Miscellaneous Remarks about Orthogonality

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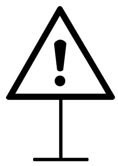
rencontre de réalisabilité, Marseille, June 2018

Warning!



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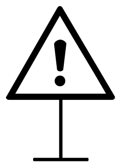

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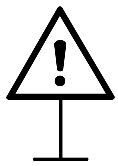

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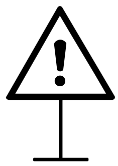

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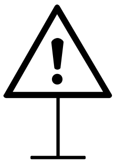

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orthogonality.

(non) definition

The second line$^{\perp}$ of Wikipedia's entry for orthogonality is

> *In mathematics,* **orthogonality** *is the generalization of the notion of perpendicularity to the linear algebra of bilinear forms.*

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Orthogonality is a tool used to define (sometimes) interesting models.

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Definition

Orthogonality is a tool used to define (sometimes) interesting models.

$^{\perp}$: the first line is *"'Orthogonal' redirects here. For the trilogy of novels by Greg Egan, see Orthogonal (novel)."* but this isn't really relevant.

Why "Perpendicularity"?

For a finitely dimensional complex vector spaces E, we have

- $u \perp v$ is defined by $u \cdot v = 0$, ("perpendicularity")
- If A is a subvector space, then $A^\perp$ is defined by $A^\perp = \{v \mid \forall u \in A, u \perp v\}$.
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Idea

Orthogonality: defining interesting "spaces" as sets of "things" T satisfying $T = T^{\perp\perp}$, for an appropriate relation "$\perp$" between "things".

Plan

1. Framework
2. Realizability
3. Linear Logic

Relations and Orthogonality

Definition

Given a relation $\perp$ between sets X and Y, we define the following operator from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$:

$$x^{\perp} \quad = \quad \left\{ b \in Y \;\middle|\; \forall a \in x, a \perp b \right\}$$

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ϕ is of the form $x \mapsto x^{\perp}$ iff ϕ transforms arbitrary unions into intersections.

Note that any such ϕ is antitonic...

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Proof idea: define $a \perp b$ iff $b \in \phi(\{a\})$. We have

$$b \in \phi(x) \quad \Leftrightarrow \quad b \in \phi\Big(\bigcup_{a \in x} \{a\}\Big) \quad \Leftrightarrow \quad b \in \bigcap_{a \in x} \phi(\{a\}) \quad \Leftrightarrow \quad b \in x^\perp \ .$$

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...but I am not going to say anything about that...

Two Orthogonalities

Lemma

Any monotonic $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ *can be factorized as* $x \mapsto x^{\perp_1 \perp_2}$ *for some set* Z *and relations* $\perp_1 \subset X \times Z$ *and* $\perp_2 \subset Z \times Y$.

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Proof: define $Z = \mathcal{P}(X)$ and

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&\Leftrightarrow b \in \phi(x) && \text{monotonicity of } \phi
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(comment for L. R.: this is impredicative...)

Closure Operators

Definition (Closure operator)

A closure operator on $\mathcal{P}(X)$ is an operator ϕ satisfying

1. *ϕ is monotonic,*
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For any relation $\perp \subset X \times Y$, $x \mapsto x^{\perp\perp}$ is a closure operator on $\mathcal{P}(X)$.

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Proposition

Any closure operator $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $\phi(x) = x^{\perp\perp}$ for some relations $\perp \subset X \times Z$.

Partial Proof

Write $\mathsf{Fix}(\phi)$ for the set of fixpoint of ϕ.

