathbf{v}_k) = 0, \tag{2.2}$$

$$\partial_t(\rho_k \mathbf{v}_k) + \nabla \cdot (\rho_k \mathbf{v}_k \otimes \mathbf{v}_k) + \nabla p_k = \nabla \cdot \mathbb{S}_k, \tag{2.3}$$

$$p_k = p_k(\rho_k) \tag{2.4}$$

where the viscous tensor $\mathbb{S}_k : \mathbb{R}_+ \times \Omega_{D,k} \to \mathcal{M}_3(\mathbb{R})$ associated with fluid k is of the form

$$\mathbb{S}_k = 2\mu_k \underbrace{\frac{(\nabla \otimes \mathbf{v}_k + (\nabla \otimes \mathbf{v}_k)^T)}{2}}_{\mathbb{D}_k} + \lambda_k (\nabla \cdot \mathbf{v}_k)\mathbb{I}$$

and the equations of state (2.4) are regular and increasing. Equation (2.2) illustrates the conservation of mass. The next one is the momentum equation, which is obtained thanks to Newton's second principle.

2.2.2 Boundary conditions

Denote $\mathbf{v}_{k,h}$ the horizontal component of $\mathbf{v}_k$ for $k \in \{1,2\}$ and

$$\mathbb{S}_k = \begin{pmatrix} \mathbb{S}_{k,h} \\ \mathbb{S}_{k,z} \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{S}_{k,hh} & \mathbb{S}_{k,hz} \\ \hline \mathbb{S}_{k,zh} & \mathbb{S}_{k,zz} \end{array}\right), \text{ where } \mathbb{S}_{k,hh} \in \mathcal{M}_2(\mathbb{R}).$$

Navier boundary conditions are prescribed on the top and bottom of the domain. These boundary conditions for compressible flows have received limited attention in the literature, yet they appear in a handful of publications, including [Ers16, FJN12, Sue14]. They read

$$\begin{cases} w_1|_{z=0} = 0, & (2.5a) \\ (-\mathbb{S}_{1,hz} + \kappa_1 \mathbf{v}_{1,h})|_{z=0} = 0. & (2.5b) \end{cases} \qquad \begin{cases} w_2|_{z=D} = 0, & (2.6a) \\ (\mathbb{S}_{2,hz} + \kappa_2 \mathbf{v}_{2,h})|_{z=D} = 0. & (2.6b) \end{cases}$$

The first equation of each boundary condition depicts the impermeability of the domain. The second one describes the horizontal friction of the fluid on the boundary. The quantities κ_1 and κ_2 are constant friction coefficients at the bottom and top boundary respectively.

2.2.3 Interface conditions

To complete the system description, interface conditions are required. Denote $[f]_{z=D\alpha_1}$ the jump of a quantity f across the interface:

$$[f]_{z=D\alpha_1} = f_2|_{z=D\alpha_1} - f_1|_{z=D\alpha_1}.$$

We start by prescribing the immiscibility of the fluids, or in other words the fact that the interface is and remains a material interface through time. This is expressed by the continuity of the normal component of the velocity field and the normal stress across the interface. Defining the total stress tensor

$$\sigma_k = -p_k\mathbb{I} + \mathbb{S}_k = (-p_k + \lambda_k(\nabla \cdot \mathbf{v}_k))\mathbb{I} + 2\mu_k\mathbb{D}_k,$$

the conditions read
$$[\sigma \mathbf{n}_{\Gamma_D}]_{z=D\alpha_1} = 0 \tag{2.7}$$

and
$$[\mathbf{v} \cdot \mathbf{n}_{\Gamma_D}]_{z=D\alpha_1} = 0, \tag{2.8}$$

where
$$\mathbf{n}_{\Gamma_D} = \frac{1}{\sqrt{\|\nabla_h(D\alpha_1)\|^2 + 1}}\begin{pmatrix}-\nabla_h(D\alpha_1) & 1\end{pmatrix}^T \tag{2.9}$$

is the unitary normal vector to the interface pointing upwards (see figure 2) and ∇_h is the horizontal gradient operator $\begin{pmatrix}\partial_x & \partial_y\end{pmatrix}^T$.

The dynamics of the interface is described thanks to the volume fraction equation
$$\partial_t(D\alpha_1) + \mathbf{v}_{1,h}(D\alpha_1) \cdot \nabla_h(D\alpha_1) = w_1(D\alpha_1). \tag{2.10}$$

This equation means that $D\alpha_1$ is transported by the velocity field at the interface. Note that thanks to the continuity condition (2.8), the same equation is obtained if the velocity of the second phase is considered. The following lemma provides a physical interpretation of equation (2.10).

Lemma 1. *The domain occupied by each fluid being advected by the flow, the characteristic functions $\chi_{D,1}$ and $\chi_{D,2}$ satisfy the following equations:*

$$\forall\, k \in \{1, 2\}, \qquad \partial_t\chi_{D,k} + \mathbf{v}_k \cdot \nabla\chi_{D,k} = 0. \tag{2.11}$$

This equation is compatible with the volume fraction equation (2.10)*, meaning that the evolution of the interface and of the domains occupied by each fluid are consistent with each other.*

Proof. Integrating equation (2.11) for the first fluid over Ω_D yields

$$\int_{\Omega_D} (\partial_t\chi_{D,1} + \mathbf{v}_1 \cdot \nabla\chi_{D,1})\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$$

On the one hand,

$$\int_{\Omega_D} \partial_t\chi_{D,1}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\mathbb{T}_L^2} \partial_t \left(\int_{(0,D)} \chi_{D,1}\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y, = \int_{\mathbb{T}_L^2} \partial_t(D\alpha_1)\mathrm{d}x\mathrm{d}y.$$

On the other hand,

$$\int_{\Omega_D} \mathbf{v}_1 \cdot \nabla\chi_{D,1} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\Omega_D} \nabla\cdot(\mathbf{v}_1\chi_{D,1})\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{\Omega_{D,1}} \nabla\cdot\mathbf{v}_1 \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Thanks to the divergence theorem and the impermeability condition (2.5a),

$$\int_{\Omega_D} \mathbf{v}_1 \cdot \nabla\chi_{D,1} \mathrm{d}x\mathrm{d}y\mathrm{d}z = -\int_{\Gamma_D} \mathbf{v}_1 \cdot \mathbf{n}_{\Gamma_D} \mathrm{d}S,$$

where $\mathbf{n}_{\Gamma_D}$ is defined by (2.9). Since the surface Γ_D is parametrized by the function $\varphi(x,y) = (x, y, D\alpha_1(x,y))$, a change of variables gives

$$\int_{\Gamma_D} \mathbf{v}_1 \cdot \mathbf{n}_{\Gamma_D} \mathrm{d}S = -\int_{\mathbb{T}^2_L} (\mathbf{v}_{1,h}(D\alpha_1)\cdot\nabla_h(D\alpha_1) - w_1(D\alpha_1))(x,y)\mathrm{d}x\mathrm{d}y = 0.$$

Putting everything together,

$$\begin{aligned}&\int_{\Omega_D} (\partial_t\chi_{D,1} + \mathbf{v}_1\cdot\nabla\chi_{D,1})\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\qquad = \int_{\mathbb{T}^2_L} (\partial_t(D\alpha_1) + \mathbf{v}_{1,h}(D\alpha_1)\cdot\nabla_h(D\alpha_1) - w_1(D\alpha_1))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Hence the compatibility between the advection of the domain (2.11) and the volume fraction equation (2.10). The reasoning is the same for the second fluid. □

There remains to prescribe the behaviour of the tangential component of the velocity fields at the interface. In order to obtain a wide range of two-phase flow models at the thin-layer limit, we allow for possible slip at the interface between the two fluids. To this end, we first consider fluid 2 as a moving rigid body and impose a Navier-slip condition on fluid 1 at interface Γ_D, in the same manner as [NRRS22] — derivations of slip conditions for compressible flows can be found in [ABH$^+$17, JW25]. We then proceed symmetrically to impose a Navier-slip condition on fluid 2 at interface Γ_D. In practice, since n_{Γ_D} is the outward-pointing normal vector to the domain $\Omega_{D,1}(t)$, the slip condition on fluid 1 at interface Γ_D reads

$$\mathbf{v}_1 \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1} = \mathbf{v}_i \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1}, \tag{2.12a}$$
$$(\mathbb{S}_1\mathbf{n}_{\Gamma_D} + \kappa_{i,1}(\mathbf{v}_1 - \mathbf{v}_i))_{\tan}|_{z=D\alpha_1} = 0. \tag{2.12b}$$

where $X_{\tan}|_{z=D\alpha_1}$ denotes the component of the vector X which is tangential to the interface and $\kappa_{i,1}$ is the friction coefficient of the first fluid at the interface. Similarly, the outward-pointing normal vector to the domain $\Omega_{D,2}(t)$ being $-\mathbf{n}_{\Gamma_D}$, the Navier condition for the second fluid at the interface reads

$$\mathbf{v}_2 \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1} = \mathbf{v}_i \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1}, \tag{2.13a}$$
$$(-\mathbb{S}_2\mathbf{n}_{\Gamma_D} + \kappa_{i,2}(\mathbf{v}_2 - \mathbf{v}_i))_{\tan}|_{z=D\alpha_1} = 0. \tag{2.13b}$$

Firstly, combining the normal components of the Navier conditions at the interface (2.12a) and (2.13a) recovers the continuity of the normal component of the velocity field at the interface (2.8). Secondly, combining the tangential components of the Navier conditions at the interface (2.12b) and (2.13b) enables to obtain a single condition on the relative velocity of the fluids at the interface. Indeed, multiplying (2.12b) by $\kappa_{i,2}$ and (2.13b) by $\kappa_{i,1}$ and substracting the two equations,

$$((\kappa_{i,2}\mathbb{S}_1 + \kappa_{i,1}\mathbb{S}_2)\mathbf{n}_{\Gamma_D} - \kappa_{i_1}\kappa_{i_2}[\mathbf{v}])_{\tan}|_{z=D\alpha_1} = 0. \tag{2.14}$$

Looking at the tangential component of equation (2.7),

$$[(\mathbb{S}\mathbf{n}_{\Gamma_D})_{\tan}]_{z=D\alpha_1} = 0. \tag{2.15}$$

Then, denoting $\kappa_i = \frac{\kappa_{i,1}\kappa_{i,2}}{\kappa_{i,1}+\kappa_{i,2}}$ the harmonic average of the friction coefficients at the interface, equation (2.14) can be rewritten as

$$(\mathbb{S}_1\mathbf{n}_{\Gamma_D} - \kappa_i[\mathbf{v}])_{\tan}|_{z=D\alpha_1} = 0. \tag{2.16}$$

Remark 1 (About the choice of interface conditions). The choice of the Navier condition at the interface is motivated by the fact that it allows a jump in the tangential component of the velocity field across the interface, which as we will see is necessary to obtain a two-velocity averaged model. In complement, the full continuity of the velocity field through the interface Γ_D is also investigated. This corresponds to imposing Dirichlet conditions for each fluid at the interface, which results in a one-velocity averaged model at the thin-layer limit. Note that this model can also be recovered by letting κ_i tend to 0.

