HOW TO CHOOSE NEW AXIOMS FOR SET THEORY?

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1. Introduction

The development of axiomatic set theory originated from the need for a rigorous investigation of the basic principles at the foundations of mathematics. The classical theory of sets ZFC offers a rich framework, nevertheless many important mathematical problems (such as the famous continuum hypothesis) cannot be solved within this theory. Set theorists have been exploring new axioms that would allow one to answer such fundamental questions that are independent from ZFC. Research in this area has led to consider several candidates for a new axiomatisation such as Large Cardinal Axioms, Forcing Axioms, Projective Determinacy and others. The legitimacy of these new axioms is, however, heavily debated and gave rise to extensive discussions around an intriguing philosophical problem: *what criteria should be satisfied by axioms?* What aspects would distinguish an axiom from a hypothesis, a conjecture and other mathematical statements? What is an axiom after all? The future of set theory very much depends on how we answer such questions. Self-evidence, intuitive appeal, fruitfulness are some of the many criteria that have been proposed. In the first part of this paper, we illustrate some classical views about the nature of axioms and the main challenges associated with these positions. In the second part, we outline a survey of the most promising candidates for a new axiomatization for set theory and we discuss to what extent those criteria are met. We assume basic knowledge of the theory ZFC.

2. Ordinary mathematics

Before we start our analysis of the axioms of set theory and the discussion about what criteria can legitimate those axioms, we should address a quite radical view based on the belief that 'ordinary mathematics needs *much less than ZFC or ZF*'. This claim suggests that strong axioms such as the Axiom of Choice, or Infinity are not really needed for standard mathematical results, and certainly ordinary mathematics does not need new strong axioms such as Large cardinals axioms, Forcing Axioms etc. If so, then our goal of securing the axioms of ZFC and the new axioms would simply be irrelevant or a mere set theoretic concern (where set theory is not considered standard mathematics).

The issue with this view is to clarify what counts as 'ordinary mathematics'. In fact, the Axiom of Choice is heavily used in many fields such as Algebra, General Topology, Measure Theory and Functional Analysis. For instance, the Axiom of Choice is indispensable for the following claims and theorems (they are actually equivalent to the Axiom of Choice):

- For every equivalence relation there is a set of representatives.
- Every vector space has a basis.
- Krull's theorem: Every unital ring other than the trivial ring contains a maximal ideal.
- Tychonoff theorem.
- In the product topology, the closure of a product of subsets is equal to the product of the closures.
- Every connected graph has a spanning tree.
- Every surjection has a right inverse.

Other weaker consequences of the Axiom of Choice cannot be proven within ZF:

- Baire category theorem (which is equivalent to the Axiom of Dependent Choice).
- Hahn Banach theorem.
- Every Hilbert space has an orthonormal basis.
- Every field has an algebraic closure.
- Stone's representation theorem for Boolean algebras.
- Nielse-Schreier theorem: every subgroup of a free group is free.
- Vitali theorem: there exists a set of reals which is not Lebesgues measurable.
- The existence of a set of reals which does not have the Baire property.
- The existence of a set of reals which does not have the perfect set property.
- Every set can be linearly ordered.

Thus, important applications of the Axiom of Choice can be found in many areas, hence rejecting the Axiom of Choice would come with a big price for the scope of 'ordinary mathematics'.

The analogous claim that ordinary mathematics does not need *more than ZFC* runs into a similar problem, as it is often the case that natural questions, that were raised in what one might consider a standard mathematical framework, turned out to be independent from ZFC, thus requiring stronger additional axioms to be settled. It is the case, for instance, for Whitehead problem in group theory: formulated in the 50's, Whitehead problem was considered one of the most important open problems in algebra for many years, until S. Shelah showed in 1974 that it was undecidable in ZFC (see [21]); Whitehead conjecture is true if we accept the axiom of constructibility, namely that every set is constructible. A similar case is the Normal Moore Space Conjecture, a topological problem whose solution was eagerly sought for many years until strong large cardinal assumptions turned out to be indispensable for its solution. The reader is certainly familiar with the famous Fermat's conjecture recently demonstrated by Wiles who won the Abel prize for his outstanding result; what the reader might not be aware of, is that Wiles's proof makes use of Grothendieck's universes whose existence requires large cardinals (strongly inaccessible cardinals). More recently, McLarty undertook a rigorous investigation of the assumptions needed for Wiles's proof and showed that finite order arithmetic suffices for the whole construction (see [15]). It is generally believed that eventually we will be able to prove Fermat's conjecture in Peano Arithmetic, yet this remains an open problem.

