A Convergent Finite Volume Numerical Scheme for the Radiative Heat Transfer System

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Abstract: We present a convergence analysis for a finite-volume discretization of the radiative heat transfer system. The model under consideration consists of the radiative transfer equation coupled with a nonlinear heat conduction equation in a grey medium, where radiative effects in the black-body limit are described by the Stefan–Boltzmann law exhibiting a quartic temperature dependence of the form T^4. Time discretization is carried out implicitly. The radiative transport equation is approximated by a standard upwind scheme, whereas the heat equation is treated with a finite volume method based on the Two-Point Flux Laplace operator, which imposes admissibility conditions on the mesh. After recalling and establishing a stability result for energy weak solutions of the continuous system, we introduce the discrete scheme. We then prove the existence of a discrete solution together with uniform estimates that mirror those obtained at the continuous level. Next, we demonstrate the convergence of a (sub)sequence of discrete solutions, corresponding to vanishing time and space steps, towards an energy weak solution of the system, thereby providing a new existence proof for weak solutions of this model. Finally, we examine the diffusive limit of the numerical scheme applied to the non-dimensionalized system as the Knudsen number approaches zero.

Keywords: Radiative transfer equation, Heat equation, Finite Volume Method, Convergence analysis.

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Contents

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1 Introduction

The transfer of heat by radiation is a classical yet demanding topic that continues to draw sustained interest in both physics and applied mathematics. Radiative transfer (RT) arises in a broad range of physical and engineering contexts, including stellar and planetary atmospheres, nuclear engineering, and industrial heat-transfer applications. It plays a pivotal role in climate science, where radiation–atmosphere interactions govern the terrestrial energy budget; see, for example, [26, 27, 23] and, more recently, [3] for comprehensive overviews. In the present study, we consider a radiative heat transfer model that characterizes the cooling process of glass, formulated as a radiative transfer equation coupled with a heat conduction equation. The model is given by

$$\begin{cases} c_m \rho_m \partial_t T = k_h \Delta T - \displaystyle\int_0^\infty \int_{\mathbb{S}^{d-1}} \kappa\left(\mathcal{B}-\psi\right) d\boldsymbol{\beta} d\nu, \quad t>0, \boldsymbol{x}\in\Omega, \\ \dfrac{1}{c}\partial_t \psi + \boldsymbol{\beta}\cdot\nabla\psi = \kappa\left(\mathcal{B}-\psi\right), \quad t>0, (\boldsymbol{x},\boldsymbol{\beta},\nu)\in\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_+. \end{cases} \tag{1.1}$$

Here Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d=2$ or 3, and $\boldsymbol{\beta}\in\mathbb{S}^{d-1}$ being the unit sphere in $\mathbb{R}^d$. The function $T=T(t,\boldsymbol{x})$ denotes the temperature of the medium and $\psi=\psi(t,\boldsymbol{x},\boldsymbol{\beta},\nu)$ is the specific radiation intensity at $\boldsymbol{x}\in\Omega$ traveling in the direction $\boldsymbol{\beta}\in\mathbb{S}^{d-1}$ with frequency $\nu>0$ at time $t>0$. The constants c_m, ρ_m, k_h, κ and c are the specific heat, the density, the thermal conductivity, the opacity coefficient, and the speed of light, respectively. Furthermore, $\mathcal{B}=\mathcal{B}(\nu,T)$ denotes the Planck's function

$$\mathcal{B}\left(\nu,T\right) := \frac{2h_p\nu^3}{c^2\left(e^{\frac{h_p\nu}{k_bT}}-1\right)}$$

for black body radiation in glass. Here h_p is the Planck's constant and k_b is the Boltzmann's constant. We refer the reader to [11, 27, 15] and references therein for more radiative heat transfer models.

Next, we give the dimensionless form of the system (1.1). We introduce the nondimensional parameter $\varepsilon = 1/\kappa_r x_r$, where x_r and κ_r are the length scale and reference absorption, respectively. Physically ε represents the ratio of a typical photon mean free path to a typical length scale of the problem (Knudsen number). The rescaled system is given by

$$\begin{cases} \varepsilon^2\partial_t T = \varepsilon^2 k\Delta T - \displaystyle\int_0^\infty \int_{\mathbb{S}^{d-1}} \kappa\left(\mathcal{B}-\psi\right) d\boldsymbol{\beta} d\nu, \quad t>0,\ x\in\Omega, \\ \varepsilon^2\dfrac{1}{c}\partial_t\psi + \varepsilon\boldsymbol{\beta}\cdot\nabla\psi = \kappa\left(\mathcal{B}-\psi\right), \quad t>0,\ (\boldsymbol{x},\boldsymbol{\beta},\nu)\in\Omega\times\mathbb{S}^{d-1}\times(0,\infty). \end{cases} \tag{1.2}$$

See [11] for more details on the derivation.We examine the cooling of glass in a grey medium, meaning that $\mathcal{B}$ is independent of the frequency ν. In this case, the specular black-body intensity $\mathcal{B}$ is given by $\mathcal{B} = \frac{\sigma}{\pi} T^4$, according to the Stefan–Boltzmann law, where σ denotes the Stefan–Boltzmann constant. For convenience, and to keep the model non-dimensionalized so that all prefactors are equal to one, we choose the constants in (1.2) to satisfy $k = c = 1$ and set $\sigma = \pi$. Because the solutions of the system above depend on ε, we introduce the notation $T_\varepsilon = T_\varepsilon(t, \boldsymbol{x})$ and $\psi_\varepsilon = \psi_\varepsilon(t, \boldsymbol{x}, \boldsymbol{\beta})$ to denote, respectively, the temperature and the radiative intensity.With these conventions, the system (1.1) can be expressed as

$$\partial_t T_\varepsilon - \Delta T_\varepsilon = \frac{1}{\varepsilon^2} \int_{\mathbb{S}^{d-1}} \psi_\varepsilon - T_\varepsilon^4 \, d\boldsymbol{\beta}, \tag{1.3a}$$

$$\partial_t \psi_\varepsilon + \frac{1}{\varepsilon} \boldsymbol{\beta} \cdot \nabla \psi_\varepsilon = -\frac{1}{\varepsilon^2} (\psi_\varepsilon - T_\varepsilon^4). \tag{1.3b}$$

where $\langle \psi_\varepsilon \rangle := \int_{\mathbb{S}^{d-1}} \psi_\varepsilon d\boldsymbol{\beta}$ denotes the radiative density. The parameter ε may be small in various practical settings and has a decisive influence on the system (1.3a)–(1.3b). Consequently, it is of substantial mathematical and physical interest to investigate the behavior of solutions in the asymptotic regime $\varepsilon \to 0$, which we refer to as the diffusive limit. The purpose of this work is to develop a finite volume scheme for the radiative heat transfer system and to carry out a rigorous convergence analysis of the resulting discrete approximation. The performance of the proposed numerical method in the diffusive regime will be examined in detail in Section 4. In the main body of the paper, however, the Knudsen number is non-dimensionalized so that there is no explicit dependence on ε.

We thus consider the following non-dimensional (normalized) system:

$$\partial_t T - \Delta T = \left(\langle \psi \rangle - T^4 \right), \qquad (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \tag{1.4a}$$

$$\partial_t \psi + \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \psi = -\left(\psi - T^4 \right), \qquad (t, \boldsymbol{x}, \boldsymbol{\beta}) \in (0, \infty) \times \Omega \times \mathbb{S}^{d-1}, \tag{1.4b}$$

The initial conditions are taken to be

$$T(t = 0, \boldsymbol{x}) = T_0(\boldsymbol{x}), \quad \text{for any } \boldsymbol{x} \in \Omega, \tag{1.5}$$

$$\psi(t = 0, \boldsymbol{x}, \boldsymbol{\beta}) = \psi_0(\boldsymbol{x}, \boldsymbol{\beta}), \quad \text{for any } \boldsymbol{x} \in \Omega, \boldsymbol{\beta} \in \mathbb{S}^{d-1}. \tag{1.6}$$

The boundary condition for the temperature T is a Dirichlet boundary condition

$$T(t, \boldsymbol{x}) = T_b(t, \boldsymbol{x}), \text{ for any } t > 0, \ x \in \partial\Omega, \tag{1.7}$$

where $T_b = T_b(t, \boldsymbol{x}) > 0$ is a non-negative function. To give the boundary condition for ψ, we define the boundary set $\Sigma = \partial\Omega \times \mathbb{S}^{d-1}$ and

$$\Sigma_- := \left\{ (x, \boldsymbol{\beta}) \in \partial\Omega \times \mathbb{S}^{d-1}, \ \boldsymbol{\beta} \cdot \boldsymbol{n}(\boldsymbol{x}) < 0 \right\}, \tag{1.8}$$

$$\Sigma_+ := \left\{ (x, \boldsymbol{\beta}) \in \partial\Omega \times \mathbb{S}^{d-1}, \ \boldsymbol{\beta} \cdot \boldsymbol{n}(\boldsymbol{x}) > 0 \right\}. \tag{1.9}$$

The boundary condition for the specific radiation intensity is given by

$$\psi(t, \boldsymbol{x}, \boldsymbol{\beta}) = \psi_b(t, \boldsymbol{x}, \boldsymbol{\beta}), \quad t > 0, \ (\boldsymbol{x}, \boldsymbol{\beta}) \in \Sigma_-, \tag{1.10}$$

where $\psi_b = \psi_b(t, \boldsymbol{x}, \boldsymbol{\beta})$ is a given function defined in the in-flow directions, which describes the radiative intensity transmitted into the medium from outside and $\boldsymbol{n}(\boldsymbol{x})$ is the outward unit normal vector to the boundary $\partial\Omega$.

In the present study, we confine our analysis to Dirichlet boundary conditions; nonetheless, alternative classes of boundary conditions can be handled within the same general framework. From the analytical perspective, the time evolution of temperature in participating media has been investigated through semigroup methods, including the framework of m-accretive operators; we refer to [1, 2, 28, 21, 6, 5, 22]. In numerous physical and engineering applications, radiative effects are coupled with thermal conduction, giving rise to radiation–conduction models in both transient and steady-state regimes [22, 30, 7, 19, 15]. In this coupled framework, elliptic and parabolic techniques yield additional regularity properties and a priori estimates. For example, stationary temperature distributions under various boundary conditions are investigated in [30, 7], where the presence of the conductivity term ΔT plays a crucial role in deriving robust estimates; see also [7, 15, 16, 17, 18, 25]. In contrast, coupling conduction with transport makes singular-limit analyses much more delicate. In diffusive-limit regimes, boundary layers and geometric corrections become significantly more intricate when the Laplacian interacts with the transport scaling [16, 24, 17, 18]. This complexity poses substantial challenges for the design of numerical schemes that accurately capture the diffusive limit in radiative heat transfer. In particular, when boundary layers are present, one must construct asymptotic-preserving schemes that remain stable and accurate at both kinetic and diffusive scales while correctly resolving the boundary-layer structure. The development of such schemes, which rigorously account for boundary layers and maintain the correct diffusive behavior, is therefore an important direction for ongoing research. Therefore, for P_1 radiative–diffusion systems, the finite-volume fractional-step scheme proposed in [13] exemplifies a classical and effective strategy. A discrete maximum principle provides uniform L^∞ bounds and, combined with a topological-degree argument, yields existence and uniqueness of discrete solutions. Furthermore, discrete estimates on space and time translates in L^2 ensure compactness, and the Kolmogorov–Riesz theorem then implies convergence (up to subsequences) toward a weak solution as the mesh size and time step tend to zero. A central ingredient in this analysis is a general estimate controlling time translates of the discrete solutions. Related developments are presented in [20], which unifies rigorous analytical properties (existence, uniqueness, and discrete maximum principles) with the construction of numerical schemes validated on representative atmospheric and solar-heating benchmark problems.

This article is concerned with the convergence analysis of a fully discrete approximation of the multidimensional radiative heat transfer system (1.4). The time discretization is performed by an implicit scheme. The radiative transfer equation is approximated by a standard upwind finite volume method, while the heat equation is discretized by a finite volume method based on the Two-Point Flux Laplace operator [8], which requires the underlying mesh to satisfy appropriate admissibility conditions. The principal contribution of the work is the demonstration that, as the temporal and spatial discretization parameters tend to zero, a (sub)sequence of the corresponding discrete solutions converges in the energy norms to an energy weak solution of the continuous system, thereby yielding a new existence proof for weak solutions of this model. The analysis is based on uniform a priori estimates and discrete compactness arguments.

The paper is structured as follows. In Section 2, we present the continuous radiative heat transfer system and define its global energy weak solutions under Dirichlet boundary conditions. We recall the existence result established in [15] and prove a stability theorem for energy weak solutions of the continuous problem with homogeneous Dirichlet boundary conditions: from any sequence of energy weak solutions satisfying a uniform energy bound, one can extract a subsequence that converges to an energy weak solution of the system. Section 3 focuses on the discretization. After introducing the numerical scheme, we prove the existence of a discrete solution and derive uniform bounds analogous to those obtained in the continuous setting. Then, using discrete compactness techniques, we establish a convergence result for a subsequence of discrete solutions associated with vanishing time and space steps, which converges to an energy weak solution of the system. This proof relies on arguments closely related to the stability result from Section 2, but adapted o the discrete setting. Finally, in Section 4, we investigate the diffusive limit of the numerical scheme applied to the non-dimensionalized system (1.3a)–(1.3b) in the asymptotic regime where the Knudsen number $\varepsilon \to 0$.

2 The continuous setting

2.1 Weak solutions

Before introducing the result of the existence of weak solutions to system (1.4a)-(1.4b), we first recall the definition of weak solutions; for more details see [15] for different types of boundary conditions. To consider the boundary condition on the radiative intensity, we introduce, for $\tau > 0$, the space

$$W^2((0,\tau)\times\Omega\times\mathbb{S}^{d-1}) := \{\psi \in \mathrm{L}^2((0,\tau)\times\Omega\times\mathbb{S}^{d-1}) \text{ s.t. } (\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\psi \in \mathrm{L}^2((0,\tau)\times\Omega\times\mathbb{S}^{d-1})\} \tag{2.1}$$

which, endowed with the scalar product

$$\langle \psi,\phi\rangle_{W^2} := \iiint_{(0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi\phi \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \iiint_{(0,\tau)\times\Omega\times\mathbb{S}^{d-1}} (\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\psi(\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\phi \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

is a Hilbert space. We then define the following trace operator which can be proved to be continuous (see [29, Appendix (B.1)]):

$$\gamma : \psi \in W^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^{d-1}) \mapsto \psi|_{\partial\Omega} \in \mathrm{L}^2(\mathbb{R}_+\times\partial\Omega\times\mathbb{S}^{d-1}, |\boldsymbol{\beta}\cdot\boldsymbol{n}| d\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t). \tag{2.2}$$

The space is endowed with the following norm:

$$\|f\|^2_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x)} := \int_{\partial\Omega}\int_{\mathbb{S}^{d-1}} f(\boldsymbol{x},\boldsymbol{\beta})^2 |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\sigma_x\,\mathrm{d}\boldsymbol{\beta}.$$

We denote

$$W^2_0((0,\tau)\times\Omega\times\mathbb{S}^{d-1}) := \{\psi \in W^2((0,\tau)\times\Omega\times\mathbb{S}^{d-1}),\ \gamma\psi = 0\}.$$

In the following we also denote by γ the usual trace operator $\gamma : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\partial\Omega)$.

Definition 2.1. *Assume $\partial\Omega \in C^1$. Let $0 \le T_0 \in \mathrm{L}^5(\Omega)$ and $0 \le \psi_0 \in \mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$. Let $0 \le T_b \in L^5_{\mathrm{loc}}([0,\infty);\mathrm{L}^5(\partial\Omega))$ and $0 \le \psi_b \in L^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Sigma_-;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x))$. We say that (T,ψ) is a weak solution of the system (1.4a)-(1.4b) with initial conditions (1.5)-(1.6) and boundary conditions (1.7), (1.10) if*

$$T \in L^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^5(\Omega)),\quad T^{\frac{5}{2}} \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^1(\Omega)),$$
$$\psi \in L^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \cap W^2_{\mathrm{loc}}([0,\infty)\times\Omega\times\mathbb{S}^{d-1}),$$

if it solves (1.4a)-(1.4b) in the sense of distributions, i.e. for any test functions $\varphi \in C_c^\infty([0,\infty)\times\Omega)$ and $\zeta \in C_c^\infty([0,\infty)\times\Omega\times\mathbb{S}^{d-1})$, the following equations hold:

$$-\iint_{[0,\infty)\times\Omega}\left(T\partial_t\varphi + T\Delta\varphi + \int_{\mathbb{S}^{d-1}}(\psi - T^4)\varphi\,\mathrm{d}\boldsymbol{\beta}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega T_0\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}, \tag{2.3}$$

$$-\iiint_{[0,\infty)\times\Omega\times\mathbb{S}^{d-1}}\left(\psi\partial_t\zeta + \psi\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\zeta - (\psi - T^4)\zeta\right)\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint_{\Omega\times\mathbb{S}^{d-1}}\psi_0\zeta(0,\cdot,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\beta}, \tag{2.4}$$

and if it satisfies the following boundary conditions:

$$\gamma T\big|_{\partial\Omega} = T_b,$$
$$\gamma\psi\big|_{\Sigma_-} = \psi_b.$$

We say that (T,ψ) is a finite energy weak solution if furthermore for almost every $\tau > 0$, there exists $C = C(\Omega,\tau) > 0$ such that

$$\begin{aligned}\|T(\tau)\|^5_{\mathrm{L}^5(\Omega)} + \|\psi(\tau)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \int_0^\tau \|\boldsymbol{\nabla}(T)^{\frac{5}{2}}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\ + \int_0^\tau \|\psi - T^4)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \|\gamma\psi - \psi_b\|^2_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|d\boldsymbol{\beta}\,\mathrm{d}\sigma_x)}\,\mathrm{d}t \\ \le C(\|T_0\|^5_{\mathrm{L}^5(\Omega)} + \|\psi_0\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}).\end{aligned} \tag{2.5}$$

Now, we recall the existence theorem that was proven in [15, Section 2].