2.3 Rescaling the equations in the thin domain assumption

Let $\varepsilon = D/L$ be a small parameter corresponding to the thin domain assumption. Let us introduce dimensionless variables $\tilde{x} = x/L$, $\tilde{y} = y/L$ and $\tilde{z} = z/D = z/(\varepsilon L)$. Denoting T the characteristic time of the flow allows the introduction of the dimensionless time $\tilde{t} = t/T$. Denoting $\mathbb{T}^2 = \mathbb{T}^2_1$, switching the variables allows to study the flow on the rescaled domain

$$\Omega = \mathbb{T}^2 \times (0,1).$$

In the same spirit, characteristic dimensions are introduced for every unknown in the system: $U = L/T$ for the horizontal velocity, $W = \varepsilon U$ for the vertical velocity, R for the density, $P = RU^2$ for the pressure, $M = RUL$ for the viscosity, and $K = RU$ for the friction coefficients. Denote the dimensionless unknowns $\tilde{u} = u/U$, $\tilde{w} = w/W$, $\tilde{\rho} = \rho/R$, $\tilde{p} = p/P$. Denote $\tilde{\mu} = \mu/M$ the inverse of the Reynolds number, $\tilde{\lambda} = \lambda/M$ and $\tilde{\kappa} = \kappa/K$. The dimensionless viscosity and friction coefficients are constant parameters which may depend on ε. Moreover, the scalings of the different coefficients are assumed to be similar for both fluids.

Denote χ_k the rescaled version of $\chi_{D,k}$ for $k \in \{1,2\}$, Ω_k the rescaled version of $\Omega_{D,k}$ and Γ the rescaled version of Γ_D. Removing the $\tilde{\cdot}$ notation for the sake of clarity, and denoting $\Delta_h = \nabla_h \cdot \nabla_h$, equations (2.1)-(2.11), (2.16) are rewritten as follows:

Lemma 2. *The rescaling of model* (2.1)-(2.11), (2.16) *results in the following set of equations on the rescaled domain* Ω.

- ***Dimensionless saturation condition.***

$$\alpha_1 + \alpha_2 = 1. \tag{2.17}$$

- ***Dimensionless advection of the domain.*** *For all* $k \in \{1, 2\}$,

$$\partial_t \chi_k + \mathbf{v}_k \cdot \nabla \chi_k = 0. \tag{2.18}$$

- ***Dimensionless volume fraction equation.***

$$\partial_t \alpha_1 + \mathbf{v}_{1,h}(\alpha_1) \cdot \nabla_h \alpha_1 = w_1(\alpha_1). \tag{2.19}$$

- ***Dimensionless mass conservation equations.*** *For all* $k \in \{1, 2\}$,

$$\partial_t \rho_k + \nabla_h \cdot (\rho_k \mathbf{v}_{k,h}) + \partial_z(\rho_k w_k) = 0. \tag{2.20}$$

- ***Dimensionless horizontal momentum equations.*** *For all* $k \in \{1, 2\}$,

$$\begin{aligned} &\partial_t(\rho_k \mathbf{v}_{k,h}) + \nabla_h \cdot (\rho_k \mathbf{v}_{k,h} \otimes \mathbf{v}_{k,h}) + \partial_z(\rho_k w_k \mathbf{v}_{k,h}) + \nabla_h p_k \\ &\qquad = (\lambda_k + \mu_k)\nabla_h(\nabla \cdot \mathbf{v}_k) + \mu_k \Delta_h \mathbf{v}_{k,h} + \frac{\mu_k}{\varepsilon^2}\partial_{zz}\mathbf{v}_{k,h}. \end{aligned} \tag{2.21}$$

- ***Dimensionless vertical momentum equations.*** *For all* $k \in \{1, 2\}$,

$$\begin{aligned} &\varepsilon(\partial_t(\rho_k w_k) + \nabla_h \cdot (\rho_k \mathbf{v}_{k,h} w_k) + \partial_z(\rho_k (w_k)^2)) + \frac{1}{\varepsilon}\partial_z p_k \\ &\qquad = \frac{1}{\varepsilon}((\lambda_k + \mu_k)\partial_z(\nabla \cdot \mathbf{v}_k) + \mu_k \partial_{zz} w_k) + \mu_k \varepsilon \Delta_h w_k. \end{aligned} \tag{2.22}$$

- ***Dimensionless Navier boundary conditions.***

$$\begin{cases} w_1|_{z=0} = 0, & (2.23\text{a}) \\ \left(-\frac{\mu_1}{\varepsilon}\partial_z \mathbf{v}_{1,h} + \kappa_1 \mathbf{v}_{1,h}\right)_{z=0} = 0. & (2.23\text{b}) \end{cases}$$

$$\begin{cases} w_2|_{z=1} = 0, & (2.24\text{a}) \\ \left(\frac{\mu_2}{\varepsilon}\partial_z \mathbf{v}_{2,h} + \kappa_2 \mathbf{v}_{2,h}\right)_{z=1} = 0. & (2.24\text{b}) \end{cases}$$

- ***Dimensionless continuity of the normal stress at the interface.***

$$\begin{aligned} \Big[&\partial_x \alpha_1 (p - \lambda \nabla \cdot \mathbf{v} - 2\mu \partial_x u) - \mu \partial_y \alpha_1 (\partial_y u + \partial_x v) \\ &\qquad + \mu \partial_x w + \frac{\mu}{\varepsilon^2}\partial_z u\Big]_{z=\alpha_1} = 0, \end{aligned} \tag{2.25}$$

$$\begin{aligned} \Big[&\partial_y \alpha_1 (p - \lambda \nabla \cdot \mathbf{v} - 2\mu \partial_y v) - \mu \partial_x \alpha_1 (\partial_y u + \partial_x v) \\ &\qquad + \mu \partial_y w + \frac{\mu}{\varepsilon^2}\partial_z v\Big]_{z=\alpha_1} = 0, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \Big[&-p + \lambda \nabla \cdot \mathbf{v} + 2\mu \partial_z w - \mu \partial_x \alpha_1 (\partial_z u + \varepsilon^2 \partial_x w) \\ &\qquad - \mu \partial_y \alpha_1 (\partial_z v + \varepsilon^2 \partial_y w)\Big]_{z=\alpha_1} = 0. \end{aligned} \tag{2.27}$$

- ***Dimensionless normal Navier interface condition.***

$$\Big[-u\partial_x\alpha_1 - v\partial_y\alpha_1 + w\Big]_{z=\alpha_1} = 0. \tag{2.28}$$

- ***Dimensionless tangential Navier interface condition.***

$$\begin{aligned}\mu_1\varepsilon\Big(\frac{1}{\varepsilon^2}\partial_z u_1 + \partial_x w_1 - 2\partial_x\alpha_1(\partial_x u_1 - \partial_z w_1) - \partial_y\alpha_1(\partial_y u_1 + \partial_x v_1)\\ -\partial_x\alpha_1\partial_y\alpha_1(\partial_z v_1 + \varepsilon^2\partial_y w_1) - (\partial_x\alpha_1)^2(\partial_z u_1 + \varepsilon^2\partial_x w_1)\Big)_{z=\alpha_1}\\ -\kappa_i[u + \varepsilon^2 w\partial_x\alpha_1]_{z=\alpha_1} = 0,\end{aligned} \tag{2.29}$$

$$\begin{aligned}\mu_1\varepsilon\Big(\frac{1}{\varepsilon^2}\partial_z v_1 + \partial_y w_1 - 2\partial_y\alpha_1(\partial_y v_1 - \partial_z w_1) - \partial_x\alpha_1(\partial_x v_1 + \partial_y u_1)\\ -\partial_y\alpha_1\partial_x\alpha_1(\partial_z u_1 + \varepsilon^2\partial_x w_1) - (\partial_y\alpha_1)^2(\partial_z v_1 + \varepsilon^2\partial_y w_1)\Big)_{z=\alpha_1}\\ -\kappa_i[v + \varepsilon^2 w\partial_y\alpha_1]_{z=\alpha_1} = 0.\end{aligned} \tag{2.30}$$

Remark 2. Note that the tangential Navier conditions (2.29)-(2.30) are written differently than the other interface conditions. Indeed, it is the sum of a trace and a jump term, whereas the other can all be written as a jump condition.

Proof. The introduction of the dimensionless variables and unknowns in the various equations, boundary conditions and interface conditions is performed using the chain rule. The characteristic function χ_k is advected by the dimensionless velocity, hence equation (2.18). The rescaling procedure immediately yields the volume fraction equation (2.19), the mass conservation equation (2.20) and the momentum equation, which is split into two components — the horizontal component (2.21) and the vertical component (2.22).

Rescaling the boundary conditions yields equations (2.23) and (2.24). Since the normal vector to the interface is now

$$\mathbf{n}_\Gamma = \frac{1}{\sqrt{1+\varepsilon^2|\nabla_h\alpha_1|^2}}\begin{pmatrix}-\varepsilon\nabla_h\alpha_1\\ 1\end{pmatrix}, \tag{2.31}$$

the rescaling of equation (2.7) detailed into every component gives (2.25), (2.26) and (2.27).

Now, the Navier interface conditions must be rescaled. Using (2.31), the continuity of the normal component of the velocity at the interface (2.8) is rescaled to obtain (2.28). The tangential condition requires more computations. First, let us introduce two tangential vectors to the interface:

$$\mathbf{t}^x_{\Gamma_D} = \begin{pmatrix}1\\0\\ \partial_x(D\alpha_1)\end{pmatrix} \quad \text{and} \quad \mathbf{t}^y_{\Gamma_D} = \begin{pmatrix}0\\1\\ \partial_y(D\alpha_1)\end{pmatrix}$$

Those two are rescaled as

$$\mathbf{t}^x_\Gamma = \begin{pmatrix}1\\0\\ \varepsilon\partial_x\alpha_1\end{pmatrix} \quad \text{and} \quad \mathbf{t}^y_\Gamma = \begin{pmatrix}0\\1\\ \varepsilon\partial_y\alpha_1\end{pmatrix}.$$

Then, the tangential component of the Navier condition (2.16) is equivalent to the system

$$\begin{cases} (\mathbb{S}_1 \mathbf{n}_{\Gamma_D} - \kappa_i[\mathbf{v}])_{z=D\alpha_1} \cdot \mathbf{t}^x_{\Gamma_D}(D\alpha_1) = 0, \\ (\mathbb{S}_1 \mathbf{n}_{\Gamma_D} - \kappa_i[\mathbf{v}])_{z=D\alpha_1} \cdot \mathbf{t}^y_{\Gamma_D}(D\alpha_1) = 0. \end{cases}$$

which is rescaled as (2.29)-(2.30). □

3 The asymptotic system in the barotropic case

This section presents the asymptotic model derived from the rescaled compressible barotropic double Navier-Stokes system (2.17)-(2.30). The first subsection introduces the fundamental notion of average that will be used later, and states the theorem presenting the asymptotic model. The second subsection develops the derivation process.