Independence results have always caused a certain embarrassment in the community of mathematicians. When a mathematical problem is proven to be independent from ZFC, suddenly it is labeled as 'just set theoretical' or 'vague' and no longer mathematical in the traditional sense. A precise definition of what 'ordinary mathematics' means should then take into account this attitude towards those problems which, at first, seem to emerge naturally as intrinsically relevant questions for mathematical research, then are dismissed after proven to require strong axioms. A simple move would be to claim that independent problems are legitimate mathematical questions that yet are 'unsolvable'. In this perspective, then, any attempt to answer such questions with stronger assumptions can only be seen as speculative. Surely, many mathematicians navigate these lines of thoughts. For instance, when Nykos [17] proved in 1980 the consistency of the Normal Moore Space Conjecture from a strongly compact cardinal, he titled his paper 'A *provisional* solution to the Normal Moore Space Conjecture' (emphasis mine). However, if any result assuming large cardinals were just 'provisional', as Nykos' choice of words suggests, then Large Cardinals Axioms would be nothing more than mere *hypotheses*. Yet, despite the general skepticism towards the legitimacy of these principles, the mathematical community seems to acknowledge them a different status, a stronger role. In fact, we can point out that Wiles's proof of Fermat's conjecture was well accepted by the community of number theorists despite its use of inaccessible cardinals. Imagine that his proof were assuming Riemann hypothesis instead, would this be considered a result even worth publishing? The supporter of the view that ordinary mathematics can all be done in ZFC, or in a much weaker system than ZF, needs to clarify what should be the status of independent problems and of the additional assumptions needed for their solution.

3. Intrinsic motivations

The word 'axiom' comes from the Greek $\alpha\xi\iota\omega\mu\alpha$ 'that which commends itself as evident'. To these days, the concept of axioms maintains the character of a self-evident statement. As Feferman pointed out (see [2]), the English Oxford Dictionary defines 'axiom' as:

"a self-evident proposition requiring no formal demonstration to prove its truth, but received and assented as soon as mentioned".

Clearly, what counts as obvious, self-evident, intuitive or inherently true is highly subjective. Thus, claiming that the truth of axioms rests on self-evidence would inevitably lead to a conception of mathematics as subjective.

"I can in no way agree to taking 'intuitively clear' as a criterion of truth in mathematics, for this criterion would mean the complete triumph of subjectivism and would lead to a break with the understanding of science as a form of social activity." (Markov 1962).

The self-evidence criterion is also quite restrictive. Not only the new axioms considered in contemporary set theory are far from self-evident (not even their strongest supporters claim they are self-evident), but even the axioms of ZFC are not strictly obvious. Certainly, the Axiom of Choice and the Axiom of Infinity were not immediately 'received and assented as soon as mentioned', on the contrary they were extensively debated and a mild skepticism still survives.

"The set theoretical axioms that sustain modern mathematics are self-evident in differing degrees. One of them – indeed, the most important of them, namely Cantor's axiom, the so-called axiom of infinity – has scarcely any claim to self-evidence at all." (Mayberry [14, p. 10])

Maddy's analysis in [9] shows that even the less controversial axioms of the theory ZF were not motivated by intrinsic reasons but rather practical ones. Consider for instance the Axiom of Foundation: first introduced in the form $A \notin A$ to block Russell's paradox, it is nowadays adopted in its stronger version "every set is well-founded". Reasons for reformulating the axiom in this way were not based on self-evidence, but originated from the belief that "no field of set theory or mathematics is in any general need of sets which are not well-founded" (Fraenkel, Bar-Hillel and Levy [[3], p. 88]) – actually, non-well founded sets can be useful in mathematics –. Today the Axiom of Foundation is better supported by the so-called '*Iterative conception*'. Roughly, this consists in the idea that sets must be obtained by an iterative process where at a first stage certain sets are secured 'immediately', then new sets can be obtained starting from the sets at the first level so to form a second level, and at each stage new sets can be defined from the ones introduced at the previous levels. Under the Axiom of Foundation, all sets can be obtained in this way, in fact the class of all sets V coincide with the *Von Neumann Universe* which is defined as follows. The level zero V_0 is the empty set, then the first level V_1 contains just the empty set, at each successor stage $\alpha + 1$, the level $V_{\alpha+1}$ is defined as the set of all subsets of V_α (in fact V_1 coincides with $\mathscr{P}(V_0)$), at limit stages λ, we let V_λ be the union of all V_α for $\alpha < \lambda$. The Von

Neumann Universe is the class obtained from the union of all V_α's. The Axiom of Foundation is equivalent to V being equal to the Von Neumann Universe which is the main expression of the iterative conception just discussed. This is often considered to be an intrinsic justification for the Axiom of Foundation, yet it is not *obvious* that such an iterative process would exhaust all possible sets. On the other hand, the Von Neumann hierarchy certainly gives a very useful and elegant description of the class of all sets, thus the Axiom of Foundation has undoubtedly strong *practical* merits.