Theorem 2.1. *Assume $\partial\Omega \in C^1$. Let $0 \le T_0 \in \mathrm{L}^5(\Omega)$ and $0 \le \psi_0 \in \mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$. Let $0 \le T_b \in L^5_{\mathrm{loc}}([0,\infty);\mathrm{L}^5(\partial\Omega))$ and $0 \le \psi_b \in L^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Sigma_-;|\boldsymbol{\beta}\cdot\boldsymbol{n}|d\boldsymbol{\beta}\,\mathrm{d}\sigma_x))$. Let the boundary condition be well prepared such that $\psi_b = T_b^4$. Then there exists a global non-negative finite energy weak solution (T,ψ) of the system (1.4a)-(1.4b) with initial conditions (1.5)-(1.6) and boundary conditions (1.7), (1.10). Moreover, $T \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^1(\Omega))$ and for almost every $\tau > 0$, there exists $C = C(\Omega,\tau) > 0$ such that*

$$\|T\|_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega))} + \|\boldsymbol{\nabla}T\|_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega))} \le C(\|T_0\|^5_{\mathrm{L}^5(\Omega)} + \|\psi_0\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}). \tag{2.6}$$

2.2 Stability of energy weak solutions

The following Theorem states that a sequence of finite energy weak solutions converges, up to extracting a subsequence, to a finite energy weak solution. The proof of this result is similar to the proof of existence given in [15]. It is crucial since the proof of convergence for numerical schemes will follow the same lines.

Theorem 2.2. *Assume $\partial\Omega \in C^1$. Consider sequences of non-negative initial conditions $(T_0^{(m)})_{m\in\mathbb{N}} \subset \mathrm{L}^5(\Omega)$ and $(\psi_0^{(m)})_{m\in\mathbb{N}} \subset \mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$ and homogeneous boundary conditions $(T_b^{(m)})^4 = \psi_b^{(m)} = 0$ for all $m \in \mathbb{N}$. Let $(T^{(m)},\psi^{(m)})_{m\in\mathbb{N}}$ be the associated sequence of finite energy weak solutions. Assume that $T_0^{(m)} \to T$ in $\mathrm{L}^5(\Omega)$ and $\psi_0^{(m)} \to \psi_0$ in $\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$ as $m\to\infty$.*

Then, there exist (T,ψ) such that:

$$T \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^5(\Omega)),\ T^{\frac{5}{2}} \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^1_0(\Omega)),$$
$$\psi \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \cap W^2_{0,\mathrm{loc}}([0,\infty)\times\Omega\times\mathbb{S}^{d-1}),$$

and up to extracting a subsequence we have, for all $\tau > 0$:

- *The sequence $(T^{(m)})_{m\in\mathbb{N}}$ converges to T strongly in $\mathrm{L}^q([0,\tau);\mathrm{L}^q(\Omega))$ for all $q \in [2,8)$, weakly in $\mathrm{L}^2([0,\tau);\mathrm{H}^1(\Omega))$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega))$.*
- *The sequence $(\psi^{(m)})_{m\in\mathbb{N}}$ converges to ψ weakly in $\mathrm{L}^q([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1}))$ for all $q \in [2,\infty)$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1}))$.*
- *The pair (T,ψ) is a finite energy weak solution of Problem (1.4).*

Proof. **Step 1: uniform with respect to m estimates:** Let $\tau > 0$. Since $(T^{(m)},\psi^{(m)})_{m\in\mathbb{N}}$ is a sequence of non-negative finite energy weak solutions, there exists $C = C(\Omega,\tau) > 0$ such that for all $m \in \mathbb{N}$,

$$\begin{aligned}\|T^{(m)}(\tau)\|^5_{\mathrm{L}^5(\Omega)} + \|\psi^{(m)}(\tau)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \int_0^\tau \|\boldsymbol{\nabla}(T^{(m)})^{\frac{5}{2}}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\ + \int_0^\tau \|\psi^{(m)} - (T^{(m)})^4\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \|\psi^{(m)}\|^2_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x)}\,\mathrm{d}t \\ \leq C(\|T_0^{(m)}\|^5_{\mathrm{L}^5(\Omega)} + \|\psi_0^{(m)}\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}),\end{aligned} \tag{2.7}$$

and since $T_0^{(m)} \to T$ in $\mathrm{L}^5(\Omega)$, $\psi_0^{(m)} \to \psi_0$ in $\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$, the right hand side of this inequality is uniformly bounded by some constant C_0 independent of m. From this estimate we obtain that

$$(T^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega)), \tag{2.8}$$

$$(\boldsymbol{\nabla}T^{(m)})^{\frac{5}{2}})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega)), \tag{2.9}$$

$$(\psi^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})), \tag{2.10}$$

$$(\psi^{(m)} - (T^{(m)})^4)_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})). \tag{2.11}$$

By the Sobolev embedding $H^1(\Omega) \subset \mathrm{L}^6(\Omega)$ we obtain that $(T^{(m)})^{\frac{5}{2}}$ is uniformly bounded in $\mathrm{L}^2([0,\tau);\mathrm{L}^6(\Omega))$ and therefore in $\mathrm{L}^2([0,\tau);\mathrm{L}^5(\Omega))$. Moreover, an easy computation gives

$$\|T^{(m)}\|^5_{\mathrm{L}^5([0,\tau);\mathrm{L}^{\frac{25}{2}}(\Omega))} = \|(T^{(m)})^{\frac{5}{2}}\|^2_{\mathrm{L}^2([0,\tau);\mathrm{L}^5(\Omega))}.$$

Using the Littlewood's interpolation inequality $\|f\|_{\mathrm{L}^8(\Omega)} \leq \|f\|^{\frac{7}{24}}_{\mathrm{L}^5(\Omega)}\|f\|^{\frac{15}{24}}_{\mathrm{L}^{\frac{25}{2}}(\Omega)}$ we get

$$\begin{aligned}&\int_0^\tau \|T^{(m)}\|^8_{\mathrm{L}^8(\Omega)}\,\mathrm{d}t \\ &\leq \int_0^\tau \|T^{(m)}\|^{\frac{7}{3}}_{\mathrm{L}^5(\Omega)}\|T^{(m)}\|^5_{\mathrm{L}^{\frac{25}{2}}(\Omega)}\,\mathrm{d}t \\ &\leq \|T^{(m)}\|^{\frac{7}{3}}_{\mathrm{L}^\infty((0,\tau);\mathrm{L}^5(\Omega))}\|(T^{(m)})^{\frac{5}{2}}\|^2_{\mathrm{L}^2((0,\tau);\mathrm{L}^5(\Omega))}\end{aligned}$$

so that

$$(T^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^8([0,\tau);\mathrm{L}^8(\Omega)). \tag{2.12}$$

Moreover, by (2.6), we have the following estimate

$$(T^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau);\mathrm{H}^1_0(\Omega)). \tag{2.13}$$

Since $T^{(m)}$ solves $\partial_t T^{(m)} = \Delta T^{(m)} + \big(\langle\psi^{(m)}\rangle - (T^{(m)})^4\big)$ in the weak sense, the above estimate together with (2.7) yields that

$$(\partial_t T^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau);\mathrm{H}^{-1}(\Omega)). \tag{2.14}$$

We may also derive uniform bounds on the sequence $(\psi^{(m)})_{m\in\mathbb{N}}$. By (2.7) we have:

$$(\psi^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})), \tag{2.15}$$

$$(\psi^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})), \tag{2.16}$$

$$((\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\psi^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1}). \tag{2.17}$$

Step 2: compactness – By the estimates (2.13) and (2.14) and the Aubin-Lions-Simon compactness theorem, there exists a function T in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$ such that up to extracting a subsequence, for all $\tau > 0$:

$$T^{(m)} \to T \quad \text{in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega)) \text{ as } m\to\infty, \tag{2.18}$$

$$\boldsymbol{\nabla}T^{(m)} \rightharpoonup \boldsymbol{\nabla}T \quad \text{weakly in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega)) \text{ as } m\to\infty. \tag{2.19}$$

Moreover, by the previously derived estimates, we also have

$$T \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^5(\Omega)) \cap \mathrm{L}^8_{\mathrm{loc}}([0,\infty);\mathrm{L}^8(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^1_0(\Omega))$$
$$T^{\frac{5}{2}} \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^1_0(\Omega))$$

and (up to extracting a new subsequence) :

$$T^{(m)} \to T \quad \text{a.e. in } [0,\tau)\times\Omega \text{ as } m\to\infty, \tag{2.20}$$

$$T^{(m)} \to T \quad \text{in } \mathrm{L}^q([0,\tau);\mathrm{L}^q(\Omega)) \text{ for all } q\in[2,8) \text{ as } m\to\infty, \tag{2.21}$$

$$T^{(m)} \rightharpoonup T \quad \text{in } \mathrm{L}^8([0,\tau);\mathrm{L}^8(\Omega)) \text{ as } m\to\infty, \tag{2.22}$$

$$T^{(m)} \rightharpoonup^* T \quad \text{in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega)) \text{ as } m\to\infty, \tag{2.23}$$

$$T^{(m)} \rightharpoonup T \quad \text{in } \mathrm{L}^2([0,\tau);\mathrm{H}^1_0(\Omega)) \text{ as } m\to\infty, \tag{2.24}$$

$$(T^{(m)})^{\frac{5}{2}} \rightharpoonup T^{\frac{5}{2}} \quad \text{in } \mathrm{L}^2([0,\tau);\mathrm{H}^1_0(\Omega)) \text{ as } m\to\infty. \tag{2.25}$$

From (2.15), there exists ψ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \cap W^2_{0,\mathrm{loc}}([0,\infty)\times\Omega\times\mathbb{S}^{d-1})$ such that (up to extracting a new subsequence) we have, for all $\tau > 0$:

$$\psi^{(m)} \rightharpoonup^* \psi \quad \text{in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \text{ as } m\to\infty, \tag{2.26}$$

$$\psi^{(m)} \rightharpoonup \psi \quad \text{in } \mathrm{L}^q([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \text{ for all } q\in[2,\infty) \text{ as } m\to\infty, \tag{2.27}$$

$$(\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\psi^{(m)} \rightharpoonup (\partial_t + \boldsymbol{\beta}\cdot\boldsymbol{\nabla})\psi \text{ in } \mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1}) \text{ as } m\to\infty. \tag{2.28}$$

Step 3: passing to the limit in the weak formulation – Let $\varphi \in C_c^\infty([0,\tau)\times\Omega)$. Since $(T^{(m)},\psi^{(m)})$ is a weak solution, we have for the considered subsequence,

$$I_1^{(m)} + I_2^{(m)} + I_3^{(m)} + I_4^{(m)} = 0 \tag{2.29}$$

where

$$\begin{aligned}
I_1^{(m)} &= \int_0^\tau \int_\Omega T^{(m)} \partial_t \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\
I_2^{(m)} &= \int_0^\tau \int_\Omega T^{(m)} \Delta \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\
I_3^{(m)} &= \int_0^\tau \iint_{\Omega\times\mathbb{S}^2} (\psi^{(m)} - (T^{(m)})^4)\varphi \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\
I_4^{(m)} &= \int_\Omega T_0^{(m)} \varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

By (2.18) and since $T_0^{(m)} \to T_0$ in $\mathrm{L}^5(\Omega)$ we get

$$\lim_{m\to+\infty} I_1^{(m)} + I_2^{(m)} + I_4^{(m)} = \int_0^\tau \int_\Omega T\partial_t\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \int_\Omega T\Delta\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega T_0\cdot\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}.$$

For the term involving $(T^{(m)})^4$, we may write

$$I_3^{(m)} = \int_0^\tau \iint_{\Omega\times\mathbb{S}^2} (\psi - T^4))\varphi\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_{3,1}^{(m)} + R_{3,2}^{(m)}$$

where

$$\begin{aligned}
R_{3,1}^{(m)} &= \int_0^\tau \iint_{\Omega\times\mathbb{S}^2} (\psi^{(m)} - \psi)\varphi\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\
R_{3,2}^{(m)} &= \int_0^\tau \iint_{\Omega\times\mathbb{S}^2} (T^4 - (T^{(m)})^4)\varphi\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}$$

By (2.27) we have $R_{3,1}^{(m)} \to 0$ as $m\to+\infty$ and by(2.20) and the dominated convergence Theorem, we have $R_{3,2}^{(m)} \to 0$ as $m\to+\infty$ and passing to the limit in (2.29), we conclude that (T,ψ) solves the temperature equation in the weak sense.

Let us now turn to the radiative intensity equation. Let $\zeta \in C_c^\infty([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$. Since $(T^{(m)},\psi^{(m)})$ is a weak solution, we have for the considered subsequence,

$$\begin{aligned}
-\int_0^\tau \iint_{\Omega\times\mathbb{S}^2} \psi^{(m)}\partial_t\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau \iint_{\Omega\times\mathbb{S}^2} \psi^{(m)}\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t&\\
-\int_0^\tau \iint_{\Omega\times\mathbb{S}^2} (\psi^{(m)} - (T^{(m)})^4)\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint_{\Omega\times\mathbb{S}^2} \psi_0^{(m)}\zeta(0,\cdot,\cdot)\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}.&
\end{aligned} \tag{2.30}$$

The weak convergence (2.27), a similar computation as above for the non linear term involving $(T^{(m)})^4$ and the strong convergence $\psi_0^{(m)} \to \psi_0$ in $\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$ allow to prove that (T,ψ) also solves the intensity equation in the weak sense. To prove that $\gamma\psi\big|_{\Sigma_-} = 0$, we first remark that by the estimate (2.7), the sequence

$$\|\psi^{(m)}{}_{|\Sigma}\|_{\mathrm{L}^2([0,\tau)\times\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t)}$$

is bounded. Therefore, up to a subsequence we have

$$\gamma\psi^{(m)} \rightharpoonup \overline{\gamma\psi} \text{ weakly in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t)).$$

We can thus take the weak limit in the equality $\gamma\psi^{(m)}\big|_{\Sigma_-} = \psi_b^{(m)} = 0$ to obtain

$$\overline{\gamma\psi}|_{\Sigma_-} = 0. \tag{2.31}$$

To prove that $\overline{\gamma\psi} = \gamma\psi$ on Σ, we use the fact that (T,ψ) solves

$$\partial_t\psi + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi = -(\psi - T^4)$$

in the weak sense. For all $\tau > 0$, we apply a test function $\zeta \in C_c^\infty((0,\tau)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ on the above equation and deduce that :

$$-\int_0^\tau \iint_{\Omega\times\mathbb{S}^{d-1}} \psi\partial_t\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} \psi\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_0^\tau \iint_\Sigma \gamma\psi\zeta(\boldsymbol{\beta}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t = -\int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} (\psi - (T)^4)\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

We can also apply the same test function on the equation

$$\partial_t\psi^{(m)} + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi^{(m)} = -(\psi^{(m)} - (T^{(m)})^4)$$

and get

$$-\int_0^\tau \iint_{\Omega\times\mathbb{S}^{d-1}} \psi^{(m)}\partial_t\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} \psi^{(m)}\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_0^\tau \iint_\Sigma \gamma\psi^{(m)}\zeta(\boldsymbol{\beta}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t = -\int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} (\psi^{(m)} - (T^{(m)})^4)\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Letting $m\to\infty$ in the above equation and comparing with the previous one leads to $\overline{\gamma\psi} = \gamma\psi$ on Σ and therefore we have

$$\gamma\psi|_{\Sigma_-} = 0. \tag{2.32}$$

Step 4: energy estimate at the limit – We have the weak limits

$$\psi^{(m)} - (T^{(m)})^4 \rightharpoonup \psi - (T)^4 \text{ in } \mathrm{L}^2([0,\tau):\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})), \\ \gamma\psi^{(m)} \rightharpoonup \gamma\psi \text{ in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t)).$$

Therefore, by the weak limits (2.23)-(2.25)-(2.26), we can take the $\liminf_{m\to+\infty}$ in (2.7) and obtain that

$$\|T(\tau)\|^5_{\mathrm{L}^5(\Omega))} + \|\psi(\tau)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \int_0^\tau \|\boldsymbol{\nabla}(T)^{\frac{5}{2}}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \\ + \int_0^\tau \|\psi - (T)^4)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \|\psi\|^2_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x)}\,\mathrm{d}t \\ \leq C(\|T_0\|^5_{\mathrm{L}^5(\Omega)} + \|\psi_0\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}),$$

which concludes the proof.

$\square$

3 An implicit Finite Volume scheme

3.1 Meshes and discrete spaces

We recall that Ω is an open, bounded, and connected subset of $\mathbb{R}^d$, with $d \in \{2, 3\}$. We now further assume that Ω is polygonal when $d = 2$ and polyhedral when $d = 3$.