3.1 Presentation of the averaged model

This section introduces some notations and presents the asymptotic barotropic model derived from the rescaled Navier-Stokes system. As explained earlier, integrating the Navier-Stokes equations causes the appearance of averaged unknowns. Since the majority of the terms in the Navier-Stokes equations are nonlinear in the compressible setting and since the multiplication and the integration do not commute, it is necessary to define different types of averages.

Definition 1 (Averages). Let f_k be a function defined on the domain filled by fluid $k \in \{1,2\}$. The average of f_k, denoted $\overline{f_k}$, is defined for all $(t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{T}^2$ by

$$\overline{f_k}(t,\mathbf{x}) = \frac{1}{\alpha_k(t,\mathbf{x})} \int_{h_{k-1}(t,\mathbf{x})}^{h_k(t,\mathbf{x})} f_k(t,\mathbf{x},z)\mathrm{d}z.$$

The Favre average of f_k with respect to ρ_k, denoted $\langle f_k \rangle$, is defined for all $(t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{T}^2$ by

$$\langle f_k \rangle(t,\mathbf{x}) = \frac{\overline{\rho_k f_k}(t,\mathbf{x})}{\overline{\rho_k}(t,\mathbf{x})} = \frac{1}{\alpha_k(t,\mathbf{x})\overline{\rho_k}(t,\mathbf{x})} \int_{h_{k-1}(t,\mathbf{x})}^{h_k(t,\mathbf{x})} (\rho_k f_k)(t,\mathbf{x},z)\mathrm{d}z.$$

Let us state the first result on the thin-domain asymptotics of two-layer models. The following theorem shows that under some scaling conditions on the viscosity and friction coefficients, the averages of solutions to the Navier-Stokes system with Navier boundary and interface conditions asymptotically satisfy the barotropic version of model (1.1)-(1.4) given in the introdution. In the theorem below, the differences between the averaged Navier-Stokes system and the limit model in terms of the small parameter ε are provided. This allows to understand the necessary conditions for the model to be valid when ε goes to 0 and to quantify the approximation errors. In the rest of the paper, the approximation errors will be denoted thanks to the Landau notation, which is defined as follows.

Definition 2. Let f be a function valued in $\mathbb{R}^d$, $d \in \mathbb{N}^*$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function. We say that $f = \mathcal{O}(g(\varepsilon))$ if there exists a positive constant C independent of ε such that

$$\|f\| \leqslant C g(\varepsilon)$$

for all ε small enough, uniformly on the domain of definition of f.

Let us now state the first result of this paper.

Theorem 1. *Assume the following scalings for viscosity and friction coefficients:*

$$\forall\, k \in \{1,2\}, \qquad \left\{ \begin{array}{c} \mu_k = \widehat{\mu_k}\varepsilon^\tau, \quad \lambda_k = \widehat{\lambda_k}\varepsilon^\tau, \quad 0 < \tau < 2; \\ \kappa_i = \widehat{\kappa_i}\varepsilon, \quad \kappa_k = \widehat{\kappa_k}\varepsilon^\xi, \quad \xi \geqslant 1, \end{array} \right. \tag{3.1}$$

where the coefficients $\widehat{\mu_k}$, $\widehat{\lambda_k}$, $\widehat{\kappa_i}$ and $\widehat{\kappa_k}$ are positive constants independent of ε, and the exponents τ and ξ are independent.

Define
$$\delta_\xi = \begin{cases} 1 & \text{if } \xi = 1, \\ 0 & \text{if } \xi > 1. \end{cases}$$

Let $(\chi_1, \alpha_1, \rho_1, \mathbf{v}_1, p_1, \chi_2, \alpha_2, \rho_2, \mathbf{v}_2, p_2)$ be a strong solution to the Navier-Stokes system (2.17)-(2.22) *with Navier boundary conditions* (2.23)-(2.24) *and interface conditions* (2.25)-(2.30).
The functions $(\alpha_1, \overline{\rho_1}, \langle \mathbf{v}_{1,h}\rangle, \overline{p_1}, \alpha_2, \overline{\rho_2}, \langle \mathbf{v}_{2,h}\rangle, \overline{p_2})$ satisfy the following system:

- ***Averaged mass conservation equations.*** *For all $k \in \{1,2\}$,*

$$\partial_t(\alpha_k \overline{\rho_k}) + \nabla_h \cdot (\alpha_k \overline{\rho_k} \langle \mathbf{v}_{k,h}\rangle) = 0. \tag{3.2}$$

- ***Averaged momentum equations.*** *For all $k \in \{1,2\}$,*

$$\begin{aligned} \partial_t(\alpha_k \overline{\rho_k}\langle \mathbf{v}_{k,h}\rangle) + \nabla_h \cdot (\alpha_k \overline{\rho_k} \langle \mathbf{v}_{k,h}\rangle \otimes \langle \mathbf{v}_{k,h}\rangle) + \nabla_h(\alpha_k \overline{p_k}) - p_i \nabla_h \alpha_k \\ = -\delta_\xi \widehat{\kappa_k}\langle \mathbf{v}_{k,h}\rangle + (-1)^k \widehat{\kappa_i}(\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle) \\ + \mathcal{O}(\varepsilon^\tau + \varepsilon^{2-\tau} + (1-\delta_\xi)\varepsilon^{\xi-1}). \end{aligned} \tag{3.3}$$

- ***Averaged equations of state.*** *For all $k \in \{1,2\}$,*

$$\overline{p_k} = p_k(\overline{\rho_k}) + \mathcal{O}(\varepsilon^\tau). \tag{3.4}$$

- ***Equality of the pressures.***

$$\overline{p_1} = \overline{p_2} = p_i \text{ up to an error } \mathcal{O}(\varepsilon^\tau). \tag{3.5}$$

- ***Saturation of the domain.***

$$\alpha_1 + \alpha_2 = 1. \tag{3.6}$$

Remark 3. This work keeps track of the approximation errors. They will enable the deduction of the necessary and sufficient conditions for the asymptotic model to be valid (see section 3.2.3), which are equivalent to the scaling prescription (3.1). Our derivation process then results in an averaged model that is compressible and inviscid. In other words, our asymptotic limit corresponds to moderate Mach numbers and high Reynolds numbers.

3.2 Proof of theorem 1: derivation of the asymptotic model

The proof of theorem 1 is divided into several steps. Starting from the rescaled equations (2.17)-(2.30), we first simplify the vertical momentum equation and the interface conditions by neglecting terms of high order in ε. Then, we integrate the mass conservation equation and the horizontal momentum equation in the vertical direction. This procedure causes boundary terms to appear in the equations, which must be closed using the rescaled boundary and interface conditions. Finally, we prescribe specific profiles for the viscosity and friction coefficients for the asymptotic model to be valid.

3.2.1 Simplification of the vertical momentum equation and interface conditions

Some of the rescaled equations and conditions listed in lemma 2 are very complex. However, some terms are of high order in ε, and therefore become very small in our setting ($\varepsilon \ll 1$). Dropping these terms allows a first simplification of the system. To make sure that the derived system is really an approximation of the initial double Navier-Stokes system, the computations keep trace of the errors caused along the way thanks to Landau notations. Since the scaling of the viscosity and friction coefficients are assumed to be independent of the fluid, the subscript k is dropped in the Landau notations.

Performing the asymptotic analysis impacts the vertical momentum equation (2.22) and interface conditions (2.27), (2.29) and (2.30) in the following way.

- **Asymptotic vertical momentum equation.** For all $k \in \{1, 2\}$,

$$\partial_z p_k = (\lambda_k + \mu_k)\partial_z(\nabla \cdot \mathbf{v}_k) + \mu_k \partial_{zz} w_k + \mathcal{O}(\varepsilon^2). \tag{3.7}$$

- **Asymptotic continuity of the vertical component of the normal stress at the interface.**

$$\Big[- p + \lambda \nabla \cdot \mathbf{v} + 2\mu \partial_z w - \mu \nabla_h \alpha_1 \cdot \partial_z \mathbf{v}_h \Big]_{z=\alpha_1} = \mathcal{O}(\mu\varepsilon^2). \tag{3.8}$$

- **Asymptotic tangential Navier interface condition.**

$$\left(\frac{\mu_1}{\varepsilon}\partial_z \mathbf{v}_{1,h}\right)_{z=\alpha_1} - \kappa_i [\mathbf{v}_h]_{z=\alpha_1} = \mathcal{O}(\mu\varepsilon + \kappa_i \varepsilon^2) \tag{3.9}$$

Remark 4. In the previous paragraph, we did not simplify the horizontal momentum equation (2.21) since the terms of order ε^2 are not negligible. Indeed, multiplying (2.21) by ε^2 and neglecting the quadratic terms in ε yields

$$\mu_k \partial_{zz} \mathbf{v}_{k,h} = \mathcal{O}(\varepsilon^2). \tag{3.10}$$

The term $\mu_k \partial_{zz} \mathbf{v}_{k,h}/\varepsilon^2$ is thus of order 1 and cannot be neglected in the equation.

3.2.2 Averaging the equations

Now, the averaging of the mass conservation equation and momentum equations is performed for each fluid in the vertical direction, therefore obtaining a two-dimensional system. The boundary terms which appear during the integration are approximated by terms which only depend on the averaged unknowns in order to close the system. The approximation errors are written explicitly during the whole averaging process. The averaging is performed for the first fluid, the second fluid being treated in the same way.

First, let us average the rescaled mass conservation equations (2.20). Multiply equation (2.20) by χ_1 and integrate it with respect to $z \in (0,1)$ — which is the same as integrating the equation with respect to $z \in (0,\alpha_1)$:

$$\int_0^1 \chi_1 \left(\partial_t \rho_1 + \nabla_h \cdot (\rho_1 \mathbf{v}_{1,h}) + \partial_z(\rho_1 w_1)\right)(z)\mathrm{d}z = 0.$$

Since $\chi_1 \partial_t \rho_1 = \partial_t(\chi_1 \rho_1) - \rho_1 \partial_t \chi_1$ with $\partial_t \chi_1(t,\mathbf{x},z) = \partial_t \alpha_1(t,\mathbf{x}) \delta_{\alpha_1(t,\mathbf{x})}(z)$,

$$\int_0^1 \chi_1 \partial_t \rho_1(z)\mathrm{d}z = \partial_t \int_0^{\alpha_1} \rho_1(z)\mathrm{d}z - \rho_1(\alpha_1)\partial_t \alpha_1 = \partial_t(\alpha_1 \overline{\rho_1}) - \rho_1(\alpha_1)\partial_t \alpha_1.$$

Proceeding similarly for the second term of the equation,

$$\begin{aligned}\partial_t(\alpha_1 \overline{\rho_1}) - \rho_1(\alpha_1)\partial_t\alpha_1 + \nabla_h \cdot (\alpha_1 \overline{\rho_1 \mathbf{v}_{1,h}}) - \rho_1 \mathbf{v}_{1,h}(\alpha_1) \cdot \nabla_h(\alpha_1) \\ + \rho_1 w_1(\alpha_1) - \rho_1 w_1(0) = 0.\end{aligned}$$

Using the volume fraction equation (2.19), the bottom impermeability condition (2.23a) and the relation $\overline{\rho_1 \mathbf{v}_{1,h}} = \overline{\rho_1}\langle \mathbf{v}_{1,h}\rangle$ finally gives the averaged mass conservation equation (3.2) for the first fluid. The reasoning is the same for the second fluid, the equation being multiplied this time by χ_2 and using the Navier condition (2.24a). Note that the averaged mass conservation equations are exact as no approximation is needed in the averaging process.