Let us reformulate the self-evidence requirement and consider the following criterion that we may call '*intrinsic necessity*':

An axioms must have some intuitive appeal, however the axiom may not be immediately obvious, but it should ultimately occur to us that what the axiom states is true and it could not be otherwise.

This reformulation may legitimate those controversial axioms that were not immediately accepted, such as the Axiom of Choice, but were progressively welcome and employed. While the well-ordering principle mainly encountered reluctance, the equivalent statement 'the cartesian product of a collection of non-empty sets is non-empty' seems to be better accepted by the mathematical community as a fundamental truth. Unfortunately, the criterion of intrinsic necessity is still problematic. There is no strong reason for believing that what the Axiom of Choice states could not be otherwise. In fact, as proven by Banach and Tarski, the Axiom of Choice is actually paradoxical as it implies a quite counterintuitive statement (Banach-Tarski paradox): given a solid ball in a 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Thus the Axiom of Choice challenges basic geometric intuition, leaving a shadow on its alleged intrinsic necessity. Even the Axiom of Foundation can hardly be justified by the criterion of intrinsic necessity, as there is no compelling reason for taking the cumulative hierarchy as a necessary feature or the universe of sets.

4. Extrinsic motivations

We argued that intrinsic motivations such as self-evidence, intrinsic necessity etc. are subjective and restrictive. Those considerations led Maddy to claim that axioms are mainly supported by *extrinsic motivations*, namely by their success, or fruitfulness. The roots of this idea already appeared in Gödel [5]:

"Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success", that is, its fruitfulness in consequences and in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs." (Gödel [5, p. 521] 1947)

It is important to stress that Gödel and Maddy mean different things with the term 'fruitful'. Let us discuss first Gödel's view. For Gödel, the fruitfulness of an axiom is measured in terms of '*verifiable consequences*'. This is a delicate notion that deserves several comments. How can we verify a mathematical statement? Is this verification the result of an empirical process? Gödel was known to believed in some sort of perception of mathematical entities analogous to our perception of physical objects. Thus, in Gödel's view, a mathematical statements can be verified to the extent that our intuition provides us with an immediate perception of the mathematical objects involved; other statements cannot be verified directly, but they can be supported by strong enough extrinsic evidence as long as their consequences can be 'verified'.

In more recent work, Magidor considers this mathematical verification to be directly connected to our empirical knowledge of the physical world:

"As far as verifiable consequences, I consider the fact that these axioms [large cardinals] provide new Π^1_0 sentences which so far were not refuted. In some sense we can consider these Π^1_0 sentences as physical facts about the world that so far are confirmed by the experience." (Magidor [13, p. 13])

Whatever meaning we accord to the expressions 'mathematical verification' and 'mathematical evidence', we should note that, as for natural sciences, a plurality of verifiable consequences cannot secure the theory with certainty, since even inconsistent mathematical theories can, at first, appear to have many verifiable consequences. Thus, we can only say of a given axiom or theory that it was not refuted *so far*. In other words, paraphrasing Popper, mathematical theories are not strictly speaking verifiable, they are only *falsifiable*; but this is not different from physics, chemistry or other sciences.

A more challenging remark is that mutually incompatible theories can all be 'successful' in the sense of both Gödel's and Magidor's quote. Consider, for instance, the theory ZFC plus the axiom of measurable cardinals versus ZF+ V=L (more details about these axioms will be provided in the second part of this paper). These two theories are incompatible, yet none of their consequences was 'refuted' so far.