Definition 3.1 (Admissible space discretization). *An admissible discretization of Ω is a finite family $\mathcal{M}$ composed of non empty convex polygons ($d = 2$) or polyhedra ($d = 3$) which form a partition of Ω : $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K, which is the union of cell faces. We denote by $\mathcal{E}$ the set of faces of the mesh, and we suppose that two neighboring cells share a whole face: for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \sigma$; we denote in the latter case $\sigma = K|L$. We denote by $\mathcal{E}_{\rm ext}$ and $\mathcal{E}_{\rm int}$ the set of external and internal faces: $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$ and $\mathcal{E}_{\rm int} = \mathcal{E} \setminus \mathcal{E}_{\rm ext}$. For $K \in \mathcal{M}$, $\mathcal{E}(K)$ stands for the set of faces of K. The unit vector normal to $\sigma \in \mathcal{E}(K)$ outward K is denoted by $\boldsymbol{n}_{K,\sigma}$.*

The mesh is admissible in the following sense (see also [8]): there exists a family of points $(\boldsymbol{x}_K)_{K \in \mathcal{M}}$ such that:

- *$\boldsymbol{x}_K \in K$ for all $K \in \mathcal{M}$,*
- *for $\sigma = K|L \in \mathcal{E}_{\rm int}$, the segment $[\boldsymbol{x}_K, \boldsymbol{x}_L]$ is orthogonal to σ*

In the following, the notation $|K|$ or $|\sigma|$ stands indifferently for the d-dimensional or the $(d-1)$-dimensional measure of the subset K of $\mathbb{R}^d$ or σ of $\mathbb{R}^{d-1}$ respectively. We denote d_σ the distance $d(\boldsymbol{x}_K, \boldsymbol{x}_L)$ if $\sigma = K|L \in \mathcal{E}_{\rm int}$ and the distance between $\boldsymbol{x}_K$ and σ if $\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}$.

*For $\mathcal{M}$ an admissible mesh, and for every $K \in \mathcal{M}$, we denote h_K the diameter of K (*i.e.* the 1D measure of the largest line segment included in K). The size of the discretization is defined by:*

$$h_{\mathcal{M}} = \max_{K \in \mathcal{M}} \; h_K. \tag{3.1}$$

For $\mathcal{M}$ an admissible mesh, and for every $K \in \mathcal{M}$, we denote ϱ_K the radius of the largest ball included in K. The regularity parameter of the discretization is defined by:

$$\theta_{\mathcal{M}} = \max\Big\{\frac{h_K}{\varrho_K},\, K \in \mathcal{M}\Big\} \cup \Big\{\frac{h_K}{h_L},\, \frac{h_L}{h_K},\, \sigma = K|L \in \mathcal{E}_{\rm int}\Big\}. \tag{3.2}$$

Definition 3.2 (Discrete spaces). *Let $\mathcal{M}$ be an admissible discretization of Ω. We denote $\mathrm{H}_{\mathcal{M}}(\Omega)$ the space of scalar functions that are constant on each cell $K \in \mathcal{M}$. For $T \in \mathrm{H}_{\mathcal{M}}(\Omega)$ and $K \in \mathcal{M}$, we denote T_K the constant value of T on K. The space $\mathrm{H}_{\mathcal{M}}(\Omega)$ is endowed with the following inner product, for $(T, \phi) \in \mathrm{H}_{\mathcal{M}}(\Omega)^2$:*

$$[T, \phi]_{\mathcal{M}} = \sum_{\substack{\sigma \in \mathcal{E}_{\rm int} \\ \sigma = K|L}} \frac{|\sigma|}{d_\sigma} (T_L - T_K)(\phi_L - \phi_K) + \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\rm ext}} \frac{|\sigma|}{d_\sigma} T_K \phi_K.$$

Thanks to the Poincaré inequality stated below, this inner product defines a norm on $\mathrm{H}_{\mathcal{M}}(\Omega)$:

$$\|T\|_{1,\mathcal{M}} = [T, T]_{\mathcal{M}}^{\frac{1}{2}}.$$

The proof of the following Lemma can be found in [9, "*Discrete functional analysis*" appendix].

Lemma 3.1 (Discrete Sobolev and Poincaré inequalities)**.** *Let $\mathcal{M}$ be an admissible discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.2)) for some positive constant θ_0. Then, for all $q \in [1, +\infty)$ if $d = 2$ and for all $q \in [1, 6]$ if $d = 3$, there exists $C = C(q, d, \theta_0) > 0$ such that:*

$$\|T\|_{\mathrm{L}^q(\Omega)} \le C\, \|T\|_{1,\mathcal{M}}, \quad \forall T \in \mathrm{H}_{\mathcal{M}}(\Omega).$$

As a consequence we have the following Poincaré inequality: there exists $C = C(d, \theta_0)$ such that

$$\|T\|_{\mathrm{L}^2(\Omega)} \le C\, \|T\|_{1,\mathcal{M}}, \quad \forall T \in \mathrm{H}_{\mathcal{M}}(\Omega).$$

Remark 3.1. *Note that for d=2, one has $C(q, d, \theta_0) \to \infty$ as $q \to \infty$.*

The following Lemma is proven in [8, Lemma 3.3].

Lemma 3.2. *Let $\mathcal{M}$ be an admissible discretization of Ω. For any $T \in \mathrm{H}_{\mathcal{M}}(\Omega)$, we define its extension (still denoted T) to the whole space $\mathbb{R}^d$ by setting $T = 0$ on $\mathbb{R}^d \setminus \Omega$. Then there exists C, only depending on Ω such that*

$$\|T(\cdot + \boldsymbol{\eta}) - T\|^2_{\mathrm{L}^2(\mathbb{R}^d)^d} \le C\|T\|^2_{1,\mathcal{M}}\, |\boldsymbol{\eta}|\, (|\boldsymbol{\eta}| + h_{\mathcal{M}}), \quad \forall \boldsymbol{\eta} \in \mathbb{R}^d,\ \forall T \in \mathrm{H}_{\mathcal{M}}(\Omega). \tag{3.3}$$

Finally, an important consequence of Lemma 3.2 is the following compactness result, the proof of which is similar to that of [8, Theorem 3.10].

Lemma 3.3 (Discrete Rellich theorem)**.** *Let $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations in the sense of Definition 3.1 which satisfies $h_{\mathcal{M}^{(m)}} \to 0$ as $m \to \infty$. For $m \in \mathbb{N}$, let $T^{(m)} \in \mathrm{H}_{\mathcal{M}^{(m)}}(\Omega)$ and assume that there exists $C \in \mathbb{R}$ such that, for all $m \in \mathbb{N}$, $\|T^{(m)}\|_{1,\mathcal{M}^{(m)}} \le C$. Then, there exists T in $\mathrm{H}^1_0(\Omega)$ and a subsequence of $(T^{(m)})_{m\in\mathbb{N}}$ (not relabeled) such that $T^{(m)}$ converges strongly in $\mathrm{L}^2(\Omega)$ towards T as $m \to \infty$.*

Let us now turn to the angular discretization and we adopt a similar approach as for the space discretization.

Definition 3.3 (Angular discretization)**.** *A discretization of the unit sphere $\mathbb{S}^{d-1}$ is a set $\mathcal{A}$ composed of solid angles $\alpha \in \mathcal{A}$ (which are subsets of $\mathbb{S}^{d-1}$) forming a partition of $\mathbb{S}^{d-1}$. For every $\alpha \in \mathcal{A}$, let be given $\boldsymbol{\beta}_\alpha$ some element in α. We denote $|\alpha|$ the $(d-1)$-Lebesgue measure of α.*

*For $\mathcal{A}$ a discretization of $\mathbb{S}^{d-1}$, and for every $\alpha \in \mathcal{A}$, we denote h_α the diameter of α (*i.e. *$h_\alpha = \sup\{d_{\mathbb{S}^{d-1}}(\boldsymbol{\beta}, \boldsymbol{\beta}'),\ \boldsymbol{\beta}, \boldsymbol{\beta}' \in \alpha\}$ where $d_{\mathbb{S}^{d-1}}$ is the natural metric on the sphere). The size of the angular discretization is then defined by:*

$$h_{\mathcal{A}} = \max_{\alpha\in\mathcal{A}}\ h_\alpha. \tag{3.4}$$

We can now define the discrete functional space for functions defined on $\Omega \times \mathbb{S}^{d-1}$. In the following, we will denote $\mathcal{D} = (\mathcal{M}, \mathcal{A})$ a discretization of $\Omega \times \mathbb{S}^{d-1}$ where $\mathcal{M}$ is an admissible mesh of Ω and $\mathcal{A}$ is a mesh of $\mathbb{S}^{d-1}$.

Definition 3.4. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{A})$ be a discretization of $\Omega \times \mathbb{S}^{d-1}$. We denote $\mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ the space of scalar functions that are constant on each couple $(K, \alpha) \in \mathcal{M} \times \mathcal{A}$. For $\psi \in \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ and $(K, \alpha) \in \mathcal{M} \times \mathcal{A}$, we denote $\psi_{K,\alpha}$ the constant value of ψ on (K, α).*

3.2 The numerical scheme

Let $\tau > 0$ and let be given a regular partition of the interval $(0, \tau)$ with a constant time step $\delta t = \tau/N$. The approximate solution at time $t^n = n\delta t$ for $n \in \{0, \dots, N\}$ is a pair $(T^n, \psi^n) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$. The implicit finite volume scheme is defined as follows.

Initialization: The initial approximation $(T^0, \psi^0) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ is given by:

$$T_K^0 = \frac{1}{|K|} \int_K T_0 \,\mathrm{d}\boldsymbol{x}, \qquad K \in \mathcal{M}, \tag{3.5a}$$

$$\psi_{K,\alpha}^0 = \frac{1}{|K|} \frac{1}{|\alpha|} \int_K \int_\alpha \psi_0 \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\boldsymbol{x}, \qquad (K, \alpha) \in \mathcal{M} \times \mathcal{A}. \tag{3.5b}$$

Time advancement: *For $n \in \{0, \dots, N-1\}$, knowing $(T^n, \psi^n) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$, solve for $(T^{n+1}, \psi^{n+1}) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ the system:*

$$\frac{1}{\delta t}\left(T^{n+1} - T^n\right) - \Delta_{\mathcal{M}}(T^{n+1}) = \left(\langle \psi^{n+1} \rangle - |T^{n+1}|(T^{n+1})^3\right), \tag{3.6a}$$

$$\frac{1}{\delta t}\left(\psi^{n+1} - \psi^n\right) + \mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\psi^{n+1}) = -\left(\psi^{n+1} - (T^{n+1})^4\right). \tag{3.6b}$$

The discretization of the non linear T^4 term in the discrete heat equation (3.6a) is taken from [13] and ensures the positivity of the discrete temperature. The discrete space differential operators involved in (3.6) as well as their main properties, are described in the following lines.

Temperature laplacian – In (3.6a) $\Delta_{\mathcal{M}}$ is the discrete Finite Volume Two-Point Flux Laplace operator associated with the Dirichlet boundary conditions. For $T \in \mathrm{H}_{\mathcal{M}}(\Omega)$, $\Delta_{\mathcal{M}} T$ is an element of $\mathrm{H}_{\mathcal{M}}(\Omega)$ defined by:

$$\Delta_{\mathcal{M}}(T)(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} (\Delta T)_K \mathbb{1}_K(\boldsymbol{x})$$

where

$$(\Delta T)_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{d_\sigma}(T_{K,\sigma} - T_K)$$

with

$$T_{K,\sigma} = \begin{cases} T_L, & \text{if } \sigma \in \mathcal{E}_{\mathrm{int}}, \quad \sigma = K|L, \\ \dfrac{1}{|\sigma|} \displaystyle\int_\sigma T_b \,\mathrm{d}\sigma_x, & \text{in } \sigma \in \mathcal{E}_{\mathrm{ext}}. \end{cases}$$

Upwind convection operator – In (3.6b), $\mathrm{div}_{\mathcal{M},\mathrm{up}}$ is the upwind convection operator defined for $\psi \in \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ as

$$\mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\psi)(\boldsymbol{x}) = \sum_{\alpha \in \mathcal{A}} \sum_{K \in \mathcal{M}} \frac{1}{|K|} \Big(\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \psi_{\sigma,\alpha} \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \Big) \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}). \tag{3.7}$$

The radiative intensity at the face σ is approximated by the upwind technique. For $\sigma = K|L \in \mathcal{E}_{\rm int}$ we have

$$\psi_{\sigma,\alpha} = \begin{cases} \psi_{K,\alpha}, & \text{if } \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \geq 0, \\ \psi_{L,\alpha}, & \text{if } \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} < 0, \end{cases}$$

and for $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)$, we have:

$$\psi_{\sigma,\alpha} = \begin{cases} \psi_{K,\alpha}, & \text{if } \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \geq 0, \\ \dfrac{1}{|\sigma|}\displaystyle\int_\sigma \psi_b \, \mathrm{d}\sigma_x, & \text{if } \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} < 0. \end{cases}$$

With the discrete unknowns computed by induction through the scheme, we associate piecewise constant functions on each time interval $(t_{n-1}, t_n]$ as follows:

$$\begin{aligned} T(t,\boldsymbol{x}) &= \sum_{n=1}^{N} \sum_{K\in\mathcal{M}} T_K^n \, \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_{(t^{n-1},t^n]}(t), \\ \psi(t,\boldsymbol{x},\boldsymbol{\beta}) &= \sum_{n=1}^{N} \sum_{\alpha\in\mathcal{A}} \sum_{K\in\mathcal{M}} \psi_{K,\alpha}^n \, \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1},t^n]}(t) \end{aligned} \tag{3.8}$$

3.3 Existence of a solution and estimates

In order to simplify the analysis of the numerical scheme, we assume homogeneous Dirichlet boundary conditions for the temperature. Consequently, the incoming radiative intensity is also taken to be homogeneous on Σ_-, namely

$$T_b^4 = \psi_b = 0.$$

We begin with a straightforward but useful lemma:

Lemma 3.4. *For every $a, b \in \mathbb{R}$ we have the three following identities:*

$$\begin{aligned} &(a-b)a = \frac{1}{2}(a^2-b^2) + \frac{1}{2}(a-b)^2, \\ &(a-b)a^4 = \frac{1}{5}(a^5-b^5) + \frac{1}{5}(a-b)^2(4a^3+3a^2b+2ab^2+b^3), \\ &(a^2-b^2)(a^8-b^8) = \frac{16}{25}(a^5-b^5)^2 + \frac{1}{25}(a-b)^4(4ab+3a^2+3b^2)(8a^3b+3a^4+8ab^3+8a^2b^2+3b^4). \end{aligned}$$

3.3.1 Energy estimates

Proposition 3.5. *Assume $T_b^4 = \psi_b = 0$. There exists a unique solution $(T^n, \psi^n)_{0\leq n\leq N}$ to the numerical scheme (3.5)-(3.6). This solution satisfies $T^n \geq 0$, $\psi^n \geq 0$ for all $n \in \{0, \ldots, N\}$.*

Moreover, it satisfies the following energy estimate: for all $n \in \{0, \dots, N\}$,

$$\frac{1}{5}\|T^n\|^5_{\mathrm{L}^5(\Omega)} + \frac{1}{2}\|\psi^n\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \frac{16}{25}\sum_{k=1}^{n}\delta t\|(T^k)^{\frac{5}{2}}\|^2_{1,\mathcal{M}} + \sum_{k=1}^{n}\delta t\|\psi^k - (T^k)^4\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\sigma\in\mathcal{E}^{(m)}_{\mathrm{ext}}\cap\mathcal{E}(K)} |\sigma|(\psi^k_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|(\psi^k_{L,\alpha}-\psi^k_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|$$
$$\le \frac{1}{5}\|T^0\|^5_{\mathrm{L}^5(\Omega)} + \frac{1}{2}\|\psi^0\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}. \tag{3.9}$$

Remark 3.2. *Estimate (3.9) is a discrete equivalent of the energy estimate satisfied at the continuous level. The last two terms in the left hand side of (3.9) are numerical estimates coming from the upwind discretization of the convection term. They will be useful for proving the convergence of the numerical scheme.*

Proof. The outline of the proof is as follows. We first prove *a priori* estimates *i.e.* we prove that if a solution exists then it is non negative and satisfies the energy estimate (3.9). We then prove the existence of a solution thanks to Brouwer's topological degree. Finally, we prove the uniqueness of the solution.

Step 1: positivity – We have $T^0 \ge 0$ and $\psi^0 \ge 0$ by (3.5). Assume that for some n we have $T^n \ge 0$ and $\psi^n \ge 0$. Equation (3.6b) can be written:

$$(1+\delta t)\,\psi^{n+1}_{K,\alpha} + \delta t \mathrm{div}(\boldsymbol{\beta}\psi^{n+1})^{\mathrm{up}}_{K,\alpha} = \psi^n_{K,\alpha} + \delta t(T^{n+1}_K)^4, \qquad \forall (K,\alpha)\in\mathcal{M}\times\mathcal{A}.$$

This is a linear system for $(\psi^{n+1}_{K,\alpha})_{(K,\alpha)\in\mathcal{M}\times\mathcal{A}}$ the matrix of which is an M-matrix thanks to the upind discretization of the convection term (see [14, Lemma 2.1]). Since the right hand side terms in this system are non negative, we obtain the non negativity of ψ^{n+1}. Then, equation (3.6a) can be written:

$$\left(1+\delta t|T^{n+1}_K|(T^{n+1}_K)^2\right)T^{n+1}_K - \delta t(\Delta T^{n+1})_K = T^n_K + \delta t\langle\psi^{n+1}_K\rangle, \qquad \forall K\in\mathcal{M}.$$

This system is a non linear system for $(T^{n+1}_K)_{K\in\mathcal{M}}$ which can be viewed as a matrix system where the matrix depends on the unknown but is also an M-matrix for any value of the unknown. Since we have proven that $\psi^{n+1}\ge 0$, the right hand side term is non negative and therefore so is T^{n+1}. We conclude by induction on n.