Let us now prove the pressure equilibrium (3.5). A Taylor expansion and the asymptotic rescaled vertical momentum equation (3.7) give that

$$\begin{cases} \forall\, z \in (0,\alpha_1), & p_1(z) = p_1(\alpha_1) + \mathcal{O}(\mu), \\ \forall\, z \in (\alpha_1, 1), & p_2(z) = p_2(\alpha_1) + \mathcal{O}(\mu). \end{cases}$$

Taking the average of these relation gives that for all $k \in \{1,2\}$,

$$\overline{p_k} = p_k(\alpha_1) + \mathcal{O}(\mu). \tag{3.11}$$

The continuity of the normal stress at the interface (3.8) implies the relation $p_1(\alpha_1) = p_2(\alpha_1) + \mathcal{O}(\mu)$. Combining these equations together yields

$$\overline{p_1} = \overline{p_2} + \mathcal{O}(\mu).$$

In order to be consistent with the literature on two-phase flows [DP99, IH11], we introduce the so-called interfacial pressure p_i, which is an approximation of the pressure at the interface. The previous equation along with (3.11) justify that we set

$$p_i = \overline{p_k} + \mathcal{O}(\mu), \qquad \forall\, k \in \{1,2\}.$$

The pressure equilibrium (3.5) stated in the theorem is thus proven.

Proposition 1. *For all $k \in \{1,2\}$, the averaged horizontal momentum equation reads*

$$\begin{aligned}\partial_t(\alpha_k\overline{\rho_k}\langle\mathbf{v}_{k,h}\rangle) + \nabla_h\cdot(\alpha_k\overline{\rho_k}\langle\mathbf{v}_{k,h}\rangle\otimes\langle\mathbf{v}_{k,h}\rangle) + \nabla_h(\alpha_k\overline{p_k}) - p_i\nabla_h\alpha_k \\ = -\frac{\kappa_k}{\varepsilon}\langle\mathbf{v}_{k,h}\rangle + (-1)^k\frac{\kappa_i}{\varepsilon}(\langle\mathbf{v}_{1,h}\rangle - \langle\mathbf{v}_{2,h}\rangle) + \mathcal{R},\end{aligned} \tag{3.12}$$

where $$\mathcal{R} = \mathcal{O}\Big(\mu + \kappa_i\varepsilon + \frac{\kappa_i(\kappa+\varepsilon)+\kappa(\kappa+\varepsilon)}{\mu} + \frac{(\varepsilon\kappa+\varepsilon^2)^2}{\mu^2}\Big).$$

Proof. The proof is performed for $k=1$. Multiply the horizontal momentum equation (2.21) by χ_1 and integrate it with respect to $z \in (0,1)$. Using the volume fraction equation (2.19) and the impermeability condition (2.23a) yields

$$\begin{aligned}\partial_t(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\rangle) + \nabla_h\cdot(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\otimes\mathbf{v}_{1,h}\rangle) + \nabla_h(\alpha_1\overline{p_1}) - p_1(\alpha_1)\nabla_h\alpha_1 \\ = \frac{\mu_1}{\varepsilon^2}\partial_z\mathbf{v}_{1,h}(\alpha_1) - \frac{\mu_1}{\varepsilon^2}\partial_z\mathbf{v}_{1,h}(0) \\ + \int_0^{\alpha_1}\big((\lambda_1+\mu_1)\nabla_h(\nabla\cdot\mathbf{v}_1) + \mu_1\Delta_h\mathbf{v}_{1,h}\big)(z)\mathrm{d}z.\end{aligned}$$

Thanks to the horizontal Navier condition on the bottom boundary (2.23b), the equation reads

$$\begin{aligned}\partial_t(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\rangle) + \nabla_h\cdot(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\otimes\mathbf{v}_{1,h}\rangle) + \nabla_h(\alpha_1\overline{p_1}) - p_1(\alpha_1)\nabla_h\alpha_1 \\ = -\frac{\kappa_1}{\varepsilon}\mathbf{v}_{1,h}(0) + \frac{\mu_1}{\varepsilon^2}\partial_z\mathbf{v}_{1,h}(\alpha_1) \\ + \int_0^{\alpha_1}\big((\lambda_1+\mu_1)\nabla_h(\nabla\cdot\mathbf{v}_1) + \mu_1\Delta_h\mathbf{v}_{1,h}\big)(z)\mathrm{d}z.\end{aligned} \tag{3.13}$$

The asymptotic Navier condition (3.9) then gives

$$\frac{\mu_1}{\varepsilon^2}\partial_z\mathbf{v}_{1,h}(\alpha_1) = \frac{\kappa_i}{\varepsilon}(\mathbf{v}_{2,h}(\alpha_1) - \mathbf{v}_{1,h}(\alpha_1)) + \mathcal{O}(\mu + \kappa_i\varepsilon).$$

Injecting this new term into (3.13) causes an approximation of order μ. The integral term of the equation being of the same order as the approximation error,

$$\begin{aligned}\partial_t(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\rangle) + \nabla_h\cdot(\alpha_1\overline{\rho_1}\langle\mathbf{v}_{1,h}\otimes\mathbf{v}_{1,h}\rangle) + \nabla_h(\alpha_1\overline{p_1}) - p_1(\alpha_1)\nabla_h\alpha_1 \\ = -\frac{\kappa_1}{\varepsilon}\mathbf{v}_{1,h}(0) + \frac{\kappa_i}{\varepsilon}(\mathbf{v}_{2,h}(\alpha_1) - \mathbf{v}_{1,h}(\alpha_1)) + \mathcal{O}(\mu + \kappa_i\varepsilon).\end{aligned} \tag{3.14}$$

To close this equation, the pointwise values of the velocity and the pressure must be expressed thanks to their averaged values. Using estimate (3.11) and the pressure equilibrium (3.5), we approximate $p_1(\alpha_1)$ by p_i, causing an approximation of order μ.

The pointwise values of the velocity is detailed in the following lemma.

Lemma 3. *The pointwise values of the velocity $\mathbf{v}_{1,h}$ can be approximated as follows:*

$$\forall\, z \in (0,\alpha_1), \qquad \mathbf{v}_{1,h}(z) = \langle\mathbf{v}_{1,h}\rangle + \mathcal{O}\left(\frac{\varepsilon\kappa + \varepsilon^2}{\mu}\right). \tag{3.15}$$

The same approximation remain valid for $\mathbf{v}_{2,h}$.

Proof. Recall that remark 4 justifies that

$$\forall\, z \in (0, \alpha_1), \qquad \mu_1 \partial_{zz} \mathbf{v}_{1,h}(z) = \mathcal{O}(\varepsilon^2).$$

Moreover, the horizontal component of the bottom Navier boundary condition (2.23b) gives

$$\mu_1 \partial_z \mathbf{v}_{1,h}(0) = \mathcal{O}(\kappa\varepsilon).$$

Thanks to a Taylor-Lagrange expansion,

$$\forall\, z \in (0, \alpha_1), \qquad \mu_1 \partial_z \mathbf{v}_{1,h}(z) = \mathcal{O}(\kappa\varepsilon + \varepsilon^2). \tag{3.16}$$

Another Taylor-Lagrange expansion gives

$$\begin{aligned}
\forall (z, z') \in (0, \alpha_1)^2, \qquad \mathbf{v}_{1,h}(z) &= \mathbf{v}_{1,h}(z') + (z - z')\partial_z \mathbf{v}_{1,h}(z') + \mathcal{O}(\varepsilon^2/\mu), \\
&= \mathbf{v}_{1,h}(z') + \mathcal{O}\left(\frac{\varepsilon\kappa + \varepsilon^2}{\mu}\right).
\end{aligned}$$

Equation (3.15) is then obtained by taking the Favre average of the previous equation with respect to z'. The reasoning is the same for $\mathbf{v}_{2,h}$ with $z \in (\alpha_1, 1)$, using the top Navier condition (2.24b) to get the estimate. ☐

Applying the different approximations for the interfacial pressure and boundary and interfacial velocity yields

$$\begin{aligned}
\partial_t(\alpha_1 \overline{\rho_1} \langle \mathbf{v}_{1,h} \rangle) + \nabla_h \cdot (\alpha_1 \overline{\rho_1} \langle \mathbf{v}_{1,h} \otimes \mathbf{v}_{1,h} \rangle) + \nabla_h(\alpha_1 \overline{p_1}) - p_i \nabla_h \alpha_1 \\
= -\frac{\kappa_1}{\varepsilon} \langle \mathbf{v}_{1,h} \rangle + \frac{\kappa_i}{\varepsilon}(\langle \mathbf{v}_{2,h} \rangle - \langle \mathbf{v}_{1,h} \rangle) \\
+ \mathcal{O}\left(\mu + \kappa_i \varepsilon + \frac{\kappa_i(\kappa + \varepsilon) + \kappa(\kappa + \varepsilon)}{\mu}\right).
\end{aligned} \tag{3.17}$$

Finally, the Favre average $\langle \mathbf{v}_{1,h} \otimes \mathbf{v}_{1,h} \rangle$ is going to be expressed with respect to the tensor product of the Favre averages. This way, the resulting system will only depend on the averages of the density, the velocity and the pressure.