In Maddy, the concepts of extrinsic justification and fruitfulness changes over her various writings. In [10] she argues that mathematical entities such as sets are not abstract but concrete objects; it follows that an immediate empirical verification of mathematical statements is possible. In [11], she describes extrinsic justifications as an inductive process, where a first level mathematics is secured intrinsically, then new axioms can be justified via their consequences demonstrable in lower-level theories. Thus, in this view, the verification can be regarded as a proof that the mathematical statement in question can be derived from a more basic theory for which we have some sort of intrinsic evidence. Later, Maddy developed a wider conception of extrinsic justifications that she describes as based on *practical, inter-theoretic motivations*. In [12], she explains the 'success' of an axiom or a theory on the basis of its effectiveness to achieve specific mathematical goals, Maddy refers to these sorts of motivations as the '*proper methods*'. So, for instance, the Axiom of Choice allows one to solve natural outstanding problems in various areas of mathematics, Projective Determinacy came into considerations as the result of a broader research for new axioms that might settle certain problems in analysis and set theory that could not be solved within ZFC, accordingly certain Large Cardinals axioms are justified as they imply Projective Determinacy and settle other problems that are independent from ZFC, and so on.

Once again, distinct incompatible theories can be equally successful, even in this sense. For instance, if many strong Large Cardinals Axioms such as the axiom of measurable cardinals can be justified in this way, even the axiom of constructibility V=L can be viewed as an effective mean to achieve specific mathematical goals: V=L settles the continuum problem as it implies the Continuum Hypothesis (and even GCH), it also implies the Axiom of Choice (which can be used itself for proving classical fundamental theorems in mathematics), it settles many other questions that are independent from ZF, for instance it implies the negation of the Suslin's hypothesis. Thus, even this approach is not immune to set theoretical pluralism; if one wants to defend the view that only one theory of sets is legitimate, then additional motivations are needed for choosing a specific theory over the others. Maddy's suggestion is to appeal on the *maximality principle*. Roughly, this consists in the idea that we should prefer the theory that maximizes the concept of set. For instance, the concept of set underlying large cardinals seems to be wider than the one associated with the axioms of constructibility which is often ruled out as 'too restrictive'. Reference to this 'maximize rule' can be found for instance in Drake [1], Moschovakis [16] and Scott [20]. Nevertheless, the alleged restrictiveness of the axiom of constructibility was recently refuted by Hamkins [6] who proved, roughly, that the axiom of constructibility is reach enough to allow one to talk about the concept of sets in the sense of large cardinals within a model of $V = L$. We will discuss this further in Section 5.

Finally, we can remark that the 'extrinsic approach' makes axioms depend on their consequences. This conflicts with the traditional view that considers axioms to be *the starting point for demonstration* from which, ideally, the truth of the other mathematical statements can be derived. Here, the situation is reversed: the consequences of an axiom legitimate the axiom, or they lead us to reject it when we have some 'counter-evidence' for such mathematical consequences. In this picture, then, mathematics resembles Quine's web of belief, namely any part of mathematics, including axioms, could be altered in the light of 'evidence'.

5. The axiom of constructibility

We now illustrate the main candidates for new axioms considered in contemporary set theory. The oldest one is certainly the *Axiom of Constructibility* V=L, that asserts that every set is constructible, namely every set belongs to Gödel's constructible universe L. L is inductively defined as follows:

- $L_0 = \emptyset$;
- $L_{\alpha+1}$ is the set of all subsets a of L_α that are definable with parameters in L_α (i.e. there is a formula $\varphi(x, a_1, ..., a_n)$ with parameters $a_i \in L_\alpha$ such that $a = \{x \in L_\alpha;\ L_\alpha \models \varphi(x, a_1, ..., a_n)\}$);
- when λ is a limit ordinal, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

Finally, $L := \bigcup_{\alpha \in Ord} L_\alpha$ (L is a class). The constructible universe was introduced by Gödel in 1938 to prove the consistency of the continuum hypothesis. In fact, the axiom of constructibility implies the generalised continuum hypothesis. Moreover, it implies the Axiom of Choice, and it settles many other questions that are independent from ZF, for instance it implies the negation of the Suslin's hypothesis.

Sentiment in favour of the Axiom of Constructibility can be found for example in Fraenkel, Bar-Hillel and Levy [3]. Nevertheless, the axiom of constructibility counts very few supporters among contemporary set theorists. Gödel him self did not consider $V = L$ as a valid candidate axiom for set theory, as he believed that CH was actually false. On the other hand, there is no clear evidence for the Continuum Hypothesis or its negation that would count as a corroboration or a falsification for the Axiom of Constructibility. Here we have a lucid example of how the extrinsic approach prioritizes the consequences over the axiom. Similarly, Maddy suggests that V=L should be rejected on the basis of the fact that it implies the existence of a an analytic (Δ^1_2) non-measurable set of reals, but there is no evidence for the claim that every set of reals is Lebesgue measurable (we will discuss Lebesgue measurability in Section 8).