Step 2: energy estimate – Assuming that there exists a solution to the scheme, we multiply (3.6a) by $\delta t\,(T^{n+1})^4$ and we integrate on Ω. Since $T^n\ge 0$ and $T^{n+1}\ge 0$, the second identity of Lemma 3.4 (applied with $a = T^{n+1}_K$ and $b = T^n_K$ on every cell K) yields, for all $n = 0,\cdots,N-1$:

$$\frac{1}{5}\|T^{n+1}\|^5_{\mathrm{L}^5(\Omega)} - \frac{1}{5}\|T^n\|^5_{\mathrm{L}^5(\Omega)} - \delta t\int_\Omega \Delta_{\mathcal{M}}(T^{n+1})\,(T^{n+1})^4\,\mathrm{d}\boldsymbol{x} \le \delta t\int_\Omega \left(\langle\psi^{n+1}\rangle - (T^{n+1})^4\right)(T^{n+1})^4\,\mathrm{d}\boldsymbol{x}.$$

The third term of the left-hand side can be written as:

$$
\begin{aligned}
&-\delta t \int_\Omega \Delta_{\mathcal{M}}(T^{n+1})\,(T^{n+1})^4 \,\mathrm{d}\boldsymbol{x} \\
&= \delta t \sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1}-T_{\mathrm{L}}^{n+1})(T_K^{n+1})^4 + \delta t \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1})^5 \\
&= \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1}-T_{\mathrm{L}}^{n+1})\left((T_K^{n+1})^4-(T_{\mathrm{L}}^{n+1})^4\right) + \delta t \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1})^5
\end{aligned}
$$

where we have performed a discrete integration by parts in the first term consisting in summing over the edges and for each edge on summing over the two adjacent cells. Since $T^{n+1}\geq 0$ the third identity of Lemma 3.4 with $a=(T_K^{n+1})^{\frac{1}{2}}$ and $b=(T_{\mathrm{L}}^{n+1})^{\frac{1}{2}}$ yields

$$
\begin{aligned}
&-\delta t \int_\Omega \Delta_{\mathcal{M}}(T^{n+1})\,(T^{n+1})^4 \,\mathrm{d}\boldsymbol{x} \\
&= \frac{16}{25}\,\delta t\Big(\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma}\left((T_K^{n+1})^{\frac{5}{2}}-(T_{\mathrm{L}}^{n+1})^{\frac{5}{2}}\right)^2 + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1})^5\Big) + \frac{9}{25} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1})^5 \\
&= \frac{16}{25}\,\delta t \|(T^{n+1})^{\frac{5}{2}}\|_{1,\mathcal{M}}^2 + \frac{9}{25} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_\sigma}(T_K^{n+1})^5.
\end{aligned}
$$

Thus, we obtain (since $T^{n+1}\geq 0$):

$$
\frac{1}{5}\|T^{n+1}\|_{\mathrm{L}^5(\Omega)}^5 - \frac{1}{5}\|T^{n}\|_{\mathrm{L}^5(\Omega)}^5 + \frac{16}{25}\,\delta t\|(T^{n+1})^{\frac{5}{2}}\|_{1,\mathcal{M}}^2 \leq \delta t \int_\Omega \left(\langle\psi^{n+1}\rangle - (T^{n+1})^4\right)(T^{n+1})^4 \,\mathrm{d}\boldsymbol{x}. \tag{3.10}
$$

We then multiply (3.6b) by $\delta t\,\psi^{n+1}$ and we integrate over $\Omega\times\mathbb{S}^{d-1}$. Using the first identity of Lemma 3.4 with $a=\psi_{K,\alpha}^{n+1}$ and $b=\psi_{K,\alpha}^{n}$ in the first term we get:

$$
\begin{aligned}
\frac{1}{2}\|\psi^{n+1}\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2 - \frac{1}{2}\|\psi^{n}\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2 + \delta t \int_{\Omega\times\mathbb{S}^{d-1}} \mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\psi^{n+1})\psi^{n+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\beta} \\
\leq -\delta t \int_\Omega \left(\langle\psi^{n+1}\rangle - (T^{n+1})^4\right)\langle\psi^{n+1}\rangle \,\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{3.11}
$$

The third term of the left-hand side can be written as:

$$
\begin{aligned}
&\delta t \int_{\Omega\times\mathbb{S}^{d-1}} \mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\psi^{n+1})\psi^{n+1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\beta} \\
&= \delta t \sum_{\alpha\in\mathcal{A}} |\alpha| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\psi_{\sigma,\alpha}^{n+1}\psi_{K,\alpha}^{n+1}\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma} \\
&= \frac{\delta t}{2} \sum_{\alpha\in\mathcal{A}} |\alpha| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\Big((\psi_{\sigma,\alpha}^{n+1})^2 + (\psi_{K,\alpha}^{n+1})^2 - (\psi_{\sigma,\alpha}^{n+1}-\psi_{K,\alpha}^{n+1})^2\Big)\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma} \\
&= \frac{\delta t}{2} \sum_{\alpha\in\mathcal{A}} |\alpha| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|(\psi_{\sigma,\alpha}^{n+1})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma} - \frac{\delta t}{2} \sum_{\alpha\in\mathcal{A}} |\alpha| \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|(\psi_{\sigma,\alpha}^{n+1}-\psi_{K,\alpha}^{n+1})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}
\end{aligned}
$$

since $\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\boldsymbol{n}_{K,\sigma}=0$ for all $K\in\mathcal{M}$. Reordering the first sum and using the upwind definition of $\psi^{n+1}_{\sigma,\alpha}$ we get:

$$\begin{aligned}
&\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{\sigma,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}\\
&=\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}|\sigma|(\psi^{n+1}_{\sigma,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot(\boldsymbol{n}_{K,\sigma}+\boldsymbol{n}_{L,\sigma})+\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{\sigma,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}\\
&=\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{K,\alpha})^2\,(\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})^+.
\end{aligned}$$

where for $a\in\mathbb{R}$, $a^+=\max(a,0)$ and $a^-=-\min(a,0)$. Reordering the second sum and using the upwind definition of $\psi^{n+1}_{\sigma,\alpha}$ we get:

$$\begin{aligned}
&-\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{\sigma,\alpha}-\psi^{n+1}_{K,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}\\
=&-\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}|\sigma|\Big((\psi^{n+1}_{\sigma,\alpha}-\psi^{n+1}_{K,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}+(\psi^{n+1}_{\sigma,\alpha}-\psi^{n+1}_{L,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{L,\sigma}\Big)\\
&-\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{\sigma,\alpha}-\psi^{n+1}_{K,\alpha})^2\,\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}\\
=&\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}|\sigma|(\psi^{n+1}_{L,\alpha}-\psi^{n+1}_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|+\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{K,\alpha})^2\,(\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})^-
\end{aligned}$$

Replacing in (3.11) we thus get:

$$\begin{aligned}
&\frac{1}{2}\|\psi^{n+1}\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}-\frac{1}{2}\|\psi^{n}\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\\
&+\frac{\delta t}{2}\sum_{\alpha\in\mathcal{A}}|\alpha|\Big(\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}|\sigma|(\psi^{n+1}_{L,\alpha}-\psi^{n+1}_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|+\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|\Big)\\
&\qquad\le-\delta t\int_\Omega\big(\langle\psi^{n+1}\rangle-(T^{n+1})^4\big)\,\langle\psi^{n+1}\rangle\,\mathrm{d}\boldsymbol{x}
\end{aligned}\tag{3.12}$$

Summing (3.10) and (3.12) we get:

$$\begin{aligned}
&\frac{1}{5}\|T^{n+1}\|^5_{\mathrm{L}^5(\Omega)}+\frac{1}{2}\|\psi^{n+1}\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}+\frac{16}{25}\delta t\|(T^{n+1})^{\frac{5}{2}}\|^2_{1,\mathcal{M}}+\delta t\|\psi^{n+1}-(T^{n+1})^4\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\\
&\quad+\frac{1}{2}\delta t\sum_{\alpha\in\mathcal{A}}|\alpha|\Big(\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}|\sigma|(\psi^{n+1}_{L,\alpha}-\psi^{n+1}_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|+\sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)}|\sigma|(\psi^{n+1}_{K,\alpha})^2\,|\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|\Big)\\
&\qquad\le\frac{1}{5}\|T^{n}\|^5_{\mathrm{L}^5(\Omega)}+\frac{1}{2}\|\psi^{n}\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})},
\end{aligned}$$

and then summing over $n = 0, \cdots, N-1$ yields (3.9).

Step 3: existence of a solution – Let be given $(T^n, \psi^n) \in \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$. We want to prove that (3.6) has a solution (T^{n+1}, ψ^{n+1}). Let $V = \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$. Endowed with $\|(T,\psi)\|_V = \frac{1}{5}\|T\|_{\mathrm{L}^5(\Omega)} + \frac{1}{2}\|\psi\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}$, V is a finite dimensional normed space. Consider the function $\mathcal{F}: V \times [0,1] \to V$ given by:

$$\mathcal{F}((T,\psi),\xi) = \left| \begin{array}{l} \dfrac{1}{\delta t}(T - T^n) - \Delta_{\mathcal{M}}(T) - \xi\left(\langle\psi\rangle - |T|(T)^3\right), \\ \dfrac{1}{\delta t}(\psi - \psi^n) + \mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\psi) + \xi\left(\psi - (T)^4\right). \end{array} \right. \tag{3.13}$$

Solving the problem $\mathcal{F}((T,\psi),\xi) = 0$ is equivalent to solving a numerical scheme similar to (3.6) (the right hand side relaxation term is multiplied by $\xi \in [0,1]$). In particular, a straightforward adaptation of step 2 shows that any solution of $\mathcal{F}((T,\psi),\xi) = 0$ satisfies $\|(T,\psi)\|_V < 2\|(T^n,\psi^n)\|_V$. Moreover $\mathcal{F}$ is continuous and the system $\mathcal{F}((T,\psi),0) = 0$ has a unique solution (it is a one-to-one linear system). The existence of at least one solution to the system $\mathcal{F}((T,\psi),1) = 0$ follows from a classical topological degree argument stated for instance in [12, Theorem 2.5].

Step 4: uniqueness of the solution – Assume there exist two solutions (T^{n+1}, ψ^{n+1}) and $(\tilde{T}^{n+1}, \tilde{\psi}^{n+1})$ to the scheme (3.6) and denote $(\delta T, \delta\psi) = (T^{n+1} - \tilde{T}^{n+1}, \psi^{n+1} - \tilde{\psi}^{n+1})$. By the identity $(a^4 - b^4) = (a-b)g(a,b)$ with $g(a,b) = a^3 + a^2 b + a b^2 + b^3$, we have $\mathcal{G}(\delta T, \delta\psi) = (0,0)$ where the operator $\mathcal{G}: \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1}) \longrightarrow \mathrm{H}_{\mathcal{M}}(\Omega) \times \mathrm{L}_{\mathcal{M},\mathcal{A}}(\Omega \times \mathbb{S}^{d-1})$ is defined as

$$\mathcal{G}\begin{pmatrix}\delta T \\ \delta\psi\end{pmatrix} = \begin{pmatrix} \left(\dfrac{1}{\delta t} + g(T^{n+1}, \tilde{T}^{n+1})\right) Id_{\mathrm{H}_{\mathcal{M}}} - \Delta_{\mathcal{M}} & -\langle\cdot\rangle \\ -g(T^{n+1}, \tilde{T}^{n+1}) Id_{\mathrm{H}_{\mathcal{M}}} & \left(\dfrac{1}{\delta t} + 1\right) Id_{\mathrm{L}_{\mathcal{M},\mathcal{A}}} + \mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\,\cdot) \end{pmatrix} \begin{pmatrix}\delta T \\ \delta\psi\end{pmatrix}.$$

The operator $\mathcal{G}$ can be written as a matrix operator with a matrix which depends on T^{n+1} and $\tilde{T}^{n+1}$. Since $g(T^{n+1}, \tilde{T}^{n+1}) \geq 0$, classical properties of the discrete Laplace operator $-\Delta_{\mathcal{M}}$ (its matrix is symetric with non negative extra diagonal terms the sum of which equals the diagonal term) and of the discrete upwind convection operator $\mathrm{div}_{\mathcal{M},\mathrm{up}}(\boldsymbol{\beta}\,\cdot)$ (see for instance [14, Lemma 2.1]) allow to prove that the transpose of the matrix defining operator $\mathcal{G}$ is a strictly diagonally dominant matrix with non-negative extra-diagonal entries and is therefore a non singular M-matrix. Hence, the matrix defining operator $\mathcal{G}$ is also a non singular M-matrix. Since $\mathcal{G}(\delta T, \delta\psi) = (0,0)$ we have $(\delta T, \delta\psi) = (0,0)$.

□

3.3.2 Other estimates

We can also derive additional estimates similar to those obtained in the continuous setting. We first define a discrete equivalent of the $\mathrm{L}^p(\mathrm{H}^1)$ norm of the temperature as follows. For T a piecewise constant function as defined in (3.8) we denote:

$$\|T\|^p_{\mathrm{L}^p((0,\tau);\mathrm{H}_{\mathcal{M}})} = \sum_{n=1}^{N} \delta t \|T^n\|^p_{1,\mathcal{M}}.$$

Proposition 3.6. *Assume $T_b^4 = \psi_b = 0$. Assume that the regularity parameter of the discretization satisfies $\theta_{\mathcal{M}} \leq \theta_0$ for some $\theta_0 > 0$. The unique solution to the numerical scheme (3.5)-(3.6) satisfies the following estimate: there exists $C_1 = C_1(T_0, \psi_0, \theta_0)$ such that*

$$\|T\|_{\mathrm{L}^8((0,\tau)\times\Omega)} + \|T\|_{\mathrm{L}^\infty((0,\tau);\mathrm{L}^2(\Omega))} + \|T\|_{\mathrm{L}^2((0,\tau);\mathrm{H}_{\mathcal{M}})} \leq C_1. \tag{3.14}$$

Proof. The proof of the $\mathrm{L}^8(\mathrm{L}^8)$ estimate is very similar to that in the continuous setting. It relies on the discrete Sobolev inequality which depends on the regularity parameter of the discretization. The two other estimates are obtained by multiplying the discrete temperature equation (3.6a) by T^{n+1} and integrating over $[0,\tau)\times\Omega$. After a discrete integration by parts in the diffusion term, the results is obtained thanks to the already proven estimate on $\|\psi - T^4\|_{\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})}$. □

We also have an $\mathrm{L}^2(\mathrm{L}^2)$ estimate on the time translations of the discrete temperature.

Proposition 3.7. *Assume $T_b^4 = \psi_b = 0$. Assume that the regularity parameter of the discretization satisfies $\theta_{\mathcal{M}} \leq \theta_0$ for some $\theta_0 > 0$. The unique solution to the numerical scheme (3.5)-(3.6) satisfies the following estimate: There exists $C_2 = C_2(T_0, \psi_0, \tau, \theta_0)$ such that*

$$\int_\eta^\tau \|T(t,\cdot) - T(t-\eta,\cdot)\|^2_{\mathrm{L}^2(\Omega)} \leq C_2(\eta^{\frac{1}{2}} + \delta t^{\frac{1}{2}}), \qquad \forall \eta \in (0, \min(\tau, 1)). \tag{3.15}$$

Proof. Let $\eta \in (0, \min(\tau,1))$. For every $t \in (\eta, \tau)$ we define n, k as the two integers satisfying $T(t,\cdot) = T^n$ and $T(t-\eta,\cdot) = T^{n-k}$. We can see that $n = \lceil \frac{t}{\delta t} \rceil$, $n-k = \lceil \frac{t-\eta}{\delta t} \rceil$ so that $1 \leq n-k \leq n \leq N$ and $k\delta t \leq \eta + \delta t$. By the first equation of the scheme we have

$$T^n - T^{n-k} = \sum_{p=n-k}^{n-1} \delta t \Big(\Delta_{\mathcal{M}}(T^{p+1}) + \big(\langle \psi^{p+1} \rangle - (T^{p+1})^4\big) \Big)$$

where the sum is empty (equals zero) if $k = 0$ (*i.e.* if $t - \eta \in (t^{n-1}, t^n]$). Let $\phi(t,\cdot)$ be a time dependent element of $\mathrm{H}_{\mathcal{M}}(\Omega)$ which we denote $\phi(t,\boldsymbol{x}) = \sum_{K\in\mathcal{M}} \phi_K(t)\, \mathbb{1}_K(\boldsymbol{x})$. Multiplying the above equality by $\phi(t,\cdot)$ and integrating over Ω, we get:

$$\int_\Omega (T(\boldsymbol{x},t) - T(\boldsymbol{x},t-\eta))\, \phi(\boldsymbol{x},t)\, \mathrm{d}\boldsymbol{x} = I_1(t) + I_2(t)$$

where

$$I_1(t) = \sum_{p=n-k}^{n-1} \delta t \int_\Omega \Delta_{\mathcal{M}}(T^{p+1}) \phi(t,\cdot)\, \mathrm{d}\boldsymbol{x},$$
$$I_2(t) = \sum_{p=n-k}^{n-1} \delta t \int_\Omega \big(\langle \psi^{p+1} \rangle - (T^{p+1})^4\big)\, \phi(t,\cdot)\, \mathrm{d}\boldsymbol{x}.$$