Lemma 4. *The Favre average of the tensor product of the velocity $\mathbf{v}_{1,h}$ with itself is approximated by the tensor product of its Favre averages.*

$$\langle \mathbf{v}_{1,h} \otimes \mathbf{v}_{1,h} \rangle = \langle \mathbf{v}_{1,h} \rangle \otimes \langle \mathbf{v}_{1,h} \rangle + \mathcal{O}\Big(\frac{(\varepsilon\kappa + \varepsilon^2)^2}{\mu^2}\Big). \tag{3.18}$$

Proof. As shown in lemma 3,

$$\forall\, z \in (0, \alpha_1), \qquad \mathbf{v}_{1,h}(z) = \langle \mathbf{v}_{1,h} \rangle + f(z),$$

where
$$f = \mathcal{O}\Big(\frac{\varepsilon\kappa + \varepsilon^2}{\mu}\Big) \quad \text{and} \quad \langle f \rangle = 0.$$

The tensor product $\mathbf{v}_{1,h} \otimes \mathbf{v}_{1,h}$ is thus developed as

$$\mathbf{v}_{1,h} \otimes \mathbf{v}_{1,h} = \langle \mathbf{v}_{1,h} \rangle \otimes \langle \mathbf{v}_{1,h} \rangle + 2\langle \mathbf{v}_{1,h} \rangle \otimes f + f \otimes f.$$

Since $f \otimes f = \mathcal{O}((\kappa\varepsilon + \varepsilon^2)^2/\mu^2)$, taking the Favre average of the equation above yields (3.18). ☐

Using approximation (3.18) in equation (3.17) yields the averaged momentum equation (3.12). For the second fluid, the reasoning is the same once the tangential Navier condition is rewritten with $\mathbb{S}_2$ instead of $\mathbb{S}_1$ — which is made possible by (2.15). ☐

3.2.3 Closing the system

To get the set of equations that is presented in theorem 1, it is necessary that the momentum equations (3.12) remain valid when ε becomes very small. Three conditions arise:

1. $\mathcal{R} \xrightarrow[\varepsilon\to 0]{} 0$, which means that the approximation error vanishes when ε tends to 0;
2. $\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle$ must be well defined (i.e finite) when ε tends to 0;
3. the friction terms in the momentum equations must be independent of ε at the thin-layer limit.

These conditions are the reason for the particular scaling requirements for the viscosity and friction coefficients presented in (3.1). Indeed, let us prescribe viscosity and friction scalings as follows:

$$\forall\, k \in \{1,2\}, \qquad \mu_k = \widehat{\mu_k}\varepsilon^{\tau}, \quad \kappa_k = \widehat{\kappa_k}\varepsilon^{\xi}, \quad \kappa_i = \widehat{\kappa_i}\varepsilon^{\zeta},$$

where $\tau, \xi, \zeta \in \mathbb{R}_+$. Rewriting the approximation error $\mathcal{R}$ with the previous scalings yields

$$\mathcal{R} = \mathcal{O}(\varepsilon^{\tau} + \varepsilon^{1+\zeta} + \varepsilon^{\xi+\zeta-\tau} + \varepsilon^{1+\zeta-\tau} + \varepsilon^{2\xi-\tau} + \varepsilon^{1+\xi-\tau} + \varepsilon^{2+2\xi-2\tau} + \varepsilon^{3+\xi-2\tau} + \varepsilon^{4-2\tau}).$$

Satisfying the first condition adds the following constraints on the viscosity and friction scalings:

$$\zeta + \xi > \tau, \quad 1+\zeta > \tau, \quad 1+\xi > \tau, \quad 2\xi > \tau \text{ and } 2 > \tau > 0.$$

In addition to the remainder $\mathcal{R}$ going to 0 at the limit, the difference between the averaged velocities must remain finite when ε becomes very small for the model to be valid. Rearrange the Navier interface condition (3.9) as

$$\mathbf{v}_{h,1}(\alpha_1) - \mathbf{v}_{h,2}(\alpha_1) = -\frac{\mu_1}{\varepsilon\kappa_i}\partial_z \mathbf{v}_{1,h}(\alpha_1) + \mathcal{O}\left(\frac{\mu\varepsilon}{\kappa_i} + \varepsilon^2\right).$$

Replacing the pointwise values of the velocities by their Favre average and using (3.16) yields

$$\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle = \mathcal{O}\left(\frac{\kappa + \varepsilon + \mu\varepsilon}{\kappa_i} + \frac{\kappa\varepsilon + \varepsilon^2}{\mu} + \varepsilon^2\right).$$

Replace the viscosities and friction coefficients by their scalings to obtain

$$\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle = \mathcal{O}(\varepsilon^{\xi-\zeta} + \varepsilon^{1-\zeta} + \varepsilon^{1+\tau-\zeta} + \varepsilon^{1+\xi-\tau} + \varepsilon^{2-\tau} + \varepsilon^{2}). \tag{3.19}$$

The second condition is thus satisfied if and only if

$$\xi \geqslant \zeta, \quad 1 \geqslant \zeta \text{ and } 1+\tau \geqslant \zeta.$$

Finally, the model must be independent of ε at the main order. Namely, the friction terms $\frac{\kappa_i}{\varepsilon}(\langle \mathbf{v}_{2,h}\rangle - \langle \mathbf{v}_{1,h}\rangle)$ and $\frac{\kappa_k}{\varepsilon}\langle \mathbf{v}_{k,h}\rangle$ for $k \in \{1,2\}$ must be well defined at the limit $\varepsilon \to 0$. Since $\langle \mathbf{v}_{1,h} \rangle - \langle \mathbf{v}_{2,h} \rangle$ is finite, this condition is satisfied if and only if

$$\xi \geqslant 1 \text{ and } \zeta \geqslant 1.$$

Taking into account all the different constraints results in the scalings (3.1):

$$\forall\, k \in \{1,2\}, \qquad \mu_k = \widehat{\mu_k}\varepsilon^\tau,\ 0 < \tau < 2, \quad \kappa_k = \widehat{\kappa_k}\varepsilon^\xi,\ \xi \geqslant 1 \text{ and } \kappa_i = \widehat{\kappa_i}\varepsilon.$$

Applying the scalings above yields equations (3.2), (3.3) and (3.5). Equation (3.6) is given as (2.19) still holds.

At this stage, this system is underdetermined as it includes eight unknowns but only six equations. It remains to adapt the barotropy laws to the new set of equations to get equations (3.4). Let us prove this result in the case $k = 1$. Using the rescaled vertical momentum equation (3.7), $\partial_z(p_1(\rho_1)) = \mathcal{O}(\mu)$. Let $z^* \in (0, \alpha_1)$ such that $\rho_1(z^*) = \overline{\rho_1}$. Thanks to a Taylor-Lagrange expansion,

$$\forall\, z \in (0, \alpha_1), \qquad p_1(\rho_1(z)) = p_1(\rho_1(z^*)) + \mathcal{O}(\mu).$$

Averaging the previous equation with respect to z and expliciting the scaling of the viscosity yields (3.4).

Remark 5. Compared to the Navier-Stokes system of section 2, the limit of the volume fraction equation (2.10) is not considered. Indeed, the volume fractions α_1 and α_2 are deduced from the equality of the averaged pressures (3.5) and the saturation condition (3.6).

3.3 Recovering the one-velocity one-pressure model

The prescription of a Navier friction condition at the interface between the fluids is motivated by the desire to derive a two-velocity model. As mentioned in remark 1, other choices for the interface conditions are possible. If the friction coefficient κ_i decreases slower than ε (i.e $\kappa_i = \widehat{\kappa_i}\varepsilon^\zeta$ with $\zeta < 1$), then the difference between the averaged velocities (3.19) tends to zero when ε becomes small, resulting in the model which would have stemmed from the derivation with Dirichlet interface conditions with respect to the relative velocity:

$$[\mathbf{v}]_{z=D\alpha_1} = 0. \tag{3.20}$$

This condition imposes the continuity of the velocity at the interface and leads to a one-velocity, one-pressure model, which reads:

Theorem 2. *Assume the same scalings for the viscosity and friction coefficients as in theorem 1 (conditions* (3.1)*). Let* $(\chi_1, \alpha_1, \rho_1, \mathbf{v}_1, p_1, \chi_2, \alpha_2, \rho_2, \mathbf{v}_2, p_2)$ *be a strong solution to the Navier-Stokes system* (2.17)-(2.22) *with Navier boundary conditions* (2.23)-(2.24) *and interface conditions* (2.25)-(2.27)*,* (3.20)*.*
Then the averaged velocities satisfy

$$\langle \mathbf{v}_{1,h} \rangle = \langle \mathbf{v}_{2,h} \rangle + \mathcal{O}(\varepsilon^{2-\tau}).$$

Denoting $\langle \mathbf{v}_h \rangle$ *this velocity, the functions* $(\alpha_1, \alpha_2, \overline{\rho_1}, \overline{\rho_2}, \langle \mathbf{v}_h \rangle, \overline{p_1}, \overline{p_2})$ *satisfy the* *following system:*

$$\partial_t(\alpha_k\overline{\rho_k}) + \nabla_h \cdot (\alpha_k\overline{\rho_k}\langle \mathbf{v}_h\rangle) = 0, \tag{3.21}$$

$$\partial_t(\overline{\rho}\langle \mathbf{v}_h\rangle) + \nabla_h \cdot (\overline{\rho}\langle \mathbf{v}_h\rangle \otimes \langle \mathbf{v}_h\rangle) + \nabla_h(\alpha_1\overline{p_1} + \alpha_2\overline{p_2}) = -\delta_\xi(\widehat{\kappa_1} + \widehat{\kappa_2})\langle \mathbf{v}_h\rangle, \tag{3.22}$$

$$\overline{p_k} = p_k(\overline{\rho_k}), \tag{3.23}$$

$$\overline{p_1} = \overline{p_2}, \tag{3.24}$$

$$\alpha_1 + \alpha_2 = 1 \tag{3.25}$$

where $k \in \{1,2\}$ and $\overline{\rho} = \alpha_1\overline{\rho_1} + \alpha_2\overline{\rho_2}$. Same as before, the momentum equation is satified up to an error of order $\varepsilon^\tau + \varepsilon^{2-\tau} + (1-\delta_\xi)\varepsilon^{\xi-1}$ and the equality of the pressures is satisfied up to an error of order ε^τ. This system is symmetrizable and therefore hyperbolic.

Remark 6. The equality of the velocities up to an error of order $\varepsilon^{2-\tau}$ stems from the Dirichlet condition associated with the estimate (3.15). Indeed, since the velocities are equal at the interface, and since they don't vary much vertically,

$$\left\{\begin{array}{rcl} \mathbf{v}_{1,h}(\alpha_1) & = & \mathbf{v}_{2,h}(\alpha_1), \\ \forall\, k \in \{1,2\}, \qquad \mathbf{v}_{k,h}(\alpha_1) & = & \langle \mathbf{v}_{k,h}\rangle + \mathcal{O}\Big(\frac{\kappa\varepsilon + \varepsilon^2}{\mu}\Big). \end{array}\right.$$

Once combined, and developing every coefficient with respect to its scaling,

$$\langle \mathbf{v}_{1,h}\rangle = \langle \mathbf{v}_{2,h}\rangle + \mathcal{O}(\varepsilon^{1+\xi-\tau} + \varepsilon^{2-\tau}).$$

Since $\xi \geqslant 1$, the equation is satisfied up to an error of order $\varepsilon^{2-\tau}$.