"There are also extrinsic reasons for rejecting V = L, most prominently that it implies the existence of a a Δ^1_2 well-ordering of the reals, and hence that there is a Δ^1_2 set which is not Lebesgue measurable." (Maddy [9])

The strongest objections to the Axiom of Constructibility are related to its apparent restrictiveness. In fact, L is provably the smallest inner model of ZFC (i.e. the smallest class satisfying the axioms of ZFC and containing all the ordinals). It follows that, if we consider other axioms such as the Axiom of Measurable cardinals (a large cardinal axiom that establishes the existence of a measurable cardinal), despite the incompatibility of this axiom with V=L, it is always possible to talk about the concept of set in the sense of V=L within the theory ZFC+$\exists\kappa$ measurable, while the converse seems not possible.

"The language of set theory as used by the believer in V=L can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation $\varphi \mapsto \varphi^L$. There is no translation in the other direction."(Steel [2])

Actually, as Hamkins showed in [6], there is a sense in which the converse is also possible.

"even if we have very strong large cardinal axioms in our current set-theoretic universe V, there is a much larger universe V^+ in which the former universe V is a countable transitive set and the axiom of constructibility holds."

This means that, even the Axiom of Constructibility is reach enough to allow us to talk about the concept of sets in the sense of large cardinals within a model of $V = L$.

6. Large cardinals axioms

Let us discuss now *Large Cardinals Axioms* –there is a whole hierarchy of large cardinals axioms, we will discuss just some notions–. A large cardinal is any uncountable cardinal κ which is at least *weakly inaccessible*, namely which satisfies the following two properties:

(1) for every cardinal $\gamma < \kappa$, we have $\gamma^+ < \kappa$;
(2) for every subset $X \subseteq \kappa$ of size $< \kappa$, we have $\sup X < \kappa$.

If we replace the condition 1 with the stronger 'for every cardinal $\gamma < \kappa$, we have $2^\gamma < \kappa$', then we have the notion of *strong inaccessible* cardinal; we will simply call *inaccessible* the strong inaccessible cardinals. We can observe that the properties above are all satisfied by $\aleph_0$, thus the axioms of weakly and strongly inaccessible cardinals establish that there are other cardinals than $\aleph_0$ with those properties. There is no precise definition of what a large cardinal axiom is, but we can say that all large cardinals axioms establish or imply the existence of inaccessible cardinals.

The existence of a weakly inaccessible cardinal yields the consistency of ZFC, as if κ is a weakly compact cardinal, then L_k is a model of ZFC. Since, by Gödel's incompleteness theorem, ZFC cannot prove its own consistency, it follows that the existence of large cardinals cannot be proven within ZFC; neither the *consistency* of large cardinals axioms can be proven from the consistency of ZFC. This marks an important difference between large cardinals axioms and other kind of axioms such as $V = L$

whose consistency can be proven relative to the consistency of ZFC.

Let us discuss some intrinsic motivations for the existence of inaccessible cardinals.

- *Uniformity.* Roughly, this is the belief that the universe of sets should be uniform, in the sense that "it doesn't change its character substantially as one goes over from smaller to larger sets or cardinals, i.e., the same or analogous states of affairs reappear again and again (perhaps in more complicated versions)" (Wang [23, pp. 189-90], see also Kanamori and Magidor [8], Solovay, Reinhardt and Kanamori [22], and Reinhardt [19]); $\aleph_0$ is the first cardinal with the properties (1) and (2) above, hence a cardinal with the same property must reappear at higher levels.
- *Inexhaustibility.* The universe of all sets is too rich to be exhausted by some basic operations such as power set or replacement, therefore there must be a cardinal which is not generated by those operations (see e.g. Gödel [5], Wang [23] or Drake [1]); such a cardinal can be proven to be inaccessible.
- *Reflection.* The universe of sets is too complex to be completely described by some property, hence anything that is true of the entire universe, must be true also at some initial segment of it, it must "reflect" at some V_κ. In particular there must be a V_κ which is also closed by the power set and replacement operations; then κ can be proven to be inaccessible.