For the first term, we perform a discrete integration by parts, apply the Cauchy-Schwarz inequality on the space integral and then on the sum to get:

$$\begin{aligned}
I_1(t) &= \sum_{p=n-k}^{n-1} -\delta t \left[T^{p+1}, \phi(t,\cdot)\right]_{\mathcal{M}} \\
&\le \sum_{p=n-k}^{n-1} \delta t \|T^{p+1}\|_{1,\mathcal{M}} \, \|\phi(t,\cdot)\|_{1,\mathcal{M}} \\
&\le (k\delta t)^{\frac{1}{2}} \Big(\sum_{p=n-k}^{n-1} \delta t \|T^{p+1}\|_{1,\mathcal{M}}^2 \Big)^{\frac{1}{2}} \|\phi(t,\cdot)\|_{1,\mathcal{M}} \\
&\le (\eta+\delta t)^{\frac{1}{2}} \, \|T\|_{\mathrm{L}^2((0,\tau);\mathrm{H}_{\mathcal{M}})} \, \|\phi(t,\cdot)\|_{1,\mathcal{M}}.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
I_2(t) &= \sum_{p=n-k}^{n-1} \delta t \int_\Omega \left(\langle \psi^{p+1}\rangle - (T^{p+1})^4 \right) \phi(t,\cdot) \, \mathrm{d}\boldsymbol{x} \\
&\le \sum_{p=n-k}^{n-1} \delta t \| \langle \psi^{p+1}\rangle - (T^{p+1})^4 \|_{\mathrm{L}^2(\Omega)} \|\phi(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \\
&\le (k\delta t)^{\frac{1}{2}} \Big(\sum_{p=n-k}^{n-1} \delta t \| \langle \psi^{p+1}\rangle - (T^{p+1})^4 \|_{\mathrm{L}^2(\Omega)}^2 \Big)^{\frac{1}{2}} \|\phi(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \\
&\le (\eta+\delta t)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \delta t \| \psi^k - (T^k)^4 \|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2 \Big)^{\frac{1}{2}} \|\phi(t,\cdot)\|_{1,\mathcal{M}}.
\end{aligned}$$

By the estimates (3.9) and (3.14), and since $(\eta+\delta t)^{\frac{1}{2}} \le \eta^{\frac{1}{2}} + \delta t^{\frac{1}{2}}$, there exists $C = C(T_0, \psi_0, \theta_0)$ such that

$$\int_\Omega \left(T(\boldsymbol{x},t) - T(\boldsymbol{x},t-\eta)\right) \phi(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \le C\,(\eta^{\frac{1}{2}} + \delta t^{\frac{1}{2}}) \|\phi(t,\cdot)\|_{1,\mathcal{M}}.$$

Selecting $\phi(t,\cdot) = T(t,\cdot) - T(t-\eta,\cdot)$ and integrating for $t\in(\eta,\tau)$ yields:

$$\int_\eta^\tau \|T(t,\cdot) - T(t-\eta,\cdot)\|_{\mathrm{L}^2(\Omega)}^2 \, \mathrm{d}t \le 2C\,(\eta^{\frac{1}{2}} + \delta t^{\frac{1}{2}}) \int_0^\tau \|T(t,\cdot)\|_{1,\mathcal{M}} \, \mathrm{d}t.$$

Using the Cauchy-Schwarz inequality in the time integral of the right hand side and invoking once again the estimate (3.14) yields the result.

□

3.4 Convergence of the scheme

Definition 3.5 (Regular sequence of discretizations)**.** *Let $(\mathcal{D}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations and time steps. For $m \in \mathbb{N}$, let $h_{\mathcal{M}^{(m)}}$, $\theta_{\mathcal{M}^{(m)}}$ be the space step size and the regularity parameter associated with $\mathcal{M}^{(m)}$ by equations (3.1) and (3.2) respectively, and let $h_{\mathcal{A}^{(m)}}$ be the step size of the angular discretization. Then this sequence $(\mathcal{D}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ is said regular if:*

(i) for all $m \in \mathbb{N}$, $\theta^{(m)} \le \theta_0$ for some positive real number θ_0,

(ii) the sequences $(\delta t^{(m)})_{m\in\mathbb{N}}$, $(h_{\mathcal{M}^{(m)}})_{m\in\mathbb{N}}$ and $(h_{\mathcal{A}^{(m)}})_{m\in\mathbb{N}}$ tend to zero as m tends to $+\infty$.

Theorem 3.8 (Convergence of the scheme)**.** *Let Ω be a polyhedral connected open subset of $\mathbb{R}^d$. Let $\tau > 0$. Let $T_0 \in \mathrm{L}^5(\Omega)$ and $\psi_0 \in \mathrm{L}^2(\Omega)$. Assume homogeneous boundary conditions $(T_b)^4 = \psi_b = 0$. Let $(\mathcal{D}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a regular sequence of staggered discretizations. For all $m \in \mathbb{N}$, there exists a solution $(T^{(m)}, \psi^{(m)})$ to the numerical scheme (3.6) with the discretization $(\mathcal{D}^{(m)}, \delta t^{(m)})$. Then, there exists (T, ψ) such that:*

$$\begin{aligned}&T \in \mathrm{L}^\infty([0,\tau); \mathrm{L}^5(\Omega)),\ T^{\frac{5}{2}} \in \mathrm{L}^2([0,\tau); \mathrm{H}^1_0(\Omega)),\\ &\psi \in \mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})) \cap W^2_0([0,\tau) \times \Omega \times \mathbb{S}^{d-1}),\end{aligned}$$

and up to extracting a subsequence we have :

- *The sequence $(T^{(m)})_{m\in\mathbb{N}}$ converges to T strongly in $\mathrm{L}^q([0,\tau); \mathrm{L}^q(\Omega))$ for all $q \in [2, 8)$, weakly in $\mathrm{L}^2([0,\tau); \mathrm{H}^1(\Omega))$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau); \mathrm{L}^5(\Omega))$.*
- *The sequence $(\psi^{(m)})_{m\in\mathbb{N}}$ converges to ψ weakly in $\mathrm{L}^q([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1}))$ for all $q \in [2,\infty)$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1}))$.*
- *The pair (T, ψ) is a finite energy weak solution of Problem (1.4).*

Theorem 3.8 is a consequence of Proposition 3.9 (compactness and regularity of the limit), Proposition 3.10 (passing to the limit in the discrete radiative equation), Proposition 3.11 (passing to the limit in the discrete temperature equation) and Proposition 3.12 (energy estimate at the limit).

Proposition 3.9. *Under the assumptions of Theorem 3.8, there exists (T, ψ) such that:*

$$\begin{aligned}&T \in \mathrm{L}^\infty([0,\tau); \mathrm{L}^5(\Omega)),\ T^{\frac{5}{2}} \in \mathrm{L}^2([0,\tau); \mathrm{H}^1_0(\Omega)),\\ &\psi \in \mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})),\end{aligned}$$

and up to extracting a subsequence we have :

- *The sequence $(T^{(m)})_{m\in\mathbb{N}}$ converges to T strongly in $\mathrm{L}^q([0,\tau); \mathrm{L}^q(\Omega))$ for all $q \in [2, 8)$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau); \mathrm{L}^5(\Omega))$.*
- *The sequence $(\psi^{(m)})_{m\in\mathbb{N}}$ converges to ψ weakly in $\mathrm{L}^q([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1}))$ for all $q \in [2,\infty)$ and weakly-$*$ in $\mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1}))$.*

Proof. By (3.9),

$$(T^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty([0,\tau); \mathrm{L}^5(\Omega)), \tag{3.16}$$

$$(\psi^{(m)})_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})), \tag{3.17}$$

$$(\psi^{(m)} - (T^{(m)})^4)_{m\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^2([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})). \tag{3.18}$$

We therefore obtain the weak compactness of the sequence of discrete intensities: there exists $\psi \in \mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1}))$ and a subsequence of $(\mathcal{D}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ (not relabeled) such that:

$$\psi^{(m)} \rightharpoonup^* \psi \quad \text{in } \mathrm{L}^\infty([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})) \text{ as } m \to \infty. \tag{3.19}$$

$$\psi^{(m)} \rightharpoonup \psi \quad \text{in } \mathrm{L}^q([0,\tau); \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1})) \text{ for all } q \in [2,\infty) \text{ as } m \to \infty. \tag{3.20}$$

To obtain the strong compactness of $(T^{(m)})_{m\in\mathbb{N}}$ in $\mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega))$ as in the continuous case, since we don't have a control on $\partial_t T^{(m)}$ we cannot directly use the Aubin-Lions-Simon Theorem. We rather use the estimate on the time-translations of the discrete temperature to apply Kolmogorov's compactness Theorem, which is recalled in appendix A.1 (Theorem A.1). In our setting, the Banach space B of Theorem A.1 is $\mathrm{L}^2(\Omega)$, $p=2$, and the subset A is defined by $A=\cup_{m\in\mathbb{N}}\{T^{(m)}\}$. Any $T^{(m)}\in A$ satisfies $\|T^{(m)}\|_{\mathrm{L}^2((0,\tau);\mathrm{L}^2(\Omega))}\le C\|T^{(m)}\|_{\mathrm{L}^2((0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})}\le CC_1$ by the Poincaré inequality of Lemma (3.1) and estimate (3.14) so that $A\subset \mathrm{L}^2((0,\tau);\mathrm{L}^2(\Omega))$. We now check the three assumptions (H_1)-(H_2)-(H_3) of the Theorem.

- (H_1) – The operator P is defined by extending any function $T\in A$ by zero outside the interval $(0,\tau)$. Clearly, one has $\|PT\|_{\mathrm{L}^2(\mathbb{R};\mathrm{L}^2(\Omega))}=\|T\|_{\mathrm{L}^2((0,\tau);\mathrm{L}^2(\Omega))}\le CC_1$ for all $T\in A$.

- (H_2) – For all $\phi\in\mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$, and $T^{(m)}\in A$, the quantity $\int_{\mathbb{R}}(PT^{(m)})\phi\,\mathrm{d}t$ is an element of $\mathrm{H}_{\mathcal{M}^{(m)}}(\Omega)$ with:

$$\Big\|\int_{\mathbb{R}}(PT^{(m)})\phi\,\mathrm{d}t\Big\|_{1,\mathcal{M}^{(m)}}\le\int_0^\tau\|T^{(m)}(t,\cdot)\|_{1,\mathcal{M}^{(m)}}|\phi(t)|\,\mathrm{d}t\le CC_1\|\phi\|_{\mathrm{L}^2((0,\tau);\mathbb{R})},$$

  by the Cauchy-Schwarz inequality and estimate (3.14). Hence, by the discrete Rellich Theorem 3.3, the family $\{\int_{\mathbb{R}}(PT)\phi\,\mathrm{d}t,\,T\in A\}$ is relatively compact in $\mathrm{L}^2(\Omega)$.

- (H_3) – It remains to prove that $\|PT-PT(\Box-\eta)\|_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\to 0$ as $\eta\to 0^+$ uniformly with respect to $T\in A$, where the square stands for the time variable t. For all $T\in A$ and $\eta\in(0,\tau)$ we have:

$$\begin{aligned}&\|PT-PT(\Box-\eta)\|^2_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\\&\quad=\int_0^\eta\|T(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t+\int_\eta^\tau\|T(t,\cdot)-T(t-\eta,\cdot)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t+\int_{\tau-\eta}^\tau\|T(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t.\end{aligned}$$

  Thanks to the $\mathrm{L}^\infty(\mathrm{L}^2)$-estimate on T (which comes from the $\mathrm{L}^\infty(\mathrm{L}^5)$-estimate), the first and third terms are each controlled by $CC_0^2\,\eta$, and the second term is controlled by $C_2(\eta^{\frac12}+(\delta t^{(m)})^{\frac12})$ thanks to Proposition 3.7 on the time translations of the velocity. Let $\delta>0$ be a small real number (smaller than 1). There exists $M\in\mathbb{N}$ such that, $(\delta t^{(m)})^{\frac12}\le\delta$ for all $m\ge M$. Let $\bar\eta$ be a real number in $(0,\min(\tau,1))$ such that $\bar\eta^{\frac12}\le\delta$. Thanks to Proposition 3.7, we have, for all $m\ge M$:

$$\|PT^{(m)}-PT^{(m)}(\Box-\eta)\|^2_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\le(2C_2+CC_0^2)\,\delta,\qquad\forall\eta\in(0,\bar\eta).$$

  Now, for a fixed $m<M$, we have, by a classical result that $\|PT^{(m)}-PT^{(m)}(\Box-\tau)\|^2_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}$ tends to zero as $\eta\to 0^+$. Hence, for all $m<M$, there exists $\eta^{(m)}>0$ such that

$$\|PT^{(m)}-PT^{(m)}(\Box-\eta)\|^2_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\le\delta,\qquad\forall\eta\in(0,\eta^{(m)}).$$

  Defining $\tilde\eta=\min(\bar\eta,\min\{\eta^{(m)},\,0\le m<M\})$ and $C=\max(1,2C_2+CC_0^2)$, two numbers which are independent of $T\in A$, we have proven that:

$$0<\eta\le\tilde\eta\qquad\Longrightarrow\qquad\|\,PT-PT(\Box-\eta)\,\|^2_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\le C\delta,\quad\forall T\in A,$$

  which expresses that $\|\,PT-PT(\Box-\eta)\,\|_{\mathrm{L}^2(\mathbb{R},\mathrm{L}^2(\Omega))}\to 0$ as $\eta\to0^+$ uniformly for $T\in A$.

Hence, Kolmogorov's Theorem A.1 applies and there exists $T \in \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega))$ and a subsequence $(\mathcal{D}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ (not relabeled) such that $T^{(m)} \to T$ in $\mathrm{L}^2([0,\tau);\mathrm{L}^2(\Omega))$ as m tends to infinity. Moreover since $(T^{(m)})_{m\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\mathrm{L}^5)$ and in $\mathrm{L}^8(\mathrm{L}^8)$ we have that the limit $T \in \mathrm{L}^8([0,\tau);\mathrm{L}^8(\Omega)) \cap \mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega))$ and up to extracting a subsequence, we have weak convergence in $\mathrm{L}^8([0,\tau);\mathrm{L}^8(\Omega))$ of $T^{(m)}$ and weak-$*$ convergence in $\mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega))$. By interpolation, since we have strong convergence in $\mathrm{L}^2(\mathrm{L}^2)$, $T^{(m)}$ strongly converges towards T in $\mathrm{L}^q([0,\tau);\mathrm{L}^q(\Omega))$ for all $q \in [2,8)$, and $(T^{(m)})^{\frac{5}{2}} \to (T)^{\frac{5}{2}}$ in $\mathrm{L}^q([0,\tau);\mathrm{L}^q(\Omega))$ for all $q \in [1, \frac{16}{5})$.

We now prove the regularity on T. Extending $T^{(m)}$ by zero outside Ω and integrating equation (3.3) of Lemma 3.2 along the time variable, we get :

$$\|T^{(m)}(\square, \cdot + \boldsymbol{\eta}) - T^{(m)}(\square, \cdot)\|^2_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\mathbb{R}^d))} \leq C(\Omega)\, \|T^{(m)}\|^2_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})}\, |\boldsymbol{\eta}|\, (|\boldsymbol{\eta}| + h_{\mathcal{M}^{(m)}}),$$
$$\forall \boldsymbol{\eta} \in \mathbb{R}^d,\ \forall m \in \mathbb{N},$$

where the dot stands for the space variable $\boldsymbol{x}$, the square stands for the time variable t. Extending T by zero outside Ω, we have $T^{(m)} \to T$ in $\mathrm{L}^2([0,\tau), \mathrm{L}^2(\mathbb{R}^d))$. Since $h_{\mathcal{M}^{(m)}} \to 0$, the limit T (extended by zero outside Ω) satisfies:

$$\|T(\square, \cdot + \boldsymbol{\eta}) - T(\square, \cdot)\|^2_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\mathbb{R}^d))} \leq C(\Omega) \liminf_{m\to+\infty} \|T^{(m)}\|^2_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})}\, |\boldsymbol{\eta}|^2 \quad \forall \eta \in \mathbb{R}^d.$$

Since $\|T^{(m)}\|_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})} \leq C_1$ by estimate (3.14), this implies that $\boldsymbol{\nabla} T \in \mathrm{L}^2([0,\tau);\mathrm{L}^2(\mathbb{R}^d))$ (see [8, Theorem 3.10]) and that $T = 0$ on $\partial\Omega$ since $T = 0$ outside Ω. Hence, $T \in \mathrm{L}^2([0,\tau);\mathrm{H}^1_0(\Omega))$. Moreover, taking $\boldsymbol{\eta} = \eta \boldsymbol{e}_i$ for $i = 1, .., d$ and letting $\eta \to 0$ yields

$$\|\boldsymbol{\nabla} T\|_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\mathbb{R}^d))} \leq C(\Omega) \liminf_{m\to+\infty} \|T^{(m)}\|_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})}.$$

By (3.9) the sequence $(\|(T^{(m)})^{\frac{5}{2}}\|_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})})_{m\in\mathbb{N}}$ is also bounded and the same argument yields that $T^{\frac{5}{2}} \in \mathrm{L}^2([0,\tau);\mathrm{H}^1_0(\Omega))$ with

$$\|\boldsymbol{\nabla}(T)^{\frac{5}{2}}\|_{\mathrm{L}^2([0,\tau);\mathrm{L}^2(\mathbb{R}^d))} \leq C(\Omega) \liminf_{m\to+\infty} \|(T^{(m)})^{\frac{5}{2}}\|_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})}. \tag{3.21}$$

□

Proposition 3.10. *Under the assumptions of Theorem 3.8, the limit pair (T, ψ) of the sequence $(T^{(m)}, \psi^{(m)})_{m\in\mathbb{N}}$ given by Prop. 3.9 satisfies the radiative intensity equation in the weak sense:*

$$- \iiint_{[0,\tau)\times\Omega\times\mathbb{S}^2} \left(\psi \partial_t \zeta + \psi \boldsymbol{\beta} \cdot \nabla \zeta - (\psi - T^4)\zeta\right) \mathrm{d}\boldsymbol{\beta}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$$
$$= \iint_{\Omega\times\mathbb{S}^2} \psi_0 \zeta(0, \cdot, \cdot)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{\beta}, \qquad \forall \zeta \in \mathcal{C}^\infty_c([0,\tau) \times \Omega \times \mathbb{S}^{d-1}). \tag{3.22}$$