4 The Navier-Stokes-Fourier case

This final part extends the results of the previous section to the non-barotropic case. The microscopic model must therefore be completed with two energy equations, and the equations of state must now include the specific internal energies. We reproduce the averaging process on these two energy equations and pay special attention to the heat transfer between the fluids. This will enable the derivation of a two-velocity, one-pressure and two-temperature averaged model.

4.1 The Navier-Stokes-Fourier system

Let $k \in \{1,2\}$. First, recall the equations describing the evolution of the domain, namely the advection of $\chi_{D,k}$, the fraction equation and the saturation condition:

$$\partial_t\chi_{D,k} + \mathbf{v}_k \cdot \nabla\chi_{D,k} = 0, \tag{4.1}$$

$$\partial_t(D\alpha_1) + \mathbf{v}_{1,h}(D\alpha_1) \cdot \nabla_h(D\alpha_1) = w_1(D\alpha_1), \tag{4.2}$$

$$\alpha_1 + \alpha_2 = 1. \tag{4.3}$$

Now, let us introduce the Navier-Stokes-Fourier equations. Let $\theta_k : \mathbb{R}_+ \times \Omega_{D,k} \to \mathbb{R}$ be the temperature of fluid k, $e_k : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ its specific internal energy and $E_k = (1/2)\left|\mathbf{v}_k\right|^2 + e_k$ its specific total energy. Let β_k be the thermal conductivity of fluid k. It is a constant parameter of the flow that might depend on ε, similarly to the viscosity and friction coefficients. The flow is now described by two compressible Navier-Stokes-Fourier systems and two equations of state:

$$\partial_t \rho_k + \nabla \cdot (\rho_k \mathbf{v}_k) = 0, \tag{4.4}$$

$$\partial_t(\rho_k \mathbf{v}_k) + \nabla \cdot (\rho_k \mathbf{v}_k \otimes \mathbf{v}_k) + \nabla p_k = \nabla \cdot \mathbb{S}_k, \tag{4.5}$$

$$\partial_t(\rho_k E_k) + \nabla \cdot ((\rho_k E_k + p_k)\mathbf{v}_k) = \nabla \cdot (\mathbb{S}_k \mathbf{v}_k) + \nabla \cdot (\beta_k \nabla \theta_k), \tag{4.6}$$

$$p_k = p_k(\rho_k, \rho_k e_k), \tag{4.7}$$

$$\theta_k = \theta_k(\rho_k, \rho_k e_k). \tag{4.8}$$

The pressure p_k and the temperature θ_k are defined thanks to a complete equation of state which assumes the existence of a specific entropy s_k such that

$$\theta_k \mathrm{d}s_k = p_k \mathrm{d}\left(\frac{1}{\rho_k}\right) + \mathrm{d}e_k. \tag{4.9}$$

Navier boundary conditions are kept on the top and bottom of the domain. In addition, the domain is thermically isolated. The boundary conditions thus read

$$\begin{cases} w_1|_{z=0} = 0, & (4.10\text{a}) \\ (-\mathbb{S}_{1,hz} + \kappa_1 \mathbf{v}_{1,h})|_{z=0} = 0, & (4.10\text{b}) \\ \beta_1 \partial_z \theta_1|_{z=0} = 0. & (4.10\text{c}) \end{cases} \qquad \begin{cases} w_2|_{z=D} = 0, & (4.11\text{a}) \\ (\mathbb{S}_{2,hz} + \kappa_2 \mathbf{v}_{2,h})|_{z=D} = 0, & (4.11\text{b}) \\ \beta_2 \partial_z \theta_2|_{z=D} = 0, & (4.11\text{c}) \end{cases}$$

The interface conditions are completed with a Robin-type condition for the heat transfer, see for instance [HO12]. The interface conditions thus read

$$[\sigma \mathbf{n}_{\Gamma_D}]_{z=D\alpha_1} = 0, \tag{4.12a}$$

$$[\mathbf{v} \cdot \mathbf{n}_{\Gamma_D}]_{z=D\alpha_1} = 0, \tag{4.12b}$$

$$(\mathbb{S}_1 \mathbf{n}_{\Gamma_D} - \kappa_i [\mathbf{v}])_{\tan}|_{z=D\alpha_1} = 0, \tag{4.12c}$$

$$\beta_1 \nabla \theta_1 \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1} = h_c[\theta]_{z=D\alpha_1}, \tag{4.12d}$$

$$\beta_2 \nabla \theta_2 \cdot \mathbf{n}_{\Gamma_D}|_{z=D\alpha_1} = h_c[\theta]_{z=D\alpha_1}, \tag{4.12e}$$

where h_c is the contact conductance for the interface. Note that the heat flux at the interface is continuous and is expressed thanks to (4.12d) and (4.12e).

4.2 The averaged model

In what follows, we apply the same approach as in sections 2 and 3. Starting from a rescaled version of the two-layer Navier-Stokes-Fourier equations, an asymptotic analysis is performed in order to obtain a system of equations for the section-averaged variables which is independent of the vertical variable. By using the relevant regimes relative to the domain thickness, we may obtain the well-known six-equation model (1.1)-(1.4), which has been extensively studied in the literature and industry [DP99, IH11, Bes90, Ndj07].

Theorem 3. *Assume the following scalings for the viscosity, friction and heat transfer coefficients:*

$$\forall\, k \in \{1,2\}, \qquad \left\{ \begin{array}{lll} \mu_k = \widehat{\mu_k}\varepsilon^\tau, & \lambda_k = \widehat{\lambda_k}\varepsilon^\tau, & 0 < \tau < 2; \\ \kappa_i = \widehat{\kappa_i}\varepsilon, & \kappa_k = \widehat{\kappa_k}\varepsilon^\xi, & \xi \geqslant 1; \\ \beta_k = \widehat{\beta_k}\varepsilon^\gamma, & & 0 \leqslant \gamma < 2. \end{array} \right. \tag{4.13}$$

Define $\delta_\xi = \left\{ \begin{array}{ll} 1 & \text{if } \xi = 1, \\ 0 & \text{if } \xi > 1, \end{array} \right.$ *and* $\delta_\gamma = \left\{ \begin{array}{ll} 1 & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma > 0. \end{array} \right.$

Let $(\chi_1, \alpha_1, \rho_1, \mathbf{v}_1, p_1, E_1, \theta_1, \chi_2, \alpha_2, \rho_2, \mathbf{v}_2, p_2, E_2, \theta_2)$ *be a strong solution to the rescaled version of the Navier-Stokes model* (4.1)-(4.9) *with boundary conditions* (4.10)-(4.11) *and interface conditions* (4.12). *Assume that*

$$\partial_z \nabla_h \theta_k = \mathcal{O}(\varepsilon^2/\beta). \tag{4.14}$$

The functions $(\alpha_1, \overline{\rho_1}, \langle \mathbf{v}_{1,h}\rangle, \overline{p_1}, \langle E_1\rangle, \overline{\theta_1}, \alpha_2, \overline{\rho_2}, \langle \mathbf{v}_{2,h}\rangle, \overline{p_2}, \langle E_2\rangle, \overline{\theta_2})$ *then satisfy the following system at the main order in* ε*:*

$$\partial_t(\alpha_k\overline{\rho_k}) + \nabla_h \cdot (\alpha_k\overline{\rho_k}\langle \mathbf{v}_{k,h}\rangle) = 0, \tag{4.15}$$

$$\begin{aligned} &\partial_t(\alpha_k\overline{\rho_k}\langle \mathbf{v}_{k,h}\rangle) + \nabla_h \cdot (\alpha_k\overline{\rho_k}\langle \mathbf{v}_{k,h}\rangle \otimes \langle \mathbf{v}_{k,h}\rangle) + \nabla_h(\alpha_k\overline{p_k}) - p_i\nabla_h\alpha_k \\ &\qquad = -\delta_\xi\widehat{\kappa_k}\langle \mathbf{v}_{k,h}\rangle + (-1)^k\widehat{\kappa_i}(\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle), \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\partial_t(\alpha_k\overline{\rho_k}\langle E_{k,h}\rangle) + \nabla_h \cdot (\alpha_k(\overline{\rho_k}\langle E_{k,h}\rangle + \overline{p_k})\langle \mathbf{v}_{k,h}\rangle) + p_i\partial_t\alpha_k \\ &\qquad = -\delta_\xi\widehat{\kappa_k}\,|\langle \mathbf{v}_{k,h}\rangle|^2 + (-1)^k\widehat{\kappa_i}(\langle \mathbf{v}_{1,h}\rangle - \langle \mathbf{v}_{2,h}\rangle) \cdot \langle \mathbf{v}_{k,h}\rangle \\ &\qquad\qquad + \delta_\gamma(\widehat{\beta_k}\nabla_h \cdot (\alpha_k\nabla_h\overline{\theta_k})) + (-1)^k\widehat{h_c}(\overline{\theta_1} - \overline{\theta_2}), \end{aligned} \tag{4.17}$$

$$\overline{p_k} = p_k(\overline{\rho_k}, \overline{\rho_k}\langle e_k\rangle) \tag{4.18}$$

$$\overline{\theta_k} = \theta_k(\overline{\rho_k}, \overline{\rho_k}\langle e_k\rangle), \tag{4.19}$$

$$\overline{p_1} = \overline{p_2} = \overline{p_i}, \tag{4.20}$$

$$\alpha_1 + \alpha_2 = 1, \tag{4.21}$$

where $k \in \{1,2\}$ *and* $\widehat{h_c} = h_c/\varepsilon$.

As in the barotropic case, the averaged momentum equation (4.16) *is satisfied up to an error of order* $\varepsilon^\tau + \varepsilon^{2-\tau} + (1-\delta_\xi)\varepsilon^{\xi-1}$. *The averaged energy equation* (4.17) *is satisfied up to an error of order* $\varepsilon^\tau + \varepsilon^{2-\tau} + \varepsilon^{2-\gamma} + (1-\delta_\gamma)\varepsilon^\gamma$. *The equation of state for the pressure* (4.18) *and the pressure equality* (4.20) *are satisfied up to an error of order* ε^τ. *The equation of state for the temperature-* (4.19) *is satisfied up to an error of order* $\varepsilon^{2-\gamma}$.

Remark 7. Replacing the equations of state (2.4) by the energy equations (4.6) and the equations of state (4.7)-(4.8) does not change the fact that the asymptotic averaged system has only one pressure.

Remark 8. The scaling of the thermal conductivity can be expressed in terms of the Péclet number Pe = RePr, where Pr $= c_p\mu/\beta$ is the Prandtl number, c_p is the specific heat and Re is the Reynolds number. The scalings prescribed in (4.13) mean that the Reynolds number is of order $\varepsilon^{-\tau}$ and is therefore very large. Similarly, the Péclet number is of order $\varepsilon^{-\gamma}$, which is consistent with engineering applications where it is often much larger than one. Finally, the Prandtl number is of order $\varepsilon^{\tau-\gamma}$. The profiles (4.13) thus do not prescribe any constraint on the value of the Prandtl number.