All these arguments seem to rest on mathematical platonism in an essential manner, as they appeal on some specific conception of "the universe of sets" as uniform, inexhaustible, indescribable and so on. But even assuming a platonic point of view, what reasons do we have to believe that the universe of sets has such features? Some issues arise, for instance, with the claim of uniformity. In fact, there are properties that do hold at $\aleph_0$ and do not occur at higher cardinals. For instance, Ramsey's theorem establishes that for every $n, m < \aleph_0$ and for every coloring of the n-tuples of $\aleph_0$ into m colors, we can find a set $H \subseteq \aleph_0$ of size $\aleph_0$ such that all the n-tuples of H have the same color, this is called a *homogeneous set*; on the other hand, it can be proven that no uncountable cardinal can satisfy the same property: if we replace $\aleph_0$ with an uncountable κ, we get a statement that is provably false in ZFC.

Typically, large cardinals generalize properties of $\aleph_0$. For instance, the notions of *Ramsey cardinal, Erdös cardinal, weakly compact cardinals* and others can be defined as special generalizations of the theorem of Ramsey that we just mentioned; some limitations are necessary because as we said the direct generalization of Ramsey Theorem to an uncountable cardinal is provably false in ZFC. We consider, for example, the axiom of weakly compact cardinals which establishes the existence of an uncountable cardinal κ such that for every coloring of the pairs of ordinals of κ into less than κ many colors there is a homogeneous set of size κ. Once again, we stress the fact that generalizations are dangerous as they may lead to inconsistencies as in the case above.

The axiom of weakly compact cardinals can also be defined as a generalization of Compactness theorem to the infinitary language $\mathscr{L}_{\kappa,\kappa}$. Given two infinite cardinals κ, λ, we denote by $\mathscr{L}_{\kappa,\lambda}$ the infinitary language that roughly allows conjunctions and disjunctions of less than κ many formulas, and quantifications over less than λ many variables. Thus, for instance $\mathscr{L}_{\omega,\omega}$ corresponds to first order logic. An uncountable cardinal κ is weakly compact if, and only if, whenever we have a theory T in $\mathscr{L}_{\kappa,\kappa}$ with at most κ non logical symbols, if T is $< \kappa$-satisfiable (i.e. every family of less than κ many sentences of T is satisfiable), then T is satisfiable. If we remove the restriction to 'theories that have at most κ non-logical symbols, we have the notion of *strongly compact cardinal*. Other large cardinals axioms can be defined as generalizations of compactness theorem. Such generalizations imply interesting 'compactness results', namely given some structure, we assume that all its smaller substructures satisfy a certain property and we deduce that the whole structure satisfy the same property. For instance assuming a strongly compact cardinal κ it is possible to prove that every abelian group of size at least κ is free abelian whenever all its smaller subgroups are free abelian. The axiom of constructibility on the other hand is the 'cemetery of compactness properties' : for instance, compactness for the freeness of abelian groups is actually false in $V = L$. The analysis of such compactness or incompactness results gives us no strong motivation to support one theory over the other, as there is no cogent reason to deem compactness more suitable than incompactness or the converse. Not even Uniformity helps us in this case, as ZFC proves both compactness and incompactness results: for example, König's lemma can be regarded as a compactness result[1], but on the other hand its generalization to $\aleph_1$ is provably false in ZFC (there are Aronszajn trees).

We can see that the notion of weakly compact cardinal can be defined both as a combinatorial and model-theoretic notion. The same occur for other large cardinals, namely it is often the case that certain mathematical problems arising in completely different contexts and fields lead to the same large cardinal notions. This fact is sometimes considered to be an intrinsic motivation for large cardinals, but however remarkable this might seem, it is not clear how it can can actually be considered as evidence for these axioms, rather than just a practical advantage.

The most powerful large cardinals axioms are the ones that can be defined as elementary embeddings of V into some inner model of ZFC. We discuss some of these notions in the next section.

[1]Given a tree of height ω whose levels are finite, if every finite subtree has a branch of the same length as the height subtree, then the whole tree also has a branch of the same length as the height of the tree

7. Measurable cardinals and elementary embeddings

In the history of large cardinals axioms the introduction of *measurable cardinals* was probably the most crucial step as it lead to the theory of elementary embeddings that are extremely useful in solving set theoretical problems and answering other mathematical questions. Let us discuss, then, these notions.