1. Because ϕ is a closure operator, $\mathsf{Fix}(\phi)$ is the set of pre-fixpoints of ϕ: $\mathsf{Fix}(\phi) = \{x \mid \phi(x) \subset x\}$.
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&\Leftrightarrow \forall x' \in \mathrm{Fix}(\phi), x' \in x^{\perp} \Rightarrow a \in x' && \text{definition of } \perp\\
&\Leftrightarrow \forall x' \in \mathrm{Fix}(\phi), \left(\forall a \in x, a \perp x'\right) \Rightarrow a \in x' && \text{definition of } \_^{\perp}\\
&\Leftrightarrow \forall x' \in \mathrm{Fix}(\phi), \left(\forall a \in x, a \in x'\right) \Rightarrow a \in x' && \text{definition of } \perp\\
&\Leftrightarrow \forall x' \in \mathrm{Fix}(\phi), x \subset x' \Rightarrow a \in x' && \text{simplification}\\
&\Leftrightarrow a \in \bigcap\{x' \in \mathrm{Fix}(\phi) \mid x \subset x'\} && \text{definition}\\
&\Leftrightarrow a \in \phi(x) && \text{lemma}
\end{aligned}
$$

Partial Proof

Write $\mathrm{Fix}(\phi)$ for the set of fixpoint of ϕ.

1. Because ϕ is a closure operator, $\mathrm{Fix}(\phi)$ is the set of pre-fixpoints of ϕ: $\mathrm{Fix}(\phi) = \{x \mid \phi(x) \subset x\}$.
2. By the Knaster-Tarski theorem, $\big(\mathrm{Fix}(\phi), \bigcap\big)$ is complete inf-lattice.

Lemma

If ϕ is a closure operator, we have $\phi(x) = \bigcap\{x' \in \mathrm{Fix}(\phi) \mid x \subset x'\}$.

Define $\underline{Z = \mathrm{Fix}(\phi)}$ and $\perp \subset X \times Z$ by $a \perp x \Leftrightarrow a \in x$.

(comment for L. R.: this is impredicative...)

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- Closure operators are equivalent to "bi-orthogonals".

Note however that not all closure operators can be obtained from a "homogeneous" relation $\perp \subset X \times X$.

counter example: $x \subset X \mapsto \begin{cases} x & \text{if } X \text{ is cofinite} \\ x & \text{otherwise} \end{cases}$

Plan

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Constant atomic types are easy: take all terms with that type.

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  "$O \subset \mathcal{SN}$", "$O = O^{\perp\perp}$"

  The relation $t \perp C$ depends on the model.

Remark: reducibility candidates and saturated sets are "closed" for some operations. In theory, they can be obtained using an orthogonality relation.

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In many models, strong normalization isn't important!

- "$t \perp \pi$" when $\langle t, \pi \rangle$ loops or reduces to $\mathtt{0}$ or $\mathtt{1}$. (Mellies & Vouillon, 2005)
- "$t \perp \pi$" if $\langle t, \pi \rangle \to^* \langle \mathtt{stop}, \widehat{n} \cdot \pi' \rangle$ for some n s.t. $f(n) = 0$. (Miquel, 2009)

Here f is an arbitrary primitive recursive function.

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More about that in Rodolphe's talk...

Personal Remarks

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Question: are there interesting "realizability" models without computational content? (except forcing models)

Plan

1. Framework
2. Realizability
3. Linear Logic

Interpreting Formulas

This is easy:

- boolean algebras (classical logic)
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Interpreting Formulas and Proofs

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For linear logic, more care is needed...

Interpreting proofs for classical logic also requires more care...

Phase Semantics

Definition (Girard, "Linear Logic", 1987)

A phase space is given by:

- *a commutative monoid (whose elements are called "phases"),*
- *a set $\perp$ of phases.*

Two phases are orthogonal, written $p \perp q$ when $pq \in \perp$, and fixpoints for $\_^{\perp\perp}$ are called facts.

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This gives a (complete) provability semantics for linear logic, where the connectives are given by

connective	$x \,\&\, y$	$x \oplus y$	$x \otimes y$	$x \parr y$	
"basis"	$x \cap y$	$x \cup y$	$x \cdot y$	???	
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The proof of completeness uses the free commutative monoid on formulas (finite multisets) with $\Gamma \perp \Delta$ iff $\vdash \Gamma, \Delta$ is provable.

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This is more "abstract" than realizability models.