4.3 Proof of theorem 3

The proof of theorem 3 is similar to the one in the barotropic case. In particular, the averaging of the mass conservation and momentum equations is the same. This proof only details the specificities of the Navier-Stokes-Fourier case: rescaling the new unknowns and the energy equations and averaging of the energy equations and of the new equations of state.

4.3.1 Rescaling the energy equations and the thermal flux conditions

Following the procedure of the barotropic case, it is now necessary to rescale and simplify (by neglecting some terms, as in section 3.2.1) the energy equation (4.6). We introduce the following characteristic dimensions: $\mathcal{E} = U^2$ for the specific energy, Θ for the temperature, $B = RU^3L/\Theta$ for the thermal conductivity and $H = RU^3/\Theta$ for the contact conductance. Consider the following rescaled variables: $\tilde{E} = E/\mathcal{E}$, $\tilde{\theta} = \theta/\Theta$, $\tilde{\beta} = \beta/B$ and $\tilde{h_c} = h_c/H$. In the following, the $\tilde{}$ notation is removed for the sake of clarity.

The rescaling procedure can be done term by term. Let us start with the left-hand side of the energy equation (4.6). Denoting $E_{k,h} = (1/2)\,|\mathbf{v}_{k,h}|^2 + e_k$ for $k \in \{1,2\}$, the rescaled left-hand side of the energy equation reads

$$\partial_t(\rho_k E_{k,h}) + \nabla\cdot((\rho_k E_{k,h} + p_k)\mathbf{v}_k) + \varepsilon^2(\partial_t(\rho_k w_k^2/2) + \nabla\cdot((\rho_k w_k^2/2))\mathbf{v}_k).$$

The terms of order ε^2 are neglected in the rest of the derivation.

Rescaling the thermal diffusion term is straightforward. The rescaled version reads

$$\beta_k \Delta_h \theta_k + \frac{\beta_k}{\varepsilon^2}\partial_{zz}\theta_k.$$

The viscous term is the most technical one.

Lemma 5. *After simplification, the rescaled viscous term in the energy equation* (4.6) *reads*

$$\frac{\mu_k}{\varepsilon^2}\partial_z(\mathbf{v}_{k,h}\cdot\partial_z\mathbf{v}_{k,h}) + \mathcal{O}(\mu). \tag{4.22}$$

Proof. Developping the viscous term of the energy equation (4.6) yields

$$\nabla\cdot(\mathbb{S}_k\mathbf{v}_k) = \nabla\cdot(\lambda_k(\nabla\cdot\mathbf{v}_k)\mathbf{v}_k) + \mu_k\nabla\cdot(2\mathbb{D}_k\mathbf{v}_k)$$

The bulk term does not need to be detailed as the rescaling does not affect its expression. Since

$$2\mathbb{D}_k\mathbf{v}_k = \begin{pmatrix} 2u\partial_x u + v(\partial_y u + \partial_x v) + w(\partial_z u + \partial_x w) \\ u(\partial_x v + \partial_y u) + 2v\partial_y v + w(\partial_z v + \partial_y w) \\ u(\partial_x w + \partial_z u) + v(\partial_y w + \partial_z v) + 2w\partial_z w \end{pmatrix},$$

the rescaled version of its divergence reads

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \frac{1}{\varepsilon}\partial_z \end{pmatrix}\cdot 2\mathbb{D}_k\mathbf{v}_k = \nabla_h\cdot((\nabla_h\otimes\mathbf{v}_{k,h} + (\nabla_h\otimes\mathbf{v}_{k,h})^T)\mathbf{v}_{k,h}) + \nabla_h\cdot(w_k\partial_z\mathbf{v}_{k,h})$$

$$+ \nabla_h\cdot(\varepsilon^2 w_k\nabla_h w_k) + \partial_z(\mathbf{v}_{k,h}\cdot\nabla_h w_k) + \frac{1}{\varepsilon^2}\partial_z(\mathbf{v}_{k,h}\cdot\partial_z\mathbf{v}_{k,h}) + 2w_k\partial_z w_k.$$

Multiplying the previous term by μ_k and adding the bulk term yields the following expression for the rescaled viscous term:

$$\begin{aligned}
&\lambda_k \nabla \cdot ((\nabla \cdot \mathbf{v}_k)\mathbf{v}_k) + \mu_k \nabla_h \cdot ((\nabla_h \otimes \mathbf{v}_{k,h} + (\nabla_h \otimes \mathbf{v}_{k,h})^T)\mathbf{v}_{k,h}) \\
&\qquad + \mu_k \nabla_h \cdot (w_k \partial_z \mathbf{v}_{k,h} + \varepsilon^2 w_k \nabla_h w_k) + \mu_k \partial_z (\mathbf{v}_{k,h} \cdot \nabla_h w_k) \\
&\qquad\qquad + \mu_k \frac{1}{\varepsilon^2} \partial_z (\mathbf{v}_{k,h} \cdot \partial_z \mathbf{v}_{k,h}) + 2\mu_k w_k \partial_z w_k.
\end{aligned}$$

The prevailing term in the previous expression is of order μ/ε^2. The rest of the terms are of order μ. Knowing that the use of Navier interface conditions causes an approximation error of order μ, those terms are neglected right away to ensure simpler equations. Hence (4.22). □

Finally, the rescaled version of the energy equation reads

$$\begin{aligned}
\partial_t(\rho_k E_{k,h}) + \nabla \cdot ((\rho_k E_{k,h} + p_k)\mathbf{v}_k) &= \frac{\mu_k}{\varepsilon^2} \partial_z (\mathbf{v}_{k,h} \cdot \partial_z \mathbf{v}_{k,h}) \\
&\quad + \beta_k \Delta_h \theta_k + \frac{\beta_k}{\varepsilon^2} \partial_{zz} \theta_k + \mathcal{O}(\mu + \varepsilon^2). \quad (4.23)
\end{aligned}$$

The rescaled adiabatic boundary conditions (4.10c) (4.11c) and the rescaled heat transfer interface conditions (4.12d)-(4.12e) are

$$\beta_1 \partial_z \theta_1 |_{z=0} = 0, \tag{4.24}$$
$$\beta_2 \partial_z \theta_2 |_{z=1} = 0, \tag{4.25}$$
$$\big(-\varepsilon \beta_1 \nabla_h \alpha_1 \cdot \nabla_h \theta_1 + \frac{\beta_1}{\varepsilon} \partial_z \theta_1 \big)\big|_{z=\alpha_1} = h_c [\theta]_{z=\alpha_1}, \tag{4.26}$$
$$\big(-\varepsilon \beta_2 \nabla_h \alpha_1 \cdot \nabla_h \theta_2 + \frac{\beta_2}{\varepsilon} \partial_z \theta_2 \big)\big|_{z=\alpha_1} = h_c [\theta]_{z=\alpha_1}. \tag{4.27}$$

Remark 9. Multiply equation (4.23) by ε^2, to get that for $k \in \{1, 2\}$,

$$\beta_k \partial_{zz} \theta_k(z) = -\mu_k \partial_z (\mathbf{v}_{k,h} \cdot \partial_z \mathbf{v}_{k,h}) + \mathcal{O}(\varepsilon^2 + \mu\varepsilon^2 + \varepsilon^4).$$

Since $\mu_k \partial_z (\mathbf{v}_{k,h} \cdot \partial_z \mathbf{v}_{k,h}) = \mu_k \left|\partial_z \mathbf{v}_{k,h}\right|^2 + \mu_k \mathbf{v}_{k,h} \cdot \partial_{zz} \mathbf{v}_{k,h}$, the estimates (3.10) and (3.16) and the scalings for the friction and the viscosity coefficients (4.13) enable the obtention of the following estimate for $k \in \{1, 2\}$:

$$\beta_k \partial_{zz} \theta_k(z) = \mathcal{O}(\varepsilon^2) \qquad \text{in } \Omega_k. \tag{4.28}$$

Using the adiabatic boundary conditions (4.24) and (4.25) gives that for $k \in \{1, 2\}$,

$$\beta_k \partial_z \theta_k(z) = \mathcal{O}(\varepsilon^2) \qquad \text{in } \Omega_k. \tag{4.29}$$

Therefore, for $k \in \{1, 2\}$,

$$\theta_k(z) = \overline{\theta_k} + \mathcal{O}(\varepsilon^2/\beta) \qquad \text{in } \ \Omega_k. \tag{4.30}$$

These estimates along with the interface conditions (4.26) and (4.27) give

$$h_c [\theta]_{z=\alpha_1} = \mathcal{O}(\varepsilon). \tag{4.31}$$

4.3.2 Averaging the energy equations

Let $k = 1$. Since the integral is linear, the averaging of the different terms of the rescaled energy equation (4.23) will be detailed separately.

First, integrate the left-hand side of the equation between $z = 0$ and $z = \alpha_1$. Similarly to section 3.2.2, use the rescaled volume fraction equation (2.19) and impermeability condition (2.23) to get

$$\begin{aligned}\int_0^{\alpha_1} \big(\partial_t(\rho_1 E_{1,h}) + \nabla\cdot((\rho_1 E_{1,h} + p_1)\mathbf{v}_1)\big)\mathrm{d}z &= \partial_t(\alpha_1 \overline{\rho_1}\langle E_{1,h}\rangle) \\ &+ \nabla_h\cdot(\alpha_1(\overline{\rho_1}\langle E_{1,h}\mathbf{v}_{1,h}\rangle + \overline{p_1\mathbf{v}_{1,h}})) - p_1(\alpha_1)\nabla_h\alpha_1 + p_1(\alpha_1)w_1(\alpha_1).\end{aligned}$$

Using once again the rescaled volume fraction equation (2.19),

$$\begin{aligned}\int_0^{\alpha_1} \big(\partial_t(\rho_1 E_{1,h}) + \nabla\cdot((\rho_1 E_{1,h} + p_1)\mathbf{v}_1)\big)\mathrm{d}z &= \partial_t(\alpha_1 \overline{\rho_1}\langle E_{1,h}\rangle) \\ &+ \nabla_h\cdot(\alpha_1(\overline{\rho_1}\langle E_{1,h}\mathbf{v}_{1,h}\rangle + \overline{p_1\mathbf{v}_{1,h}})) + p_1(\alpha_1)\partial_t\alpha_1.\end{aligned}$$

Replace the velocities by their Favre averages thanks to (3.15) to get

$$\begin{aligned}\int_0^{\alpha_1} \big(\partial_t(\rho_1 E_{1,h}) + \nabla\cdot((\rho_1 E_{1,h} + p_1)\mathbf{v}_1)\big)\mathrm{d}z &= \partial_t(\alpha_1 \overline{\rho_1}\langle E_{1,h}\rangle) \\ &+ \nabla_h\cdot(\alpha_1(\overline{\rho_1}\langle E_{1,h}\rangle + \overline{p_1})\langle\mathbf{v}_{1,h}\rangle) + p_1(\alpha_1)\partial_t\alpha_1 + \mathcal{O}\Big(\frac{\varepsilon\kappa+\varepsilon^2}{\mu}\Big). \quad (4.32)\end{aligned}$$