In 1902, Lebesgue formulated the measure problem: he asked whether there is a function that associates to every bounded set of reals a real number between 0 and 1 and such that the function is not identically 0, it is translation invariant and countably additive. Motivated by this question, he introduced his famous Lebesgue measure (a function with these properties) and asked whether every bounded set of reals was Lebesgue measurable, namely whether his measure was defined over every bounded set of reals. Vitali soon found a counterexample under the Axiom of Choice, the problem was then reformulated by replacing the condition of translation invariance with 'every singleton must have measure 0', the minimal request for avoiding trivial solutions. The problem was still proven to be independent from ZF, in fact a counterexample can be built under CH. At this point Banach realized that the problem did not depend on the structure of $\mathbb{R}$, and it could be reformulated for a general set S : is there a function $\mu : \mathscr{P}(S) \to [0, 1]$ which is not identically 0, assigns to every singleton the value 0 and is countably additive? The solution of this problem comes down to the existence of certain large cardinal, the *real valued measurable cardinals*. A cardinal κ is real valued measurable if every set of size κ has a measure μ with the properties above which moreover is κ-additive, namely for every family $\{X_\alpha\}_\alpha$ of less than κ many sets, $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. This is an example of a notion that can be justified extrinsically by Maddy's 'proper methods', namely it arose naturally as the solution to a specific mathematical problem.

Now, if we require that not only every set of size κ has a measure, but also the measure takes just two values 0 or 1, then we have *measurable* cardinals. In fact this notion has an extremely powerful characterization: κ is measurable if and only if one can define a non-trivial elementary embedding[2] $j : V \to M$ where M is a transitive class, such that κ is the least cardinal that is moved by j. By using this characterization, Scott was able to prove that if there is a measurable cardinal, then $V \neq L$. Thus, measurable cardinals, as well as any other stronger large cardinal, are incompatible with the axiom of constructibility.

Many powerful large cardinal notions can be defined in terms of elementary embeddings where we require the transitive class M to be 'closer' to V. These notions have weak intrinsic justifications, in fact the ultimate large cardinal notion expressible in

[2]A function $j : V \to M$ is an elementary embedding if for every formula φ and parameters $a_1, ..., a_n$ one has $V \models \varphi(a_1, ..., a_n)$ if and only if $M \models \varphi(j(a_1), ..., j(a_n))$.

terms of elementary embeddings is provably inconsistent with ZFC. This is the notion of *Reinhardt cardinal*, an uncountable cardinal κ for which there is a non trivial embedding j of V into itself where κ is the least cardinal which is moved by j.

Large cardinals axioms that establish the existence of elementary embeddings are more successfully justified by their fruitfulness, as they settle a number of questions that are independent from ZFC. The mort remarkable application of such cardinals is the theory of projective sets that under these cardinals gets a very elegant and exhaustive analysis. In fact, the existence of infinitely many Woodin cardinals implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, the so-called *regularity properties*. These considerations brings us to discuss Determinacy hypotheses, which is the object of the next section.

8. Determinacy Hypotheses

The study of regularity properties dates back to the earliest 20th century from the work of the french analysts Borel, Baire and Lebesgue. Research in this area led to the development of an independent discipline, known as descriptive set theory. About 40 years later, it was shown that the open questions that descriptive set theorists were trying to solve (namely whether every set of real has the regularity properties above) could not be answered within ZFC (as we have seen for Lebesgue mesurability). In 1962 Mycielski and Steinhaus introduced the Axiom of Determinacy AD which was proven to solve such problems. AD is the assertion that every set of reals is *determined*, that means that for every set of reals A, one of the two players has a winning strategy in the following game of length ω. We regard A as a subset of ${}^{\omega}\omega$ (in set theory a real is an omega sequence of natural numbers), the two players I and II alternatively choose natural numbers $n_0, n_1, n_2, ...$ At the end of the game a sequence $\langle n_i; i \in \mathbb{N}\rangle$ is generated, player I wins if and only if the sequence belongs to A.

The Axiom of Determinacy implies that all sets of reals are Lebesgue measurable, have the perfect set property and the Baire property. Moreover, the statement that every set of reals has the perfect set property implies a weak form of the continuum hypothesis: every uncountable set of reals has the same cardinality as the full set of reals. On the other hand, AD implies the negation of of the generalised continuum hypothesis. Despite its fruitfulness, AD was never seriously considered as a valid candidate new axiom for set theory as it contradicts the Axiom of Choice –once again the priority goes on the consequences rather than the axiom, but the consequence (here the Axiom of Choice) is itself in need for a justification–. This led to investigate two distinct directions. The first approach was to assume AD in a quite natural subuniverse, namely $L(\mathbb{R})$, together with AC in the full universe V (L($\mathbb{R}$) is the smallest transitive inner model of ZF containing all the ordinals and the reals). The second approach was to consider a weakening of AD, called *Projective Determinacy*, PD. Projective Determinacy is the statement that every *projective* set of reals is determined. PD implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, and unlike AD, Projective Determinacy is not known to contradict the Axiom of Choice. Projective Determinacy follows from the existence of infinitely Woodin cardinals and this is the reason why this large cardinal assumption implies that every projective set of reals has the regularity properties above.