Moreover, the function ψ satisfies the boundary condition:

$$\gamma\psi|_{\Sigma_-} = \psi_b = 0. \tag{3.23}$$

Proof. To prove the result we pass to the limit $m \to +\infty$ in the weak formulation of the discrete radiative intensity equation. Let $\zeta \in \mathcal{C}_c^\infty([0,\tau) \times \Omega \times \mathbb{S}^{d-1})$ and for $m \in \mathbb{N}$ we define a discretized test function $\zeta^{(m)}$ as follows:

$$\zeta^{(m)}(t, \boldsymbol{x}, \boldsymbol{\beta}) = \sum_{n=1}^{N^{(m)}} \sum_{\alpha \in \mathcal{A}^{(m)}} \sum_{K \in \mathcal{M}^{(m)}} \zeta_{K,\alpha}^{n-1} \, \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1}, t^n]}(t) \tag{3.24}$$

where $\zeta_{K,\alpha}^n = \zeta(t^n, \boldsymbol{x}_K, \boldsymbol{\beta}_\alpha)$ for all $n \in \{0, .., N^{(m)}\}$, $K \in \mathcal{M}^{(m)}$ and $\alpha \in \mathcal{A}^{(m)}$, with $\boldsymbol{x}_K$ the mass center of K. We extend $\zeta^{(m)}$ to $t = 0$ by continuity. We also define a discrete time derivative and a discrete gradient of $\zeta^{(m)}$ by:

$$\begin{aligned}
&\eth_t \zeta^{(m)}(t, \boldsymbol{x}, \boldsymbol{\beta}) = \sum_{n=1}^{N^{(m)}} \sum_{\alpha \in \mathcal{A}^{(m)}} \sum_{K \in \mathcal{M}^{(m)}} \frac{1}{\delta t} \left(\zeta_{K,\alpha}^n - \zeta_{K,\alpha}^{n-1} \right) \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1}, t^n]}(t), \\
&\overline{\boldsymbol{\nabla}} \zeta^{(m)}(t, \boldsymbol{x}, \boldsymbol{\beta}) = \sum_{n=1}^{N^{(m)}} \sum_{\alpha \in \mathcal{A}^{(m)}} \sum_{K \in \mathcal{M}^{(m)}} (\boldsymbol{\nabla} \zeta)_{K,\alpha}^{n-1} \, \mathbb{1}_K(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1}, t^n]}(t), \\
&\text{with } (\boldsymbol{\nabla} \zeta)_{K,\alpha}^{n-1} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \zeta_{\sigma,\alpha}^{n-1} \, \boldsymbol{n}_{K,\sigma} \text{ and } \zeta_{\sigma,\alpha}^{n-1} = |\sigma|^{-1} \int_\sigma \zeta(t^{n-1}, \boldsymbol{x}, \boldsymbol{\beta}_\alpha) \, \mathrm{d}\sigma_x.
\end{aligned}$$

From the regularity of ζ, it is easy to prove that for all q in $[1, \infty]$, there exists $C = C(\tau, \Omega, \theta_0, q, \zeta)$ such that for all $m \in \mathbb{N}$:

$$\begin{aligned}
&\|\eth_t \zeta^{(m)} - \partial_t \zeta\|_{\mathrm{L}^q([0,\tau) \times \Omega \times \mathbb{S}^{d-1})} + \|\overline{\boldsymbol{\nabla}} \zeta^{(m)} - \boldsymbol{\nabla} \zeta\|_{\mathrm{L}^q([0,\tau) \times \Omega \times \mathbb{S}^{d-1})} \leq C(\delta t^{(m)} + h_{\mathcal{M}^{(m)}} + h_{\mathcal{A}^{(m)}}), \\
&\|\zeta^{(m)}(0, \cdot, \cdot) - \zeta(0, \cdot, \cdot)\|_{\mathrm{L}^q(\Omega \times \mathbb{S}^{d-1})} \leq C(h_{\mathcal{M}^{(m)}} + h_{\mathcal{A}^{(m)}}).
\end{aligned} \tag{3.25}$$

$$\tag{3.26}$$

Multiplying the discrete intensity equation (3.6b) by $\delta t \, |\alpha| \, |K| \, \zeta_{K,\alpha}^n$, summing over $K \in \mathcal{M}^{(m)}$, $\alpha \in \mathcal{A}^{(m)}$ and $n \in \{0, \ldots, N^{(m)} - 1\}$ we get

$$I_1^{(m)} + I_2^{(m)} + I_3^{(m)} = 0, \tag{3.27}$$

where

$$\begin{aligned}
I_1^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{K \in \mathcal{M}^{(m)}} |K| (\psi_{K,\alpha}^{n+1} - \psi_{K,\alpha}^n) \zeta_{K,\alpha}^n, \\
I_2^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{K \in \mathcal{M}^{(m)}} \Big(\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \psi_{\sigma,\alpha}^{n+1} \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \Big) \zeta_{K,\alpha}^n, \\
I_3^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{K \in \mathcal{M}^{(m)}} |K| (\psi_{K,\alpha}^{n+1} - (T_K^{n+1})^4) \zeta_{K,\alpha}^n.
\end{aligned}$$

We observe that, since $\delta t^{(m)} \to 0$ as $\to +\infty$, for m large enough, $\zeta_{K,\alpha}^{N^{(m)}} = 0$ for all $K \in \mathcal{M}^{(m)}, \alpha \in \mathcal{A}^{(m)}$ since ζ has a compact support in $[0,\tau) \times \Omega \times \mathbb{S}^{d-1}$. In a similar way, for m large enough we have $\zeta_{K,\alpha}^n = 0$ for all $\alpha \in \mathcal{A}^{(m)}$ and $n \in \{0,\dots,N^{(m)}-1\}$ and for all $K \in \mathcal{M}^{(m)}$ such that $\partial K \cap \partial\Omega \neq \emptyset$. We suppose throughout this proof that we are in this case. Performing a discrete integration by parts in $I_1^{(m)}$, we get:

$$\begin{aligned} I_1^{(m)} &= -\sum_{n=1}^{N^{(m)}} \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} |K| \psi_{K,\alpha}^n (\zeta_{K,\alpha}^n - \zeta_{K,\alpha}^{n-1}) - \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} |K| \psi_{K,\alpha}^0 \zeta_{K,\alpha}^0 \\ &= -\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi^{(m)}\, \eth_t \zeta^{(m)} \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \iint_{\Omega\times\mathbb{S}^{d-1}} \psi_0 \zeta^{(m)}(0,\cdot,\cdot)\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

By (3.25), since $\delta t^{(m)} \to 0$, $h_{\mathcal{M}^{(m)}} \to 0$ and $h_{\mathcal{A}^{(m)}} \to 0$ as $m \to \infty$, we obtain that $\eth_t \zeta_{\mathcal{M}^{(m)}}$ strongly converges towards $\partial_t \zeta$ in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$ as $m \to \infty$; in addition, by (3.26), $\zeta^{(m)}(0,\cdot,\cdot)$ strongly converges to $\zeta(0,\cdot,\cdot)$ in $\mathrm{L}^1(\Omega\times\mathbb{S}^{d-1})$. Since $\psi^{(m)}$ weakly converges towards ψ in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$, we obtain:

$$\lim_{m\to\infty} I_1^{(m)} = -\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi\, \partial_t \zeta \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \iint_{\Omega\times\mathbb{S}^{d-1}} \psi_0 \zeta(0,\cdot,\cdot)\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}.$$

Let us now turn to the second term $I_2^{(m)}$. Rearranging the terms, we get:

$$\begin{aligned} I_2^{(m)} &= -\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} |\sigma| \psi_{\sigma,\alpha}^n (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \\ &= -\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} |\sigma| \frac{\psi_{K,\alpha}^n + \psi_{L,\alpha}^n}{2} (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} + R_{2,1}^{(m)}, \end{aligned} \tag{3.28}$$

where:

$$R_{2,1}^{(m)} = \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} |\sigma| \Big(\frac{\psi_{K,\alpha}^n + \psi_{L,\alpha}^n}{2} - \psi_{\sigma,\alpha}^n \Big) (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma}.$$

Reordering the sum in the first term of (3.28), we get:

$$\begin{aligned} I_2^{(m)} &= -\frac{1}{2} \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma| (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \Big) + R_{2,1}^{(m)} \\ &= -\frac{1}{2} \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \boldsymbol{\beta}_\alpha \cdot \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma| (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{n}_{K,\sigma} \Big) + R_{2,1}^{(m)}. \end{aligned} \tag{3.29}$$

Then we observe that

$$I_2^{(m)} = -\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \,\boldsymbol{\beta}_\alpha \cdot \Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \,\tfrac{1}{2}(\zeta_{L,\alpha}^{n-1} + \zeta_{K,\alpha}^{n-1})\, \boldsymbol{n}_{K,\sigma}\Big) + R_{2,1}^{(m)} + R_{2,2}^{(m)}$$

where

$$R_{2,2}^{(m)} = \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \,\boldsymbol{\beta}_\alpha\, \zeta_{K,\alpha}^{n-1} \cdot \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \boldsymbol{n}_{K,\sigma}\Big).$$

Introducing $\zeta_{\sigma,\alpha}^{n-1}$, we get

$$I_2^{(m)} = -\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \,\boldsymbol{\beta}_\alpha \cdot \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma| \,\zeta_{\sigma,\alpha}^{n-1}\, \boldsymbol{n}_{K,\sigma}\Big) + R_{2,1}^{(m)} + R_{2,2}^{(m)} + R_{2,3}$$

where

$$R_{2,3}^{(m)} = \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{K\in\mathcal{M}^{(m)}} \psi_{K,\alpha}^n \,\boldsymbol{\beta}_\alpha \cdot \Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \,(\zeta_{\sigma,\alpha}^{n-1} - \tfrac{1}{2}(\zeta_{L,\alpha}^{n-1} + \zeta_{K,\alpha}^{n-1}))\, \boldsymbol{n}_{K,\sigma}\Big),$$

and finally,

$$I_2^{(m)} = -\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi^{(m)}\, \boldsymbol{\beta}\cdot\overline{\boldsymbol{\nabla}}\zeta^{(m)} \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \sum_{i=1}^{4} R_{2,i}^{(m)}$$

where

$$R_{2,4}^{(m)} = \iint_{[0,\tau)\times\Omega} \sum_{\alpha\in\mathcal{A}^{(m)}} \int_\alpha \psi^{(m)}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_\alpha)\cdot\overline{\boldsymbol{\nabla}}\zeta^{(m)} \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Since $\psi^{(m)}$ weakly converges to ψ in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$ and $\overline{\boldsymbol{\nabla}}\zeta^{(m)} \to \boldsymbol{\nabla}\zeta$ strongly in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$ (by (3.25)) we have

$$\lim_{m\to+\infty} -\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi^{(m)}\, \boldsymbol{\beta}\cdot\overline{\boldsymbol{\nabla}}\zeta^{(m)} \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi\, \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\zeta \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

It remains to prove that $\sum_{i=1}^4 R_{2,i}^{(m)} \to 0$ as $m\to+\infty$. In the following, in order to ease the notations, we denote $A^{(m)} \lesssim B^{(m)}$ when there is a constant C, independent of m, such that $A^{(m)} \le C\,B^{(m)}$. Recalling the upwind definition of $\psi_{\sigma,\alpha}^n$, we get:

$$|R_{2,1}^{(m)}| \le \frac{1}{2}\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)}\\ \sigma=K|L}} |\sigma|\, |\psi_{K,\alpha}^n - \psi_{L,\alpha}^n|\, |\zeta_{L,\alpha}^{n-1} - \zeta_{L,\alpha}^{n-1}|\, |\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma}|.$$

Applying the Cauchy-Schwarz inequality, we get:

$$
\begin{aligned}
|R_{2,1}^{(m)}| \leq \frac{1}{2} \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma \in \mathcal{E}_{\rm int}^{(m)} \\ \sigma = K|L}} |\sigma| \, |\psi_{K,\alpha}^n - \psi_{L,\alpha}^n|^2 \, |\boldsymbol{\beta} \cdot \boldsymbol{n}_{K,\sigma}| \Big)^{\frac{1}{2}} \\
\times \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma \in \mathcal{E}_{\rm int}^{(m)} \\ \sigma = K|L}} |\sigma| \, |\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}|^2 \, |\boldsymbol{\beta} \cdot \boldsymbol{n}_{K,\sigma}| \Big)^{\frac{1}{2}} \\
\lesssim \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma \in \mathcal{E}_{\rm int}^{(m)} \\ \sigma = K|L}} |\sigma| \, |\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}|^2 \, |\boldsymbol{\beta} \cdot \boldsymbol{n}_{K,\sigma}| \Big)^{\frac{1}{2}}
\end{aligned}
$$

by estimate (3.9). By Taylor's inequality applied to the smooth function ζ and the regularity of the discretization, we have $|\zeta_{L,\alpha}^{n-1} - \phi_{K,\alpha}^{n-1}|^2 \lesssim h_{\mathcal{M}^{(m)}}/|\sigma|(|K|+|L|)\, \|\boldsymbol{\nabla}\zeta\|^2_{\mathbf{L}^\infty([0,\tau)\times\Omega\times\mathbb{S}^{d-1})}$. Hence:

$$
|R_{2,1}^{(m)}| \lesssim \Big(h_{\mathcal{M}^{(m)}} \times \tau \, \times 4\pi \times 2|\Omega| \|\boldsymbol{\nabla}\zeta\|^2_{\mathbf{L}^\infty([0,\tau)\times\Omega\times\mathbb{S}^{d-1})} \Big)^{\frac{1}{2}} \lesssim h_{\mathcal{M}^{(m)}}^{\frac{1}{2}}.
$$

Thus, $R_{2,1}^{(m)} \to 0$ as $m \to +\infty$. The second remainder term satisfies $R_{2,2}^{(m)} = 0$ since $\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \boldsymbol{n}_{K,\sigma} = 0$ for all $K \in \mathcal{M}^{(m)}$. For the control of $R_{2,3}^{(m)}$, we have

$$
R_{2,3}^{(m)} = \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{K \in \mathcal{M}^{(m)}} \psi_{K,\alpha}^n \, \boldsymbol{\beta}_\alpha \cdot \Big(\sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \, (\zeta_{\sigma,\alpha}^{n-1} - \hat{\zeta}_{\sigma,\alpha}^{n-1}) \, \boldsymbol{n}_{K,\sigma} \Big).
$$

where $\hat{\zeta}_{\sigma,\alpha}^{n-1} = \frac{1}{2}(\zeta_{L,\alpha}^{n-1} + \zeta_{K,\alpha}^{n-1})$. Reordering the sum we get

$$
R_{2,3}^{(m)} = \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha \in \mathcal{A}^{(m)}} |\alpha| \sum_{K \in \mathcal{M}^{(m)}} |\sigma| \, (\psi_{K,\alpha}^n - \psi_{L,\alpha}^n) \, (\zeta_{\sigma,\alpha}^{n-1} - \hat{\zeta}_{\sigma,\alpha}^{n-1}) \, \boldsymbol{\beta} \cdot \boldsymbol{n}_{K,\sigma}.
$$

Hence $R_{2,3}^{(m)}$ can be controlled the same way as $R_{2,1}^{(m)}$ and we obtain $R_{2,3}^{(m)} \to 0$ as $m \to +\infty$. For $R_{2,4}^{(m)}$, observing that $|\boldsymbol{\beta} - \boldsymbol{\beta}_\alpha| \leq h_{\mathcal{A}^{(m)}}$ for all $\boldsymbol{\beta} \in \alpha$ and using the Cauchy-Schwarz inequality we get

$$
|R_{2,4}^{(m)}| \leq h_{\mathcal{A}^{(m)}} \|\psi^{(m)}\|_{\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})} \|\overline{\boldsymbol{\nabla}}\zeta^{(m)}\|_{\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})} \lesssim h_{\mathcal{A}^{(m)}}.
$$

Thus, $R_{2,4}^{(m)} \to 0$ as $m \to +\infty$ and therefore

$$
\lim_{m\to\infty} I_2^{(m)} = - \iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} \psi \, \boldsymbol{\beta} \cdot \boldsymbol{\nabla}\zeta \, \mathrm{d}\boldsymbol{\beta} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.
$$

It remains to identify the limit of $I_3^{(m)}$. We observe that

$$
I_3^{(m)} = \iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} (\psi^{(m)} - (T^{(m)})^4) \zeta^{(m)} \, \mathrm{d}\boldsymbol{\beta} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.
$$

Since $\zeta^{(m)}$ strongly converge towards ζ in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$ and $\psi^{(m)} \rightharpoonup \psi$ and $(T^{(m)})^4 \rightharpoonup T^4$ weakly in $\mathrm{L}^2([0,\tau)\times\Omega\times\mathbb{S}^{d-1})$ and $\mathrm{L}^2([0,\tau)\times\Omega)$ respectively, we have

$$\lim_{m\to+\infty} I_3^{(m)} = \iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} (\psi-(T)^4)\zeta \,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Hence, letting $m\to+\infty$ in (3.27) yields

$$-\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^2} \left(\psi\partial_t\zeta+\psi\boldsymbol{\beta}\cdot\nabla\zeta-(\psi-T^4)\zeta\right)\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$=\iint_{\Omega\times\mathbb{S}^2}\psi_0\zeta(0,\cdot,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\beta}, \qquad \forall\zeta\in\mathcal{C}_c^\infty([0,\tau)\times\Omega\times\mathbb{S}^{d-1}).$$