Next is the averaging of the viscous term. Using the asymptotic Navier interface condition (3.9) and the rescaled bottom impermeability condition (2.23b) yields

$$\begin{aligned}\int_0^{\alpha_1} \frac{\mu_1}{\varepsilon^2}\partial_z(\mathbf{v}_{1,h}\cdot\partial_z\mathbf{v}_{1,h})\mathrm{d}z &= \frac{\mu_1}{\varepsilon^2}(\mathbf{v}_{1,h}(\alpha_1)\cdot\partial_z\mathbf{v}_{1,h}(\alpha_1) - \mathbf{v}_{1,h}(0)\cdot\partial_z\mathbf{v}_{1,h}(0)) \\ &= -\frac{\kappa_i}{\varepsilon}(\mathbf{v}_{1,h}(\alpha_1) - \mathbf{v}_{2,h}(\alpha_1))\cdot\mathbf{v}_{1,h}(\alpha_1) - \frac{\kappa_1}{\varepsilon}\left|\mathbf{v}_{1,h}(0)\right|^2 + \mathcal{O}(\mu + \kappa_i\varepsilon).\end{aligned}$$

Replacing the pointwise values of the velocities by their Favre average using estimate (3.15) gives

$$\begin{aligned}\int_0^{\alpha_1} \frac{\mu_1}{\varepsilon^2}\partial_z(\mathbf{v}_{1,h}\cdot\partial_z\mathbf{v}_{1,h})\mathrm{d}z &= -\frac{\kappa_i}{\varepsilon}(\langle\mathbf{v}_{1,h}\rangle - \langle\mathbf{v}_{2,h}\rangle)\cdot\langle\mathbf{v}_{1,h}\rangle - \frac{\kappa_1}{\varepsilon}\left|\langle\mathbf{v}_{1,h}\rangle\right|^2 \\ &+ \mathcal{O}\Big(\mu + \kappa_i\varepsilon + \frac{(\kappa+\kappa_i)(\kappa+\varepsilon)}{\mu}\Big). \quad (4.33)\end{aligned}$$

Last, integrate the thermal diffusion term. Using the adiabatic boundary condition (4.24) and the interface condition (4.26) yields

$$\begin{aligned}\int_0^{\alpha_1} \big(\beta_1\Delta_h\theta_1 + \frac{\beta_1}{\varepsilon^2}\partial_{zz}\theta_1\big)\mathrm{d}z &= \beta_1\nabla_h\cdot(\alpha_1\overline{\nabla_h\theta_1}) \\ &- \beta_1\nabla_h\alpha_1\cdot\nabla_h\theta_1(\alpha_1) + \frac{\beta_1}{\varepsilon^2}\partial_z\theta_1(\alpha_1), \\ &= \beta_1\nabla_h\cdot(\alpha_1\overline{\nabla_h\theta_1}) - \frac{h_c}{\varepsilon}(\theta_1(\alpha_1) - \theta_2(\alpha_1)).\end{aligned}$$

Approximate the pointwise values of the temperatures by their averages thanks to (4.30) to get

$$\int_0^{\alpha_1} \big(\beta_1 \Delta_h \theta_1 + \frac{\beta_1}{\varepsilon^2} \partial_{zz} \theta_1\big) \mathrm{d}z = \beta_1 \nabla_h \cdot (\alpha_1 \overline{\nabla_h \theta_1}) - \frac{h_c}{\varepsilon} (\overline{\theta_1} - \overline{\theta_2}) + \mathcal{O}(\varepsilon^2/\beta). \quad (4.34)$$

The term $\overline{\nabla_h \theta_1}$ is unsatisfactory since it introduces a new unknown. To solve this problem, two strategies are possible. The first one is to express this term in terms of the averaged temperature θ_1 using the lemma below.

Lemma 6. *Assume that for $k \in \{1,2\}$*

$$\partial_z \nabla_h \theta_k = \mathcal{O}(\varepsilon^2/\beta) \quad \textit{in } \Omega_k.$$

Then, for $k \in \{1,2\}$,

$$\nabla_h \cdot (\alpha_k \overline{\nabla_h \theta_k}) = \nabla_h \cdot (\alpha_k \nabla_h \overline{\theta_k}) + \mathcal{O}(\varepsilon^2/\beta). \quad (4.35)$$

Proof. The proof is done with $k = 1$. Since

$$\alpha_1 \nabla_h \overline{\theta_1} = \alpha_1 \overline{\nabla_h \theta_1} + \nabla_h \alpha_1 (\theta_1(\alpha_1) - \overline{\theta_1}), \quad (4.36)$$

taking the horizontal divergence of the previous equation yields

$$\begin{aligned} \nabla_h \cdot (\alpha_1 \nabla_h \overline{\theta_1}) = \nabla_h \cdot ((\alpha_1 \overline{\nabla_h \theta_1})) + \Delta_h \alpha_1 (\theta_1(\alpha_1) - \overline{\theta_1}) \\ + \nabla_h \alpha_1 \cdot ((\nabla_h \theta_1)(\alpha_1) + \partial_z \theta_1(\alpha_1) \nabla_h \alpha_1 - \nabla_h \overline{\theta_1}). \end{aligned}$$

Using estimates (4.28) and (4.30),

$$\nabla_h \cdot (\alpha_1 \nabla_h \overline{\theta_1}) = \nabla_h \cdot ((\alpha_1 \overline{\nabla_h \theta_1})) + \nabla_h \alpha_1 \cdot ((\nabla_h \theta_1)(\alpha_1) - \nabla_h \overline{\theta_1}) + \mathcal{O}(\varepsilon^2/\beta).$$

Reusing (4.36) and estimate (4.30), $\qquad \nabla_h \overline{\theta_1} = \overline{\nabla_h \theta_1} + \mathcal{O}(\varepsilon^2/\beta).$

Therefore,

$$\nabla_h \cdot (\alpha_1 \nabla_h \overline{\theta_1}) = \nabla_h \cdot (\alpha_1 \overline{\nabla_h \theta_1}) + \nabla_h \alpha_1 \cdot ((\nabla_h \theta_1)(\alpha_1) - \overline{\nabla_h \theta_1}) + \mathcal{O}(\varepsilon^2/\beta).$$

Finally, assumption (4.14) enables the obtention of equation (4.35). ☐

The second way to solve the introduction of the unknown $\overline{\nabla_h \theta_1}$ is to scale the thermal conductivities as $\beta_k = \widehat{\beta_k} \varepsilon^\gamma$ for $k \in \{1,2\}$ with $0 < \gamma < 2$. This allows to get rid of the heat diffusion term up to an error of order ε^γ. Note that in this case, no assumption about the vertical variation of $\nabla_h \theta_k$ is required. To synthesize the results, we assume that the thermal conductivities are scaled with $0 \leqslant \gamma < 2$ as in (3.1) and that the hypothesis (4.14) is satisfied.

Assembling (4.32), (4.33), (4.34) and (4.35) yields the averaged energy equation

$$\begin{aligned} \partial_t (\alpha_1 \overline{\rho_1} \langle E_{1,h} \rangle) + \nabla_h \cdot (\alpha_1 (\overline{\rho_1} \langle E_{1,h} \rangle + \overline{p_1}) \langle \mathbf{v}_{1,h} \rangle) + p_1(\alpha_1) \partial_t \alpha_1 = -\frac{\kappa_1}{\varepsilon} \left| \langle \mathbf{v}_{1,h} \rangle \right|^2 \\ + \frac{\kappa_i}{\varepsilon} (\langle \mathbf{v}_{1,h} \rangle - \langle \mathbf{v}_{2,h} \rangle) \cdot \langle \mathbf{v}_{1,h} \rangle + \beta_1 \nabla_h (\alpha_1 \nabla_h \overline{\theta_1}) - \frac{h_c}{\varepsilon} (\langle \theta_1 \rangle - \langle \theta_2 \rangle) + \mathcal{R}, \end{aligned}$$

where $$\mathcal{R} = \mathcal{O}\Big(\mu + \kappa_i \varepsilon + \varepsilon^2 + \frac{(\kappa + \kappa_i + \varepsilon)(\kappa + \varepsilon)}{\mu} + \frac{\varepsilon^2}{\beta}\Big).$$

Since the averaged pressures are equal up to an error of order μ, they are still indistiguishable in the averaged energy equation. Moreover, similarly to the momentum equation, the pointwise value $p_1(\alpha_1)$ is replaced by p_i.

Using the scalings (4.13), the averaged energy equation (4.17) holds up to an error of order $\varepsilon^\tau + \varepsilon^{2-\tau} + \varepsilon^{2-\gamma} + (1 - \delta_\gamma)\varepsilon^\gamma$.

Remark 10. The scaling for the contact conductance h_c has been prescribed in order to get a two-temperature model. Note that the only constraint on h_c is given by (4.31). Chosing a contact conductance that decreases slower than ε would therefore imply that the difference of temperatures at the interface decreases to zero when ε becomes very small, resulting in a one-temperature averaged model (since the temperatures do not vary sharply in the vertical direction).

4.3.3 Obtaining the averaged equations of state

The equations of state are averaged in a similar way than in the barotropic case. However, now the pressures and the temperatures depend on two variables, so the proof needs a little adaptation. Let us prove the result for the pressure p_1 (the proof for p_2 and the temperatures are similar).

Recall that thanks to the asymptotic vertical momentum equation,

$$\partial_z p_1(\rho_1, \rho_1 e_1) = \mathcal{O}(\mu).$$

Let $z^* \in (0, \alpha_1)$ such that $\rho_1(z^*) = \overline{\rho_1}$. Thanks to a Taylor-Lagrange expansion,

$$\forall\, z \in (0, \alpha_1), \qquad p_1(\rho_1(z), \rho_1 e_1(z)) = p_1(\rho_1(z^*), \rho_1 e_1(z)) + \mathcal{O}(\mu).$$

Then, consider $z' \in (0, \alpha_1)$ such that $\rho_1 e_1(z') = \overline{\rho_1 e_1} = \overline{\rho_1}\langle e_1\rangle$. A second Taylor-Lagrange expansion yields

$$\forall\, z \in (0, \alpha_1), \qquad p_1(\rho_1(z^*), \rho_1 e_1(z)) = p_1(\rho_1(z^*), \rho_1 e_1(z')) + \mathcal{O}(\mu).$$

Therefore, $$\forall\, z \in (0, \alpha_1), \qquad p_1(\rho_1(z), \rho_1 e_1(z)) = p_1(\overline{\rho_1}, \overline{\rho_1}\langle e_1\rangle) + \mathcal{O}(\mu).$$

Averaging the previous equation with respect to z and expliciting the scaling of the viscosity yields (4.18).

The procedure is the same for the temperature, but using estimate (4.29), the approximation error is of order ε^2/β.

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