9. Ultimate L and Forcing Axioms

As we said, the Axiom of Constructibility and the Axiom of Determinacy both decide the continuum problem (the former implies GCH, the latter implies a weak form of CH, but it also implies the negation of the generalised continuum hypothesis). Large cardinals axioms, on the other hand, do not decide the size of the continuum. So, a quite promising direction of research was considered which combine large cardinals with L and it may shed light on the size of the continuum, this approach is known as *V=Ultimate L*.

To understand this view, consider the intuition behind the Axiom of Constructibility: L is build up from a cumulative process where each stage is obtained from the previous one by a canonical operation, namely by taking the definable subsets of the previous stage. The universe of sets resulting from this process is quite restrictive as only few 'canonical' sets are accepted at each stage. The idea behind V=Ultimate L is that, while we want large cardinals to exist in the universe of sets, we only want to include sets that are canonical or necessary after a fashion. Ultimate L, proposed by Woodin, is the alleged inner model for supercompact cardinals. Roughly this is an L-like model where lives a supercompact cardinal. Such a model was not build yet and it is an open problem whether it can actually be found, but it can be proven that if the construction of the Ultimate L is successful, then it would contain also all the stronger large cardinals (i.e. stronger than supercompact cardinals). More importantly, V=Ultimate L would imply CH.

Magidor, however, expressed some doubts about this approach:

"It is very likely that the Ultimate L, like the old L, will satisfy many of the combinatorial principles like $\diamondsuit_{\omega_1}$. These principles are usually the reason that "L is the paradise of counter examples". They allow one to construct counter examples to many elegant conjectures. (The Suslin Hypothesis is a famous case)." (Magidor [13])

As for the Axiom of Constructibility, V=Ultimate L rests on the idea of a limitation of the concept of set through a cumulative process, while other views rely on the opposite slogan that the concept of set should be as rich as possible. The most important example of such a liberal view is given by *Forcing axioms.* Forcing is the main tool for proving independence results in set theory. There are essentially two main approaches for building models of set theory and proving consistency results: one is through inner models which are obtained roughly by 'restricting' V into a subclass; the other is by using the Forcing technique where, conversely, V is expanded to a larger universe. Forcing axioms roughly establish that anything that can be forces by some 'nice' forcing notions (a forcing is simply a partially ordered set) is a set in the universe. For instance, the two most fruitful Forcing Axioms, PFA and MM, are the following statements.

The Proper Forcing Axiom PFA states that if P is a forcing notion that is proper and D is a collection of $\aleph_1$ many dense subsets of P, then there is a generic filter that meets all the dense sets in D.

Roughly, this says that anything that can be forced by a *proper forcing* is a set in the universe.

Martin's Maximum MM asserts that if P is a forcing notion that preserves stationary subsets of ω_1 and D is a collection of $\aleph_1$ many dense subsets of P, then there is a generic filter that meets all the dense sets in D.

Roughly, this says that anything that can be forced by a forcing that *preserves stationary subsets of* ω_1 is a set in the universe. We will not discuss these notions in the details, we should only point out that MM is the strongest possible version of a Forcing Axiom and it was proven to be consistent relative to the existence of a supercompact cardinal (this provides another motivation for large cardinals axioms). Forcing Axioms settle many important questions that cannot be answered within ZFC, but more importantly they find remarkable applications in cardinal arithmetic. In fact, Foreman Magidor and Shelah proved in 1988 that Martin's Maximum settles the size of the continuum, it implies that the $2^{\aleph_0} = \aleph_2$. Later in 1992, Todorčević and Veličkovič showed that even the weaker axiom PFA implies that the size of the continuum is $\aleph_2$. Other remarkable applications of Forcing Axioms include the singular cardinals hypothesis (from PFA), the Axiom of Determinacy in L($\mathbb{R}$), the statement that any two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic (from PFA), every automorphism of the Boolean algebra $\mathcal{P}(\omega)/fin$ is trivial (from PFA), the $\aleph_2$-saturation of the ideal of non stationary sets on ω_1 (from MM), and the reflection of stationary subsets of κ for any regular cardinal $\kappa \geq \omega_2$ (from MM).

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