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Finiteness Spaces

They give a denotational semantics for differential proofs (Ehrhard, 2003)

Definition

A finiteness space over X is a $C \subset \mathcal{P}(X)$ such that $C = C^{\perp\perp}$,
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Other connectives are defined by de Morgan duality...

Finiteness spaces give a model of differential λ-calculus...

"The operations of identification could be seen as formal derivation or formal primitive. The interest of this approach was to propose, at the the theoretical level, to replace brutal beta-conversion by iterated linear conversions."
Girard, "Linear Logic", 1987

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- quantum coherent spaces (Girard, 2004) — $u \perp v$ iff $0 \leqslant \mathrm{tr}(uv) \leqslant 1$

  where u and v are self-adjoint operators on a finite dimensional Hilbert space

  "One of the wild hopes that this suggests is the possibility of a direct connection with quantum mechanics... but let's not dream too much!", (Girard, "Linear Logic", 1987)

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- geometry of interaction (Girard, 1989) — $u \perp v$ iff uv is nilpotent

  ie $(uv)^n = 0$ for some n, where u and v are operators or matrices

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- probabilistic coherent spaces (Girard, 2004) — $u \perp v$ iff $\sum_{a \in X} u_x v_x \leqslant 1$
- quantum coherent spaces (Girard, 2004) — $u \perp v$ iff $0 \leqslant \mathrm{tr}(uv) \leqslant 1$
- geometry of interaction (Girard, 1989) — $u \perp v$ iff uv is nilpotent
- ludics (Girard, 2001) — $u \perp v$ iff interaction of uv "goes well" where u and v are abstract proof / terms

Old Fashioned Coherent Spaces

The original presentation of coherent spaces uses (reflexive) graphs.

Definition

- *A coherent space over X is a reflexive graph,* *$a \frown b$ means that a and b are related*
- *a coherent set, or clique is a complete subgraph,*
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- Don't forget to check the transformations are inverse to each other.

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In particular

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3. "&" and "$\oplus$" coincide for finitary sets.

With that in mind, the following is surprising

Theorem

There is a canonical "inclusion" of Coh *into* Fin *that preserves the linear structure.*

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$\mathcal{F}(C)$ is a finiteness space over X.

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Quite surprisingly, we have

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  - $x \cap y \subset y \in \mathcal{F}(C)$: doesn't contain infinite anticliques,
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  - $x \cap y \subset x \in C^{\perp_f}$: doesn't contain infinite cliques,

  by Ramsey's theorem, $x \cap y$ is finite and thus $x \in C^{\perp_c\perp_f\perp_f}$. We have $C^{\perp_f} \subset C^{\perp_c\perp_f\perp_f}$.

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For $C \otimes D$, we need to show "r contains an infinite anticlique iff $\pi_1(r)$ or $\pi_2(r)$ contain an infinite anticlique".

- Suppose $\pi_1(r)$ contains an infinite anticlique A. For each $a \in A$, take b such that $(a, b) \in A$.

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  The set of those (a, b) is an infinite anticlique of $C \otimes D$.

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  The set of those (a, b) is an infinite anticlique of $C \otimes D$.
- If r contains an infinite anticlique A, then at least one of $\pi_i(A)$ is infinite.
  Suppose $\pi_1(A)$ is infinite, but doesn't contain an infinite anticlique.

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  Suppose $\pi_1(A)$ is infinite, but doesn't contain an infinite anticlique.
  By Ramsey's theorem, it contains an infinite clique C. For each $a \in C$, take b such that $(a, b) \in A$.

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  The set of those (a, b) is an infinite anticlique of $C \otimes D$.
- If r contains an infinite anticlique A, then at least one of $\pi_i(A)$ is infinite.
  Suppose $\pi_1(A)$ is infinite, but doesn't contain an infinite anticlique.
  By Ramsey's theorem, it contains an infinite clique C. For each $a \in C$, take b such that $(a, b) \in A$.
  The set of those b is an infinite anticlique in $\pi_2(A) \subset \pi_2(r)$.

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Unfortunately

Lemma

This doesn't extend to the exponentials.