Let us now prove that $\gamma\psi|_{\Sigma_-}=\psi_b=0$. Let $\Psi^{(m)}$ be the function defined on $[0,\tau)\times\Sigma$ by:

$$\Psi^{(m)}(t,\boldsymbol{x},\boldsymbol{\beta})=\begin{cases}\psi_{K,\alpha}^n|\boldsymbol{\beta}\cdot\boldsymbol{n}_{K,\sigma}|^{\frac{1}{2}}, & \text{if } (t,\boldsymbol{x},\boldsymbol{\beta})\in(t^{n-1},t^n]\times\Sigma_+\cap(t^{n-1},t^n]\times\partial K\times\alpha,\\ 0, & \text{if } (t,\boldsymbol{x},\boldsymbol{\beta})\in(t^{n-1},t^n]\times\Sigma_-.\end{cases} \tag{3.30}$$

By (3.9) the sequence $(\Psi^{(m)})_{m\in\mathbb{N}}$ is uniformly bounded in $\mathrm{L}^2([0,\tau)\times\Sigma)$. Hence, there exists $\overline{\Psi}$ such that $\Psi^{(m)}$ weakly converges in $\mathrm{L}^2([0,\tau)\times\Sigma)$ towards $\overline{\Psi}$. Taking the weak limit in the equality $\Psi^{(m)}|_{\Sigma_-}=0$ yields

$$\overline{\Psi}|_{\Sigma_-}=0.$$

We have proved that (T,ψ) solve

$$\partial_t\psi+\beta\cdot\nabla\psi=-(\psi-T^4)$$

in the weak sense. Thus, applying a test function $\zeta\in C_c^\infty((0,\tau)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ on the above equation, we deduce that :

$$-\int_0^\tau\iint_{\Omega\times\mathbb{S}^{d-1}}\psi\partial_t\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_0^\tau\int_{\Omega\times\mathbb{S}^{d-1}}\psi\boldsymbol{\beta}\cdot\nabla\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+\int_0^\tau\iint_\Sigma\gamma\psi\zeta(\boldsymbol{\beta}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\sigma_x\,\mathrm{d}t=-\int_0^\tau\int_{\Omega\times\mathbb{S}^{d-1}}(\psi-(T)^4)\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{3.31}$$

Defining a discretized test function $\zeta^{(m)}$ as in (3.24), multiplying the discrete intensity equation (3.6b) by $\delta t\,|\alpha|\,|K|\,\zeta_{K,\alpha}^n$ and summing over $K\in\mathcal{M}^{(m)}$, $\alpha\in\mathcal{A}^{(m)}$ and $n\in\{0,\dots,N^{(m)}-1\}$ we get $I_1^{(m)}+I_2^{(m)}+I_3^{(m)}=0$ where $I_1^{(m)}$, $I_2^{(m)}$ and $I_3^{(m)}$ have the same expressions as in (3.27). Since ζ has now compact support in $(0,\tau)$, the time boundary terms arising from $I_1^{(m)}$ cancel, and we have:

$$\lim_{m\to\infty}I_1^{(m)}=-\iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}}\psi\,\partial_t\zeta\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

The term $I_3^{(m)}$ has the same limit as above. Performing a discrete integration by parts in $I_2^{(m)}$, there is now a new boundary term:

$$\begin{aligned} I_2^{(m)} = &- \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\text{int}}^{(m)}\\ \sigma=K|L}} |\sigma| \psi_{\sigma,\alpha}^n (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \\ &+ \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\sigma\in\mathcal{E}_{\text{ext}}^{(m)}\cap\mathcal{E}(K)} |\sigma| \psi_{K,\alpha}^n \zeta_{K,\alpha}^{n-1} (\boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma})^+ \\ = &- \sum_{n=1}^{N^{(m)}} \delta t \sum_{\alpha\in\mathcal{A}^{(m)}} |\alpha| \sum_{\substack{\sigma\in\mathcal{E}_{\text{int}}^{(m)}\\ \sigma=K|L}} |\sigma| \psi_{\sigma,\alpha}^n (\zeta_{L,\alpha}^{n-1} - \zeta_{K,\alpha}^{n-1}) \boldsymbol{\beta}_\alpha \cdot \boldsymbol{n}_{K,\sigma} \\ &+ \iint_{[0,\tau)\times\Sigma} \Psi^{(m)} S^{(m)} \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\sigma_x \,\mathrm{d}t + \iint_{[0,\tau)\times\Sigma} R^{(m)} \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\sigma_x \,\mathrm{d}t \end{aligned} \tag{3.32}$$

where

$$S^{(m)}(t,\boldsymbol{x},\boldsymbol{\beta}) = \sum_{n=1}^{N^{(m)}} \sum_{\alpha\in\mathcal{A}^{(m)}} \sum_{\sigma\in\mathcal{E}_{\text{ext}}^{(m)}\cap\mathcal{E}(K)} \zeta_{K,\alpha}^{n-1} |\boldsymbol{\beta}\cdot\boldsymbol{n}_{K,\sigma}|^{\frac{1}{2}} \mathbb{1}_\sigma(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1},t^n]}(t),$$

and

$$\begin{aligned} &R^{(m)}(t,\boldsymbol{x},\boldsymbol{\beta}) = \\ &\quad \sum_{n=1}^{N^{(m)}} \sum_{\alpha\in\mathcal{A}^{(m)}} \sum_{\sigma\in\mathcal{E}_{\text{ext}}^{(m)}\cap\mathcal{E}(K)} \psi_{K,\alpha}^n \zeta_{K,\alpha}^{n-1} \left((\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})^+ - (\boldsymbol{\beta}\cdot\boldsymbol{n}_{K,\sigma})^+ \right) \mathbb{1}_\sigma(\boldsymbol{x}) \mathbb{1}_\alpha(\boldsymbol{\beta}) \mathbb{1}_{(t^{n-1},t^n]}(t). \end{aligned}$$

The limit of the first term on the right hand side of (3.32) can be computed as previously. For the second term, we have the weak convergence $\Psi^{(m)} \rightharpoonup \overline{\Psi}$ in $\mathrm{L}^2([0,\tau)\times\Sigma)$ and by the regularity of ζ, we easily prove that $S^{(m)} \to \zeta|_{[0,\tau]\times\partial\Omega\times\mathbb{S}^{d-1}} |\boldsymbol{\beta}\cdot\boldsymbol{n}|^{\frac{1}{2}}$ strongly in $\mathrm{L}^2([0,\tau)\times\Sigma)$ (we use the dominated convergence theorem). The third term is a remainder term that tends to zero as $m\to+\infty$ due to the inequality $|(\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})^+ - (\boldsymbol{\beta}\cdot\boldsymbol{n}_{K,\sigma})^+| \le |\boldsymbol{\beta}-\boldsymbol{\beta}_\alpha| \le h_{\mathcal{A}^{(m)}}$ for $\boldsymbol{\beta}\in\alpha$, combined with the dominated convergence theorem.

Hence, letting $m\to+\infty$ in $I_1^{(m)} + I_2^{(m)} + I_3^{(m)} = 0$, we get

$$\begin{aligned} &- \int_0^\tau \iint_{\Omega\times\mathbb{S}^{d-1}} \psi \partial_t \zeta \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t - \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} \psi \boldsymbol{\beta}\cdot\nabla\zeta \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \\ &\qquad + \int_0^\tau \iint_\Sigma \overline{\Psi} \zeta |\boldsymbol{\beta}\cdot\boldsymbol{n}|^{\frac{1}{2}} \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\sigma_x \,\mathrm{d}t = - \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} (\psi - (T)^4) \zeta \,\mathrm{d}\boldsymbol{\beta} \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t. \end{aligned}$$

Comparing with (3.31) and recalling that $\overline{\Psi}|_{\Sigma_-} = 0$ we obtain that

$$\gamma\psi|_{\Sigma_-} = 0.$$

This concludes the proof of Proposition 3.10.

$\square$

Proposition 3.11. *Under the assumptions of Theorem 3.8, the limit pair (T,ψ) of the sequence $(T^{(m)},\psi^{(m)})_{m\in\mathbb{N}}$ given by Prop. 3.9 satisfies the temperature equation in the weak sense:*

$$-\iint_{[0,\tau)\times\Omega}\left(T\partial_t\varphi+T\Delta\varphi+\int_{\mathbb{S}^{d-1}}(\psi-T^4)\varphi\,\mathrm{d}\boldsymbol{\beta}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\int_\Omega T_0\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}. \tag{3.33}$$

Proof. To prove the result we pass to the limit $m\to+\infty$ in the weak formulation of the discrete temperature equation. Let $\varphi\in\mathcal{C}_c^\infty([0,\tau)\times\Omega)$ and for $m\in\mathbb{N}$ we define a discretized test function $\varphi^{(m)}$ as follows:

$$\varphi^{(m)}(t,\boldsymbol{x})=\sum_{n=1}^{N^{(m)}}\sum_{K\in\mathcal{M}^{(m)}}\varphi_K^{n-1}\,\mathbb{1}_K(\boldsymbol{x})\mathbb{1}_{(t^{n-1},t^n]}(t)$$

where $\varphi_K^n=\varphi(t^n,\boldsymbol{x}_K)$ for all $n\in\{0,..,N^{(m)}\}$ and $K\in\mathcal{M}^{(m)}$. We extend $\varphi^{(m)}$ to $t=0$ by continuity. We observe that, since $\delta t^{(m)}\to 0$ as $\to+\infty$, for m large enough, $\varphi_K^{N^{(m)}}=0$ for all $K\in\mathcal{M}^{(m)}$ since φ has a compact support in $[0,\tau)\times\Omega$. In a similar way, for m large enough we have $\varphi_K^n=0$ for all $n\in\{0,\dots,N^{(m)}-1\}$ and for all $K\in\mathcal{M}^{(m)}$ such that $\partial K\cap\partial\Omega\neq\emptyset$. We suppose throughout this proof that we are in this case. We now define a discrete time derivative of $\varphi^{(m)}$ by:

$$\eth_t\varphi^{(m)}(t,\boldsymbol{x})=\sum_{n=1}^{N^{(m)}}\sum_{K\in\mathcal{M}^{(m)}}\frac{1}{\delta t}\left(\varphi_K^n-\varphi_K^{n-1}\right)\ \mathbb{1}_K(\boldsymbol{x})\mathbb{1}_{(t^{n-1},t^n]}(t).$$

From the regularity of φ, we can prove that for all q in $[1,\infty]$, there exists $C=C(\tau,\Omega,\theta_0,q,\varphi)$ such that for all $m\in\mathbb{N}$:

$$\|\eth_t\varphi^{(m)}-\partial_t\varphi\|_{\mathrm{L}^q([0,\tau)\times\Omega)}\le C(\delta t^{(m)}+h_{\mathcal{M}^{(m)}}), \tag{3.34}$$

$$\|\varphi^{(m)}(0,\cdot)-\varphi(0,\cdot)\|_{\mathrm{L}^q(\Omega)}\le Ch_{\mathcal{M}^{(m)}}. \tag{3.35}$$

Multiplying the discrete intensity equation (3.6a) by $\delta t\,|K|\,\varphi_K^n$, summing over $K\in\mathcal{M}^{(m)}$, $\alpha\in\mathcal{A}^{(m)}$ and $n\in\{0,\dots,N^{(m)}-1\}$ we get

$$I_1^{(m)}+I_2^{(m)}=I_3^{(m)}, \tag{3.36}$$

where

$$I_1^{(m)}=\sum_{n=0}^{N^{(m)}-1}\sum_{K\in\mathcal{M}^{(m)}}|K|(T_K^{n+1}-T_K^n)\varphi_K^n,$$

$$I_2^{(m)}=-\sum_{n=0}^{N^{(m)}-1}\delta t\sum_{K\in\mathcal{M}^{(m)}}\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}}^{(m)}\\ \sigma=K|L}}\frac{|\sigma|}{d_\sigma}(T_L^{n+1}-T_K^{n+1})\Big)\varphi_K^n,$$

$$I_3^{(m)}=\sum_{n=0}^{N^{(m)}-1}\delta t\sum_{K\in\mathcal{M}^{(m)}}|K|\Big(\sum_{\alpha\in\mathcal{A}^{(m)}}|\alpha|\psi_{K,\alpha}^{n+1}-(T_K^{n+1})^4\Big)\varphi_K^n.$$

Performing a discrete integration by parts in $I_1^{(m)}$, we get:

$$\begin{aligned}
I_1^{(m)} &= -\sum_{n=1}^{N^{(m)}} \sum_{K\in\mathcal{M}^{(m)}} |K| T_K^n (\varphi_K^n - \varphi_K^{n-1}) - \sum_{K\in\mathcal{M}^{(m)}} |K| T_K^0 \phi_K^0 \\
&= -\iint_{[0,\tau)\times\Omega} T^{(m)}\, \eth_t \varphi^{(m)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega T_0 \varphi^{(m)}(0,\cdot)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

By (3.34), since $\delta t^{(m)} \to 0$ and $h_{\mathcal{M}^{(m)}} \to 0$ as $m\to\infty$, we obtain that $\eth_t \varphi_{\mathcal{M}^{(m)}}$ strongly converges towards $\partial_t \varphi$ in $\mathrm{L}^2([0,\tau)\times\Omega)$ as $m\to\infty$; in addition, by (3.35), $\varphi^{(m)}(0,\cdot)$ strongly converges to $\varphi(0,\cdot)$ in $\mathrm{L}^1(\Omega\times\mathbb{S}^{d-1})$. Since $T^{(m)}$ weakly converges towards T in $\mathrm{L}^2([0,\tau)\times\Omega)$, we get

$$\lim_{m\to\infty} I_1^{(m)} = -\iint_{[0,\tau)\times\Omega} T\,\partial_t\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega T_0 \varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}.$$

Let us now turn to the second term $I_2^{(m)}$. Rearranging the terms and recalling that $\varphi_K^n = 0$ is $\partial K \cap \partial\Omega \neq \emptyset$ we get:

$$\begin{aligned}
I_2^{(m)} &= -\sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K\in\mathcal{M}^{(m)}} \Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (T_L^{n+1} - T_K^{n+1}) \Big) \varphi_K^n \\
&= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (T_L^{n+1} - T_K^{n+1})(\varphi_L^n - \varphi_K^n) \\
&= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} (T_L^{n+1} - T_K^{n+1}) \int_\sigma \boldsymbol{\nabla}\varphi(t^n,\cdot)\cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}) + R_{2,1}^{(m)} \\
&= -\sum_{n=1}^{N^{(m)}} \delta t \sum_{K\in\mathcal{M}^{(m)}} T_K^n \int_K \mathrm{div}(\boldsymbol{\nabla}\varphi(t^{n-1},\cdot))\,\mathrm{d}\boldsymbol{x} + R_{2,1}^{(m)} \\
&= -\iint_{[0,\tau)\times\Omega} T^{(m)} \Delta\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + R_{2,1}^{(m)} + R_{2,2}^{(m)}
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
R_{2,1}^{(m)} &= -\sum_{n=1}^{N^{(m)}} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} (T_L^n - T_K^n) \int_\sigma \Big(\boldsymbol{\nabla}\varphi(t^{n-1},\cdot)\cdot\boldsymbol{n}_{K,\sigma} - \frac{1}{d_\sigma}(\varphi_L^{n-1} - \varphi_K^{n-1}) \Big) \,\mathrm{d}\sigma(\boldsymbol{x}), \\
R_{2,2}^{(m)} &= \sum_{n=1}^{N^{(m)}} \iint_{(t^{n-1},t^n)\times\Omega} T^{(m)}(t,\cdot)(\Delta\varphi(t,\cdot) - \Delta\varphi(t^{n-1},\cdot))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}$$

Let us prove that $R_{2,1}^{(m)} + R_{2,2}^{(m)} \to 0$ as $m\to+\infty$. We first observe that by the Cauchy-Schwarz inequality, one has

$$|R_{2,1}^{(m)}| \leq \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)} \\ \sigma=K|L}} \frac{|\sigma|}{d_\sigma} (T_L^n - T_K^n)^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)}} \frac{d_\sigma}{|\sigma|} (R_\sigma^{n-1})^2 \Big)^{\frac{1}{2}}$$

$$= \|T^{(m)}\|_{\mathrm{L}^2((0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})} \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)}} \frac{d_\sigma}{|\sigma|} (R_\sigma^{n-1})^2 \Big)^{\frac{1}{2}}$$

where

$$R_\sigma^{n-1} = \int_\sigma \Big(\boldsymbol{\nabla}\varphi(t^{n-1},\cdot)\cdot \boldsymbol{n}_{K,\sigma} - \frac{1}{d_\sigma}(\varphi_L^{n-1} - \varphi_K^{n-1}) \Big) \, \mathrm{d}\sigma(\boldsymbol{x}).$$

From the regularity of φ and the admissibilty condition on the space discretization (the segment $[\boldsymbol{x}_K, \boldsymbol{x}_L]$ is colinear to $\boldsymbol{n}_{K,\sigma}$), we have $R_\sigma^{n-1} \leq C(\theta_0)|\sigma| h_{\mathcal{M}^{(m)}} \|\varphi\|_{\mathrm{W}^{2,\infty}([0,\tau)\times\Omega)}$. Hence

$$|R_{2,1}^{(m)}| \lesssim h_{\mathcal{M}^{(m)}} \|T^{(m)}\|_{\mathrm{L}^2((0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})} \Big(\sum_{n=1}^{N^{(m)}} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(m)}} d_\sigma |\sigma| \Big)^{\frac{1}{2}}.$$

By the estimate $\|T^{(m)}\|_{\mathrm{L}^2([0,\tau);\mathrm{H}_{\mathcal{M}^{(m)}})} \leq C_1$ and the fact that $\sum_{\mathcal{E}_{\mathrm{int}}^{(m)}} d_\sigma|\sigma| \leq d|\Omega|$ we obtain that $R_{2,1}^{(m)} \to 0$ as $m \to +\infty$. The convergence $R_{2,2}^{(m)} \to 0$ as $m \to +\infty$ follows from a uniform control on $T^{(m)}$ in $\mathrm{L}^1([0,\tau)\times\Omega)$ and the fact that for all $(t,\boldsymbol{x}) \in [0,\tau)\times\Omega$, $|\Delta\varphi(t,\boldsymbol{x}) - \Delta\varphi(t^{n-1},\boldsymbol{x})| \leq \delta t^{(m)} \|\varphi\|_{\mathrm{W}^{3,\infty}((0,\tau)\times\Omega)}$. By the convergence of $T^{(m)}$ towards T in $\mathrm{L}^2([0,\tau)\times\Omega)$, letting $m \to +\infty$ in (3.37) yields

$$\lim_{m\to\infty} I_2^{(m)} = -\iint_{[0,\tau)\times\Omega} T\Delta\varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Observing that $I_3^{(m)}$ is the same term as in the proof of Proposition 3.10 (for a test function which does not depend on the angular variable $\boldsymbol{\beta}$) we obtain that

$$\lim_{m\to+\infty} I_3^{(m)} = \iiint_{[0,\tau)\times\Omega\times\mathbb{S}^{d-1}} (\psi - (T)^4)\varphi \, \mathrm{d}\boldsymbol{\beta} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t.$$

Hence, letting $m \to +\infty$ in (3.36) concludes the proof of Proposition 3.11. $\square$

Proposition 3.12. *Under the assumptions of Theorem 3.8, the limit pair (T,ψ) of the sequence $(T^{(m)},\psi^{(m)})_{m\in\mathbb{N}}$ satisfies the following energy estimate: there exists $C = C(\Omega)$ such that*

$$\begin{aligned} \|T(\tau)\|_{\mathrm{L}^5(\Omega))}^5 + \|\psi(\tau)\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2 + \int_0^\tau \|\nabla(T)^{\frac{5}{2}}\|_{\mathrm{L}^2(\Omega)}^2 \, \mathrm{d}t \\ + \int_0^\tau \|\psi - (T)^4\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2 \, \mathrm{d}t + \int_0^\tau \|\psi\|_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta} d\sigma_x)}^2 \, \mathrm{d}t \\ \leq C(\|T_0\|_{\mathrm{L}^5(\Omega)}^5 + \|\psi_0\|_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}^2). \end{aligned}$$

Proof. By (3.9) we have

$$\frac{1}{5}\|T^{(m)}(\tau)\|^5_{\mathrm{L}^5(\Omega)} + \frac{1}{2}\|\psi^{(m)}(\tau)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \frac{16}{25}\int_0^\tau \|(T^{(m)})^{\frac{5}{2}}\|^2_{1,\mathcal{M}^{(m)}}\,\mathrm{d}t + \int_0^\tau \|\psi^{(m)} - (T^{(m)})^4\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \|\Psi^{(m)}\|^2_{\mathrm{L}^2(\Sigma)} \le \frac{1}{5}\|T^{(m)}(0)\|^5_{\mathrm{L}^5(\Omega)} + \frac{1}{2}\|\psi^{(m)}(0)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}. \tag{3.38}$$

where $\Psi^{(m)}$ is defined in (3.30), and $T^{(m)}(0)$ and $\psi^{(m)}(0)$ are the piecewise constant functions defined by (3.5a) and (3.5a). Moreover, we have the weak limits

$$\begin{aligned}
&T^{(m)} \rightharpoonup^* T \quad \text{in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^5(\Omega)) \text{ as } m\to\infty,\\
&\psi^{(m)} \rightharpoonup^* \psi \quad \text{in } \mathrm{L}^\infty([0,\tau);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})) \text{ as } m\to\infty,\\
&\psi^{(m)} - (T^{(m)})^4 \rightharpoonup \psi - (T)^4 \text{ in } \mathrm{L}^2([0,\tau):\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})),\\
&\Psi^{(m)} \rightharpoonup \gamma\psi|\boldsymbol{\beta}\cdot\boldsymbol{n}| \text{ in } \mathrm{L}^2([0,\tau);\mathrm{L}^2(\Sigma)).
\end{aligned}$$

We also have $T^{(m)}(0) \to T_0$ in $\mathrm{L}^5(\Omega)$ and $\psi^{(m)}(0) \to \psi_0$ in $\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})$. Hence, taking the $\liminf_{m\to+\infty}$ in the left hand side of (3.38) and recalling (3.21) we obtain that there exists $C = C(\Omega)$ such that

$$\|T(\tau)\|^5_{\mathrm{L}^5(\Omega))} + \|\psi(\tau)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})} + \int_0^\tau \|\nabla(T)^{\frac{5}{2}}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \int_0^\tau \|\psi - (T)^4)\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}\,\mathrm{d}t + \int_0^\tau \|\psi\|^2_{\mathrm{L}^2(\Sigma;|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,\mathrm{d}\boldsymbol{\beta}d\sigma_x)}\,\mathrm{d}t \le C(\|T_0\|^5_{\mathrm{L}^5(\Omega)} + \|\psi_0\|^2_{\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})}).$$

□

4 Comments on the Diffusive Limit of the Scheme

In [15], the authors investigate the diffusion limit $\varepsilon \to 0$ of the non-dimensionalized formulation of the radiative heat transfer system:

$$\partial_t T_\varepsilon - \Delta T_\varepsilon = \frac{1}{\varepsilon^2}\left(\langle\psi_\varepsilon\rangle - T_\varepsilon^4\right), \qquad (t,\boldsymbol{x}) \in (0,\infty)\times\Omega, \tag{4.1a}$$

$$\partial_t \psi_\varepsilon + \frac{1}{\varepsilon}\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi_\varepsilon = -\frac{1}{\varepsilon^2}\left(\psi_\varepsilon - T_\varepsilon^4\right), \qquad (t,\boldsymbol{x},\boldsymbol{\beta}) \in (0,\infty)\times\Omega\times\mathbb{S}^{d-1}. \tag{4.1b}$$

Physically ε represents the ratio of a typical photon mean free path to a typical length scale of the problem, associated with the initial and boundary conditions. We now recall how to formally derive the diffusive limit system given in [15]. By equation (4.1b),

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon. \tag{4.2}$$

Therefore, we have

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\left(T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon\right) - \varepsilon^2\partial_t\left(T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon\right).$$

Combing the terms with the same order and assuming $T_\varepsilon \in C^2_{t,x}$ and $\psi_\varepsilon \in C^3_{t,x,\boldsymbol{\beta}}$, we get:

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}T_\varepsilon^4 - \varepsilon^2\left(\partial_t T_\varepsilon^4 - \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\psi_\varepsilon\right)\right) + \mathcal{O}(\varepsilon^3).$$

We can use (4.2) again in the fourth term on the right of the above equation and obtain that:

$$\begin{aligned}\psi_\varepsilon =&\ T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}T_\varepsilon^4 - \varepsilon^2\left(\partial_t T_\varepsilon^4 - \boldsymbol{\beta}\cdot\boldsymbol{\nabla}(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}T_\varepsilon^4)\right) + \mathcal{O}(\varepsilon^3)\\ =&\ T_\varepsilon^4 - \varepsilon\boldsymbol{\beta}\cdot\boldsymbol{\nabla}T_\varepsilon^4 - \varepsilon^2\partial_t T_\varepsilon^4 + \varepsilon^2\boldsymbol{\beta}\otimes\boldsymbol{\beta} : \mathrm{Hess}(T_\varepsilon^4) + \mathcal{O}(\varepsilon^3)\end{aligned} \tag{4.3}$$

Assuming $T_\varepsilon \to \overline{T}, \quad \psi_\varepsilon \to \overline{\psi} \quad$ as $\varepsilon \to 0$, we can pass to the limit $\varepsilon \to 0$ in (4.3) and get $\overline{\psi} = \overline{T}^4$. We can also use (4.3) combined with

$$\langle\boldsymbol{\beta}\rangle = 0, \qquad \langle\boldsymbol{\beta}\otimes\boldsymbol{\beta}\rangle = \frac{1}{d}I_d \tag{4.4}$$

to find that the radiative density satisfies

$$\langle\psi_\varepsilon\rangle = \ T_\varepsilon^4 - \varepsilon^2\partial_t T_\varepsilon^4 + \varepsilon^2\frac{1}{d}\Delta T_\varepsilon^4 + \mathcal{O}(\varepsilon^3)$$

This enables us to pass to the limit on the last term in (4.1a):

$$\frac{1}{\varepsilon^2}(\langle\psi_\varepsilon - T_\varepsilon^4\rangle) \to -\partial_t\overline{T}^4 + \frac{1}{d}\Delta\overline{T}^4.$$

We can then pass to the limit in (4.1a) and derive the following nonlinear limit diffusion equation:

$$\partial_t\left(\overline{T} + \overline{T}^4\right) - \Delta\left(\overline{T} + \frac{1}{d}\overline{T}^4\right) = 0. \tag{4.5}$$

This diffusive limit from (4.1) to (4.5) was rigorously justified in [15] by means of weak compactness and relative entropy methods.

A natural question to ask is whether the numerical scheme proposed in this paper satisfies the *asymptotic preservation* (AP) property, ensuring its compatibility with the limit $\varepsilon \to 0$. More precisely, we use the following definition:

Definition 4.1 (AP schemes)**.** *Let P^ε be the problem* (4.1) *and P^0 be its diffusive limit* (4.5)*.*

- *A numerical scheme $P^\varepsilon_{h,\delta t}$ (h is the space step and δt the time step) for P^ε is said to be asymptotic preserving if for every $(h, \delta t)$ the solution of $P^\varepsilon_{h,\delta t}$ converges as $\varepsilon \to 0$ to a solution of some numerical scheme $P^0_{h,\delta t}$ which is a consistent and stable approximation of P^0.*
- *The numerical scheme $P^\varepsilon_{h,\delta t}$ is said to be uniformly asymptotic preserving if moreover, $P^\varepsilon_{h,\delta t}$ is consistent with P^ε uniformly in ε.*

The proof of the asymptotic–preserving property can be carried out by means of compactness arguments and is naturally formulated in the framework of approximations of weak solutions. In contrast, the uniform asymptotic–preserving property relies on additional regularity assumptions on the data of the considered systems and is more appropriately established in the setting of strong solutions.

For our scheme, we can carry out the same formal derivation as above to identify the formal limit of the numerical scheme (3.5)-(3.6) (applied to the nondimensionalized model (4.1)) as ε goes to zero while keeping the discretization fixed. Establishing rigorously that the scheme is asymptotic-preserving would then require ε-uniform estimates derived from the energy, in the spirit of the continuous analysis in [15]. We begin by assuming that the angular discretization fulfills a discrete counterpart of (4.4), which can indeed be enforced in practice:

$$\sum_{\alpha\in\mathcal{A}} |\alpha|\boldsymbol{\beta}_\alpha = 0, \qquad \sum_{\alpha\in\mathcal{A}} |\alpha|\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\alpha = \frac{1}{d} I_d. \tag{4.6}$$

Then, assuming we have bounds on the discrete solution $(T_K^n, \psi_{K,\alpha}^n)_{0\le n\le N, K\in\mathcal{M},\alpha\in\mathcal{A}}$ that are independent of ε, we may derive a discrete analogue of the identity (4.3) :

$$\psi_{K,\alpha}^{n+1} = (T_K^{n+1})^4 - \varepsilon\boldsymbol{\beta}_\alpha \cdot (\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1} + \varepsilon^2 \boldsymbol{\beta}_\alpha \cdot (\boldsymbol{\nabla}(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} T^4))_{K,\alpha}^{n+1} + \mathcal{O}(\varepsilon^3) \tag{4.7}$$

where the upwind gradient corresponding to (3.7) is given by:

$$\begin{aligned}(\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1} = \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{(T_K^{n+1})^4 + (T_L^{n+1})^4}{2}\right) \boldsymbol{n}_{K,\sigma} \\ + \frac{\Theta_{\varepsilon,h}}{|K|} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{(T_K^{n+1})^4 - (T_L^{n+1})^4}{2} \operatorname{sgn}(\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})\right) \boldsymbol{n}_{K,\sigma}\end{aligned} \tag{4.8}$$

with $\Theta_{\varepsilon,h} = 1$, and where

$$\begin{aligned}\boldsymbol{\beta}_\alpha \cdot (\boldsymbol{\nabla}(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} T^4))_{K,\alpha}^{n+1} = \boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\alpha : \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{(\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1} + (\boldsymbol{\nabla} T^4)_{L,\alpha}^{n+1}}{2}\right) \otimes \boldsymbol{n}_{K,\sigma} \\ + \boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\alpha : \frac{\Theta_{\varepsilon,h}}{|K|} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{(\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1} - (\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1}}{2} \operatorname{sgn}(\boldsymbol{\beta}_\alpha\cdot\boldsymbol{n}_{K,\sigma})\right) \otimes \boldsymbol{n}_{K,\sigma}.\end{aligned} \tag{4.9}$$

Hence, at the discrete level, even if we enforce the identity (4.6), we do not have that $\langle\boldsymbol{\beta}_\alpha \cdot (\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1}\rangle = 0$ since the upwind gradient $(\boldsymbol{\nabla} T^4)_{K,\alpha}^{n+1}$ depends on $\boldsymbol{\beta}_\alpha$ precisely because of the upwinding. Numerically, following classical techniques [4] in the context of AP schemes for the low Mach number limit for fluids, we can try to correct this behavior in the limit $\varepsilon \to 0$ by damping the numerical diffusion of the upwind scheme choosing $\Theta_{\varepsilon,h}$ such that

$$\Theta_{\varepsilon,h} = \frac{\varepsilon^{1+\eta}}{\varepsilon^{1+\eta} + h_{\mathcal{M}}}, \quad \eta \in (0,1).$$

In the limit $\varepsilon \to 0$ we formally recover a centered gradient which does not depend on α:

$$(\boldsymbol{\nabla} T^4)_K^{n+1} = \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{(T_K^{n+1})^4 + (T_L^{n+1})^4}{2}\right) \boldsymbol{n}_{K,\sigma}$$

Applying $\langle\cdot\rangle$ to (4.7) we thus get after a little bit of algebra (and using the second part of (4.6)):

$$\langle\psi_{K,\alpha}^{n+1}\rangle = (T_K^{n+1})^4 - \varepsilon^2\frac{(T_K^{n+1})^4-(T_K^n)^4}{\delta t} + \varepsilon^2\frac{1}{d}(\widetilde{\Delta}T^4)_K^{n+1} + \mathcal{O}(\varepsilon^{2+\eta})$$

where

$$(\widetilde{\Delta}T^4)_K^{n+1} = \frac{1}{|K|}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\left(\frac{(\boldsymbol{\nabla}T^4)_K^{n+1}+(\boldsymbol{\nabla}T^4)_L^{n+1}}{2}\right)\cdot\boldsymbol{n}_{K,\sigma}. \tag{4.10}$$

Therefore, passing to the limit $\varepsilon\to 0$ in the numerical scheme, we formally obtain the following discretization of the limit diffusion equation (4.5):

$$\frac{T_K^{n+1}-T_K^n}{\delta t} + \frac{(T_K^{n+1})^4-(T_K^n)^4}{\delta t} - (\Delta T^{n+1})_K - \frac{1}{d}(\widetilde{\Delta}T^4)_K^{n+1} = 0. \tag{4.11}$$

Several observations should be made concerning this discrete diffusive limit. When the numerical diffusion inherent to the upwind scheme is reduced by the factor $\Theta_{\varepsilon,h}$, it is no longer possible to establish a maximum principle (in particular, the positivity property) for the discrete radiative intensity ψ_K^{n+1}, because the matrix associated with the implicit scheme ceases to satisfy the defining characteristics of an M-matrix. Consequently, this obstruction prevents the derivation of the energy estimate (3.9). Therefore, compactness cannot be ensured either in the asymptotic regime $h_{\mathcal{M}},\delta t\to 0$ or in the limit $\varepsilon\to 0$. Furthermore, the discrete Laplacian operator $\widetilde{\Delta}$ arising in the discrete diffusion limit is known to be unstable on cartesian grids, as it decouples odd and even modes and may induce numerical instabilities commonly referred to as checkerboard modes [10]. For these reasons, the numerical scheme proposed in this paper does not appear to be the most suitable option for preserving the asymptotic behavior linked to the diffusive limit.

A Appendices

A.1 Kolomogorov's compactness theorem

Theorem A.1 (Kolmogorov's Theorem)**.** *Let B be a Banach space, p a real number such that $1\le p<+\infty$ and $\tau>0$. Let $A\subset \mathrm{L}^p((0,\tau);B)$. The subset A is relatively compact in $\mathrm{L}^p((0,\tau);B)$ if A satisfies the three following conditions:*

(H_1) *For all $u\in A$, there exists $Pu\in\mathrm{L}^p(\mathbb{R};B)$ such that $Pu=u$ almost everywhere in $(0,\tau)$ and $\|Pu\|_{\mathrm{L}^p(\mathbb{R};B)}\le C$, where C depends only on A.*

(H_2) *For all $\phi\in\mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$, the family $\{\int_{\mathbb{R}}(Pu)\phi\,\mathrm{d}t,\ u\in A\}$ is relatively compact in B.*

(H_3) *$\|Pu-Pu(.-\eta)\|_{\mathrm{L}^p(\mathbb{R};B)}\to 0$, as $\eta\to 0^+$, uniformly with respect to $u\in A$.*

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